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A sharp estimate of the number of integral points in a 4-dimensional tetrahedra

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§ 1. Introduction

The general problem of counting the number Q of nonnegative integral points satisfying

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\leqq 1 \tag{1.1}$$

where a, b and c are positive integers, had been a challenge for many years. In 1951, Mordell [Mo] gave a formula for Q, expressed in terms of three Dedekind sums, in the case that a, b, c are pairwise relatively prime. Recently Pommersheim [Po], using the technique of toric varieties, has given a formula for Q for arbitrary positive integers a, b and c. More generally, let Δ be a polytope of dimension n in the lattice $\mathbb{Z}^n$. Denote $l_\Delta(k)$ the number of lattice points in Δ dilated by a factor of the integer k:

$$l_\Delta(k)=\#(k\Delta\cap\mathbb{Z}^n),\quad k\in\mathbb{Z}^n. \tag{1.2}$$

It was proved in [Eh] that $l_\Delta(k)$ is a polynomial in k of degree n with rational coefficients

$$l_\Delta(k)=a_nk^n+a_{n-1}k^{n-1}+\cdots+a_0.$$

It is called the Ehrhart polynomial of Δ. One would like to find the coefficients a_i in terms of the geometry of Δ. Ehrhart showed that

$a_n =$ volume of Δ,

$a_{n-1} =$ half the sum of the volumes of $n-1$ dimensional faces of Δ

(measured with respect to the $(n-1)$-dimensional lattice in the $(n-1)$-plane),

$a_0 = 1$.

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For a three-dimensional integral convex polytope Δ, Ehrhart's results give

$$l_\Delta(k) = \mathrm{vol}(\Delta)k^3 + \frac{1}{2}S(\Delta)k^2 + a_1 k + 1 \tag{1.3}$$

where $S(\Delta)$ denotes the sum of the lattice volumes of the two dimensional faces of Δ. The contribution of [Po] is the explicit description of a_1 in (1.3). Later, Kantor and Khovanskii [KK] succeeded in computing a_{n-2} in general. In fact, they gave a general formula for the number of integral points in any integral polytope in $\mathbb{R}^4$. Most recently Cappell and Shaneson [CS] have announced a fantastic result with which they can compute all the a_i's in (1.2) explicitly. Unfortunately, as in the case of [Po], they need to assume the vertex points of Δ are in $\mathbb{Z}^n$.

However, for the sake of applications in number theory and geometry which we shall explain below, we are interested in the problem of estimating the number P of positive integral points satisfying (1.1), where a, b and c are positive real numbers. Of course one can deduce the estimate of Q once the estimate of P is known. The novelty in our problem is that we count the lattice points in a polytope whose vertices are not necessarily integer points (or even rational points). In [XY] we proved the following theorem.

Theorem 1.1. *Let $a \geqq b \geqq c \geqq 2$ be real numbers. Then*

$$6P \leqq (a-1)(b-1)(c-1) - c + 1$$

and the equality is attained if and only if $a = b = c =$ integer.

It is a natural question to ask whether one can generalize Theorem 1.1 to higher dimension. Namely, we like to estimate the number P_n of positive integral points satisfying

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leqq 1 \tag{1.4}$$

where $a_1 \geqq a_2 \geqq \cdots \geqq a_n > 0$ are positive real numbers. There are at least two reasons to consider this problem. Firstly, we were told by Professor Granville [Gr] that this is an extremely important question in number theory; it would have many applications to current problems in analytic number theory, primality testing and in factoring. Given a set $\mathscr{P}$ of primes $p_1 < p_2 < \cdots < p_n \leqq y$. Number theorists are interested in counting the number of integers $m \leqq x$ where $m = p_1^{\ell_1} p_2^{\ell_2} \cdots p_n^{\ell_n}$ is composed only of primes from the $\mathscr{P}$, and $x = y^u$ for u not too large ($\forall u > 2$ would be nice, but $\forall u > \log y$ would still be interesting). Thus they wish to count the number of $(\ell_1, \ldots, \ell_n) \in (\mathbb{Z}^+ \cup \{0\})^n$ such that

$$\frac{\ell_1}{a_1} + \frac{\ell_2}{a_2} + \cdots + \frac{\ell_n}{a_n} \leqq 1 \quad \text{where} \quad a_i = \frac{\log x}{\log p_i} \geqq u.$$

This is, of course, the problem that we consider above. Readers may want to consult a special issue of the Philosophical Transactions of the Royal Society Vol. 345, 1993, which is all about this subject and applications. They may also want to look at the excellent reviews by Hildebrand and Tenenbaum [HT] and by Morton [Mo]. Perhaps the best reference for the application is Carl Pomerance's lecture [Pol] at the Zurich ICM. Secondly the problem has interesting application in geometry. Let $f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of complex analytic functions with an isolated critical point at the origin. The Milnor number μ of the singularity is $\dim \mathbb{C}\{z_1, \dots, z_n\}/(f_{z_1}, \dots, f_{z_n})$. Let $\pi: M \to V$ be a resolution of $V = \{(z_1, \dots, z_n) : f(z_1, \dots, z_n) = 0\}$. The geometric genus p_g of the singularity $(V, 0)$ is the dimension of $H^{n-1}(M, \mathcal{O})$. In 1978, Durfee [Du] has made the following conjecture.

Durfee Conjecture. $n!\,p_g \leqq \mu$ with equality only when $\mu = 0$.

The connection between Durfee conjecture and the proposed problem is as follows. A polynomial in $(z_1, \dots, z_n)$ is weighted homogeneous of type $(w_1, \dots, w_n)$, where $w_1, \dots, w_n$ are fixed positive rational numbers if it can be expressed as a linear combination of monomials $z_1^{i_1}, \dots, z_n^{i_n}$ for which $\frac{i_1}{w_1} + \cdots + \frac{i_n}{w_n} = 1$. If f is a weighted homogeneous polynomial of type $(a_1, \dots, a_n)$ with isolated singularity at the origin, then Milnor and Orlik [MO] proved that $\mu = (a_1 - 1)(a_2 - 1) \cdots (a_n - 1)$. On the other hand, Merle and Teissier [MT] showed that p_g is exactly the number P_n of positive integral points satisfying (1.2). Thus Durfee conjecture provide us a guidance for the upper estimate of P_n. Unfortunately Durfee conjecture is not sharp as it can be seen in our Theorem 1.1. The purpose of this paper is to prove the following theorem:

Main Theorem. *Let $a \geqq b \geqq c \geqq d \geqq 2$ be real numbers. Let P_4 be the number of positive integral points satisfying*

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leqq 1. \tag{1.5}$$

Then

$$24P_4 \leqq abcd - \frac{3}{2}(abc + abd + acd + bcd) + \frac{11}{3}(ab + ac + bc) - 2(a + b + c)$$

and the equality is attained if and only if $a = b = c = d =$ integer.

It was pointed out to one of us by Pommersheim that even for 3 dimensional tetrahedron, he cannot deduce our result [XY1] by his formula. In fact the special case of our estimate gives a sharp polynomial upper bound of the Dedekind sum appearing in his formula. Likewise, our sharp estimate of number of integral points in 4-dimensional tetrahedron gives a sharp polynomial upper bound of the generalized Dedekind sum appearing in the formula of Cappell and Shaneson [CS]. Although the strategy of the proof of our theorem is very simple, our proof is quite delicate. This perhaps reflects the fact that sharp inequality is sometimes harder than equality. We first give an estimate of the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leqq 1$ where $a \geqq b \geqq c \geqq 1$ are real numbers (cf. Theorem 2.1). We then apply this result to estimate the number of positive integral solutions of (1.3) on hyperplanes parallel to xyz-plane and sum this estimates up. Unlike the case $n = 3$, we have to face an additional difficulty due to the fact that the right hand side of (2.1) may be negative. Our main theorem follows from a careful analysis of this sum.

We would like to thank professor Granville for providing us various number theoretical references.

§ 2. Sharp upper estimate of the number of integral points in a tetrahedron of dimension 3

The following Theorem 2.1 which is a slight generalization of the theorem in [Xu-Ya] is needed for our later computation.

Theorem 2.1. *Let $a \geqq b \geqq c \geqq 1$ be real numbers. Let P_3 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leqq 1$ i.e.*

$$P_3 = \# \left\{ (x, y, z) \in \mathbb{Z}_+^3 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leqq 1 \right\}$$

where $\mathbb{Z}_+$ is the set of positive integers. If $P_3 > 0$, then

$$6P_3 \leqq (a-1)(b-1)(c-1) - c + 1 \tag{2.1}$$

and the equality is attained if and only if $a = b = c =$ integer.

