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American Journal of Mathematics, Volume 118, Number 2, April 1996,
pp. 389-399 (Article)

Published by The Johns Hopkins University Press
DOI: 10.1353/ajm.1996.0020

MICRO-LOCAL CHARACTERIZATION OF QUASI-HOMOGENEOUS SINGULARITIES

By YI-JING XU and STEPHEN S.-T. YAU

Abstract. A moduli algebra $A(V)$ of hypersurface singularity $(V, 0)$ is a finite dimensional **C**-algebra. In 1982, Mather and Yau proved that two germs of complex analytic hypersurfaces of the same dimension with isolated singularities are biholomorphically equivalent if and only if their moduli algebra are isomorphic. It is a natural question to ask for a necessary and sufficient condition for a complex analytic isolated hypersurface singularity to be quasi-homogeneous in terms of its moduli algebra. In this paper we prove that $(V, 0)$ admits a quasi-homogeneous structure if and only if its moduli algebra is isomorphic to a finite dimensional nonnegatively graded algebra. In 1983, Yau introduced a finite dimensional Lie algebra $L(V)$ to an isolated hypersurface singularity $(V, 0)$. $L(V)$ is defined to be the algebra of derivations of the moduli algebra $A(V)$ and is finite dimensional. We prove that $(V, 0)$ is quasi-homogeneous singularity if (1) $L(V)$ is isomorphic to a nonnegatively graded Lie algebra without center, (2) There exists E in $L(V)$ of degree zero such that $[E, D_i] = iD_i$ for any D_i in $L(V)$ of degree i, and (3) For any element $a \in m - m^2$ where m is the maximal ideal of $A(V)$, aE is not in degree zero part of $L(V)$.

0. Introduction. A polynomial $f(z_0, \ldots, z_n)$ is weighted homogeneous of type $(q_0, \ldots, q_n; d)$, where $q_0, \ldots, q_n$ and d are fixed positive integers, if it can be expressed as a linear combination of monomials $z_0^{i_0} z_i^{i_1} \ldots z_n^{i_n}$ for which $q_0 i_0 + q_1 i_1 + \cdots + q_n i_n = d$. In this case, we say that z_i has weight q_i and f has weight d. Recall that an isolated hypersurface singularity $(V, 0) = \{(z_0, \ldots, z_n) : f(z_0, \ldots, z_n) = 0\} \subseteq \mathbf{C}^{n+1}$ is quasi-homogeneous if f is in the Jacobian ideal of f, i.e., $f \in (\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})$. By a theorem of Saito [Sa1], if f is quasi-homogeneous with isolated critical point at 0, then after a biholomorphic change of coordinates, f becomes a weighted homogeneous polynomial.

A moduli algebra $A(V)$ of hypersurface singularity

$$(V, 0) = \{(z_0, \ldots, z_n) : f(z_0, \ldots, z_n) = 0\} \subseteq \mathbf{C}^{n+1}$$

is a finite dimensional **C**-algebra $\mathbf{C}(z_0, z_1, \ldots, z_n)/(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})$. In 1982, Mather and the second author [MY] proved that two germs of complex analytic hypersurfaces of the same dimension with isolated singularities are biholomorphically equivalent if and only if their moduli algebra are isomorphic. It is natural to ask that, as a necessary and sufficient condition, a complex analytic isolated

Manuscript received March 28, 1995; revised July 27, 1995.
Research of both authors supported in part by NSF grant DMS-8822747.
American Journal of Mathematics 118 (1996), 389–399.

hypersurface singularity be quasi-homogeneous in terms of its moduli algebra. In this paper, we prove the following theorem.

THEOREM A. *Let* $(V,0) = \{(z_0,\dots,z_n) : f(z_0,\dots,z_n) = 0\} \subseteq \mathbf{C}^{n+1}$ *be an isolated hypersurface singularity. Then* $(V,0)$ *admits a quasi-homogeneous structure if and only if its moduli algebra* $A(V)$ *is isomorphic to a finite dimensional nonnegatively graded algebra* $\bigoplus_{i\geq 0} A_i$*, with* $A_0 = \mathbf{C}$.

