Math. Nachr. 170 (1994) 183–200

Algebraic Classification and Obstructions to Embedding of Strongly Pseudoconvex Compact 3-dimensional CR Manifolds in $\mathbb{C}^3$

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Dedicated to Professor F. HIRZEBRUCH *on his sixty-fifth birthday*

(Received August 3, 1993)

0. Introduction

It is well-known that the problem of complete classification of compact strongly pseudoconvex CR manifolds is in general quite difficult. For example, let $X_{a,b}$ be the ellipsoid

$$\sum_{i=1}^{n} \frac{x_i^2}{a_i^2} + \frac{y_i^2}{b_i^2} = 1 \text{ in } \mathbb{C}^n$$

where $\{z_i = x_i + \sqrt{-1}y_i\}$ is the coordinate system of $\mathbb{C}^n$. WEBSTER [We] showed that $X_{a,b}$ and $X_{a,a}$ are CR isomorphic if and only if $a = b$ except for some trivial cases. On the other hand, for any compact strongly pseudoconvex CR manifold X of dimension $2n - 1$ in $\mathbb{C}^N$, in view of a result of HARVEY-LAWSON [Ha-La], X is the boundary of a complex variety $V \subseteq \mathbb{C}^n$ with only isolated singularities at Y. Following the point of view of [Lu-Ya], which was presented by the second author in the A.M.S. Summer Institute on Several Complex Variables in 1990, we are interested in the following notion of equivalence of CR manifolds.

Definition 0.1. Let X_1, X_2 be two connected compact strongly pseudoconvex embeddable CR manifolds of dimension $2n - 1$. We say that X_1 and X_2 are algebraically equivalent if there exist CR embeddings of X_1 and X_2 into $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively such that the corresponding varieties V_1 and V_2, which are bounded by X_1 and X_2 in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively, have isomorphic singularities Y_1 and Y_2, i.e., $(V_1, Y_1) \cong (V_2, Y_2)$ as germs of varieties.

Obviously, in order to understand the classification problem of CR manifolds, a first step is to understand the classification problem of CR manifolds up to algebraic equivalence. In 1974, BOUTET DE MONVEL [Bo] (cf. [Kol] also) proved that if X is a compact C^∞ strongly pseudoconvex CR manifold of dimension $2n - 1$ and $n \geq 3$, then M is CR embeddable in $\mathbb{C}^N$. H. GRAUERT has constructed compact 3-dimensional strongly pseudoconvex CR manifolds which are not embeddable. Such examples were also studied by H. ROSSI [Ro] and D. BURNS [Bu]. In this paper, we shall only consider connected compact 3-dimensional embeddable strongly pseudoconvex CR manifolds. We shall study the topology and analytic structure of these manifolds up to algebraic equivalence. It can be shown that the embeddable CR manifolds considered in BURNS-EPSTEIN [Bu-Ep] (i.e., those CR manifolds obtained by deforming the CR structure of the standard sphere in $\mathbb{C}^2$) are rational CR manifolds (cf. Definition 4.1). We conjecture that if X is a Burns-Epstein CR manifold, then the CR invariant $m_Z(X)$ (cf. Proposition 2.7) is actually zero. In this paper, we introduce numerical CR invariants: $m_Z(X)$, $p_f(X)$, $p_a(X)$, $p_g(X)$, q(X), $\chi(X)$, $\omega(X)$ and Γ_X for strongly pseudoconvex compact 3-dimensional CR manifolds X. It is an important open problem to find intrinsic definitions for them. One significant application of this theory is that it provides us with obstructions to embedding strongly pseudoconvex compact 3-dimensional CR manifolds in $\mathbb{C}^3$ (cf. Theorem 5.1 – Theorem 5.4). Theorem 5.1 – Theorem 5.4 are sharp in the sense that there are numerous examples of compact strongly pseudoconvex CR manifolds which are not embeddable in $\mathbb{C}^3$ and they do not satisfy the conditions stated in Theorem 5.1 – Theorem 5.4.

In § 1 we recall some basic notations and prove that CR equivalence implies algebraic equivalence. In § 2 we introduce the notion of topological algebraic equivalence. We define Γ_X which gives a complete classification of all topological algebraic equivalence classes. We also introduce some numerical invariants $m_Z(X)$, $p_f(X)$ and $p_a(X)$. $p_f(X)$ and $p_a(X)$ depend only on the topological algebraic equivalence class of X while $m_Z(X)$ depends on the algebraic equivalence class of X. In § 3, we introduce more numerical invariants $p_g(X)$, $\chi(X)$, $\omega(X)$ and $q(X)$ under algebraic equivalence. We show that $0 \leq p_f(X) \leq p_a(X) \leq p_g(X)$. In case X admits a transversal holomorphic S^1-action, we show that $q(X) \leq p_g(X)$. In § 4 we give the topological algebraic classification of strongly pseudoconvex 3-dimensional CR manifolds embeddable in $\mathbb{C}^3$ with $p_a = 0$, $p_a = 1$ and $p_g = 1$ respectively. In § 5, we point out that the classification results in § 4 give obstructions to embedding CR manifolds in $\mathbb{C}^3$. In Theorem 5.4, numerical obstruction is obtained to embedding CR manifolds in $\mathbb{C}^3$.

1. Preliminary

In this section, we shall recall some basic materials that will be needed for later discussion. We show that the CR isomorphic equivalence class of a strongly pseudoconvex compact connected embeddable manifold is smaller than the algebraic equivalence class of the manifold.

Definition 1.1. Let X be a compact, connected orientable real manifold of dimension $2n - 1$. A CR *strucuture on* X is an $(n - 1)$-dimensional subbundle S of $\mathbb{C}TX$ such that

(1) $S \cap \bar{S} = \{0\}$.

(2) If L, L' are local sections of S, then so is $[L, L']$.

Definition 1.2. Let $L_1, \ldots, L_{n-1}$ be a local basis for sections of S over an open subset of X so that $\bar{L}_1, \ldots, \bar{L}_{n-1}$ is a local basis for sections of $\bar{S}$. Since $S \oplus \bar{S}$ has complex codimension one in $\mathbb{C}TX$, we may choose a local section N of $\mathbb{C}TX$ such that $L_1, \ldots, L_{n-1}, \bar{L}_1, \ldots, \bar{L}_{n-1}$,

N span $\mathbb{C}TX$. We may assume that N is purely imaginary. Then the matrix (c_{ij}) defined by

$$[L, \overline{L}_j] = \sum a_{ij}^k L_k + \sum b_{ij}^k \overline{L}_k + c_{ij} N$$

is Hermitian and is called the *Levi form.*

The Levi form is noninvariant; however its essential features are invariant. (cf. Fo-Ko])

Proposition 1.3. *The number of nonzero eigenvalues and the absolute value of the signature of (c_{ij}) at each point are independent of the choice of $L_1, \ldots, L_{n-1}, N$.*

Definition 1.4. Let X be a CR manifold. Then X is *strongly pseudoconvex* if the Hermitian matrix (c_{ij}) obtained in Definition 1.2. is always nonsingular and its eigenvalues are of the same sign.