Proof. We have shown (2.1) is true if $a \geqq b \geqq c \geqq 2$. So we only need to consider the case $1 < c < 2$. Observe that in the latter case, $a \geqq 4$. $\frac{x}{a} + \frac{y}{b} \leqq 1 - \frac{1}{c}$ implies

$$\frac{x}{a\left(1 - \frac{1}{c}\right)} + \frac{y}{b\left(1 - \frac{1}{c}\right)} \leqq 1 . \tag{2.2}$$

By (2.2) of [Xu-Ya], we have

$$\begin{aligned} 6P_3 &\leqq 6 \left\{ \frac{1}{2} a \left(1 - \frac{1}{c}\right) \left[b \left(1 - \frac{1}{c}\right) - 1 \right] + \frac{a}{8b} \right\} \\ &= \left\{ \frac{3a}{c} \left[\frac{b}{c}(c-1) - 1 \right] + \frac{3a}{4b(c-1)} \right\} (c-1) . \end{aligned}$$

In order to prove the theorem, it suffices to prove

$$\frac{3a}{c} \left[\frac{b}{c}(c-1) - 1 \right] + \frac{3a}{4b(c-1)} \leqq (a-1)(b-1) - 1 . \tag{2.3}$$

From (2.2), we may assume that $b\left(1-\frac{1}{c}\right) \geqq 1$, otherwise $P_3 = 0$. So the left hand side of (2.3) has the following estimates:

$$\begin{aligned}\frac{3a}{c}\left[\frac{b}{c}(c-1)-1\right]+\frac{3a}{4b(c-1)} &\leqq \frac{3ab}{c}\left(1-\frac{1}{c}\right)-\frac{3a}{c}+\frac{3a}{4c}\\ &= \frac{3ab}{c}\left(1-\frac{1}{c}\right)-\frac{9a}{4c}\\ &\leqq \frac{3ab}{3c}\left(1-\frac{1}{c}\right)-\frac{9}{8}a\\ &\leqq \frac{3ab}{4}-\frac{9}{8}a\,.\end{aligned}$$

To prove (2.3), it remains to show that

$$\frac{3ab}{4}-\frac{9}{8}a \leqq ab-a-b\,.$$

This is equivalent to show that

$$b \leqq \frac{1}{4}ab+\frac{a}{8}$$

which is obviously true since $a \geqq 4$. Q.E.D.

Corollary 2.2. *Let $a \geqq b \geqq c > 0$ be real numbers. Let Q_3 be the number of non-negative integral points satisfying $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} \leqq 1$. If $Q > 0$, then*

$$Q \leqq \frac{1}{6abc}\left(s^2+(a+b)s\right) \quad \textit{where} \quad s = abc+ab+ac+bc$$

and the equality is attained if and only if $a = b = c =$ integer.

Proposition 2.3. *Let $a \geqq b \geqq c \geqq 1$ be real numbers and*

$$P_3 = \#\left\{(x,y,z)\in\mathbb{Z}_+^3 : \frac{x}{a}+\frac{y}{b}+\frac{z}{c} \leqq 1\right\}.$$

If the R.H.S. of (2.1) $(a-1)(b-1)(c-1)-c+1$ is negative, then $P_3 = 0$ and $b < 2$.

Proof. The R.H.S. of $(2.1) = (a-1)(b-1)(c-1)-c+1 = (c-1)(ab-a-b)$ is negative if and only if $ab-a-b<0$ i.e. $a < \frac{b}{b-1}$. Therefore $b \leqq a < \frac{b}{b-1}$ which implies $b < 2$. Since $b < 2$ implies $P_3 = 0$, it follows that $(c-1)(ab-a-b) < 0$ implies $P_3 = 0$. Q.E.D.

§ 3. Proof of the Main Theorem

We shall first prove that if $a \geqq b \geqq c \geqq d \geqq 2$, then

$$(3.1) \quad 24P_4 \leqq abcd - \frac{3}{2}(abc + abd + acd + bcd) + \frac{11}{3}(ab + ac + bc) - 2(a + b + c).$$

Let $d = [d] + \beta$ with $0 \leqq \beta < 1$. At the level of $w = [d] = d - \beta$, we need to consider the inequality

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{d-\beta}{d} \leqq 1$$

which is equivalent to

$$(3.2) \qquad \frac{x}{\frac{a}{d}\beta} + \frac{y}{\frac{b}{d}\beta} + \frac{z}{\frac{c}{d}\beta} \leqq 1.$$

There are two cases to be considered:

Case (i). If $\frac{a}{d}\beta \geqq 3$, $\frac{b}{d}\beta > 2$ and $\frac{c}{d}\beta > 1$, then (3.2) has positive integral solutions. Therefore in the following estimate (3.3), we have to take $k = [d] = d - \beta$.

Case (ii). If $\frac{a}{d}\beta < 3$ or $\frac{b}{d}\beta \leqq 2$ or $\frac{c}{d}\beta \leqq 1$, then (3.2) has no positive integral solutions. Therefore in the estimate (3.3) below, we need to take $k = [d] - 1 = d - (1 + \beta)$.

In view of the 3-dimensional estimate (cf. Theorem 2.1) we have

$$\begin{aligned}
(3.3) \quad & 6P_4 \\
&\leqq \sum_{u=1}^{k} \left[abc\left(1 - \frac{u}{d}\right)^3 - (ab + ac + bc)\left(1 - \frac{u}{d}\right)^2 + (a+b)\left(1 - \frac{u}{d}\right)\right] \\
&= \sum_{u=1}^{k} \left[abc\left(1 - \frac{3u}{d} + \frac{3u^2}{d^2} - \frac{u^3}{d^3}\right) - (ab + ac + bc)\left(1 - \frac{2u}{d} + \frac{u^2}{d^2}\right) + (a+b)\left(1 - \frac{u}{d}\right)\right] \\
&= \sum_{u=1}^{k} \left[-\frac{abc}{d^3}u^2 + \left(\frac{3abc}{d^2} - \frac{ab + ac + bc}{d^2}\right)u^2 + \left(-\frac{3abc}{d} + \frac{2(ab + ac + bc)}{d} - \frac{a+b}{d}\right)u\right] \\
&\quad + k(abc - ab - ac - bc + a + b) \\
&= -\frac{abc}{d^3}\frac{k^2(k+1)^2}{4} + \frac{3abc - ab - ac - bc}{d^2} \cdot \frac{k(k+1)(2k+1)}{6} \\
&\quad + \frac{-3abc + 2(ab + ac + bc) - (a+b)}{d} \cdot \frac{k(k+1)}{2} + k(abc - ab - ac - bc + a + b)
\end{aligned}$$

$$= -\frac{abc}{4d^3}(k^4+2k^3+k^2) + \frac{3abc-ab-ac-bc}{6d^2}(2k^3+3k^2+k)$$
$$+\frac{-3abc+2ab+2ac+2bc-a-b}{2d}(k^2+k) + (abc-ab-ac-bc+a+b)k$$
$$= -\frac{abc}{4d^3}k^4 + \left(-\frac{abc}{2d^3} + \frac{3abc-ab-ac-bc}{3d^2}\right)k^3$$
$$+\left(-\frac{abc}{4d^3} + \frac{3abc-ab-ac-bc}{2d^2} + \frac{-3abc+2ab+2ac+2bc-a-b}{2d}\right)k^2$$
$$+\left(\frac{3abc-ab-ac-bc}{6d^2} + \frac{-3abc+2ab+2ac+2bc-a-b}{2d}\right.$$
$$\left. + abc-ab-ac-bc+a+b\right)k$$
$$= -\frac{abcd}{4}\left(\frac{k}{d}\right)^4 + \frac{6abcd-3abc-2abd-2acd-2bcd}{6}\left(\frac{k}{d}\right)^3$$
$$+\left(-\frac{abc}{4d} + \frac{-3abcd+3abc+2abd+2acd+2bcd-ab-ac-bc-ad-bd}{2}\right)\left(\frac{k}{d}\right)^2$$
$$+\left(\frac{2abcd-3abc-2abd-2acd-2bcd+2ab+2ac+2bc+2ad+2bd-a-b}{2}\right.$$
$$\left. + \frac{3abc-ab-ac-bc}{6d}\right)\left(\frac{k}{d}\right).$$

However, in ligth of Proposition 2.3, we have to consider when a negative amount is added on the right hand side of (3.3).

