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Nonexistence of Negative Weight Derivation of Moduli Algebras of Weighted Homogeneous Singularities

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INTRODUCTION

A moduli algebra $A(V)$ of hypersurface singularity $(V, 0) = \{(z_0, \dots, z_n): f(z_0, \dots, z_n) = 0\} \subseteq \mathbb{C}^{n+1}$ is a finite dimensional $\mathbb{C}$-algebra $\mathbb{C}\{z_0, z_1, \dots, z_n\}/(f, \partial f/\partial z_0, \dots, \partial f/\partial z_n)$. In 1982, Mather and the third author [Ma-Ya] proved that two germs of complex analytic isolated hypersurface singularities of the same dimension are biholomorphically equivalent if and only if their moduli algebras are isomorphic. This raises a natural and important question, the so-called recognition problem: give a necessary and sufficient condition for a commutative local Artinian algebra to be a moduli algebra. In 1983, the third author [Ya1] introduced a finite dimensional Lie algebra $L(V)$ to an isolated hypersurface singularity $(V, 0)$. $L(V)$ is defined to be the algebra of derivations of the moduli algebra $A(V)$ and is finite dimensional. It was proved in [Ya2] (for $n \le 2$), [Ya3] (for $n \le 4$), and [Ya4] that $L(V)$ is actually a solvable Lie algebra. In [Se-Ya], Seeley and the third author have shown how to use the Lie algebra structures to distinguish complex analytic structures of isolated

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hypersurface singularities. In fact they have constructed continuous numerical invariants out of these Lie algebras.

If $A(V)$ is a graded algebra $\oplus_{i=0}^{\infty} A_i$, then $L(V)$ is naturally a graded Lie algebra $\oplus L_k$ where L_k is the set of derivations of A which sends A_i to A_{i+k} for all i. In [Ya5], the third author conjectured that $L_k = 0$ for all $k < 0$, i.e., there is no negative weight derivation of moduli algebra. The immediate consequence of this conjecture is Theorem 2.4 the microlocal characterization of quasi-homogeneous singularity as shown in [Ya5]. The purpose of this paper is to prove this conjecture for $n \leq 2$ (cf. Theorem 2.2 and Theorem 2.3).

1. Notations and Known Results

Definition 1.1. Suppose that f: $(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a holomorphic function. Then f is said to be quasi-homogeneous if f is in the Jacobian ideal of f, i.e., $f \in (\partial f/\partial z_0, \dots, \partial f/\partial z_n)\mathscr{O}_{\mathbb{C}^{n+1},0}$.

Definition 1.2. A polynomial $f(z_0, \dots, z_n)$ is weighted homogeneous of type $(\alpha_0, \dots, \alpha_n; d)$, where $\alpha_0, \dots, \alpha_n$ and d are fixed positive integers, if it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} \dots z_n^{i_n}$ for which $\alpha_0 i_0 + \alpha_1 i_1 + \cdots + \alpha_n i_n = d$.

By a fundamental theorem of Saito [Sa], if f is quasi-homogeneous with isolated critical point at the origin, then after a biholomorphic change of coordinates, f becomes a weighted homogeneous polynomial.

Let f be a weighted homogeneous polynomial of type $(\alpha_0, \dots, \alpha_n; d)$ with isolated singularity at the origin. Then by a result of Saito [Sa], we shall always assume without loss of generality that $2\alpha_i \leq d$ for all $0 \leq i \leq n$. If we give z_i variable weight α_i for $0 \leq i \leq n$, then the moduli algebra $A(V)$ is a graded algebra $\oplus_{i=0}^{\infty} A_i$ and the algebra of derivations $L(V)$ of $A(V)$ is also graded. In fact we have the following lemmas.