Proposition 1.5. *Let X_1 and X_2 be two strongly pseudoconvex compact connected CR manifolds in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively. If X_1 is Cr equivalent to X_2, then X_1 is algebraically equivalent to X_2.*

Proof. By a result of HARVEY and LAWSON [Ha-La], X_1 and X_2 are boundaries of complex analytic subvarieties V_1 and V_2 in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively. By replacing V_1 and V_2 by their normalizations, we may assume without loss of generality that V_1 and V_2 are normal varieties. We need to show that V_1 is biholomorphic to V_2. Let φ_1 be the CR isomorphism from the boundary X_1 of V_1 to the boundary X_2 of V_2. By the strong pseudoconvexity of $X_1 = \partial V_1$ and the normality of V_1, it is easy to see that φ_1 extends to a holomorphic map $\overline{\varphi} : V_1 \to \mathbb{C}^{N_2}$. Clearly $\overline{\varphi}_1(V_1)$ and V_2 are both complex varieties in $\mathbb{C}^{N_2}$ which have the same boundary. By the uniqueness of the complex Plateau problem (see [Ha-La]), we see that $\overline{\varphi}_1(V_1) = V_2$. Similarly let φ_2 be the inverse mapping of φ_1 which is a CR isomorphism from X_2 to X_1. The same argument as before shows that φ_2 extends to a holomorphic map $\overline{\varphi}_2 : V_2 \to \mathbb{C}^{N_1}$ such that $\overline{\varphi}_2(V_2) = V_1$. $\overline{\varphi}_2 \circ \overline{\varphi}_1 : V_1 \to V_1$ is a holomorphic mapping which extends the identity map $\mathrm{Id} : \partial V_1 \to \partial V_1$. By the uniqueness of the extension, we conclude that $\overline{\varphi}_2 \circ \overline{\varphi}_1 : V_1 \to V_1$ is an identity map. Similarly, $\overline{\varphi}_1 \circ \overline{\varphi}_2 : V_2 \to V_2$ is an identity map. Q.E.D.

2. Topology of 3-dimensional CR manifolds up to algebraic equivalence

Definition 2.0 Let X_1, X_2 be two compact connected strongly pseudoconvex embeddable CR manifolds of dimension $2n - 1$. We say that X_1 and X_2 are topologically algebraic equivalent or have the same topology up to algebraic equivalence if there exist CR embeddings of X_1 and X_2 into $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively such that the corresponding varieties V_1 and V_2, which are bounded by X_1 and X_2 in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively, have topologically isomorphic singularities Y_1 and Y_2, i.e., $(V_1, Y_1) \cong (V_2, Y_2)$ topologically as germs of varieties.

In this section, we shall discuss the topology of compact connected 3-dimensional CR manifolds up to algebraic equivalence.

Remark 2.1.

(a) Obviously two strongly pseudoconvex embeddable CR manifolds are topologically algebraic equivalent if they are algebraic equivalent.

(b) However, it is an open problem whether two homeomorphic strongly pseudoconvex embeddable CR manifolds have the same topology up to algebraic equivalence.

In what follows, we shall introduce a weighted graph Γ to a compact connected 3-dimensional CR manifold X. This graph Γ determines the topological algebraic equivalence class of X. Conversely Γ is constant on the topological algebraic equivalence class of X except in two special cases.

Definition 2.2. Let X be a connected compact strongly pseudoconvex CR manifold of real dimension 3 which is embeddable in $\mathbb{C}^n$. Let V be the subvariety in $\mathbb{C}^n$ such that the boundary of V is X in the C^∞ sense. Then V has isolated singularities at $Y = \{p_1, \ldots, p_m\}$. Let $\pi: M \to V$ be a resolution of singularities of V such that the exceptional set $A = \pi^{-1}(Y)$ has normal crossing, i.e., irredicible components A_i of A are nonsingular, they intersect transversely and no three meet at a point. The topological nature of the embedding of the exceptional set A in M is described by the weighted dual graph Γ_M. The vertices of Γ_M correspond to the A_i's. The edges of Γ_M connecting the vertices corresponding to A_i and A_j, $i \neq j$, correspond to the points of $A_i \cap A_j$. Finally, associated to each A_i is its genus g_i as a Riemann surface, and its weight $A_i \cdot A_i$, the topological self intersection number. Among all the resolutions of V such that the exceptional sets have normal crossings, there is a unique minimal one M_0, which is called the *minimal good resolution*. Any resolution M of V with normal crossing exceptional set is obtained by applying quadratic transformations successively on M_0. The graph Γ_X of the CR manifold X is defined to be Γ_{M_0}.

Remark 2.3. It is well known that Γ_X is characterized by the following condition: Vertex with genus 0 and self intersection number -1 must connect with other three distinct vertices (cf. La1]).

Theorem 2.4. *Let X_1 and X_2 be strongly pseudoconvex compact connected embeddable CR manifolds of dimension 3. Then*

(a) $\Gamma_{X_1} = \Gamma_{X_2}$ *implies that X_1 is topologically algebraic equivalent to X_2.*

(b) *If X_1 is algebraically equivalent to X_2, then $\Gamma_{X_1} = \Gamma_{X_2}$. In fact, if X_1 is topologically algebraic equivalent to X_2, then $\Gamma_{X_1} = \Gamma_{X_2}$ except for the following two cases:*

Case (i). *Both Γ_{X_1} and Γ_{X_2} are exactly those of the form below with all a_i equal to or smaller than -2. The genus of each vertex is zero.*

Case (ii). *Both Γ_{X_1} and Γ_{X_2} are exactly those of the form below with all a_i equal to or smaller than -2 and one a_i equal to or smaller than -3. The genus of each vertex is zero.*

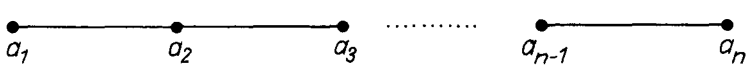

Abb. 1

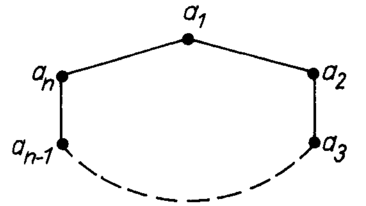

Abb. 2

Proof. We shall use the notation in Definition 2.2. Let $\tilde{X}_i$ be the boundary of a strongly pseudoconvex tubular neighborhood of Y_i in V_i, $i = 1, 2$. By taking the normalization if necassary, we shall assume that both V_i are normal analytic varieties. Suppose $\Gamma_{X_1} = \Gamma_{X_2}$. Then by a result of MUMFORD [Mu], $\tilde{X}_1$ is homeomorphically equivalent to $\tilde{X}_2$. It follows that (V_1, Y_1) is homeomorphically equivalent to (V_2, Y_2) as germs of varieties in view of a result of MILNOR [Mi]. So statement (a) follows.

If X_1 is algebraically equivalent to X_2, then by the proof of Proposition 1.5, one sees easily that (V_1, Y_1) is analytically equivalent to (V_2, Y_2) as germs of varieties. It follows that $\Gamma_{X_1} = \Gamma_{X_2}$. On the other hand, if we only assume that X_1 is topologically algebraic equivalent to X_2, then (V_1, Y_1) is homeomorphically equivalent to (V_2, Y_2) as germs of varieties, i.e., $\tilde{X}_1$ is homeomorphically equivalent to $\tilde{X}_2$. So the last part of (b) is a consequence of a theorem of NEUMANN [Ne]. Q.E.D.