In 1983 [Ya1], the second author introduced a finite dimensional Lie algebra $L(V)$ to an isolated hypersurface singularity $(V,0)$. Defined to be the algebra of derivations of the moduli algebra $A(V)$, $L(V)$ is finite dimensional. It was proved in [Ya3], [Ya2] and [Ya4] that $L(V)$ is actually a solvable Lie algebra. In [S-Y], Seeley and the second author have shown how to use the Lie algebra structures to distinguish complex analytic structures of isolated hypersurface singularities. In fact, they have constructed continuous numerical invariants out of these Lie algebras. Then it is natural to ask if it is possible to characterize the quasi-homogeneity of an isolated hypersurface singularity in terms of its Lie algebra. This question was also asked independently by Lê Dũng Tráng.

Let A be a graded algebra $\bigoplus_{i=0}^{\infty} A_i$. A derivation D of A is said to have weight k if D sends A_i to A_{i+k} for all i. The Lie algebra $L(V)$ of a quasi-homogeneous isolated hypersurface singularity is obviously a graded Lie algebra over $\mathbf{C}$. It also has the left module structure over the moduli algebra $A(V)$. With this in mind, we have the following micro-local characterization of quasi-homogeneity of isolated hypersurface singularity.

THEOREM B. *Let* $(V,0) = \{(z_0,\dots,z_n) \in \mathbf{C}^{n+1} : f(z_0,\dots,z_n) = 0\}$ *be an isolated hypersurface singularity. Let* $A(V) = \mathbf{C}\{z_0,z_1,\dots,z_n\}/(f, \frac{\partial f}{\partial z_0},\dots,\frac{\partial f}{\partial z_n})$ *be the moduli algebra and* $L(V)$ *be the Lie algebra of derivations of* $A(V)$*. Then* $(V,0)$ *is a quasi-homogeneous singularity, i.e.,* $f \in (\frac{\partial f}{\partial z_0},\dots,\frac{\partial f}{\partial z_n})$ *if*

(1) $L(V)$ *is isomorphic to a nonnegatively graded Lie algebra* $\bigoplus_{i=0}^{k} L_i$ *without center.*

(2) *There exists* $E \in L_0$ *such that* $[E, D_i] = iD_i$ *for any* $D_i \in L_i$.

(3) *For any element* $\alpha \in m - m^2$ *where m is the maximal ideal of* $A(V)$*,* αE *is not in* L_0.

It is easy to see that the conditions (2) and (3) above are necessary for $(V,0)$ to be quasi-homogeneous. We expect that (1) above is also a necessary condition. In [C-X-Y], we prove that this is indeed the case for $n = 2$, i.e., $L(V)$ has no negative graded part if $n = 2$. The fact that (1) above is also a necessary condition is a special case of the Halperin conjecture.

The Halperin conjecture is one of the most important open problems in rational homotopy theory. Let $A = F[z_0,\dots,z_n]/I$ where $I = (f_0,\dots,f_n)$ is generated by a regular sequence $f_0,\dots,f_n$ of length $n+1$, and $f_0,\dots,f_n$ are weighted homogeneous with respect to the weights $wt(z_i) = q_i \in Z_+$ for $0 \leq i \leq n$, and F is a field of characteristic zero. Clearly A is a graded algebra and $L(A) := Der(A)$ is also graded (cf. Lemma 2.1).

The Halperin Conjecture. If A is as above, then $L(A)$ is nonnegatively graded.

The topological interpretation of the Halperin conjecture is the following. If the cohomological algebra with coefficient F of a topological space X is isomorphic to A as a graded algebra, then it is known that the vanishing of the negatively graded part of $L(A)$ implies the collapsing at the E_2-term of the Serre spectral sequence with $F-$coefficients of any orientable fibration having X as fibre. Actually, the vanishing of the negatively graded part of $L(A)$ and the collapsing of the E_2-term of the Serre spectral sequence are equivalent for $F = Q$ and X a rational space (see e.g. [Me]). In fact, Proposition 2.6 below confirms the Halperin conjecture in a special case (see also [Th]).