Lemma 1.1 (cf. [Ya5]). *Let $A = \oplus_{i=0}^{t} A_i$ be a graded commutative Artinian local algebra. Let $L(A)$ be the algebra of derivations of A. Then $L(A)$ is a graded Lie algebra $\oplus_{k=-t}^{t} L_k$ where $L_k = \{D \in L(A)\colon D(A_i) \subseteq A_{i+k}$ for all $i\}$.*

Lemma 1.2 (cf. [Ya2, Ya5]). *Let A be a commutative Artinian local algebra. Let $D \in L(A)$ be any derivation of A. Then D preserves the m-adic filtration of A, i.e., $D(m) \subseteq m$ where m is the maximal ideal of A.*

In case f is actually an homogeneous polynomial, then it was shown in [Ya5] that $L(V)$ is a graded Lie algebra without negative weight, i.e., $L(V) = \oplus_{k=0}^{t} L_k$. In fact we have the following proposition.

PROPOSITION 1.3 (cf. [Ya5]). *Let* $A = \oplus_{i=0}^{t} A_i$ *be a commutative Artinian local algebra with* $A_0 = \mathbb{C}$. *Suppose the maximal ideal of* A *is generated by* A_j *for some* $j > 0$. *Then* $L(A)$ *is a nonnegatively graded Lie Algebra* $\oplus_{k=0}^{t} L_k$.

2. NONEXISTENCE OF NEGATIVE WEIGHT DERIVATION OF MODULI ALGEBRAS OF WEIGHTED HOMOGENEOUS SINGULARITIES

We begin with the following lemma.

LEMMA 2.1. *Let* f *be a weighted homogeneous polynomial with isolated singularity in* $z_0, \dots, z_n$ *variables of type* $(\alpha_0, \dots, \alpha_n; d)$.

Assume $\mathrm{wt}(z_0) = \alpha_0 \geq \mathrm{wt}(z_1) = \alpha_1 \geq \cdots \geq \mathrm{wt}(z_n) = \alpha_n$. Then f must be in one of the following two cases:

Case 1.

$$f = z_0^m + a_1(z_1, \dots, z_n) z_0^{m-1} + \cdots + a_{m-1}(z_1, \dots, z_n) z_0 + a_m(z_1, \dots, z_n).$$

Case 2.

$$f = z_0^m z_i + a(z_1, \dots, z_n) z_0^{m-1} + \cdots + a_{m-1}(z_1, \dots, z_n) z_0 + a_m(z_1, \dots, z_n).$$

Proof. We write

$$f = a_0(z_1, \dots, z_n) z_0^m + a_1(z_1, \dots, z_n) z_0^{m-1} + \cdots + a_m(z_1, \dots, z_n).$$

If $a_0(z_1, \dots, z_n)$ is a constant, then we are in Case 1. So we shall assume that $a_0(z_1, \dots, z_n)$ is not a constant. By the weight consideration we conclude that $a_i(z_1, \dots, z_n)$ vanishes at $(0, 0, \dots, 0)$ for $1 \leq i \leq m$. We claim that $(\partial a_i / \partial z_j)(0, 0, \dots, 0) \neq 0$ for some $0 \leq i \leq m$ and $1 \leq j \leq n$. Otherwise the multiplicity of $a_i(z_1, \dots, z_n)$ at $(0, 0, \dots, 0)$ would be at least two and hence f would be singular along the z_0-axis which contradicts our assumption that f has only isolated singularity. Let i, j be such that $(\partial a_i / \partial z_j)(0, 0, \dots, 0) \neq 0$. Then $a_i(z_1, \dots, z_n) = cz_j +$ other terms where $c \neq 0$. Since the term $z_0^{m-i} z_j$ is in f, we have $(m - i)\alpha_0 + \alpha_j = d$. It follows from $m\alpha_0 \leq d$ and $\alpha_j \leq \alpha_0$ that $i = 0$ or 1.

If $i = 0$, after a change of coordinates which respect the weights, then f is of the form $f = z_0^m z_j + a_1(z_1, \dots, z_n) z_0^m + \cdots + a_m(z_1, \dots, z_n)$.

If $i = 1$, then $\alpha_j = \alpha_0$ and $f = a_0(z_1, \dots, z_n)z_0^m + (cz_j + \dots)z_0^{m-1} + \cdots + a_m(z_1, \dots, z_n)$. Since $\mathrm{wt}(z_0^m) = \mathrm{wt}(z_j z_0^{m-1})$, we conclude that $a_0(z_1, \dots, z_n)$ is a constant. Then f is in Case 1 again. Q.E.D.