We shall introduce some numerical invariants which will be useful in studying the classification of strongly pseudoconvex 3-dimensional CR manifolds up to algebraic equivalence.

Definition 2.5. With the notation in Definition 2.2, let $(A_i \cdot A_j)$ be the intersection matrix of the exceptional set. By a theorem of GRAUERT [Gr] and MUMFORD [Mu], this matrix $(A_i \cdot A_j)$ is negative definite. It follows that there exists a unique minimal positive divisor Z on M with support on A such that $Z \cdot A_i \leq 0$ for every irreducible component A_i of A. (For the existence of Z, see p. 131 of [Ar]; for explicit computation of Z, see [La 2, 3].) We define the first numerical invariant $m_Z(M) = Z \cdot Z$. For any positive divisor D with support contained in A, let $\mathcal{O}_D = \mathcal{O}_M/\mathcal{O}_M(-D)$ and $\chi(\mathcal{O}_D)$ be the Euler characteristic $\sum_{i=0}^{2} (-1)^i \dim H^i(M, \mathcal{O}_D)$. We define the *fundamental genus of* X to be $p_f(M) = 1 - \chi(\mathcal{O}_Z)$ and the *arithmetic genus of* X to be $p_a(M) = \sup_D (1 - \chi(\mathcal{O}_D))$ where D ranges over all positive divisors with support contained in A.

Remark 2.6. For any positive divisor $\mathcal{D}$ with support contained in A

$$p_D(M) = 1 - \chi(\mathcal{O}_D)$$

will be called the arithmetic genus of $\mathcal{D}$. It can be computed via Riemann-Roch theorem by the formula

$$p_D(M) = \tfrac{1}{2}(D^2 + D \cdot K) + 1,$$

where K is the canonical divisor on M.

Proposition 2.7. *The above definitions of $m_Z(M)$, $p_f(M)$ and $p_a(M)$ depend only on the CR manifold X.* Henceforth, these invariants will be denoted by $m_Z(X)$, $p_f(X)$ *and* $p_a(X)$ *respectively. In fact, if X_1 is topologically algebraic equivalent to X_2, then $p_f(X_1) = p_f(X_2)$ and $p_a(X_1) = p_a(X_2)$.*

Proof. We shall first prove that the definitions of $m_Z(M)$, $p_f(M)$ and $p_a(M)$ are independent of the choice of M. By HIRONAKA [Hi], given two resolutions of V, we can find a third resolution of V which dominates both of these two resolutions. In view of Theorem 5.7 of [La1] which says that any birational morphism between connected 2-dimensional manifolds is obtained by a finite number of iterated quadratic transformations at points, we only need to prove

$$m_Z(M) = m_{\tilde{Z}}(\overline{M}), \qquad p_f(M) = p_f(\overline{M}), \quad \textit{and} \quad p_a(M) = p_a(\tilde{M}),$$

where $\sigma : \tilde{M} \to M$ is obtained by blowing up one point q in the exceptional set A of M. There are two cases to consider.

Case 1. The center q of the blowing up is in the smooth part of A. Without loss of generality, we shall assume that q is in A_1 and not in the other irreducible components of A. Denote

$$\tilde{A}_0 = \sigma^{-1}(q) \quad \text{and} \quad \tilde{A}_i = \text{proper transform of } A_i.$$

Then $\tilde{A}_0 \cdot \tilde{A}_0 = -1$, $\tilde{g}_0 = 0$, $\tilde{A}_1 \cdot \tilde{A}_1 = A_1 \cdot A_1 - 1$ and $\tilde{A}_i \cdot \tilde{A}_i = A_i \cdot A_i$ for $i \geq 2$. Let

$$\tilde{Z} = \sum_{i \geq 0} \tilde{z}_i \tilde{A}_i \quad \text{and} \quad Z = \sum_{i \geq 1} z_i A_i$$

be the fundamental cycles on $\tilde{M}$ and M respectively. Observe that

$$\left(\sum_{i \geq 1} \tilde{z}_i A_i\right) \cdot A_j = \tilde{Z} \cdot \tilde{A}_j \leq 0 \quad \text{for} \quad j \geq 2$$

and

$$\begin{aligned}\left(\sum_{i \geq 1} \tilde{z}_i A_i\right) \cdot A_1 &= \left(\sum_{i \geq 2} \tilde{z}_i \tilde{A}_i\right) \cdot \tilde{A}_1 + \tilde{z}_1 A_1 \cdot A_1 \\ &= \left(\sum_{i \geq 2} \tilde{z}_i \tilde{A}_i\right) \cdot \tilde{A}_1 + \tilde{z}_1(\tilde{A}_1 \cdot \tilde{A} + 1) = \left(\sum_{i \geq 1} \tilde{z}_i A_i\right) \cdot \tilde{A}_1 + \tilde{z}_1 \\ &= \left(\sum_{i \geq 1} \tilde{z}_i A_i\right) \cdot \tilde{A}_1 - \tilde{z}_0 + \tilde{z}_1 = \tilde{Z} \cdot \tilde{A}_1 + \tilde{Z} \cdot \tilde{A}_0 \leq 0 \,.\end{aligned}$$

By the definition of the fundamental cycle, we conclude that

$$\sum_{i \geq 1} \tilde{z}_i A_i \geq Z, \text{ i.e., } \tilde{z}_i \geq z_i \quad \text{for all} \quad i \geq 1 \,.$$

On the other hand, consider the cycle $\sum_{i \geq 1} \tilde{z}_i \tilde{A}_i + z_1 \tilde{A}_0$. We have

$$\left(\sum_{i \geq 1} z_i \tilde{A}_i + z_1 \tilde{A}_0\right) \cdot \tilde{A}_j = Z \cdot A_j \leq 0 \quad \text{for} \quad j \geq 2 \,,$$

$$\Big(\sum_{i\geq 1} z_i\tilde{A}_i + z_1\tilde{A}_0\Big)\cdot\tilde{A}_1 = \Big(\sum_{i\geq 2} z_iA_i\Big)\cdot A_1 + z_1\tilde{A}_1\cdot\tilde{A}_1 + z_1$$
$$= \Big(\sum_{i\geq 2} z_iA_i\Big)\cdot A_1 + z_1(A_1\cdot A_1 - 1) + z_1$$
$$= Z\cdot A_1 \leq 0\,,$$
$$\Big(\sum_{i\geq 1} z_i\tilde{A}_i + z_1\tilde{A}_0\Big)\cdot\tilde{A}_0 = z_1\tilde{A}_1\cdot\tilde{A}_0 + z_1\tilde{A}_0\cdot\tilde{A}_0 = z_1 - z_1 = 0\,.$$