Acknowledgments. We gratefully acknowledge a stimulating discussion with Professor Bruce Bennett. We would also like to thank Professor Lê Dũng Tráng for his encouragement to work on this problem. The second author would like to thank The Johns Hopkins University and The University of Pisa for their warm hospitality while part of this paper was written. We thank Professor C. Huneke for his interest in our work, and Professor S. Halperin for drawing our attention to his beautiful conjecture, as well as for providing us with some references.

1. Characterization of quasi-homogeneity by moduli algebra. We first recall a well-known result of Rossi [Ro] on holomorphic vector fields.

LEMMA 1.1. (Rossi) *Let* $(V,0) = \{(z_0,\dots,z_n) : f(z_0,\dots,z_n) = 0\} \subseteq \mathbf{C}^{n+1}$ *be an isolated hypersurface singularity. Let* $\theta = \sum_{i=0}^{n} a_i(z)\frac{\partial}{\partial z_i}$ *be a holomorphic vector field of* $(V,0)$*. Then* $a_i(0) = 0$ *for* $0 \leq i \leq n$.

THEOREM 1.2. *Let* $(V,0) = \{(z_0,\dots,z_n) : f(z_0,\dots,z_n) = 0\} \subseteq \mathbf{C}^{n+1}$ *be an isolated hypersurface singularity. Then* $(V,0)$ *is quasi-homogeneous if and only if its moduli algebra* $A(V)$ *is isomorphic to a finite dimensional graded commutative local algebra* $\bigoplus_{i\geq 0} A_i$ *with* $A_0 = \mathbf{C}$.

Proof. One direction is obvious. If $(V,0)$ is quasi-homogeneous, then after a biholomorphic change of coordinates, f is a weighted homogeneous polynomial. So the moduli ideal $(f, \frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}) = (\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n})$ is a graded ideal and $A(V) = \bigoplus_{i\geq 0} A_i$ with $A_0 = \mathbf{C}$.

Conversely, suppose $A(V) = \bigoplus_{i\geq 0} A_i$ with $A_0 = \mathbf{C}$. Let $m = \bigoplus_{i\geq 1} A_i$ be the maximal ideal of $A(V)$. Let $\{x_0, x_1, \ldots, x_m\}$ be a $\mathbf{C}$-basis of m/m^2 with $x_i \in A_{q_i}$ for $0 \le i \le m$. Let $E : A(V) \longrightarrow A(V)$ be a linear map such that the restriction of E on A_i is just multiplication by i. Then E is a derivation of $A(V)$. E can be viewed as a derivation of $\mathbf{C}[x_0, x_1, \ldots, x_n]$ which leaves the moduli ideal $(f, \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_m})$ in $\mathcal{O}_{m+1}$ invariant. E is of the form $\sum_{i=0}^m q_i x_i \frac{\partial}{\partial x_i}$. If we let the degree of x_i be q_i for $0 \le i \le m$, then $\mathbf{C}[x_0, x_1, \ldots, x_m]$ is graded and the natural map $\mathbf{C}[x_0, x_1, \ldots, x_m] \longrightarrow A(V)$ is a graded homomorphism of degree 0. Let $\bigoplus_{r>0} J_r$ be the grading of the moduli ideal $(f, \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_m})$. As E is a graded derivation of degree 0, E leaves J_r invariant for all $r > 0$. Since E is injective on J_r and $\dim_{\mathbf{C}} J_r < \infty$, we see that $E : J_r \longrightarrow J_r$ is surjective for all $r > 0$. Hence $E : (f, \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_m}) \longrightarrow (f, \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_m})$ is bijective. Let b_i and $a_{i0}, a_{i1}, \ldots, a_{im}$ be such that

$$E\left(\frac{\partial f}{\partial x_i}\right) = b_i f + \sum_{j=0}^m a_{ij} \frac{\partial f}{\partial x_j} \qquad \text{for all } 0 \le i \le m.$$