THEOREM 2.2. *Let $f(z_0, z_1)$ be a weighted homogeneous polynomial of type $(\alpha_0, \alpha_1; d)$ with isolated singularity at the origin. Assume that $d \geq 2\alpha_0 \geq 2\alpha_1$. Let D be a derivation of the moduli algebra $\mathbb{C}[z_0, z_1]/(f, \partial f/\partial z_0, \partial f/\partial z_1)$. Then $D \equiv 0$ if D is negatively weighted.*

Proof. In view of Lemma 1.2, it is easy to see that if D is negatively weighted, then $D = z_1^k(\partial/\partial z_0)$ where $k \geq 1$ and $\mathrm{wt}\, D = k\alpha_1 - \alpha_0 < 0$. Since D is a derivation of $C[z_0, z_1]/(\partial f/\partial z_0, \partial f/\partial z_1)$, we have $D(J) \subseteq J$ where $J = (\partial f/\partial z_0, \partial f/\partial z_1)$ is the Jacobian ideal of f. Observe that $\mathrm{wt}(\partial f/\partial z_0) = d - \alpha_0 \leq d - \alpha_1 = \mathrm{wt}(\partial f/\partial z_1)$ and $\mathrm{wt}(D(\partial f/\partial z_0)) < \mathrm{wt}(\partial f/\partial z_0)$. We deduce that $D(\partial f/\partial z_0) = 0$. Let

$$f(z_0, z_1) = \sum_{\alpha_0 n_0 + \alpha_1 n_1 = d} c_{(n_0, n_1)} z_0^{n_0} z_1^{n_1}.$$

Then we have

$$\begin{aligned} D\left(\frac{\partial f}{\partial z_0}\right) &= z_1^k \frac{\partial}{\partial z_0}\left(\sum_{\alpha_0 n_0 + \alpha_1 n_1 = d} n_0 c_{(n_0, n_1)} z_0^{n_0 - 1} z_1^{n_1}\right) \\ &= \sum_{\alpha_0 n_0 + \alpha_1 n_1 = d} n_0(n_0 - 1) c_{(n_0, n_1)} z_0^{n_0 - 2} z_1^{n_1 + k} \\ &= 0. \end{aligned}$$

Hence, we conclude that $c_{(n_0, n_1)} = 0$ for $n_0 \geq 2$ and

$$f(z_0, z_1) = c_{(1,p)} z_0 z_1^p + c_{(0,q)} z_1^q,$$

where $d = \alpha_0 + p\alpha_1 = q\alpha_1$. Recall that $\alpha_0 > k\alpha_1$. We deduce from these that $q > k + p \geq 2$. In order that f has an isolated singularity at the origin, we need $p = 1$. So

$$\begin{aligned} f(z_0, z_1) &= c_{(1,1)} z_0 z_1 + c_{(0,q)} z_1^q \\ \frac{\partial f}{\partial z_0} &= c_{(1,1)} z_1. \end{aligned}$$

Hence $D = z_1^k(\partial/\partial z_0)$ is a zero derivation on $\mathbb{C}[z_0, z_1]/(\partial f/\partial z_0, \partial f/\partial z_1)$. Q.E.D.

THEOREM 2.3. *Let $f(z_0, z_1, z_2)$ be a weighted homogeneous polynomial of type $(\alpha_0, \alpha_1, \alpha_2; d)$ with isolated singularity at the origin. Assume that $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$. Let D be a derivation of the moduli algebra*

$\mathbb{C}[z_0, z_1, z_2]/(\partial f/\partial z_0, \partial f/\partial z_1, \partial f/\partial z_2)$. *Then* $D \equiv 0$ *if* D *is negatively weighted.*

Proof. In view of Lemma 1.2, it is easy to see that if D is negatively weighted, then $D = p_0(z_1, z_2)(\partial/\partial z_0) + cz_2^k(\partial/\partial z_1)$ where c is a constant. There are two cases.

Case 1. $c \neq 0$. According to Lemma 2.1, we only need to consider two subcases.