Again, by the minimality of the fundamental cycle, we deduce that

$$\sum_{i\geq 1} z_i\tilde{A}_i + z_1\tilde{A}_0 \geq \tilde{Z}.$$

It follows easily that

$$\tilde{Z} = \sum_{i\geq 1} z_i\tilde{A}_i + z_1\tilde{A}_0 = \sigma^*Z\,.$$

Therefore

$$\tilde{Z}\cdot\tilde{Z} = \Big(\sum_{i\geq 1} z_i\tilde{A}_i\Big)^2 + 2z_1\tilde{A}_0\cdot\Big(\sum_{i\geq 1} z_i\tilde{A}_i\Big) + z_1^2\tilde{A}_0\cdot\tilde{A}_0$$
$$= \Big(\sum_{i\geq 2} z_i\tilde{A}_i\Big)^2 + 2z_1\tilde{A}_1\cdot\Big(\sum_{i\geq 2} z_i\tilde{A}_i\Big) + z_1^2\tilde{A}_1\cdot\tilde{A}_1 + 2z_1^2 - z_1^2$$
$$= \Big(\sum_{i\geq 2} z_iA_i\Big)^2 + 2z_1A_1\cdot\Big(\sum_{i\geq 2} z_iA_i\Big) + z_1^2(A_1\cdot A_1 - 1) + z_1^2$$
$$= \Big(\sum_{i\geq 2} z_iA_i\Big)^2 + 2z_1A_1\cdot\Big(\sum_{i\geq 2} z_iA_i\Big) + z_1^2A_1\cdot A_1$$
$$= \Big(\sum_{i\geq 1} z_iA_i\Big)^2 = Z\cdot Z\,.$$

Observe that we have shown $m_{\tilde{Z}}(\overline{M}) = m_Z(M)$. Let K and $\tilde{K}$ be the canonical divisors on M and $\tilde{M}$ respectively. Then

$$\tilde{K} = \sigma^*K + \tilde{A}_0$$

and

$$\tilde{Z}\cdot\tilde{K} = (\sigma^*Z)\cdot(\sigma^*K + \tilde{A}_0) = Z\cdot K + \sigma^*Z\cdot\tilde{A}_0 = Z\cdot K\,.$$

Therefore

$$p_f(M) = 1 - \chi(\mathcal{O}_Z) = \tfrac{1}{2}(Z^2 + Z\cdot K) + 1$$
$$= \tfrac{1}{2}(\tilde{Z}^2 + \tilde{Z}\cdot\tilde{K}) + 1 = p_f(\tilde{M})\,.$$

For any positive $D = \sum_{i\geq 1} d_iA_i$ supported in A, consider

$$\sigma^*D = d_1\tilde{A}_0 + \sum_{i\geq 1} d_i\tilde{A}_i\,.$$

Clearly $\tilde{A}_0\cdot\sigma^*D = 0$. Hence

$$\sigma^*D\cdot\tilde{K} = \sigma^*D\cdot(\sigma^*K + \tilde{A}_0) = D\cdot K$$

and

$$p_D(M) = \tfrac{1}{2}(D^2 + D\cdot K) + 1 = \tfrac{1}{2}((\sigma^*D)^2 + \sigma^*D\cdot\tilde{K}) + 1 = p_{\tilde{D}}(\tilde{M})\,.$$

Since this is true for any positive D supported in A, it follows that $p_a(M) \leq p_a(\tilde{M})$. Conversely for any positive divisor

$$\tilde{D} = \sum_{i \geq 0} \tilde{d}_i \tilde{A}_i \text{ support in } \tilde{A},$$

consider $D = \sum_{i \geq 1} \tilde{d}_i \mathrm{A}_i$. We have

$$\begin{aligned}
\tilde{D}^2 &= \tilde{d}_0^2 \tilde{A}_0^2 + 2\tilde{d}_0 \tilde{A}_0 \cdot \left[\sum_{i \geq 1} \tilde{d}_i \tilde{A}_i\right] + \left[\sum_{i \geq 1} \tilde{d}_i \tilde{A}_i\right]^2 \\
&= -\tilde{d}_0^2 + 2\tilde{d}_0 \tilde{d}_1 + \tilde{d}_1^2 \tilde{A}_1^2 + 2\tilde{d}_1 \tilde{A}_1 \cdot \left[\sum_{i \geq 2} \tilde{d}_i \tilde{A}_i\right] + \left[\sum_{i \geq 2} \tilde{d}_i \tilde{A}_i\right]^2 \\
&= -\tilde{d}_0^2 + 2\tilde{d}_0 \tilde{d}_1 - \tilde{d}_1^2 + \tilde{d}_1^2 A_1^2 + 2\tilde{d}_1 A_1 \cdot \left[\sum_{i \geq 2} \tilde{d}_i A_i\right] + \left[\sum_{i \geq 2} \tilde{d}_i A_i\right]^2 \\
&= -(\tilde{d}_0 - \tilde{d}_1)^2 + \left[\sum_{i \geq 1} \tilde{d}_i A_i\right]^2 = -(\tilde{d}_0 - \tilde{d}_1)^2 + D^2,
\end{aligned}$$

$$\begin{aligned}
\tilde{D} \cdot \tilde{K} &= \left[\tilde{d}_0 \tilde{A}_0 + \sum_{i \geq 1} \tilde{d}_i \tilde{A}_i\right] \cdot (\sigma^* K + \tilde{A}_0) \\
&= [(\tilde{d}_0 - \tilde{d}_1)\tilde{A}_0 + \sigma^* D] \cdot (\sigma^* K + \tilde{A}_0) \\
&= -(\tilde{d}_0 - \tilde{d}_1) + D \cdot K.
\end{aligned}$$

Hence

$$\begin{aligned}
p_{\tilde{D}}(\overline{M}) &= \tfrac{1}{2}(\tilde{D}^2 + \tilde{D} \cdot \tilde{K}) + 1 \\
&= \tfrac{1}{2}[-(\tilde{d}_0 - \tilde{d}_1)^2 + D^2 - (\tilde{d}_0 - \tilde{d}_1) + D \cdot K] + 1 \\
&= -\tfrac{1}{2}[(\tilde{d}_0 - \tilde{d}_1)^2 + (\tilde{d}_0 - \tilde{d}_1)] + \tfrac{1}{2}(D^2 + D \cdot K) + 1 \leq p_D(M).
\end{aligned}$$

Since this is true for any positive $\tilde{D}$ supported in $\tilde{A}$, we have $p_a(M) \leq p_a(M)$. So we have proven $p_a(M) = p_a(M)$.

Case 2. The center q of the blow up is the singular part of A. Without loss of generality, we shall assume that q is in A_1 and A_2 and not in the other irreducible components of A. Using the same notation as above, we have

$$\tilde{A}_0 \cdot \tilde{A}_0 = -1, \qquad g_0 = 0, \qquad \tilde{A}_1 \cdot \tilde{A}_1 = A_1 \cdot A_1 - 1,$$

$$\tilde{A}_2 \cdot \tilde{A}_2 = A_2 \cdot A_2 - 1 \quad \text{and} \quad \tilde{A}_i \cdot A_i = A_i \cdot \tilde{A}_i \quad \text{for} \quad i \geq 3.$$

We see easily as before that

$$\tilde{Z} = \sigma^* Z = \sum_{i \geq 0} \tilde{z}_i \tilde{A}_i \quad \text{where} \quad \tilde{z}_i = z_i \quad \text{for} \quad i \geq 1 \quad \text{and} \quad \tilde{z}_0 = \tilde{z}_1 + \tilde{z}_2,$$

$$\tilde{K} = \sigma^* K + \tilde{A}_0,$$

and hence $m_Z(M) = m_{\tilde{Z}}(\tilde{M})$, $p_f(M) = p_f(\tilde{M})$. The proof that $p_a(M) = p_a(\tilde{M})$ is also similar to Case 1.