By the surjectivity of $E : (f, \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_m}) \longrightarrow (f, \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_m})$, there exist c_i, d_{i0}, $d_{i1}, \ldots, d_{im}$ such that

$$\begin{aligned}
\frac{\partial f}{\partial x_i} &= E\left(c_i f + \sum_{j=0}^m d_{ij} \frac{\partial f}{\partial x_j}\right) \\
&= E(c_i) f + c_i \sum_{j=0}^m q_j x_j \frac{\partial f}{\partial x_j} \\
&\quad + \sum_{j=0}^m E(d_{ij}) \frac{\partial f}{\partial x_j} + \sum_{j=0}^m d_{ij} \left(b_j f + \sum_{l=0}^m a_{jl} \frac{\partial f}{\partial x_l}\right) \\
&= \left(E(c_i) + \sum_{j=0}^m d_{ij} b_j\right) f + \sum_{j=0}^m c_i q_j x_j \frac{\partial f}{\partial x_j} + \sum_{j=0}^m E(d_{ij}) \frac{\partial f}{\partial x_j} + \sum_{j,l=0}^m d_{ij} a_{jl} \frac{\partial f}{\partial x_l} \\
&= \left(E(c_i) + \sum_{j=0}^m d_{ij} b_j\right) f + \sum_{j=0}^m \left[c_i q_j x_j + E(d_{ij}) + \sum_{l=0}^m d_{il} a_{lj}\right] \frac{\partial f}{\partial x_j}.
\end{aligned}$$

Let

$$\theta_i = \sum_{j=0}^m \left[c_i q_j x_j + E(d_{ij}) + \sum_{l=0}^m d_{il} a_{lj} - \delta_{ij}\right] \frac{\partial}{\partial x_j}.$$

Then $\theta_i(f) = -\left[E(c_i) + \sum_{j=0}^m d_{ij} b_j\right] f$. So θ_i is a holomorphic vector field of $\{f(x_1, \ldots, x_m) = 0\}$. By Lemma 1.1, $\theta_{ij}(0) = 0$ for all $0 \le j \le m$ where we write $\theta_i = \sum_{j=0}^m \theta_{ij} \frac{\partial}{\partial x_j}$. Observe that for any $g \in \mathbf{C}[x_0, \ldots, x_m]$, $E(g)$ vanishes at 0.

Therefore we conclude that

$$\left(\sum_{l=0}^{m} d_{il}a_{lj} - \delta_{ij}\right)(0) = 0 \qquad \text{for all } \ 0 \leq i \leq m.$$

This means that

$$(1.2) \qquad \begin{bmatrix} d_{00} & d_{01} & \cdots & d_{0m} \\ d_{10} & d_{11} & \cdots & d_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ d_{m0} & d_{m1} & \cdots & d_{mm} \end{bmatrix}(0) \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0m} \\ a_{10} & a_{11} & \cdots & a_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m0} & a_{m1} & \cdots & a_{mm} \end{bmatrix}(0) = I,$$

where I is the identity matrix. On the other hand, by the surjectivity of

$$E : \left(f, \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_m}\right) \longrightarrow \left(f, \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_m}\right),$$

there exist c and $d_0, d_1, \ldots, d_m$ such that

$$\begin{aligned} f &= E\left(cf + \sum_{i=0}^{m} d_i \frac{\partial f}{\partial x_i}\right) \\ &= E(c)f + c\sum_{j=0}^{m} q_j x_j \frac{\partial f}{\partial x_j} + \sum_{i=0}^{m} E(d_i)\frac{\partial f}{\partial x_i} + \sum_{i=0}^{m} d_i \left(b_i f + \sum_{j=0}^{m} a_{ij}\frac{\partial f}{\partial x_j}\right) \\ &= \left(E(c) + \sum_{i=0}^{m} b_i d_i\right) f + \sum_{j=0}^{m}\left(cq_j x_j + E(d_j) + \sum_{i=0}^{m} d_i a_{ij}\right)\frac{\partial f}{\partial x_j}. \end{aligned}$$