(a) If d is an integer, then at level k, we have

$$\frac{x}{a\left(1-\frac{k}{d}\right)} + \frac{y}{b\left(1-\frac{k}{d}\right)} + \frac{z}{c\left(1-\frac{k}{d}\right)} \leqq 1.$$

For $k = d-1$, we have

$$\frac{x}{\frac{a}{d}} + \frac{y}{\frac{b}{d}} + \frac{z}{\frac{c}{d}} \leqq 1.$$

If $b/d < 2$, a negative amount is possibly added in (3.3). For $k < d-1$, we have $b\left(1-\frac{k}{d}\right) \geqq 2\frac{b}{d} \geqq 2$. In view of Proposition 2.3, if d is an integer, we need to take special care for the case $\frac{b}{d} < 2$. In this case we only need to add up to $k = d-2$ level in (3.3).

(b) If d is not an integer, then $d=[d]+\beta$ with $0<\beta<1$. At level $k=[d]=d-\beta$, we have

$$\frac{x}{\frac{a}{d}\beta}+\frac{y}{\frac{b}{d}\beta}+\frac{z}{\frac{c}{d}\beta}\leqq 1.$$

At level $k=[d]-1=d-\beta-1$, we have

$$\frac{x}{\frac{a}{d}(1+\beta)}+\frac{y}{\frac{b}{d}(1+\beta)}+\frac{z}{\frac{c}{d}(1+\beta)}\leqq 1.$$

At all other lower levels, no negative amount problem will occur in (3.3).

Case (i): $\frac{a}{d}\beta\geqq 3$, $\frac{b}{d}\beta>2$ and $\frac{c}{d}\beta>1$. In this case, we take $k=[d]=d-\beta$ in (3.3). In view of Proposition 2.3, we have

$$\begin{aligned}
6P_4 \leqq & -\frac{1}{4}abcd+abc\beta-\frac{3}{2}\frac{abc}{d}\beta^2+\frac{abc}{d^2}\beta^3-\frac{abc}{4d^3}\beta^4 \\
& +abcd-3abc\beta+\frac{3abc}{d}\beta^2-\frac{abc}{d^2}\beta^3 \\
& -\frac{1}{2}abc+\frac{3}{2}\frac{abc}{d}\beta-\frac{3}{2}\frac{abc}{d^2}\beta^2+\frac{1}{2}\frac{abc}{d^3}\beta^3 \\
& -\frac{1}{3}abd+ab\beta-\frac{ab}{d}\beta^2+\frac{1}{3}\frac{ab}{d^2}\beta^3 \\
& -\frac{1}{3}acd+ac\beta-\frac{ac}{d}\beta^2+\frac{1}{3}\frac{ac}{d^2}\beta^3 \\
& -\frac{1}{3}bcd+bc\beta-\frac{bc}{d}\beta^2+\frac{1}{3}\frac{bc}{d^2}\beta^3 \\
& -\frac{abc}{4d}+\frac{abc}{2d^2}\beta-\frac{abc}{4d^3}\beta^2-\frac{3}{2}abcd+3abc\beta-\frac{3}{2}\frac{abc}{d}\beta^2 \\
& +\frac{3}{2}abc-3\frac{abc}{2d^4}\beta+\frac{3abc}{2d^2}\beta^2+abd-2ab\beta+\frac{ab}{d}\beta^2 \\
& +acd-2ac\beta+\frac{ac}{d}\beta^2+bcd-2bc\beta+\frac{bc}{d}\beta^2-\frac{ab}{2}+\frac{ab}{d}\beta \\
& -\frac{ab}{2d^2}\beta^2-\frac{ac}{2}+\frac{ac}{d}\beta-\frac{ac}{2d^2}\beta^2-\frac{bc}{2}+\frac{bc}{d}\beta-\frac{bc}{2d^2}\beta^2
\end{aligned}$$

$$-\frac{ad}{2}+a\beta-\frac{a}{2d}\beta^2-\frac{bd}{2}+b\beta-\frac{b}{2d}\beta^2+\frac{1}{2}\frac{abc}{d}$$

$$-\frac{1}{2}\frac{abc}{d^2}\beta-\frac{1}{6}\frac{ab}{d}+\frac{1}{6}\frac{ab}{d^2}\beta-\frac{1}{6}\frac{ac}{d}+\frac{1}{6}\frac{ac}{d^2}\beta$$

$$-\frac{1}{6}\frac{bc}{d}+\frac{1}{6}\frac{bc}{d^2}\beta+abcd-abc\beta-\frac{3}{2}abc+\frac{3}{2}\frac{abc}{d}\beta$$

$$-abd+ab\beta-acd+ac\beta-bcd+bc\beta+ab-\frac{ab}{d}\beta$$

$$+ac-\frac{ac}{d}\beta+bc-\frac{bc}{d}\beta+ad+bd-\frac{1}{2}a-\frac{1}{2}b$$

$$-a\beta-b\beta+\frac{a}{2d}\beta+\frac{b}{2d}\beta.$$

It follows that

$$(3.4)\quad 6P_4\leqq\frac{1}{4}abcd-\frac{1}{2}abc-\frac{1}{3}abd-\frac{1}{3}acd-\frac{1}{3}bcd+\frac{1}{2}ab+\frac{1}{2}ac$$
$$+\frac{1}{2}bc+\frac{1}{2}ad+\frac{1}{2}bd-\frac{1}{2}(a+b)+\frac{1}{4}\frac{abc}{d}-\frac{1}{6}\frac{ab}{d}-\frac{1}{6}\frac{ac}{d}-\frac{1}{6}\frac{bc}{d}+\Delta_1$$

where

$$(3.5)\quad \Delta_1=\left(-\frac{1}{4}\beta^4+\frac{1}{2}\beta^3-\frac{1}{4}\beta^2\right)\frac{abc}{d^3}+\left(\frac{1}{3}\beta^3-\frac{1}{2}\beta^2+\frac{1}{6}\beta\right)\left(\frac{ab}{d^2}+\frac{ac}{d^2}+\frac{bc}{d^2}\right)$$
$$+\left(\frac{1}{2}\beta-\frac{1}{2}\beta^2\right)\left(\frac{a}{d}+\frac{b}{d}\right).$$

From (3.4) we have

$$(3.6)\quad 24P_4\leqq abcd-2abc-\frac{4}{3}(abd+acd+bcd)+2(ab+ac+bc+ad+bd)$$
$$-2(a+b)+\frac{abc}{d}-\frac{2}{3}\left(\frac{ab}{d}+\frac{ac}{d}+\frac{bc}{d}\right)-\Delta_2$$

where

$$(3.7)\quad \Delta_2=(\beta^4-2\beta^3+\beta^2)\frac{abc}{d^3}-\left(\frac{4}{3}\beta^3-2\beta^2+\frac{2}{3}\beta\right)\left(\frac{ab}{d^2}+\frac{ac}{d^2}+\frac{bc}{d^2}\right)$$
$$-2(\beta-\beta^2)\left(\frac{a}{d}+\frac{b}{d}\right).$$

The difference Δ between the right hand sides of (3.1) and (3.6) is

$$\begin{aligned}(3.8)\qquad \Delta &= \text{R.H.S. of (3.1)} - \text{R.H.S. of (3.6)}\\ &= \frac{1}{2}abc - \frac{1}{6}(abd+acd+bcd) + \frac{5}{3}(ab+ac+bc) - 2ad - 2bd\\ &\quad - 2c - \frac{abc}{d} + \frac{2}{3}\left(\frac{ab}{d}+\frac{ac}{d}+\frac{bc}{d}\right) + \Delta_2 .\end{aligned}$$

We are going to show $\Delta > 0$. There are two subcases to be considered: (a) $\frac{c}{d} \geqq 2$ and (b) $\frac{c}{d} < 2$.

