STANDARD MONOMIALS AND INVARIANT THEORY OF ARC SPACES II: SYMPLECTIC GROUP

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ABSTRACT. This is the second in a series of papers on standard monomial theory and invariant theory of arc spaces. For any algebraically closed field K, we construct a standard monomial basis for the arc space of the Pfaffian variety over K. As an application, we prove the arc space analogue of the first and second fundamental theorems of invariant theory for the symplectic group.

1. INTRODUCTION

Invariant theory. Given an algebraically closed field K, an algebraic group G over K, and a finite-dimensional G-module W, a fundamental problem in invariant theory is to describe the ring of invariant polynomial functions $K[W]^G$. It is natural to also consider $K[V]^G$, where $V = W^{\oplus p} \bigoplus W^{*\oplus q}$ is the direct sum of p copies of W and q copies of the dual G-module W^*. In Weyl's terminology [18], a *first fundamental theorem of invariant theory* (FFT) for the pair (G, W) is a generating set for $K[V]^G$, and a *second fundamental theorem* (SFT) for (G, W) is a generating set for the ideal of relations among the generators of $K[V]^G$. When char $K = 0$, if G is one of the classical groups and W is the standard module, the FFTs and SFTs are due to Weyl [18]. The analogous results in arbitrary characteristic were proven much later by de Concini and Procesi using standard monomial theory [4].

In this paper, we consider the case where G is the symplectic group. For an even integer h, let $W = K^{\oplus h}$ be equipped with a non-degenerate, skew-symmetric bilinear form given by $w = \sum_{i=1}^{h/2} dz_{2i-1} \wedge dz_{2i}$. Then

$$Sp_h(K) = \{A \in SL_h(K) |\ A \text{ preserves } w\}$$

is the symplectic group over K. For $p \geq 1$, let $V = W^{\oplus p}$ be the direct sum of p copies of W. The affine coordinate ring of V is

$$K[V] = K[a_{il}^{(0)} |\ 1 \leq i \leq p,\ 1 \leq l \leq h].$$

Theorem 1.1. *(FFT and SFT for $Sp_h(K)$ and $W = K^{\oplus h}$)*

(1) The ring of invariants $K[V]^{Sp_h(K)}$ is generated by

$$X_{uv}^{(0)} = \sum_{i=1}^{h/2} (a_{u2i-1}^{(0)} a_{v2i}^{(0)} - a_{v2i-1}^{(0)} a_{u2i}^{(0)}), \qquad 1 \leq u, v \leq p.$$

(2) The ideal of relations among the generators in (1) is generated by the Pfaffians

$$\text{(1.1)} \qquad P \begin{pmatrix} X_{u_1 u_1}^{(0)} & X_{u_1 u_2}^{(0)} & \cdots & X_{u_1 u_{h+2}}^{(0)} \\ X_{u_2 u_1}^{(0)} & X_{u_2 u_2}^{(0)} & \cdots & X_{u_2 u_{h+2}}^{(0)} \\ \vdots & \vdots & \vdots & \vdots \\ X_{u_{h+2} u_1}^{(0)} & X_{u_{h+2} u_2}^{(0)} & \cdots & X_{u_{h+2} u_{h+2}}^{(0)} \end{pmatrix}, \qquad 1 \le u_i < u_{i+1} \le p.$$

Standard monomials and Pfaffian varieties. Standard monomial theory was initiated by Seshadri, Musili and Lakshmibai [17, 8, 9, 10], generalizing earlier work of Hodge [6]. It involves combinatorial bases for the coordinate rings of Schubert varieties inside quotients of classical groups by parabolic subgroups. In this paper, we only need the case of Pfaffian varieties. For a positive integer p, let

$$\text{(1.2)} \qquad R = R^p = \mathbb{Z}[x_{uv}^{(0)} \mid 1 \le u, v \le p] / (x_{uv}^{(0)} + x_{vu}^{(0)}, x_{uu}^{(0)})$$

be the ring of polynomial functions with integer coefficients on the space of $p \times p$ skew-symmetric matrices. Consider the Pfaffian $P(B)$ of the skew-symmetric matrix