We next show that $m_Z(M)$, $p_f(M)$ and $p_a(M)$ are CR invariants. Let X' be another CR manifold in $\mathbb{C}^{n'}$ which is CR isomorphic to X. Let V and V' be the subvarieties in $\mathbb{C}^n$ and $\mathbb{C}^{n'}$ such that the boundaries of V and V' are X and X' respectively. Notice that V and V' may not be normal. Let $\tilde{V}$ and $\tilde{V}'$ be the normalizations of V and V' respectively. In view of the proof of Proposition 1.5, we known that $\tilde{V}$ is biholomorphically equivalent to $\tilde{V}'$. By a theorem of HIRONAKA [Hi], we can find a resolution $\overline{M}$ with normal crossings which dominates $\tilde{V}$ and $\tilde{V}'$. By the first part of the proof, this shows that $m_Z(M)$, $p_f(M)$ and $p_a(M)$ are CR invariants.

Finally, to prove that the invariants $p_f(X)$ and $p_a(X)$ are determined by the topology of X up to algebraic equivalence, we first observe that they depend only on the graph Γ_X. Indeed, recalling the formula for $p_D(M) = 1 - \chi(\mathcal{O}_D)$ in Remark 2.6, we have

$$p_{D_1+D_2}(M) = p_{D_1}(M) + p_{D_2}(M) + D_1 \cdot D_2 - 1 .$$

In view of the above formula, we see that $p_f(M)$ and $p_a(M)$ are computable from Γ_X. We observe that $p_f(X) = 0$ and $p_a(X) = 0$ if Γ_X is of the form in Case b(i) of Theorem 2.4 while $p_f(X) = 1$ and $p_a(X) = 1$ if Γ_X is of the form in Case b(ii) of Theorem 2.4. Therefore $p_f(X)$ and $p_a(X)$ are determined by the topological algebraic equivalence class of X.

Q.E.D.

Corollary 2.8. *If X_1 is algebraically equivalent to X_2, then $m_Z(X_1) = m_Z(X_2)$. In fact, as long as Γ_{X_1} and Γ_{X_2} are not of the form in Case b(i) and Cace b(ii) of Theorem 2.4, then $m_Z(X_1) = m_Z(X_2)$ if X_1 is topologically algebraic equivalent to X_2.*

Proof. This is easily seen from the proof of Proposition 2.7. Q.E.D.

3. Numerical invariants under algebraic equivalence

Definition 3.0. Let X be a connected compact strongly pseudoconvex embeddable CR manifold of real dimension 3. With the notation in Definition 2.2, let $p_g(M)$, $\chi(M)$ and $\omega(M)$ be $\dim H^1(M, \mathcal{O})$, $\chi_T(A) + K \cdot K$ and $\dim H^1(M, \Omega^1) + K \cdot K$ respectively, where $\chi_T(A)$ is the topological Euler characteristic of A, Ω^1 is the sheaf of germs of holomorphic 1-forms on M and K is the divisor supported in A with coefficients in rational numbers such that

$$A_i \cdot K = -A_i \cdot A_i + 2g_i - 2 \quad \text{for all} \quad A_i \subseteq A .$$

Proposition 3.1. *$p_g(M)$, $\chi(M)$ and $\omega(M)$ depend only on the CR manifold X. Henceforth, these invariants will be denoted by $p_g(X)$, $\chi(X)$ and $\omega(X)$ respectively. In fact, if X_1 is algebraically equivalent to X_2, then*

$$p_g(X_1) = p_g(X_2), \chi(X_1) = \chi(X_2) \quad \text{and} \quad \omega(X_1) = \omega(X_2) .$$

Proof. See Theorem 2.2. and Theorem 2.5 of [Lu-Ya]. Q.E.D.

Remark 3.2. We shall call $p_g(X)$ the geometric genus of the CR manifold X.

Proposition 3.3. *Let X be a connected compact strongly pseudoconvex CR manifold of real dimension 3 which is embeddable in $\mathbb{C}^n$. Let V be the subvariety in $\mathbb{C}^n$ such that the* *boundary of V is X and V has isolated singularities at* $Y = \{p_1, \ldots, p_m\}$. *Let* $\pi : M \to V$ *be a resolution of singularities of V such that the exceptional sets* $A_i = \pi^{-1}(p_i)$ *has normal crossing for all* $1 \le i \le m$. *Let* U_i *be a stronly pseudoconvex neighborhood of* p_i, $1 \le i \le m$. *Then*

(i) $$p_f(X) = \sum_{i=1}^{m} p_f(\pi^{-1}(U_i)),$$

(ii) $$p_a(X) = \sum_{i=1}^{m} p_a(\pi^{-1}(U_i)) \quad \text{and}$$

and

(iii) $$p_g(X) = \sum_{i=1}^{m} p_g(\pi^{-1}(U_i)).$$

Proof. (i) and (ii) are obvious because $p_f(X)$ and $p_a(X)$ depend only on Γ_X. (iii) is a consequence of Lemma 5.3 of [La1]. Q.E.D.

We remark that $p_a(\pi^{-1}(U_i))$ and $p_g(\pi^{-1}(U_i))$ are also called the arithmetic genus and geometric genus of the singularity, (V, p_i) and are denoted by $p_a(V, p_i)$ and $p_g(V, p_i)$ respectively.

Proposition 3.4. *Let X be a connected compact strongly pseudoconvex embeddable CR manifold of real dimension 3. Then*

$$0 \le p_f(X) \le p_a(X) \le p_g(X).$$

Proof. We shall use the notations in Definition 2.2 and Definition 2.5. In view of Proposition 3.3, we may assume without loss of generality that A is connected. Following LAUFER [La 2, 3], we may compute the fundamental cycle Z via a computation sequence for Z.

$$Z_0 = 0, \quad Z_1 = A_{i_1}, \quad Z_2 = Z_1 + A_{i_2}, \ldots, \tag{3.1}$$

$$Z_j = Z_{j-1} + A_{i_j}, \ldots, \quad Z_\ell = Z_{\ell-1} + A_{i_j} = Z,$$

where A_{i_1} is arbitrary and $A_{i_j} \cdot Z_{j-1} > 0$, $1 < j \le \ell$. $\mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j)$ represents the sheaf of germs of sections of a line bundle over A_{i_j} of Chern class $-A_{i_j} \cdot Z_{j-1}$. So

$$H^0(M, \mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j)) = 0 \quad \text{for} \quad j > 1.$$

$$O \to \mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j) \to \mathcal{O}_{Z_j} \to \mathcal{O}_{Z_{j-1}} \to O \tag{3.2}$$

is an exact sheaf sequence. From the long exact cohomology sequence for (3.2), it follows by induction that

$$H^0(M, \mathcal{O}_{Z_k}) = \mathbb{C}, \qquad 1 \le k \le \ell. \tag{3.3}$$

Since M is complex two-dimensional and not compact, $H^2(M, \mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ on M. So

$$\chi(\mathcal{O}_Z) = \dim H^0(M, \mathcal{O}_Z) - \dim H^1(M, \mathcal{O}_Z) \le 1.$$

It follows that

$$p_f(X) = 1 - \chi(\mathcal{O}_Z) \ge 0.$$

Clearly we have $p_a(X) \ge p_f(X)$ by the definition of $p_a(X)$.