Case 1(a). $f = z_0^m + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2)z_0 + a_m(z_1, z_2)$. Then

$$\begin{aligned} D\left(\frac{\partial f}{\partial z_0}\right) &= p_0(z_1, z_2)\big(m(m-1)z_0^{m-2} \\ &\quad + (m-1)(m-2)a_1(z_1, z_2)z_0^{m-3} + \cdots + 2a_{m-2}(z_1, z_2)\big) \\ &\quad + cz_2^k\left((m-1)\frac{\partial a_1}{\partial z_1}(z_1, z_2)z_0^{m-2}\right. \\ &\quad \left. + (m-2)\frac{\partial a_2}{\partial z_1}(z_1, z_2)z_0^{m-3} + \cdots + \frac{\partial a_{m-1}}{\partial z_1}(z_1, z_2)\right). \end{aligned}$$

Since D is a derivation of the moduli algebra $\mathbb{C}[z_0, z_1, z_2]/(\partial f/\partial z_0, \partial f/\partial z_1, \partial f/\partial z_2)$, $D(J) \subseteq J$ where J is the ideal generated by $\partial f/\partial z_0$, $\partial f/\partial z_1$, and $\partial f/\partial z_2$. As D is negatively weighted and $\mathrm{wt}(\partial f/\partial z_0) = d - \alpha_0 \leq d - \alpha_1 = \mathrm{wt}(\partial f/\partial z_1) \leq d - \alpha_2 = \mathrm{wt}(\partial f/\partial z_2)$, we have $D(\partial f/\partial z_0) \equiv 0$. Therefore we conclude that $mp_0(z_1, z_2) = -cz_2^k(\partial a_1/\partial z_1)(z_1, z_2)$. Let us make the following coordinate change which respects the weights

$$\begin{cases} z_0 = z_0' - \dfrac{1}{m}a_1(z_1', z_2') \\ z_1 = z_1' \\ z_2 = z_2'. \end{cases}$$

Then

$$\begin{aligned} D &= -\frac{1}{m}cz_2^k\frac{\partial a_1}{\partial z_1}(z_1, z_2)\frac{\partial}{\partial z_0} + cz_2^k\frac{\partial}{\partial z_1} \\ &= cz_2^k\left(-\frac{1}{m}\frac{\partial a_1}{\partial z_1}(z_1, z_2)\frac{\partial}{\partial z_0} + \frac{\partial}{\partial z_1}\right) \\ &= cz_2'^k\frac{\partial}{\partial z_1'}. \end{aligned}$$

Let $g(z_0', z_1', z_2') = f(z_0, z_1, z_2)$. By the same argument as before, we have

$$D\left(\frac{\partial g}{\partial z_0'}\right) = c z_2'^k \frac{\partial}{\partial z_1'}\left(\frac{\partial g}{\partial z_0'}\right) = 0.$$

Now $g = z_0'^m + b_1(z_1', z_2')z_0'^{m-1} + \cdots + b_m(z_1', z_2')$. It follows easily that

$$\frac{\partial b_1}{\partial z_1'}(z_1', z_2') = \cdots = \frac{\partial b_{m-1}}{\partial z_1'}(z_1', z_2') = 0.$$

Consider

$$D\left(\frac{\partial g}{\partial z_1'}\right) = c z_2'^k \frac{\partial^2 g}{\partial z_1'^2} = c z_2'^k \frac{\partial^2 b_m}{\partial z_1'^2}(z_1', z_2').$$

Since $\partial g/\partial z_0'$ has the minimal weight in J and $\mathrm{wt}(D(\partial g/\partial z_1')) < \mathrm{wt}(\partial g/\partial z_1')$, we conclude that $D(\partial g/\partial z_1')$ is a multiple of $\partial g/\partial z_0'$, i.e., there exists h such that

$$c z_2'^k \frac{\partial^2 b_m}{\partial z_1'^2}(z_1', z_2') = h\frac{\partial g}{\partial z_0'}(z_1', z_2')$$
$$= h\left(m z_0'^{m-1} + \cdots + b_{m-1}(z_1', z_2')\right).$$