Case (i) subcase (a): $\frac{c}{d} \geqq 2$. Observe that $\beta^4 - 2\beta^3 + \beta^2 = \beta^2(\beta-1)^2 \geqq 0$ and

$$-2(\beta-\beta^2) = 2(\beta^2-\beta) = 2\left[\left(\beta-\frac{1}{2}\right)^2 - \frac{1}{4}\right] \geqq -\frac{1}{2}.$$

Let $f(\beta) = -\frac{4}{3}\beta^3 + 2\beta^2 - \frac{2}{3}\beta$. Then $f'(\beta) = -4\beta^2 + 4\beta - \frac{2}{3}$. So the critical point of $f(\beta)$ is $\frac{3\pm\sqrt{3}}{6}$. Then the minimum value of $f(\beta)$ in $[0,1)$ is attained at $\beta = \frac{3-\sqrt{3}}{6}$ with minimum value $-\frac{\sqrt{3}}{27}$. Therefore

$$\begin{aligned}\Delta &\geqq \frac{1}{2}abc - \frac{1}{6}(abd+acd+bcd) + \frac{5}{3}(ab+ac+bc) - 2ad - 2bd\\ &\quad -2c - \frac{abc}{d} + \frac{2}{3}\left(\frac{ab}{d}+\frac{ac}{d}+\frac{bc}{d}\right) - \frac{\sqrt{3}}{27}\left(\frac{ab}{d^2}+\frac{ac}{d^2}+\frac{bc}{d^2}\right) - \frac{1}{2}\left(\frac{a}{d}+\frac{b}{d}\right)\\ &\geqq \left(\frac{1}{2}abc - \frac{1}{2}abd - \frac{abc}{d} + ab\right) + \left(\frac{2}{3}ab + \frac{5}{3}ac + \frac{5}{3}bc - 2ad - 2bd\right)\\ &\quad + \frac{2}{3}\frac{ab}{d} + \frac{4}{3}(a+b) - 2c - \frac{\sqrt{3}}{27}\left(\frac{ab}{d^2}+\frac{ac}{d^2}+\frac{bc}{d^2}\right) - \frac{1}{2}\left(\frac{a}{d}+\frac{b}{d}\right)\\ &\geqq \frac{1}{2}ab\left(c-d-\frac{2c}{d}+2\right) + \left(\frac{2}{3}ab + \frac{5}{3}ac + \frac{5}{3}bc - 2ad - 2bd\right)\\ &\quad + \left(\frac{2}{3}\frac{ab}{d} - \frac{\sqrt{3}}{9}\frac{ab}{d^2}\right) + (a+b-2c) + \left[\frac{1}{3}(a+b) - \frac{1}{2}\left(\frac{a}{d}+\frac{b}{d}\right)\right].\end{aligned}$$

Notice that

$$c - d - \frac{2c}{d} + 2 = \frac{c}{d}(d-2) - (d-2) = \frac{c-d}{d}(d-2) \geqq 0$$
$$\text{since}\quad a \geqq b \geqq c \geqq d \geqq 2,$$

$$\frac{2}{3}ab + \frac{5}{3}ac + \frac{5}{3}bc - 2ad - 2bd \geqq 0 \quad \text{since} \quad a \geqq b \geqq c \geqq d,$$

$$\frac{2}{3}\frac{ab}{d} - \frac{\sqrt{3}}{9}\frac{ab}{d^2} \geqq 0 \quad \text{since} \quad d \geqq 2 > 1,$$

$$a + b - 2c \geqq 0 \quad \text{since} \quad a \geqq b \geqq c,$$

$$\frac{1}{3}(a+b) - \frac{1}{2}\left(\frac{a}{d} + \frac{b}{d}\right) = (a+b)\left(\frac{1}{3} - \frac{1}{2d}\right) > 0 \quad \text{since} \quad d \geqq 2.$$

Hence we see that $\Delta > 0$ in subcase (a) of case (i).

Case (i) subcase (b): $\frac{c}{d} < 2$. Recall that in case (i), we have assumed $\frac{c}{d}\beta > 1$. It follows that $\beta > \frac{1}{2}$. Therefore in Δ_2, we have

$$\beta^4 - 2\beta^3 + \beta^2 > 0, \quad -\frac{4}{3}\beta^3 + 2\beta^2 - \frac{2}{3}\beta > 0 \quad \text{and} \quad -2(\beta - \beta^2) > -\frac{1}{2} \quad \text{for} \quad \frac{1}{2} < \beta < 1.$$

In view of (3.8) and (3.7), we have

$$\begin{aligned}(3.9) \quad \Delta &> \frac{1}{2}abc - \frac{1}{6}(abd + acd + bcd) + \frac{5}{3}(ab + ac + bc) - 2ad - 2bd - 2c \\ &\quad - \frac{abc}{d} + \frac{2}{3}\left(\frac{ab}{d} + \frac{ac}{d} + \frac{bc}{d}\right) - \frac{1}{2}\left(\frac{a}{d} + \frac{b}{d}\right) \\ &> \left(\frac{1}{2}abc - \frac{1}{2}abd - \frac{abc}{d} + ab\right) + \left(\frac{2}{3}ab + \frac{5}{3}ac + \frac{5}{3}bc - 2ad - 2bd\right) \\ &\quad + \left(\frac{2}{3}\frac{ab}{d} - 2c\right) + \frac{2}{3}\left(\frac{ab}{d} + \frac{bc}{d}\right) - \frac{1}{2}\left(\frac{a}{d} + \frac{b}{d}\right) \\ &\geqq \left(\frac{2}{3}\frac{ab}{d} - 2c\right) + \left(\frac{2}{3}c - \frac{1}{2}\right)\left(\frac{a}{d} + \frac{b}{d}\right).\end{aligned}$$

Observe that

$$\frac{a}{d}\beta \geqq 3 \Rightarrow \frac{a}{d} \geqq \frac{3}{\beta} \geqq 3.$$

It follows from (3.9) that $\Delta > 0$ since $a \geqq b \geqq c \geqq d \geqq 2$.

Case (ii): Either $\frac{a}{d}\beta < 3$ or $\frac{b}{d}\beta \leqq 2$ or $\frac{c}{d}\beta \leqq 1$. In this case, we take $k = [d] - 1 = d - (1 + \beta)$ in (3.3). We have

$$(3.10)\quad 24P_4 \leqq abcd - 2abc - \frac{4}{3}(abd + acd + bcd) + 2(ab + ac + bc + ad + bd)$$
$$- 2(a+b) + \frac{abc}{d} - \frac{2}{3}\left(\frac{ab}{d} + \frac{ac}{d} + \frac{bc}{d}\right)$$
$$+ (-\beta^4 - 2\beta^3 - \beta^2)\frac{abc}{d^3} + \left(\frac{4}{3}\beta^3 + 2\beta^2 + \frac{2}{3}\beta\right)\left(\frac{ab}{d^2} + \frac{ac}{d^2} + \frac{bc}{d^2}\right)$$
$$- 2(\beta + \beta^2)\left(\frac{a}{d} + \frac{b}{d}\right) + \delta$$

where $\delta \geqq 0$ is an adjusting term due to the negative amount added to R.H.S. of (3.3). The difference Δ between the right hand side of (3.1) and (3.10) is

$$(3.11)\qquad \Delta = \text{R.H.S. of } (3.1) - \text{R.H.S. of } (3.10)$$
$$= \frac{1}{2}abc - \frac{1}{6}(abd + acd + bcd) + \frac{5}{3}(ab + ac + bc) - 2ad$$
$$- 2bd - 2c - \frac{abc}{d} + \frac{2}{3}\left(\frac{ab}{d} + \frac{ac}{d} + \frac{bc}{d}\right) + \Delta_3 - \delta$$

where

$$(3.12)\quad \Delta_3 = (\beta^4 + 2\beta^3 + \beta^2)\frac{abc}{d^3} - \left(\frac{4}{3}\beta^3 + 2\beta^2 + \frac{2}{3}\beta\right)\left(\frac{ab}{d^2} + \frac{ac}{d^2} + \frac{bc}{d^2}\right)$$
$$+ 2(\beta + \beta^2)\left(\frac{d}{d} + \frac{b}{d}\right)$$
$$= \beta(\beta+1)\left[\beta(\beta+1)\frac{abc}{d^3} - \frac{2}{3}(2\beta+1)\left(\frac{ab}{d^2} + \frac{ac}{d^2} + \frac{bc}{d^2}\right) + 2\left(\frac{a}{d} + \frac{b}{d}\right)\right].$$

To prove the theorem, we shall show that $\Delta \geqq 0$ for $a \geqq b \geqq c \geqq d \geqq 2$. We shall view Δ as a function of a, b and c with $a \geqq b \geqq c (\geqq d \geqq 2)$ by fixing d and β. Consider the following cases:

Case (ii) subcase (a): On $a = b = c (= d)$ i.e. on the vertex of the region under consideration

(a1) $a = b = c (= d) =$ integer.