Finally to prove that $p_g(X) \geq p_a(X)$, it suffices to prove that

$$\dim H^1(M, \mathcal{O}) \geq 1 - \dim H^0(M, \mathcal{O}_D) + \dim H^1(M, \mathcal{O}_D) \tag{3.4}$$

for any positive divisor D supported on A. By the long exact cohomology sequence for

$$O \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to O$$

it follows from $H^2(M, \mathcal{O}(-D)) = 0$ that

$$\dim H^1(M, \mathcal{O}) \geq \dim H^1(M, \mathcal{O}_D). \tag{3.5}$$

(3.4) follows immediately from (3.5) and the fact that $\dim H^0(M, \mathcal{O}_D) \geq 1$. Q.E.D.

Definition 3.5. Let X be a connected compact strongly pseudoconvex embeddable CR manifold of real dimension 3. With the notation in Definition 2.2, let $q(M)$ be

$$\dim H^0(M - A, \Omega^1)/H^0(M, \Omega^1).$$

Proposition 3.6. *The above definition of $q(M)$ depends only on the CR manifold X. Henceforth, this invariant will be denoted by $q(X)$. In fact, if X' is algebraically equivalent to X, then $q(X) = q(X')$.*

Proof. With the notation in Definition 2.2, let $\pi: M \to V_1$ be the blow-down of A in M, $Y_1 = \pi(A)$. Then V_1 is a normal analytic space with Y_1 as singular set which only consists of isolated points. Let $\overline{\Omega}^1_{V_1}$ be the sheaf of germs of holomorphic 1-forms of $V_1 - Y_1$ which are locally L^2 integrable in the sense of GRIFFITHS [Gr]. Then actually $\overline{\Omega}^1_{V_1}$ is equal to the 0-th direct image sheaf $\pi_* \Omega^1_M$ (cf. [La2]). Let $\theta: V_1 - Y_1 \hookrightarrow V_1$ be the inclusion map. Then the 0-the direct image sheaf $\overline{\overline{\Omega}}^1_{V_1} := \theta_* \Omega^1_{V_1 - Y_1}$ is coherent by SIU's theorem [Si]. Clearly we have an inclusion $\overline{\Omega}^1_{V_1} \hookrightarrow \overline{\overline{\Omega}}^1_{V_1}$. It is easy to see that

$$q(M) = \sum_{p \in Y_1} \dim (\overline{\overline{\Omega}}^1_{V_1, p}/\overline{\Omega}^1_{V_1, p}).$$

So $q(M)$ depends only on the CR manifold X.

Let X' be a strongly pseudoconvex CR manifold algebraically equivalent to X. Let $\pi: M' \to V_1'$ be the blow-down of A' in M', $Y_1' = \pi(A')$. In view of the proof of Proposition 1.5, we have $(V_1, Y_1) \cong (V_1', Y_1')$ as germs of analytic varieties. It follows that $q(X) = q(X')$. *Q.E.D.*

Remark 3.7. We call $q(X)$ the irregularity of the CR manifold X.

Proposition 3.8. *With the notation in Proposition 3.3, we have*

$$q(X) = \sum_{i=1}^{m} q(\pi^{-1}(U_i)).$$

Proof. This follows from the proof of Theorem 3.6. Q.E.D.

Definition 3.9. Let X be a CR manifold with structures S as in Definition 1.1. Since $S \cap \overline{S} = \{0\}$, there is a unique subbundle $\mathscr{H}$ of $T(X)$ such that

$$\mathbb{C}\mathscr{H} = S \oplus \overline{S},$$

i.e., $\mathscr{H}$ is the real part of $S \oplus \bar{S}$. Furthermore, there is a unique homomorphism $J: \mathscr{H} \to \mathscr{H}$ such that

$$J^2 = -I, \qquad I = \text{identity}.$$

The pair $(\mathscr{H}, J)$ is called the *real expression of S*.

Definitions 3.10. With the notation in the above definition, a smooth S^1-action on X is said to be *holomorphic* if it preserves the subbundle $\mathscr{H} \subset T(X)$ and commutes with J. It is said to be *transversal* if, in addition, the vector field V which generates the action is transversal to $\mathscr{H}$ at all points of X.

Theorem 3.11. [La-Ya]. *Let X be a strongly pseudoconvex CR manifold of dimension $2n-1>1$, and suppose that X admits a transversal holomorphic S^1-action. Then there exists a holomorphic equivariant embedding $X \hookrightarrow V$ as a hypersurface in an n-dimensional algebraic variety $V \subseteq \mathbb{C}^N$ with a linear $\mathbb{C}^*$-action. V has at most one singular point at the origin.*

Theorem 3.12. *Let X be a strongly pseudoconvex CR manifold of dimension 3, and suppose that X admits a transversal holomorphic S^1-action. Then $q(X) \le p_g(X)$, i.e., irregularity of X is bounded above by its geometric genus.*

Proof. We shall use the notation in Theorem 3.11. Let σ be the $\mathbb{C}^*$-action leaving V invariant, defined by

$$\sigma(t, (z_1, \dots, z_N)) = (t^{\ell_1} z_1, \dots, t^{\ell_N} z_N),$$

where the l_i's are positive integers. Let $\varphi: \mathbb{C}^N \to \mathbb{C}^N$ be defined by

$$\varphi(z_1, \dots, z_N) = (z_1^{\ell_1}, \dots, z_N^{\ell_N})$$

and let $V^* = \varphi^{-1}(V)$ be the cone above V. Then V^* has a natural $\mathbb{C}^*$-action defined by

$$\sigma^*(t, (z_1, \dots, z_N)) = (tz_1, \dots, tz_N)$$

and the induced map $\varphi: V^* \to V$ commutes with the $\mathbb{C}^*$-action. Let

$$A^* = (V^* - \{0\})/\mathbb{C}^* \subseteq \mathbb{P}^{N-1}.$$

Let N^* be the universal subbundle (i.e., dual of hyperplane bundle) of $\mathbb{P}^{N-1}$ restricted to A^*. Identify $\mathbb{Z}_{\ell_i}$ with the group of ℓ_ith roots of 1. $G = \mathbb{Z}_{\ell_1} \oplus \dots \oplus \mathbb{Z}_{\ell_N}$ acts on V^* by coordinatewise multiplication. G also acts on A^* and N^*. Let $\pi: A^{**} \to A^*$ be the normalization and N^{**} be the pull-back of N^* by π. Then in view of Theorem 4.19 and Theorem 4.20 of [Ya], we have the following formulas

$$q(X) = \begin{cases} 0 & \text{if } g^{**} \le 1, \\ \sum_{-\infty}^{-1} \dim H^0(A^{**}, K_{A^{**}} N^{**-n})^G & \text{if } g^{**} \ge 2, \end{cases}$$

$$p_g(X) = \begin{cases} 0 & \text{if } g^{**} = 0, \\ \dim H^0(A^{**}, K_{A^{**}})^G & \text{if } g^{**} = 1, \\ \sum_{-\infty}^{-1} \dim H^0(A^{**}, K_{A^{**}} N^{**-n-1})^G & \text{if } g^{**} \ge 2, \end{cases}$$

where g^{**} is the genus of A^{**}, $K_{A^{**}}$ is the canonical line bundle of A^{**} and $H^0(A^{**}, K_{A^{**}}N^{**-n})^G$ denotes the G-invariant sections. Therefore

$$q(X) + \dim H^0(A^{**}, K_{A^{**}})^G = p_g(X).$$

In particular, $q(X) \leq p_g(X)$. Q.E.D.