If $(\partial^2 b_m/\partial z_1'^2)(z_1', z_2') \neq 0$, then this is possible only if $m = 1$. However, if $m = 1$ then 0 is not a singular point. So we have $(\partial^2 b_m/\partial z_1'^2)(z_1', z_2') = 0$. This implies

$$b_m(z_1', z_2') = d_1 z_1' z_2'^{l_1} + d_2 z_2'^{l_2}$$

where d_1, d_2 are constants. Then g has an isolated singularity at 0 only if $l_1 = 1$. Because $b_m(z_1', z_2')$ is weighted homogeneous, we have $d = \alpha_1 + \alpha_2$. It follows from the assumption $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$ that $\alpha_0 = \alpha_1 = \alpha_2$. So g is a homogeneous polynomial. Our theorem follows from Proposition 1.3.

Case 1(b). $f = z_0^m z_i + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2)z_0 + a_m(z_1, z_2)$. If $i = 1$, then we have

$$0 = D\left(\frac{\partial f}{\partial z_0}\right) = p_0(z_1, z_2)\big(m(m-1)z_0^{m-2}z_1$$
$$+ (m-1)(m-2)a_1(z_1, z_2)z_0^{m-3} + \ldots\big)$$
$$+ c z_2^k\left(m z_0^{m-1} + (m-1)\frac{\partial a_1}{\partial z_1}(z_1, z_2)z_0^{m-2} + \ldots\right).$$

This forces $c = 0$ which contradicts our hypothesis. So this case does not occur.

If $i = 2$, there are two subcases to be considered.

If $m = 1$, then we have $\alpha_0 + \alpha_2 = d$. Thus the assumption that $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$ implies $\alpha_0 = \alpha_1 = \alpha_2$. By Proposition 1.3, we are done.

Next we assume $m > 1$. Then

$$\begin{aligned} 0 &= D\left(\frac{\partial f}{\partial z_0}\right) \\ &= p_0(z_1, z_2)\frac{\partial}{\partial z_0}\left(\frac{\partial f}{\partial z_0}\right) + cz_2^k\frac{\partial}{\partial z_1}\left(\frac{\partial f}{\partial z_0}\right) \qquad (2.1) \\ &= \frac{\partial}{\partial z_0}\left(p_0(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1}\right) \end{aligned}$$

and

$$\begin{aligned} D\left(\frac{\partial f}{\partial z_1}\right) &= p_0(z_1, z_2)\frac{\partial}{\partial z_0}\left(\frac{\partial f}{\partial z_1}\right) + cz_2^k\frac{\partial}{\partial z_1}\left(\frac{\partial f}{\partial z_1}\right) \\ &= \frac{\partial}{\partial z_1}\left(p_0(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1}\right) - \frac{\partial p_0}{\partial z_1}(z_1, z_2)\cdot\frac{\partial f}{\partial z_0} \qquad (2.2) \\ &= h\frac{\partial f}{\partial z_0}. \end{aligned}$$

Equation (2.2) implies that

$$\frac{\partial}{\partial z_1}\left(p_0(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1}\right) = \tilde{h}\frac{\partial f}{\partial z_0}. \qquad (2.3)$$

From (2.1), we know that the left hand side of (2.3) is independent of z_0 variable. Since $m > 1$, the right hand side of (2.3) is independent of z_0 variable only if $\tilde{h} = 0$. Thus we have

$$\frac{\partial}{\partial z_1}\left(p_0(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1}\right) = 0. \qquad (2.4)$$

It follows from (2.1) and (2.4) that

$$p_0(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1} = uz_2^l, \qquad (2.5)$$

where either $u = 0$ or $u \neq 0$ and $l > k$.