In this case, at level $k = d - 1$, we use estimate $(1-1)(1-1)(1-1) - (1-1) = 0$. So $\delta = 0$ and Δ becomes

$$\Delta = \frac{1}{2}a^3 - \frac{1}{2}a^3 + 5a^2 - 4a^2 - 2a - a^2 + 2a + \Delta_3$$
$$= \Delta_3 = 0\,.$$

(a2) $a = b = c (= d) = [d] + \beta$, $0 < \beta < 1$.

We first recall that in case (ii), we do not add level $d-\beta$ in (3.3). At level $k = d-\beta-1$, we use the following estimate in the R.H.S. of (3.3):

$$(1+\beta-1)(1+\beta-1)(1+\beta-1)-(1+\beta-1)=\beta^3-\beta<0.$$

Therefore the adjusting term δ is $\beta-\beta^3$. By the computation in (a1), we have

$$\begin{aligned}\Delta &= \frac{\beta(\beta+1)}{3}(3\beta^2-\beta+4)+\delta\\ &= \frac{\beta(\beta+1)}{3}(3\beta^2-\beta+4)+\beta(\beta+1)(\beta-1)\\ &= \beta(\beta+1)\left(\beta^2+\frac{2}{3}\beta+\frac{1}{3}\right)\\ &> 0.\end{aligned}$$

Case (ii) subcase (b): On $a=b=c(\geqq d)$ i.e. on a 1-dimensional edge. In this case, Δ becomes

$$\begin{aligned}\Delta|_{a=b=c} &= \frac{1}{2}a^3-\frac{1}{2}a^2d+5a^2-4ad-2a-\frac{a^3}{d}+\frac{2a^2}{d}\\ &\quad+\beta(\beta+1)\left[\beta(\beta+1)\frac{a^3}{d^3}-\frac{2}{3}(2\beta+1)\frac{3a^2}{d^2}+4\frac{a}{d}\right]-\delta,\\ \frac{d\Delta|_{a=b=c}}{da} &= \frac{3}{2}a^2-ad+10a-4d-2-\frac{3a^2}{d}+\frac{4a}{d}\\ &\quad+\beta(\beta+1)\left[\beta(\beta+1)\frac{3a^2}{d^3}-4(2\beta+1)\frac{a}{d^2}+\frac{4}{d}\right]\\ &= \frac{a(a-d)(d-2)}{d}+\frac{a^2(d-2)}{2d}+8a-4d-2+\frac{4a}{d}\\ &\quad+\beta(\beta+1)\left[3\beta(\beta+1)\frac{a^2}{d^3}-4(2\beta+1)\frac{a}{d^2}+\frac{4}{d}\right]-\frac{d\delta}{da}.\end{aligned}$$

(b1) $\beta=0$ and $\frac{a}{d}=\frac{b}{d}=\frac{c}{d}\geqq 2$. Then $\delta=0$ and

$$\begin{aligned}\frac{d\Delta|_{a=b=c}}{da} &= \frac{a(a-d)(d-2)}{d}+\frac{a^2(d-2)}{2d}+8a-4d-2+\frac{4a}{d}\\ &> 0 \quad \text{since} \quad a\geqq d\geqq 2.\end{aligned}$$

(b2) $\beta = 0$ and $\frac{a}{d} = \frac{b}{d} = \frac{c}{d} < 2$. Then at level $k = d - 1$, we have estimate

$$\left(\frac{a}{d}-1\right)\left(\frac{b}{d}-1\right)\left(\frac{c}{d}-1\right)-\left(\frac{c}{d}-1\right)=\left(\frac{a}{d}-1\right)^3-\left(\frac{a}{d}-1\right)<0.$$

So $\delta = \left(\frac{a}{d}-1\right)-\left(\frac{a}{d}-1\right)^3 > 0$ and

$$\begin{aligned}\frac{d\Delta|_{a=b=c}}{da} &= \frac{a(a-d)(d-2)}{d}+\frac{a^2(d-2)}{2d}+8a-4d-2+\frac{4a}{d}-\frac{d\delta}{da}\\ &= \frac{a(a-d)(d-2)}{d}+\frac{a^2(d-2)}{2d}+8a-4d-2+\frac{4a}{d}+\frac{3}{d}\left(\frac{a}{d}-1\right)^2-\frac{1}{d}\\ &> 0 \quad \text{since} \quad a \geqq d \geqq 2.\end{aligned}$$

(b3) $0 < \beta < 1$. Since we do not add level $d - \beta$ in (3.3), we only need to consider level $k = d - \beta - 1$. At this level, we have estimate $\left(\frac{a}{d}(1+\beta)-1\right)^3-\left(\frac{a}{d}(1+\beta)-1\right)$.

If $\frac{a}{d}(1+\beta) \geqq 2$, then $\delta = 0$ and

$$\begin{aligned}\frac{d\Delta|_{a=b=c}}{da} &\geqq \frac{a(a-d)(d-2)}{d}+\frac{a^2(d-2)}{2d}+8a-4d-2+\frac{4a}{d}\\ &\quad+\beta(\beta+1)\left\{\frac{3\beta(\beta+1)}{d^3}\left[\frac{2(2\beta+1)d}{3\beta(\beta+1)}\right]^2-\frac{4(2\beta+1)}{d^2}\,\frac{2(2\beta+1)d}{3\beta(\beta+1)}+\frac{4}{d}\right\}\\ &= \frac{a(a-d)(d-2)}{d}+\frac{a^2(d-2)}{2d}+8a-4d-2+\frac{4a}{d}\\ &\quad+\frac{\beta(\beta+1)}{d^3}\left\{\frac{-4(2\beta+1)^2}{3\beta(\beta+1)}d^2+4d^2\right\}\\ &\geqq \frac{a(a-d)(d-2)}{d}+\frac{a^2(d-2)}{2d}+8a-4d-2+\frac{4a}{d}\\ &\quad-\frac{4(\beta^2+\beta+1)}{3d}\\ &\geqq \frac{a(a-d)(d-2)}{d}+\frac{a^2(d-2)}{2d}+8a-4d-2+\frac{4a}{d}-\frac{4}{d}\\ &> 0.\end{aligned}$$

If $\frac{a}{d}(1+\beta) < 2$, then $\delta = \left(\frac{a}{d}(1+\beta)-1\right) - \left(\frac{a}{d}(1+\beta)-1\right)^3$. So

$$\begin{aligned}
\Delta|_{a=b=c} &= \frac{1}{2}a^3 - \frac{1}{2}a^2 d + 5a^2 - 4ad - 2a - \frac{a^3}{d} + \frac{2a^2}{d} \\
&\quad + \beta(\beta+1)\left[\beta(\beta+1)\frac{a^3}{d^3} - \frac{2}{3}(2\beta+1)\frac{3a^2}{d^2} + 4\frac{a}{d}\right] \\
&\quad + \left(\frac{a}{d}(1+\beta)-1\right)^3 - \left(\frac{a}{d}(1+\beta)-1\right),
\end{aligned}$$

$$\begin{aligned}
\frac{d\Delta|_{a=b=c}}{da} &= \frac{a(a-d)(d-2)}{d} + \frac{a^2(d-2)}{2d} + 8a - 4d - 2 + \frac{4a}{d} \\
&\quad + \frac{\beta(\beta+1)}{d}\left[3\beta(\beta+1)\frac{a^2}{d^2} - 4(2\beta+1)\frac{a}{d} + \frac{4}{d}\right] \\
&\quad + \frac{3(1+\beta)}{d}\left(\frac{a}{d}(1+\beta)-1\right)^2 - \frac{1+\beta}{d}.
\end{aligned}$$