Let

$$H = \sum_{j=0}^{m}\left(cq_j x_j + E(d_j) + \sum_{i=0}^{m} d_i a_{ij}\right)\frac{\partial}{\partial x_j}.$$

Then $H(f) = [1 - E(c) - b_0 d_0 - b_1 d_1 - \cdots - b_m d_m]f$. So H is a vector field of $\{f(x_1, \ldots, x_m) = 0\}$. By Lemma 1.1, $H_i(0) = 0$ for $0 \leq i \leq m$ where $H = \sum_{i=0}^{m} H_i \frac{\partial}{\partial x_i}$. Since $E(d_i)$ vanishes at the origin for $i = 0, 1, \ldots, m$, we conclude that

$$\left(\sum_{i=0}^{m} d_i a_{ij}\right)(0) = 0 \qquad \text{for all } \ 0 \leq j \leq m,$$

i.e.,

$$[d_0, d_1, \dots, d_m](0) \cdot \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0m} \\ a_{10} & a_{11} & \cdots & a_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m0} & a_{m1} & \cdots & a_{mm} \end{bmatrix} (0) = [0, 0, \dots, 0].$$

Since the matrix $[a_{ij}](0)$ is nonsingular by (1.2), we deduce that $(d_0(0), d_1(0), \dots, d_m(0)) = (0, 0, \dots, 0)$. It follows that $1 - E(c) - b_0 d_0 - b_1 d_1 - \cdots - b_m d_m$ is a unit in $\mathcal{O}_{m+1} = \mathbf{C}\{x_0, x_1, \dots, x_m\}$ since $E(c)$ vanishes at the origin. Because of the equation $(1 - E(c) - b_0 d_0 - b_1 d_1 - \cdots - b_m d_m) f = H(f)$, we conclude that $f \in (\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m})\mathcal{O}_{m+1}$. By definition of quasi-homogeneity, $(V, 0)$ is quasi-homogeneous. □

2. Some elementary properties of the Lie algebra of a graded commutative Artinian local algebra. We shall first prove some elementary properties of the Lie algebra of a graded commutative Artinian local algebra.

LEMMA 2.1. *Let $A = \bigoplus_{i=0}^{t} A_i$ be a graded commutative Artinian local algebra. Let $L(A)$ be the derivation algebra of A. Then $L(A)$ is a graded Lie algebra.*

Proof. Let $L_k = \{D \in L(A) : D(A_i) \subseteq A_{i+k} \text{ for all } i\}$. We are going to prove that $L(A) = \bigoplus_k L_k$. Let D be any derivation in $L(A)$. We can write $D = \sum_k D_k$ where $D_k : A \longrightarrow A$ is only a linear map such that $D_k(A_i) \subseteq A_{i+k}$ for all i. To finish the proof, it suffices to show that D_k is a derivation for all k. Let ℓ be the least integer such that $D_\ell \neq 0$. Let a, b be homogeneous elements in A (i.e. of pure weight). Then

$$\begin{aligned} D(ab) &= D(a) \cdot b + a \cdot D(b) \\ &\Rightarrow \sum_k D_k(ab) = \sum_k D_k(a) \cdot b + \sum_k a \cdot D_k(b). \end{aligned}$$

By considering the minimal weight on both sides, we find

$$D_\ell(ab) = D_\ell(a) \cdot b + a \cdot D_\ell(b).$$

In general, we write $a = \sum_i a_i$ and $b = \sum_j b_j$ where a_i and b_j are homogeneous elements of weight i and j respectively. It is easy to check that

$$D_\ell(ab) = a \cdot D_\ell(b) + D_\ell(a) \cdot b.$$

Therefore D_ℓ is a derivation. It follows that $D - D_\ell = \sum_{k \geq \ell+1} D_k$ is also a derivation. The previous argument shows that $D_{\ell+1}$ is a derivation. By induction, the claim is proved. □

Definition 2.2. The *socle* of a local Artinian algebra A with maximal ideal m is the complex vector subspace $\mathrm{Soc}\,A = \{a \in A : a \cdot m = 0\}$ in A. The *type* of A is the complex dimension of Soc A as a vector space. The algebra A is *Gorenstein* when its type is one.