Now (2.5) can be written as

$$p_0(z_1, z_2)\left(mz_2 z_0^{m-1} + (m-1)a_1(z_1, z_2)z_0^{m-2} + \cdots + a_{m-1}(z_1, z_2)\right)$$
$$+ cz_2^k\left(\frac{\partial a_1}{\partial z_1}(z_1, z_2)z_0^{m-1} + \frac{\partial a_2}{\partial z_1}(z_1, z_2)z_0^{m-2} + \cdots + \frac{\partial a_m}{\partial z_1}(z_1, z_2)\right)$$
$$= uz_2^l. \tag{2.6}$$

As $m > 1$, (2.6) implies

$$mp_0(z_1, z_2)z_2 + cz_2^k\frac{\partial a_1}{\partial z_1}(z_1, z_2) = 0. \tag{2.7}$$

If $cz_2^k(\partial a_1/\partial z_1)(z_1, z_2) = 0$, then (2.7) implies $p_0(z_1, z_2) = 0$ and (2.6) becomes

$$cz_2^k\left(\frac{\partial a_1}{\partial z_1}(z_1, z_2)z_0^{m-1} + \frac{\partial a_2}{\partial z_1}(z_1, z_2)z_0^{m-2} + \cdots + \frac{\partial a_m}{\partial z_1}(z_1, z_2)\right) = uz_2^l. \tag{2.8}$$

Since $c = \neq 0$, we have

$$\frac{\partial a_1}{\partial z_1}(z_1, z_2) = \cdots = \frac{\partial a_{m-1}}{\partial z_1}(z_1, z_2) = 0 \tag{2.9}$$

$$\frac{\partial a_m}{\partial z_1}(z_1, z_2) = ez_2^{l-k}, \qquad e \neq 0 \tag{2.10}$$

and hence

$$a_m(z_1, z_2) = ez_1 z_2^{l-k} + z_2^k. \tag{2.11}$$

We must have $l - k = 1$ and $e \neq 0$, otherwise f would be singular along the z_1-axis in view of (2.9) and (2.10), which is a contradiction. Since the term $z_1 z_2$ appears in f, we conclude that $\alpha_1 + \alpha_2 = d$. The assumption $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$ implies $\alpha_0 = \alpha_1 = \alpha_2$. Hence we are done by Proposition 1.3.

Next we assume $cz_2^k(\partial a_1/\partial z_1)(z_1, z_2) \neq 0$. Then (2.7) implies that $p_0(z_1, z_2) = z_2^{k-1}q_0(z_1, z_2)$. From (2.5), we have

$$q_0(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2\frac{\partial f}{\partial z_1} = uz_2^{l-k+1}. \tag{2.12}$$

Let us write

$$q_0(z_1, z_2) = \alpha z_1^s + z_2\gamma_0(z_1, z_2).$$

We claim that $\alpha = 0$. Suppose on the contrary that $\alpha \neq 0$. If we rewrite f in the form

$$f = b_0(z_0, z_1)z_2^n + b_1(z_0, z_1)z_2^{n-1} + \cdots + b_n(z_0, z_1)$$

then (2.12) becomes

$$\begin{aligned}(\alpha z_1^s + z_2\gamma_0(z_1, z_2))\biggl(&\frac{\partial b_0}{\partial z_0}(z_0, z_1)z_2^n \\ &+ \frac{\partial b_1}{\partial z_0}(z_0, z_1)z_2^{n-1} + \cdots + \frac{\partial b_n}{\partial z_0}(z_0, z_1)\biggr) \\ + cz_2\biggl(&\frac{\partial b_0}{\partial z_1}(z_0, z_1)z_2^n \\ &+ \frac{\partial b_1}{\partial z_1}(z_0, z_1)z_2^{n-1} + \cdots + \frac{\partial b_n}{\partial z_1}(z_0, z_1)\biggr) = uz_2^{l-k+1}. \quad (2.13)\end{aligned}$$

Thus forces $(\partial b_n/\partial z_0)(z_0, z_1) = 0$ and hence from (2.13) again we have

$$\begin{aligned}(\alpha z_1^s + z_2\gamma_0(z_1, z_2))\biggl(&\frac{\partial b_0}{\partial z_0}(z_0, z_1)z_2^{n-1} \\ &+ \frac{\partial b_1}{\partial z_0}(z_0, z_1)z_2^{n-2} + \cdots + \frac{\partial b_{n-1}}{\partial z_0}(z_0, z_1)\biggr) \\ + c\biggl(&\frac{\partial b_0}{\partial z_1}(z_0, z_1)z_2^n + \frac{\partial b_1}{\partial z_1}(z_0, z_1)z_2^{n-1} + \cdots + \frac{\partial b_n}{\partial z_1}(z_0, z_1)\biggr) \\ &= uz_2^{l-k}.\end{aligned}$$