Observe that the derivative of $\left(-4d + \frac{4a}{d}\right)$ with respect to d is $\frac{-4(d^2+a^2)}{d^2} < 0$, we deduce from $d \geqq 2$ that

$$-4d + \frac{4a}{d} \geqq -8 + 2a.$$

It follows that

$$\begin{aligned}
\frac{d\Delta|_{a=b=c}}{da} &\geqq \frac{a(a-d)(d-2)}{d} + \frac{a^2(d-2)}{2d} + (8a-2) + (-8+2a) \\
&\quad + \frac{\beta(\beta+1)}{d}\left\{3\beta(\beta+1)\left[\frac{2(2\beta+1)}{3\beta(\beta+1)}\right]^2 - 4(2\beta+1)\left[\frac{2(2\beta+1)}{3\beta(\beta+1)}\right] + \frac{4}{d}\right\} \\
&\quad + 3\left(\frac{a}{d}\right)^2\frac{(1+\beta)^3}{d} - 6\left(\frac{a}{d}\right)\frac{(1+\beta)^2}{d} + 3\frac{(1+\beta)}{d} - \frac{1+\beta}{d} \\
&\geqq 10a - 10 + \frac{\beta(\beta+1)}{d}\left[\frac{4(2\beta+1)^2 - 8(2\beta+1)^2}{3\beta(\beta+1)} + \frac{4}{d}\right] \\
&\quad + 3\left(\frac{a}{d}\right)\frac{(1+\beta)^3}{d} - \frac{12(1+\beta)}{d} + \frac{2(1+\beta)}{d} \\
&= 10a - 10 - \frac{16\beta^2+16\beta+4}{3d} + \frac{4\beta(\beta+1)}{d^2} + 3\left(\frac{a}{d}\right)\frac{(1+\beta)^3}{d} - \frac{10(1+\beta)}{d} \\
&\geqq 10a - 10 - \frac{16\beta^2+16\beta+4}{3d} + \frac{9(1+3\beta+3\beta^2)}{3d} - \frac{10(1+\beta)}{d}
\end{aligned}$$

$$= 10a - 10 - \frac{10(1+\beta)}{d} + \frac{5 + 11\beta + 11\beta^2}{3d}$$

$$\geqq 10a - 20 + \frac{5 + 11\beta + 11\beta^2}{3d}$$

$$> 0 \quad \text{since} \quad a \geqq d \geqq 2.$$

From the computation in (b1), (b2) and (b3), we know that $\frac{d\Delta|_{a=b=c}}{da} > 0$. Therefore the minimum value of $\Delta|_{a=b=c}$ is attained at $a = b = c = d$. In case (ii) subcase (a), we saw that $\Delta \geqq 0$ on $a = b = c = d$. Hence $\Delta \geqq 0$ on $a = b = c \geqq d$. Actually $\Delta > 0$ as long as $a = b = c > d$ or $a = b = c = d$ not an integer.

Case (ii) subcase (c): On $a = b$, i.e. on a surface boundary of the region. In this case, Δ becomes

$$\Delta|_{a=b} = \frac{1}{2}a^2c - \frac{1}{6}(a^2d + 2acd) + \frac{5}{3}(a^2 + 2ac) - 4ad - 2c - \frac{a^2c}{d} + \frac{2}{3}\left(\frac{a^2}{d} + \frac{2ac}{d}\right)$$

$$+ \beta(\beta+1)\left[\beta(\beta+1)\frac{a^2c}{d^3} - \frac{2}{3}(2\beta+1)\left(\frac{a^2}{d^2} + \frac{2ac}{d^2}\right) + 4\frac{a}{d}\right] - \delta,$$

$$\frac{\partial\Delta|_{a=b}}{\partial a} = ac - \frac{1}{3}(ad + cd) + \frac{10}{3}a + \frac{10}{3}c - 4d - \frac{2ac}{d} + \frac{4}{3}\frac{a}{d} + \frac{4}{3}\frac{c}{d}$$

$$+ \beta(\beta+1)\left[\beta(\beta+1)\frac{2ac}{d^3} - \frac{2}{3}(2\beta+1)\left(\frac{2a}{d^2} + \frac{2c}{d^2}\right) + \frac{4}{d}\right] - \frac{\partial\delta}{\partial a}.$$

We want to show that $\frac{\partial\Delta|_{a=b}}{\partial a} > 0$ on $a = b \geqq c (\geqq d)$:

$$\frac{\partial^2\Delta|_{a=b}}{\partial a^2} = c - \frac{d}{3} + \frac{10}{3} - \frac{4}{3d} + \frac{2c}{d} + \beta(\beta+1)\left[\beta(\beta+1)\frac{2c}{d^3} - \frac{2}{3}(2\beta+1)\frac{2}{d^2}\right] - \frac{\partial^2\delta}{\partial a^2}$$

$$\geqq \left(\frac{2d}{3} + \frac{4}{3}\right) + \left(c - d + 2 - \frac{2c}{d}\right) + \frac{4}{3d} + \beta(\beta+1)\cdot\frac{2}{d^2}\left(\beta^2 - \frac{1}{3}\beta - \frac{4}{3}\right) - \frac{\partial^2\delta}{\partial a^2}$$

$$= \left(\frac{2d}{3} + \frac{4}{3}\right) + \frac{(c-d)(d-2)}{d} + \frac{4}{3d} + \beta(\beta+1)\cdot\frac{2}{d^2}\left(\frac{1}{36} - \frac{1}{18} - \frac{4}{3}\right) - \frac{\partial^2\delta}{\partial a^2}$$

$$\geqq \frac{2d}{3} + \frac{4}{3} + \frac{4}{3d} - \frac{49\beta(\beta+1)}{18d^2} - \frac{\partial^2\delta}{\partial a^2}.$$

(c1) $\beta = 0$ and $\frac{a}{d} = \frac{b}{d} \geqq 2$. Then $\delta = 0$ and

$$\frac{\partial^2 \Delta|_{a=b}}{\partial a^2} \geqq \frac{2d}{3} + \frac{4}{3} + \frac{4}{3d} > 0 .$$

Now on $a(=b) = c \geqq d$,

$$\frac{\partial \Delta|_{a=b}}{\partial a} = a^2 - \frac{2}{3}ad + \frac{20}{3}a - 4d - \frac{2a^2}{d} + \frac{8}{3}\frac{a}{d}$$
$$= \frac{a(a-d)(d-2)}{d} + \frac{ad}{3} + \frac{14}{3}a - 4d + \frac{8}{3}\frac{a}{d} > 0 .$$

(c2) $\beta = 0$ and $\frac{a}{d} = \frac{b}{d} < 2$. At the level $k = d - 1$, we have used the estimate $\left(\frac{a}{d} - 1\right)\left(\frac{a}{d} - 1\right)\left(\frac{c}{d} - 1\right) - \left(\frac{c}{d} - 1\right)$ on the right hand side of (3.3). Hence $\delta = -\left(\frac{c}{d} - 1\right)\left[\left(\frac{a}{d} - 1\right)^2 - 1\right]$,

$$\frac{\partial^2 \Delta|_{a=b}}{\partial a^2} \geqq \frac{2d}{3} + \frac{4}{3} + \frac{4}{3d} + \frac{2}{d^2}\left(\frac{c}{d} - 1\right)$$
$$\geqq \frac{2d}{3} + \frac{4}{3} + \frac{4}{3d} > 0 .$$

Now on $a(=b) = c \geqq d$

$$\frac{\partial \Delta|_{a=b}}{\partial a} = a^2 - \frac{2}{3}ad + \frac{20}{3}a - 4d - \frac{2a^2}{d} + \frac{8}{3}\frac{a}{d} + \frac{2}{d}\left(\frac{a}{d} - 1\right)^2$$
$$= \frac{a(a-d)(d-2)}{d} + \frac{ad}{3} + \frac{14}{a} - 4d + \frac{8}{3}\frac{a}{d} + \frac{2}{d}\left(\frac{a}{d} - 1\right)^2$$
$$> 0 .$$

(c3) $0 < \beta < 1$. Since we do not add level $d - \beta$ in (3.3), we only need to consider level $k = d - 1$. At this level, we have used the estimate $\left(\frac{c}{d}(1+\beta) - 1\right)\left\{\left[\frac{a}{d}(1+\beta) - 1\right]^2 - 1\right\}$.