PROPOSITION 2.3. *Let $A = \bigoplus_{i=0}^{k} A_i$ be a graded commutative Artinian local algebra with $A_0 = \mathbf{C}$ and $k \geq 1$. Then $\dim_{\mathbf{C}} L(A) \geq \dim_{\mathbf{C}} A - \dim_{\mathbf{C}} SocA$.*

Proof. Let E be the Euler derivation of A, i.e., E is a weight 0 derivation which sends A_i to itself via multiplication by i. Let $\{x_1, x_2, \ldots, x_n\}$ be a $\mathbf{C}$-basis of m/m^2 with $x_i \in A_{q_i}$ for $1 \leq i \leq n$. Then $E = \sum_{i=1}^{n} q_i x_i \frac{\partial}{\partial x_i}$. Consider the natural map $\phi : A \longrightarrow L(A)$ which sends $a \in A$ to $a \cdot E$ in $L(A)$. Observe that $a \cdot E$ is a zero derivation if and only if a annihilates the maximal ideal of A. This means that $\ker \phi = \mathrm{Soc}\,A$. Hence $\dim_{\mathbf{C}} \phi(A) = \dim_{\mathbf{C}} A - \dim_{\mathbf{C}}$ Soc A. □

Example 2.4. Let $A_t = \bigoplus_{i=0}^{3} A_i$ with $A_0 = \mathbf{C}$, $A_1 = \mathbf{C}x \oplus \mathbf{C}y \oplus \mathbf{C}z$, $A_2 = \mathbf{C}xy \oplus \mathbf{C}yz \oplus \mathbf{C}zx$, and $A_3 = \mathbf{C}xyz$ where the multiplication rules are given as follows:

$$\begin{aligned} x^2 = -\frac{t}{3}yz,\ y^2 &= -\frac{t}{6}zx,\ z^2 = -\frac{t}{6}xy \\ x^2y = xy^2 = y^2z &= yz^2 = x^2z = xz^2 = 0 \\ x^3 = y^3 &= z^3 = -\frac{t}{3}xyz \\ x^iy^jz^k &= 0 \qquad \text{for} \quad i+j+k \geq 4. \end{aligned}$$

We shall assume $t \neq 0$ and $\frac{t^6}{27} - 7t^3 - 216 \neq 0$. Under these assumptions,

$$\begin{aligned} L(A_t) = \Big\langle & x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, xy\frac{\partial}{\partial x} - \frac{t}{6}zx\frac{\partial}{\partial y}, zx\frac{\partial}{\partial x} - \frac{t}{6}xy\frac{\partial}{\partial z}, xy\frac{\partial}{\partial y} - \frac{t}{6}yz\frac{\partial}{\partial x}, \\ & yz\frac{\partial}{\partial y} - \frac{t}{6}xy\frac{\partial}{\partial z}, yz\frac{\partial}{\partial z} - \frac{t}{6}zx\frac{\partial}{\partial y}, zx\frac{\partial}{\partial z} - \frac{t}{6}yz\frac{\partial}{\partial x}, xyz\frac{\partial}{\partial x}, xyz\frac{\partial}{\partial y}, xyz\frac{\partial}{\partial z} \Big\rangle \end{aligned}$$

$$\begin{aligned} \phi(A) = \Big\langle & x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, -\frac{t}{3}yz\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}, xy\frac{\partial}{\partial x} - \frac{t}{3}zx\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}, \\ & zx\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} - \frac{t}{3}xy\frac{\partial}{\partial z}, xyz\frac{\partial}{\partial x}, xyz\frac{\partial}{\partial y}, xyz\frac{\partial}{\partial z} \Big\rangle . \end{aligned}$$

The above example shows that the estimate in Theorem 2.3 is sharp.