Recall that either $u = 0$ or $u \neq 0$ and $l > k$. The above equation implies $\alpha z_1^s(\partial b_{n-1}/\partial z_0)(z_0, z_1) = -c(\partial b_n/\partial z_1)(z_0, z_1)$. Since $(\partial b_n/\partial z_0)(z_0, z_1) = 0$, we have $(\partial b_{n-1}/\partial z_0)(z_0, z_1) = c'z_1^{s'}$ where $c' = -c/\gamma \neq 0$. If $s' = 0$, then $b_{n-1}(z_0, z_1) = c'z_0 + c''z_1$ and hence z_0z_2 occurs in f. It follows again that $\alpha_0 = \alpha_1 = \alpha_2$ and we are done by Proposition 1.3. If $s' > 0$, then

$$b_{n-1}(z_0, z_1) = c'z_1^{s'}z_0 + \tilde{u}z_1^{\tau}, \qquad s' > 0, \tau > 0.$$

Now $(\partial b_n/\partial z_0)(z_0, z_1) = 0$ implies $b_n(z_0, z_1) = wz_1^t$, $t > 1$. Notice that $(\partial f/\partial z_2) = nb_0(z_0, z_1)z_2^{n-1} + (n-1)b_1(z_0, z_1)z_2^{n-2} + \cdots + b_{n-1}(z_0,$

z_1). It follows that f is singular along the z_0-axis. The contradiction comes from our hypothesis $\alpha \neq 0$.

Now we have $q_0(z_1, z_2) = z_2 r(z_1, z_2)$ and (2.12) becomes

$$r(z_1, z_2)\frac{\partial f}{\partial z_0} + c\frac{\partial f}{\partial z_1} = uz_2^{l-k}. \tag{2.14}$$

So we have

$$r(z_1, z_2)\left(\frac{\partial b_0}{\partial z_0}(z_0, z_1)z_2^n + \frac{\partial b_1}{\partial z_0}(z_0, z_1)z_2^{n-1} + \cdots + \frac{\partial b_n}{\partial z_0}(z_0, z_1)\right)$$
$$+ c\left(\frac{\partial b_0}{\partial z_1}(z_0, z_1)z_2^n + \frac{\partial b_1}{\partial z_1}(z_0, z_1)z_2^{n-1} + \cdots + \frac{\partial b_n}{\partial z_1}(z_0, z_1)\right) = uz_2^{l-k}.$$

It follows that $r(z_1, 0) \neq 0$, otherwise we would have $(\partial b_n/\partial z_1)(z_0, z_1) = 0$. Therefore f would be of the form $f = z_0^m + c_1(z_1, z_2)z_0^{m-1} + \cdots + c_m(z_1, z_2)$ which is absurd.

Now let $r(z_1, z_2) = vz_1^h + z_2\bar{r}(z_1, z_2)$, where $h > 0$ and $v \neq 0$. Then we have

$$vz_1^h\frac{\partial b_n}{\partial z_0}(z_0, z_1) + c\frac{\partial b_n}{\partial z_1}(z_0, z_1) = 0 \qquad (v \neq 0, c \neq 0). \tag{2.15}$$

Let $b_n(z_0, z_1) = d_0 z_0^k z_1^{l_0} + d_1 z_0^{k-1} z_1^{l_1} + \cdots + d_k z_1^{l_k}$, $d_0 \neq 0$. Then

$$\frac{\partial b_n}{\partial z_0}(z_0, z_1) = kd_0 z_0^{k-1} z_1^{l_0} + (k-1)d_1 z_0^{k-2} z_1^{l_1} + \cdots + d_{k-1} z_1^{l_{k-1}}$$

$$\frac{\partial b_n}{\partial z_1}(z_0, z_1) = l_0 d_0 z_1^{l_0 - 1} + (l_1 - 1)d_1 z_0^{k-1} z_1^{l_1 - 1} + \cdots + d_k z_1^{l_k - 1}.$$

Clearly (2.15) is possible only if $k = 0$ or $l_0 = 0$.