If $\frac{a}{d}(1+\beta) \geqq 2$, then $\delta = 0$,

$$\frac{\partial^2 \Delta|_{a=b}}{\partial a^2} \geqq \frac{2d}{3} + \frac{4}{3} + \frac{4}{3d} - \frac{49\beta(\beta+1)}{18d^2}$$
$$\geqq \frac{4d}{3} + \frac{4}{3d} + \frac{4}{3} - \frac{49}{36} > 0 \quad \text{since} \quad d \geqq 2 .$$

Now on $a(=b) = c \geqq d$,

$$
\begin{aligned}
\frac{\partial \Delta|_{a=b}}{\partial a} &= a^2 - \frac{2}{3}ad + \frac{20}{3}a - 4d - \frac{2a^2}{d} + \frac{8}{3}\frac{a}{d} \\
&\quad + \beta(\beta+1)\left[\beta(\beta+1)\frac{2a^2}{d^3} - \frac{2}{3}(2\beta+1)\left(\frac{4a}{d^2}\right) + \frac{4}{d}\right] \\
&= \left(a^2 - ad + 2a - \frac{2a^2}{d}\right) + \frac{ad}{3} + \frac{14}{3}a - 4d + \frac{8}{3}\frac{a}{d} \\
&\quad + \frac{2\beta(\beta+1)}{3d^3}\left[3\beta(\beta+1)a^2 - 4(2\beta+1)ad + 6d^2\right] \\
&\geqq \frac{a(a-d)(d-2)}{d} + \frac{ad}{3} + \frac{14}{3}a - 4d + \frac{8}{3}\frac{a}{d} - \frac{8\beta(\beta+1)(2\beta+1)a}{3d^2} \\
&> \frac{ad}{3} + \frac{14}{3}a - 4d + \frac{8}{3}\frac{a}{d} - \frac{16a}{d^2} \\
&\geqq \frac{a(a-d)(d-2)}{d} + \frac{ad}{3} + \frac{14}{3}a - 4d + \frac{8a}{3d} \\
&\quad + \frac{2\beta(\beta+1)}{3d^3}\left\{3\beta(\beta+1)\left[\frac{2(2\beta+1)d}{3\beta(\beta+1)}\right]^2 - 4(2\beta+1)\frac{2(2\beta+1)d^2}{3\beta(\beta+1)} + 6d^2\right\} \\
&= \frac{a(a-d)(d-2)}{d} + \frac{ad}{3} + \frac{14}{3}a - 4d + \frac{8a}{3d} \\
&\quad + \frac{2\beta(\beta+1)}{3d^3}\left\{\frac{-4(2\beta+1)^2d^2}{3\beta(\beta+1)} + 6d^2\right\} \\
&= \frac{a(a-d)(d-2)}{d} + \frac{ad}{3} + \frac{14a}{3} - 4d + \frac{8a}{3d} + \frac{4}{9d}(\beta^2+\beta-2) \\
&\geqq \frac{ad}{3} + \frac{14a}{3} - 4d + \frac{8a}{3d} - \frac{8}{9d} \\
&> 0 \quad \text{since} \quad a \geqq d \geqq 2.
\end{aligned}
$$

If $\frac{a}{d}(1+\beta) < 2$, then $\delta = -\left(\frac{c}{d}(1+\beta) - 1\right)\left[\left(\frac{a}{d}(1+\beta) - 1\right)^2 - 1\right]$,

$$
\begin{aligned}
\frac{\partial^2 \Delta|_{a=b}}{\partial a^2} &\geqq \frac{2d}{3} + \frac{4}{3} + \frac{4}{3d} - \frac{49\beta(\beta+1)}{18d^2} + \left(\frac{c}{d}(1+\beta) - 1\right)\frac{2}{d^2}(1+\beta)^2 \\
&= \frac{2d}{3} + \frac{4}{3} + \frac{4}{3d} - \frac{49\beta(\beta+1)}{18d^2} - \frac{2+4\beta+2\beta^2}{d^2} + \frac{c}{d}\,\frac{2(1+\beta)^3}{d^2} \\
&\geqq \frac{2d}{3} + \frac{4}{3} + \frac{4}{3d} - \frac{85\beta^2 + 121\beta + 36}{18d^2} + \frac{36(1+3\beta+3\beta^2)}{18d^2}
\end{aligned}
$$

$$= \frac{2d}{3} + \frac{4}{3} + \frac{4}{3d} + \frac{23\beta^2 - 13\beta}{18d^2}$$

$$> 0 \quad \text{since} \quad d \geqq 2.$$

Now on $a(=b) = c \geqq d$,

$$\frac{\partial \Delta|_{a=b}}{\partial a} = a^2 - \frac{2}{3}ad + \frac{20}{3}a - 4d - \frac{2a^2}{d} + \frac{8}{3}\frac{a}{d}$$

$$+ \beta(\beta+1)\left[\beta(\beta+1)\frac{2a^2}{d^3} - \frac{2}{3}(2\beta+1)\frac{4a}{d^2} + \frac{4}{d}\right] + \frac{2(1+\beta)}{d}\left[\frac{a}{d}(1+\beta) - 1\right]^2$$

$$\geqq a^2 - \frac{2}{3}ad + \frac{20}{3}a - 4d - \frac{2a^2}{d} + \frac{8}{3}\frac{a}{d}$$

$$+ \beta(\beta+1)\left[\beta(\beta+1)\frac{2a^2}{d^3} - \frac{2}{3}(2\beta+1)\frac{4a}{d^2} + \frac{4}{d}\right]$$

$$> 0 \quad \text{as shown before.}$$

From the computation in (c1), (c2) and (c3), we know that

$$\frac{\partial^2 \Delta|_{a=b}}{\partial a^2} > 0 \quad \text{on} \quad a = b \geqq c(\geqq d)$$

and

$$\frac{\partial \Delta|_{a=b}}{\partial a} > 0 \quad \text{on} \quad a = b = c \geqq d.$$

We deduce that $\frac{\partial \Delta|_{a=b}}{\partial a} > 0$ on $a = b \geqq c \geqq d$. It follows that the minimum value of $\Delta|_{a=b}$ on $a = b \geqq c \geqq d$ must be attained at $a = b = c \geqq d$. However we have already shown in case (ii) subcase (b) that $\Delta|_{a=b=c} \geqq 0$. So $\Delta|_{a=b} \geqq 0$. In fact from the conclusion of case (ii) subcase (b), we deduce that $\Delta > 0$ as long as we are not in the situation where $a = b = c = d =$ integer.

Case (ii) subcase (d): On $a \geqq b \geqq c \geqq d$.

$$\Delta = \frac{1}{2}abc - \frac{1}{6}(abd + acd + bcd) + \frac{5}{3}(ab + ac + bc) - 2ad$$

$$- 2bd - 2c - \frac{abc}{d} + \frac{2}{3}\left(\frac{ab}{d} + \frac{ac}{d} + \frac{bc}{d}\right)$$

$$+ \beta(\beta+1)\left[\beta(\beta+1)\frac{abc}{d^3} - \frac{2}{3}(2\beta+1)\left(\frac{ab}{d^2} + \frac{ac}{d^2} + \frac{bc}{d^2}\right)\right.$$

$$\left. + 2\left(\frac{a}{d} + \frac{b}{d}\right)\right] - \delta,$$

$$\frac{\partial \Delta}{\partial a} = \frac{1}{2}bc - \frac{1}{6}(bd+cd) + \frac{5}{3}(b+c) - 2d - \frac{bc}{d} + \frac{2}{3}\left(\frac{b}{d}+\frac{c}{d}\right)$$

$$+ \beta(\beta+1)\left[\beta(\beta+1)\frac{bc}{d^3} - \frac{2}{3}(2\beta+1)\left(\frac{b+c}{d^2}\right) + \frac{2}{d}\right] - \frac{\partial \delta}{\partial a},$$

$$\frac{\partial^2 \Delta}{\partial a \partial b} = \frac{1}{2}c - \frac{d}{6} + \frac{5}{3} - \frac{c}{d} + \frac{2}{3d} + \beta(\beta+1)\left[\beta(\beta+1)\frac{c}{d^3} - \frac{2}{3}(2\beta+1)\frac{1}{d^2}\right]$$

$$= \left(\frac{1}{2}c - \frac{d}{6} + 1 - \frac{c}{d}\right) + \frac{2}{3} + \frac{2}{3d}$$

$$+ \beta(\beta+1)\left[\beta(\beta+1)\frac{1}{d^2} - \frac{2}{3}(2\beta+1)\frac{1}{d^2}\right] - \frac{\partial^2 \delta}{\partial b \partial a}$$

$$= \frac{2cd + (c-d)(d-6)}{6d} + \frac{2}{3} + \frac{2}{3d} + \beta(\beta+1)\left(\beta^2 - \frac{1}{3}\beta - \frac{2}{3}\right)\frac{1}{d^2} - \frac{\partial^2 \delta}{\partial a \partial b}$$

$$\geqq \frac{2cd - 4(c-d)}{6d} + \frac{2}{3} + \frac{2}{3d} + \beta(\beta+1)\frac{-25}{36}\frac{1}{d^2} - \frac{\partial^2 \delta}{\partial a \partial b},$$

$$\frac{\partial^2 \Delta}{\partial a \partial b} = \frac{1}{2}b - \frac{d}{6} + \frac{5}{3} - \frac{c}{d} + \frac{2}{3d} + \beta(\beta+1)\left[\beta(\beta+1)\frac{c}{d^3} - \frac{2}{3}(2\beta+1)\frac{1}{d^2}\right] - \frac{\partial^2 \delta}{\partial a \partial c}$$