4. Algebraic classification of CR manifolds

In this section, we shall use some of the invariants, introduced in the previous section to give a rough algebraic classification of strongly pseudoconvex 3 dimensional CR manifolds embeddable in $\mathbb{C}^3$.

Definition 4.1. Let X be a connected compact strongly pseudoconvex embeddable CR manifold of real dimension 3. If $p_a(X) = 0$, then X is called a *rational CR manifold.* If $p_a(X) = 1$, then X is called an *elliptic CR-manifold.*

Theorem 4.2. *Let X be a connected compact strongly pseudoconvex embeddable CR manifold of real dimension 3. Then*

$$p_f(X) = 0 \Leftrightarrow p_a(X) = 0 \Leftrightarrow p_g(X) = 0.$$

Proof. In view of Proposition 3.4, it suffices to show that if then $p_f(X) = 0$, $p_g(X) = 0$. In view of Proposition 3.3, we shall assume that A is connected. Consider the computation sequence for the fundamental cycle Z as in (3.1). We have shown in (3.3) that

$$H^0(M, \mathcal{O}_{Z_k}) = \mathbb{C},\ 1 \leq k \leq \ell.$$

Observe that the long exact cohomology sequence of (3.2) shows that

$$\dim H^1(M, \mathcal{O}_{Z_\ell}) \geq \dim H^1(M, \mathcal{O}_{Z_{\ell-1}}).$$

By definition, $p_f(X) = 0$ is equivalent to $\chi(\mathcal{O}_Z) = 1$ which implies $H^1(M, \mathcal{O}_Z) = 0$. Hence $H^1(M, \mathcal{O}_{Z_{\ell-1}}) = 0$ and $\chi(\mathcal{O}_{Z_{\ell-1}}) = 1$. By Riemann-Roch Theorem, we have

$$1 = \chi(\mathcal{O}_Z) = \chi(\mathcal{O}_{A_{i_\ell}}) + \chi(\mathcal{O}_{Z_{\ell-1}}) - A_{i_\ell} \cdot Z_{\ell-1}$$

which implies

$$\chi(\mathcal{O}_{A_{i_\ell}}) = A_{i_\ell} \cdot Z_{\ell-1}.$$

Notice that $\chi(\mathcal{O}_{A_{i_\ell}}) \leq 1$ while $A_{i_\ell} \cdot Z_{\ell-1} \geq 1$ by the definition of the computation sequence for Z. We conclude that A_{i_ℓ} is a Riemann surface of genus zero and $A_{i_\ell} \cdot Z_{l-1} = 1$. Repeating this argument, we conclude that A_{i_j} has genus zero for all $j \geq 1$ and $A_{i_j} \cdot Z_{j-1} = 1$ for all $j \geq 2$. The exact sheaf sequence

$$O \to \mathcal{O}(-Z_1) \to \mathcal{O} \to \mathcal{O}_{Z_1} \to \mathcal{O}$$

yields

$$H^1(M, \mathcal{O}(-Z_1)) \to H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}_{Z_1}) \to O.$$

Similarly, for $2 \leq k \leq \ell$, the exact sequence

$$0 \to \mathcal{O}(-Z_k) \to \mathcal{O}(-Z_{k-1}) \to \mathcal{O}(-Z_{k-1})/\mathcal{O}(-Z_k) \to 0$$

yields

$$H^1(M, \mathcal{O}(-Z_k)) \to H^1(M\ \mathcal{O}(-Z_{k-1})) \to H^1(M, \mathcal{O}(-Z_{k-1})/\mathcal{O}(-Z_k)) \to 0\,.$$

Since $\mathcal{O}(-Z_{k-1})/\mathcal{O}(-Z_k)$ represents the sheaf of germs of sections of a line bundle over A_{i_k} of Chern class $-A_{i_k} \cdot Z_{k-1} = -1$,

$$H^1(M, \mathcal{O}(-Z_{k-1})/\mathcal{O}(-Z_k)) = 0\,.$$

Hence

$$H^1(M, \mathcal{O}(-Z_k)) \to H^1(M, \mathcal{O}(-Z_{k-1}))$$

is surjective. By composing the above maps, we have surjective map

$$H^1(M, \mathcal{O}(-Z)) \to H^1(M, \mathcal{O}).$$

Consider the exact sequence

$$0 \to \mathcal{O}(-Z - Z_k) \to \mathcal{O}(-Z - Z_{k-1}) \to \mathcal{O}(-Z - Z_{k-1})/\mathcal{O}(-Z - Z_k) \to 0\,.$$

Since $\mathcal{O}(-Z - Z_{k-1})/\mathcal{O}(-Z - Z_k)$ represents the sheaf of germs of sections of a line bundle over A_{i_k} of Chern class $-A_{i_k}(Z + Z_k) \geq -1$,

$$H^1(M, \mathcal{O}(-Z - Z_{k-1})/\mathcal{O}(-Z - Z_k)) = 0\,.$$

Hence the map $H^1(M, \mathcal{O}(-Z - Z_k) \to H^1(M, \mathcal{O})$ is surjective. Continuing the argument, we have that the map

$$H^1(M, \mathcal{O}(-nZ)) \to H^1(M, \mathcal{O})$$

is surjective for all n. Hence by [Gr, § 4, Satz 1, p. 355], $H^1(M, \mathcal{O}) = 0$ and $p_g(X) = 0$.

Q.E.D.

Theorem 4.3. *Let X be a connected compact strongly pseudoconvex CR manifold of real dimension 3 embeddable in $\mathbb{C}^3$. Suppose that X is a rational CR manifold (i.e., $p_a(X) = 0$). Then Γ_X is a direct sum of the following graphs.*

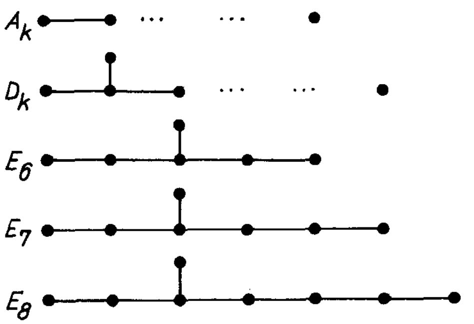

Abb. 3

where each vertex • is a rational curve with self intersection number -2.