LEMMA 2.5. *Let A be a commutative Artinian local algebra. Let $D \in L(A)$ be any derivation of A. Then D preserves the m-adic filtration of A, i.e., $D(m) \subseteq m$ where m is the maximal ideal of A.*

Proof. If the assertation is false, then there exists $a \in m$ such that $D(a) \notin m$. Since A is an Artinian algebra, we can find a smallest integer k such that $a^k \neq 0$ but $a^{k+1} = 0$. Then $0 = D(a^{k+1}) = (k+1)a^k D(a)$, which implies $a^k = 0$ because $D(a)$ is a unit. This leads to a contradiction. □

The following proposition is the motivation for Theorem 3.2 in our next section.

PROPOSITION 2.6. *Let $A = \bigoplus_{i=0}^{k} A_i$ be a graded commutative Artinian local algebra with $A_0 = \mathbf{C}$. Suppose the maximal ideal of A is generated by A_j for some $j > 0$. Then $L(A)$ is a graded Lie algebra without negative weight.*

Proof. By Lemma 2.1, we know that $L(A)$ is a graded Lie algebra. So the only thing we need to prove is that $L(A)$ does not have derivation of negative weight. Let D be such an element. Then $D(A_j) \subset A_0$. On the other hand, Lemma 2.5 says that $D(A_j) \subseteq \oplus_{i=j}^{k} A_i$. It follows that $D(A_j) = 0$. Since the maximal ideal is generated by A_j, we conclude that $D(A) = 0$. □

3. Characterization of quasi-homogeneity by Lie algebra. We begin with the following observation:

PROPOSITION 3.1. *Let $(V, 0)$ be a hypersurface singularity defined by a weighted homogeneous polynomial $f(z_0, z_1, \ldots, z_n)$ which has isolated singularity at the origin with multiplicity of at least three. Suppose that $n \geq 1$. Then the Lie algebra $L(V)$ is graded and without center.*

Proof. Since f is a weighted homogeneous polynomial, the moduli ideal $(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}) = (\frac{\partial f}{\partial z_0} \ldots, \frac{\partial f}{\partial z_n})$ is graded and hence

$$A(V) := \mathbf{C}[z_0, z_1, \ldots, z_n]/(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})$$

is graded. By Lemma 2.1, $L(V)$ is graded. Let D be an element in the center of $L(V)$. Write $D = \sum_i D_i$ where D_i is a derivation with weight i. Let $E = \sum_{i=0}^{n} q_i x_i \frac{\partial}{\partial x_i}$ be the Euler derivation where $q_i = \mathrm{wt}(x_i)$. Then

$$0 = [E, D] = \left[E, \sum_i D_i\right] = \sum_i iD_i$$

which implies $D_i = 0$ for $i \neq 0$. Hence D is a homogeneous element of weight 0. Let $D = \sum_{i=0}^{n} a_i \frac{\partial}{\partial z_i}$. Then

$$0 = [x_i E, D] = x_i[E, D] + [x_i, D]E = -a_i E.$$

This implies that $a_i \in$ Socle of $A(V)$ for all $0 \leq i \leq n$. By local duality, we know that Socle of $A(V)$ is the highest degree nonzero subspace of $A(V)$. In fact, the dimension of Socle of $A(V)$ is precisely one and is spanned by $\operatorname{Hess}(f) := \det[\frac{\partial^2 f}{\partial z_i \partial z_j}]_{i,j=0,1\dots,n}$ (cf. §3 of [Sa2]). We shall assume without loss of generality that $q_0 \geq q_1 \geq \cdots \geq q_n$. Suppose that $\frac{\partial^2 f}{\partial z_0^2} = 0$. Then $\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}$ have degree one in z_0 variable while $\frac{\partial f}{\partial z_0}$ is independent of z_0 variable. These imply that f is singular along the z_0-axis which contradicts our hypothesis that f has isolated singularity at 0. We conclude that $\frac{\partial^2 f}{\partial {z_0}^2} \neq 0$. Since the multiplicity of f is at least three, we have

$$d = \operatorname{wt}(f) \geq \operatorname{wt}(z_n) + 2\operatorname{wt}(z_0) = q_n + 2q_0.$$