$$= \frac{2bd + (b-d)(d-6)}{6d} + \frac{2}{3} + \frac{2}{3d} + \beta(\beta+1)\left(\beta^2 - \frac{1}{3}\beta - \frac{2}{3}\right)\frac{1}{d^2} - \frac{\partial^2 \delta}{\partial a \partial c}$$

$$= \frac{2bd - 4(b-d)}{6d} + \frac{2}{3} + \frac{2}{3d} - \frac{25\beta(\beta+1)}{36d^2} - \frac{\partial^2 \delta}{\partial a \partial c},$$

$$\left.\frac{\partial \Delta}{\partial a}\right|_{b=c=d} = \frac{b^2}{6} + \frac{b}{3} + \frac{4}{3} + \frac{\beta(\beta+1)}{b}\left(\beta^2 - \frac{5}{3}\beta + \frac{2}{3}\right) - \frac{\partial \delta}{\partial a}$$

$$\geqq \frac{b^2}{6} + \frac{b}{3} + \frac{4}{3} + \frac{\beta(\beta+1)}{b}\left(\frac{25}{36} - \frac{25}{18} + \frac{2}{5}\right) - \frac{\partial \delta}{\partial a}$$

$$\geqq \frac{b^2}{6} + \frac{b}{3} + \frac{4}{3} - \frac{1}{18b} - \frac{\partial \delta}{\partial a}.$$

(d1) $\beta = 0$ and $\frac{b}{d} \geqq 2$. Then $\delta = 0$ and

$$\frac{\partial^2 \Delta}{\partial a \partial b} \geqq \frac{2}{3} + \frac{2}{3d} > 0 \quad \text{since} \quad d \geqq 2,$$

$$\frac{\partial^2 \Delta}{\partial a \partial c} \geqq \frac{2}{3} + \frac{2}{3d} > 0 \quad \text{since} \quad d \geqq 2,$$

$$\left.\frac{\partial \Delta}{\partial a}\right|_{b=c=d} > \frac{b^2}{6} + \frac{b}{3} + \frac{4}{3} - \frac{1}{18b} > 0.$$

(d2) $\beta = 0$ and $\frac{b}{d} < 2$. At the level $k = d - 1$, we have used the estimate

$$\left(\frac{a}{d} - 1\right)\left(\frac{b}{d} - 1\right)\left(\frac{c}{d} - 1\right) - \left(\frac{c}{d} - 1\right)$$

on the right hand side of (3.3). If

$$\left(\frac{a}{d} - 1\right)\left(\frac{b}{d} - 1\right)\left(\frac{c}{d} - 1\right) - \left(\frac{c}{d} - 1\right) \geqq 0,$$

then $\delta = 0$ and the necessary estimates are exactly the same as (d1). If

$$\left(\frac{a}{d} - 1\right)\left(\frac{b}{d} - 1\right)\left(\frac{c}{d} - 1\right) - \left(\frac{c}{d} - 1\right) < 0,$$

then

$$\delta = -\left(\frac{a}{d} - 1\right)\left(\frac{b}{d} - 1\right)\left(\frac{c}{d} + 1\right) + \left(\frac{c}{d} - 1\right)$$

$$\begin{aligned}
\frac{\partial^2 \Delta}{\partial a \partial b} &\geqq \frac{2cd - 4(c-d)}{6d} + \frac{2}{3} + \frac{2}{3d} + \frac{1}{d^2}\left(\frac{c}{d} - 1\right) \\
&> 0 \quad \text{since} \quad c \geqq d \geqq 2, \\
\frac{\partial^2 \Delta}{\partial a \partial b} &\geqq \frac{2bd - 4(b-d)}{6d} + \frac{2}{3} + \frac{2}{3d} + \frac{1}{d^2}\left(\frac{b}{d} - 1\right) \\
&> 0 \quad \text{since} \quad b \geqq d \geqq 2, \\
\left.\frac{\partial \Delta}{\partial a}\right|_{b=c=d} &\geqq \frac{b^2}{6} + \frac{b}{3} + \frac{4}{3} - \frac{1}{18b} + \frac{1}{d}\left(\frac{b}{d} - 1\right)\left(\frac{c}{d} - 1\right) > 0.
\end{aligned}$$

(d3) $0 < \beta < 1$. Since we do not add level $d - \beta$ in (3.3), we only need to consider level $k = d - \beta - 1$. At this level, we have used the estimate

$$\left(\frac{a}{d}(1+\beta) - 1\right)\left(\frac{b}{d}(1+\beta) - 1\right)\left(\frac{c}{d}(1+\beta) - 1\right) - \left(\frac{c}{d}(1+\beta) - 1\right). \tag{3.13}$$

If the above estimate $\geqq 0$, then $\delta = 0$,

$$\begin{aligned}
\frac{\partial^2 \Delta}{\partial a \partial b} &\geqq \frac{2cd - 4(c-d)}{6d} + \frac{2}{3} + \frac{2}{3d} - \frac{25\beta(\beta+1)}{36d^2} \\
&\geqq \frac{2}{3} + \frac{2}{3d} - \frac{25}{72} > 0,
\end{aligned}$$

$$\frac{\partial^2 \Delta}{\partial a \partial c} \geqq \frac{2bd - 4(b-d)}{6d} + \frac{2}{3} + \frac{2}{3d} - \frac{25\beta(\beta+1)}{36d^2}$$

$$\geqq \frac{2}{3} + \frac{2}{3d} - \frac{25}{72} > 0,$$

$$\left.\frac{\partial \Delta}{\partial a}\right|_{b=c=d} > \frac{b^2}{6} + \frac{b}{3} + \frac{4}{3} - \frac{1}{18b} > 0.$$

If (3.13) is less than zero, then

$$\delta = -\left(\frac{a}{d}(1+\beta) - 1\right)\left(\frac{b}{d}(1+\beta) - 1\right)\left(\frac{c}{d}(1+\beta) - 1\right) + \left(\frac{c}{d}(1+\beta) - 1\right),$$

$$\frac{\partial^2 \Delta}{\partial a \partial b} \geqq \frac{2cd - 4(c-d)}{6d} + \frac{2}{3} + \frac{2}{3d} - \frac{25\beta(\beta+1)}{36d^2} + \frac{(1+\beta)^2}{d^2}\left(\frac{c}{d}(1+\beta) - 1\right)$$

$$\geqq \frac{2}{3} + \frac{2}{3d} - \frac{25}{72} > 0,$$

$$\frac{\partial^2 \Delta}{\partial a \partial c} \geqq \frac{2bd - 4(b-d)}{6d} + \frac{2}{3} + \frac{2}{3d} - \frac{25\beta(\beta+1)}{36d^2} + \frac{(1+\beta)^2}{d^2}\left(\frac{b}{d}(1+\beta) - 1\right)$$

$$\geqq \frac{2}{3} + \frac{2}{3d} - \frac{25}{72} > 0.$$

In all the cases (d1), (d2) and (d3), since $\frac{\partial^2 \Delta}{\partial a \partial b} > 0$ and $\frac{\partial^2 \Delta}{\partial a \partial c} > 0$ on $a \geqq b \geqq c \geqq d$, we deduce that $\frac{\partial \Delta}{\partial a} > 0$ on $a \geqq b \geqq c \geqq d$ also. It follows that the minimum value of Δ must be attained on the surface $a = b \geqq c \geqq d$. But we have already shown case (ii) subcase (c) that $\Delta \geqq 0$ on this surface $a = b \geqq c \geqq d$. Therefore $\Delta \geqq 0$ on $a \geqq b \geqq c \geqq d$. This completes the proof of case (ii). In view of case (ii) subcase (c), we deduce that $\Delta > 0$ as long as we are not in the situation where $a = b = c = d =$ integer.

It is obvious that if $a = b = c = d =$ integer, then (3.1) becomes an equality. Conversely if (3.1) becomes an equality, then (3.6) and (3.10) must be an equality and $\Delta = 0$. By Theorem 2.1, (3.3) is an equality only if $a = b = c$. We have shown that in case (i), we always have $\Delta > 0$. Therefore if $\Delta = 0$, we must be in case (ii). Hence if (3.1) is an equality, we must be in case (ii) and $a = b = c$ and $\Delta = 0$. The proof of case (ii) subcase (d) shows that $\Delta = 0$ only if $a = b = c = d =$ integer. Therefore we have shown that (3.1) is an equality if and only if $a = b = c = d =$ integer.

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