Proof. We shall use the notation in Definition 2.2. By our hypothesis, X is a boundary of complex hypersurface V with isolated singularities $\{p_1, \ldots, p_m\}$. Let f be the defining equation for V. Then

$$\omega = \frac{dx \wedge dy}{\frac{\partial f}{\partial z}} = \frac{dy \wedge dz}{\frac{\partial f}{\partial x}} = \frac{dz \wedge dx}{\frac{\partial f}{\partial y}}$$

is a holomorphic 2-form which is nowhere zero on $V - \{p_1, \ldots, p_m\}$. Let $\pi : M_0 \to V$ be the minimal resolution of V. Then $K :=$ divisor of $\pi^*(\omega)$ is supported on $A = \pi^{-1}\{p_1, \ldots, p_m\}$. Now since X is a rational CR manifold, $p_g(X) = 0$ by Theorem 4.2. Hence $\pi^*(\omega)$ is a holomorphic 2-form on M_0. It follows that K is an effective divisor with support on A, i.e.,

$$K = \sum n_i A_i, \qquad n_i \geq 0.$$

By the adjunction formula, we have

(4.1) $\quad A_i \cdot K \geq 0 \quad$ for all $\quad A_i \subseteq A$

because M_0 is the minimal resolution of V. Consequently we have

(4.2) $\quad k^2 = \sum n_i(A_i \cdot K) \geq 0.$

On the other hand, the intersection matrix is negative definite. It follows that $K^2 = 0$ and hence $K \cdot A_i = 0$ for all A_i by (4.1) and (4.2). The adjunction formula tells us that $A_i^2 = -2$ and the genus $g_i = 0$. Then as an easy exercise, one can show that weighted dual graph of the exceptional set is the direct sum of those from A_k, D_k, E_6, E_7 and E_8. Q.E.D.

Theorem 4.4. *Let X be a connected strongly pseudoconvex CR manifold of real dimension 3 embeddable in $\mathbb{C}^3$. Suppose $p_g(X) = 1$. Then Γ_X is a direct sum of exactly one graph listed in Table 5.1, Table 5.2 and Table 5.3 of [La3] and some of the graphs listed in Theorem 4.3.*

Proof. We shall use the notation in Definition 2.2. By our hypothesis, X is a boundary of complex hypersurface V with isolated singularities $\{p_1, \ldots, p_m\}$. Since $p_g(X) = 1$, by Proposition 3.3 it follows that there is exactly one p_i such that the geometric genus $p_g(V, p_i)$ of the singularity (V, p_i) is one and the geometric genus $p_g(V, p_j)$, for $j \neq i$, of the singularity (V, p_j) is zero. (V, p_i) is called a minimally elliptic singularity. Hence the weighted dual graph of the exceptional set corresponding to (V, p_i) is one of those listed in [La3]. (V, p_j), for $j \neq i$, is a rational hypersurface singularity. The weighted dual graph of the exceptional set corresponding to (V, p_j) is one of those listed in Theorem 4.3. Q.E.D.

Theorem 4.5. *Let X be a connected compact strongly pseudoconvex CR manifold of real dimension 3 embeddable in $\mathbb{C}^3$. Suppose that X is an elliptic CR manifold (i.e., $p_a(X) = 1$). Then Γ_X is a direct sum of exactly one graph listed in [Ya] and some of the graphs listed in Theorem 4.3.*

Proof. The proof is similar to the proof of Theorem 4.4.

5. Obstructions to embedding of strongly pseudoconvex compact manifolds in $\mathbb{C}^3$

In Theorem 3.2, Theorem 3.3, Theorem 3.6 and Theorem 3.7 of [Lu-Ya], we have obtained some obstructions to embedding strongly pseudoconvex CR manifolds in $\mathbb{C}^3$. We would like to point out that the classification theory in § 4 yields automatically obstructions to embedding CR manifolds in $\mathbb{C}^3$. Let us reformulate Theorem 4.3, Theorem 4.4. and Theorem 4.5 below.

Theorem 5.1. *Let X be a connected compact strongly pseudoconvex CR manifold of real dimension 3 with $p_a(X) = 0$. If Γ_X is not a direct sum of the graphs listed in Theorem 4.3, then X is not embeddable in $\mathbb{C}^3$.*

Theorem 5.2. *Let X be a connected compact strongly pseudoconvex CR manifold of real dimension 3 with $p_g(X) = 1$. If Γ_X is not a direct sum of exactly one graph listed in Table 5.1, Table 5.2 and Table 5.3 of [La3] and some of the graphs listed in Theorem 4.3, then X is not embeddable in $\mathbb{C}^3$.*

Theorem 5.3. *Let X be a connected strongly pseudoconvex CR manifold of real dimension 3 with $p_a(X) = 1$. If Γ_X is not a direct sum of exactly one graph listed in [Ya] and some of the graphs listed in Theorem 4.3, then X is not embeddable in $\mathbb{C}^3$.*

For an elliptic strongly pseudoconvex compact CR manifold X of real dimension 3, we have a numerical obstruction for X to be embeddable in $\mathbb{C}^3$. Recall the definition of $\chi(X)$ in Definition 3.0.

Theorem 5.4. *Let X be a connected compact strongly pseudoconvex CR manifold of real dimension 3 with $p_a(X) = 1$. If $\chi(X) \leq -4$, then X is not embeddable in $\mathbb{C}^3$.*

Proof. We need to show that if X is embeddable in $\mathbb{C}^3$, then $\chi(X) \geq -3$. Let V be the subvariety in $\mathbb{C}^3$ such that the boundary of V is X and V has isolated singularities at $Y = \{p_1, \ldots, p_m\}$. By Proposition 3.3, we may assume that

$$p_a(V, p_1) = 1 \quad \text{and} \quad p_a(V, p_j) = 0 \quad \text{for} \quad 2 \leq j \leq m\,.$$

Let $\pi: M_0 \to V$ be the minimal resolution of singularities of V such that the exceptional sets $A^j = \pi^{-1}(p_j)$ has normal crossing for all $1 \leq j \leq m$. Since p_j, $1 \leq j \leq m$, are hypersurface singularities, we deduce from the adjunction formula that the divisor K defined in Definition 3.0 is actually a canonical divisor. We also know that $K = \sum_{j=1}^{m} K_j$ where K_j is an integral divisor supported in A^j. In view of Theorem 4.1 of [Xu-Ya], we have

$$K_1^2 + \chi_T(A^1) \geq -3\,. \tag{5.1}$$

On the other hand, by the proof of Theorem 4.3, we know that $K_j = 0$ for $2 \leq j \leq m$. Therefore

$$K_j^2 + \chi_T(A^j) = n_j + 1 > 0 \quad \text{for} \quad 2 \leq j \leq m\,, \tag{5.2}$$

where n_j = number of irreducible components in the exceptional set A^j. It follows from (5.1) and (5.2) that

$$\begin{aligned}\chi(X) &= K^2 + \chi_T(A) \\ &= \sum_{j=1}^{m} K_j^2 + \sum_{j=1}^{m} \chi_T(A^j) \\ &= K_1^2 + \chi_T(A^1) + \sum_{j=2}^{m} (K_j^2 + \chi_T(A^j)) \geq -3 .\end{aligned} \qquad \text{Q.E.D.}$$

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