If $k = 0$, then $b_n(z_0, z_1) = d_0 z_1^{l_0}$, then (2.15) is not satisfied.

If $l_0 = 0$, then $b_n(z_0, z_1) = d_0 z_0^k + d_1 z_0^{k-1} z_1^{l_1} + \cdots + d_k z_1^{l_k}$. Hence f is again of the form $f = z_0^m + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_m(z_1, z_2)$ which is absurd.

This completes the proof of Case 1(b).

Case 2. $c = 0$. In this case, $D = p_0(z_1, z_2)(\partial/\partial z_0)$. By Lemma 2.1, we have to consider two cases

Case 2a. $f = z_0^m + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2)z_0 + a_m(z_1, z_2)$. Then $0 = D(\partial f/\partial z_0) = p_0(z_1, z_2)(m(m-1)z_0^{m-2} + (m-1)(m-2)a_1(z_1, z_2)z_0^{m-3} + \dots)$ implies $p_0(z_1, z_2) = 0$ or $m = 1$. If $p_0(z_1, z_2) = 0$, then $D = 0$. On the other hand, if $m = 1$, then 0 is not a singular point of f.

Case 2b. $f = z_0^m z_i + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_m(z_1, z_2)$. The fact that $D(\partial f/\partial z_0) = 0$ again implies $p_0(z_1, z_2) = 0$ or $m = 1$. We only need to consider the case that $m = 1$. So

$$f = z_0 z_i + a_1(z_1, z_2).$$

If $i = 2$, then $\alpha_0 = \alpha_1 = \alpha_2$ and f is homogeneous. The result follows from Proposition 1.3.

If $i = 1$, then we have $\alpha_0 = \alpha_1 = 2$ and

$$f = z_0 z_1 + d_1 z_1^2 + d_2 z_1 z_2^l + d_3 z_2^{2l},$$

where d_1, d_2, d_3 are constants. Since $D(\partial f/\partial z_1) = p_0(z_1, z_2)(\partial^2 f/\partial z_0 \partial z_1) = p_0(z_1, z_2) \in J$, we have $D = 0$ as a derivation of $\mathbb{C}[z_0, z_1, z_2]/(\partial f/\partial z_0, \partial f/\partial z_1, \partial f/\partial z_2)$. Q.E.D.

As an immediate consequence of Theorem 2.2, Theorem 2.3, and a result in [Ya5], we have the following theorem

THEOREM 2.4. *For $n \leq 2$, let $(V, 0) = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}: f(z_0, \ldots, z_n) = 0\}$ be an isolated hypersurface singularity. Let $A(V) = \mathbb{C}\{c_0, \ldots, z_n\}/(f, \partial f/\partial z_0, \ldots, \partial f/\partial z_n)$ be the moduli algebra and $L(V)$ be the Lie algebra of derivations of $A(V)$. Then $(V, 0)$ is a quasihomogeneous singularity, i.e., $f \in (\partial f/\partial z_0, \partial f/\partial z_1, \ldots, \partial f/\partial z_n)$ if and only if the following conditions are satisfied.*

(1) *$L(V)$ is isomorphic to a nonnegatively graded Lie algebra $\oplus_{i=0}^k L_i$ without center.*

(2) *There exists $E \in L_0$ such that $[E, D_i] = iD_i$ for any $D_i \in L_i$.*

(3) *For any element $\alpha \in m - m^2$ where m is the maximal ideal of $A(V)$, αE is not in L_0.*

Proof. It was proven in [Ya5] that conditions (1), (2), and (3) are enough to guarantee that $(V, 0)$ is a quasihomogeneous singularity.

Conversely if $(V, 0)$ is a quasihomogeneous singularity, then after a biholomorphic change of variables, we may assume that $(V, 0)$ is a weighted-homogeneous singularity. Conditions (2) and (3) are obviously satisfied where E is nothing but the Euler derivation. Condition (1) follows from Theorem 2.2 and Theorem 2.3. Q.E.D.

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