On the other hand,

$$\begin{aligned} \operatorname{wt}\operatorname{Hess}(f) &= (d - 2q_0) + (d - 2q_1) + \cdots + (d - 2q_n) \\ &\geq (q_n + 2q_0 - 2q_0) + (q_n + 2q_0 - 2q_1) + \cdots + (q_n + 2q_0 - 2q_n) \\ &\geq nq_n + q_0 \\ &> q_0. \end{aligned}$$

The fact that D is a homogeneous element of weight 0 implies that wt a_i = wt $z_i = q_i$ for all $0 \leq i \leq n$. However $a_i \in \operatorname{Soc} A(V)$ for all $0 \leq i \leq n$ imply wt $a_i = \operatorname{wt}\operatorname{Hess}(f) > q_0 \geq q_i$ for all $0 \leq i \leq n$. This would lead to a contradiction unless $a_i = 0$ for all $0 \leq i \leq n$. Hence $D = 0$. □

THEOREM 3.2. *Let* $(V, 0) = \{(z_0, \dots, z_n) \in \mathbf{C}^{n+1} : f(z_0, \dots, z_n) = 0\}$ *be an isolated hypersurface singularity. Let* $A(V) = \mathbf{C}\{z_0, \dots, z_n\}/(f, \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n})$ *be the moduli algebra and* $L(V)$ *be the Lie algebra of derivations of* $A(V)$. *Then* $(V, 0)$ *is a quasi-homogeneous singularity, i.e.,* $f \in (\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n})$ *if the following conditions are satisfied:*

(1) $L(V)$ *is isomorphic to a nonnegatively graded Lie algebra* $\bigoplus_{i=0}^{k} L_i$ *without center.*

(2) *There exists* $E \in L_0$ *such that* $[E, D_i] = iD_i$ *for any* $D_i \in L_i$.

(3) *For any element* $\alpha \in m - m^2$ *where* m *is the maximal ideal of* $A(V)$, αE *in not in* L_0.

Proof. By conditions (1) and (2), adjoint representation of $L(V)$ is faithful and ad E is semisimple. Take the Jordan decomposition of $E = S + N$ where S is semisimple and N is nilpotent as in [Sa1]. In view of the theorem on page 99 of [Hu], we know that $N = 0$. Therefore, there exists a coordinate $x_0, x_1, \dots, x_n$

such that

$$E = \alpha_0 x_0 \frac{\partial}{\partial x_0} + \alpha_1 x_1 \frac{\partial}{\partial x_1} + \cdots + \alpha_n x_n \frac{\partial}{\partial x_n}.$$

Observe that

$$[E, x_i E] = -x_i[E, E] + [E, x_i]E = \alpha_i x_i E. \tag{3.1}$$

Write $x_i E = D_0 + D_1 + \cdots + D_k$ where $D_i \in L_i$ for all $0 \leq i \leq k$. Then

$$[E, x_i E] = \sum_{j=0}^{k} [E, D_j] = \sum_{j=0}^{k} j D_j. \tag{3.2}$$

On the other hand, (3.1) says that

$$[E, x_i E] = \alpha_i \sum_{j=0}^{k} D_j. \tag{3.3}$$

If $\alpha_i = 0$, the (3.2) and (3.3) imply $D_j = 0$ for all $1 \leq j \leq k$, i.e. $x_i E \in L_0$. This contradicts hypothesis (3) of the Theorem. Therefore, $\alpha_i = j$ for some positive integer j between 1 and k in view of (3.2) and (3.3). Since E acts on $A(V)$, $A(V)$ is graded according to the eigenspace of E. $A(V)$ is nonnegatively graded because all the α_i's are positive integers. Notice that the kernel of E on $A(V)$ is precisely **C**. Hence we can apply Theorem 1.2 to conclude that $(V, 0)$ is a quasi-homogeneous singularity. □

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