$$\text{(1.3)} \qquad B = \begin{pmatrix} x_{u_1 u_1}^{(0)} & x_{u_1 u_2}^{(0)} & \cdots & x_{u_1 u_h}^{(0)} \\ x_{u_2 u_1}^{(0)} & x_{u_2 u_2}^{(0)} & \cdots & x_{u_2 u_h}^{(0)} \\ \vdots & \vdots & \vdots & \vdots \\ x_{u_h u_1}^{(0)} & x_{u_h u_2}^{(0)} & \cdots & x_{u_h u_h}^{(0)} \end{pmatrix},$$

with $1 \le u_i < u_{i+1} \le p$ and $h \in 2\mathbb{Z}_{\ge 0}$. Throughout this paper, we will represent $P(B)$ by the ordered h-tuple $|u_h, \ldots, u_2, u_1|$. There is a partial ordering on the set of these Pfaffians given by

$$|u_h, \ldots, u_2, u_1| \le |u'_{h'}, \ldots, u'_2, u'_1|, \quad \text{if } h' \le h, u_i \le u'_i.$$

R has a standard monomial basis (cf. [11]) with respect to this partially ordered set; the ordered products $A_1 A_2 \cdots A_k$ of Pfaffians A_i with $A_i \le A_{i+1}$, form a basis of R.

Similarly, let $R[h]$ be the ideal of R generated by the Pfaffians of the diagonal $h \times h$-minors, which are precisely the elements (1.1) with $h + 2$ replaced by h. Let

$$\text{(1.4)} \qquad R_h = R/R[h+2].$$

Then R_h has a basis consisting of ordered products $A_1 A_2 \cdots A_k$ of the Pfaffians A_i with $|h, \ldots, 2, 1| \le A_i \le A_{i+1}$.

For an arbitrary algebraically closed field K, let $SM_p = SM_p(K)$ be the affine space of $p \times p$ skew-symmetric matrices with entries in K. The affine coordinate ring $K[SM_p]$ is obtained from R by base change, that is, $K[SM_p] = R \otimes_{\mathbb{Z}} K$. Let $K[SM_p][h]$ be the ideal generated by the Pfaffians of the diagonal $h \times h$ minors. The Pfaffian variety $Pf_h = Pf_h(K)$ is a closed subvariety of SM_p with $K[SM_p][h]$ as the defining ideal. Then for an even integer h, the affine coordinate ring $K[Pf_h] = K[SM_p]/K[SM_p][h]$ has a standard monomial basis: the ordered products of $A_1 A_2 \cdots A_k$ with $|h - 2, \ldots, 2, 1| \le A_i \le A_{i+1}$ form a basis of $K[Pf_h]$. For $G = Sp_h(K)$ and V as in Theorem 1.1, we have $V /\!\!/ Sp_k(K) = \text{Spec } K[V]^{Sp_h(K)} \cong Pf_{h+2}$. The proof of Theorem 1.1 in [4] makes use of this standard monomial basis. The key is to consider an integral form of $K[V]$, namely, $\mathbb{Z}[a_{il}^{(0)}]$ for $1 \le i \le p$ and $1 \le l \le h$, and show that the natural map $R_h \to \mathbb{Z}[a_{il}^{(0)}]$ is injective. After tensoring with K, this yields an injective map $K[Pf_{h+2}] \to K[V]$ whose image is precisely

$K[V]^{Sp_h(K)}$. A uniform treatment of these results for all the classical results using standard monomial theory can be found in the book [11] of Lakshmibai and Raghavan.

Arc spaces. Given an irreducible scheme X of finite type over K, the arc space $J_\infty(X)$ is defined as the inverse limit of the finite jet schemes $J_n(X)$ [5]. By Corollary 1.2 of [2], it is determined by its functor of points: for every K-algebra A, we have a bijection

$$\mathrm{Hom}(\mathrm{Spec}\, A, J_\infty(X)) \cong \mathrm{Hom}(\mathrm{Spec}\, A[[t]], X).$$

If G is an algebraic group over K, $J_\infty(G)$ is again an algebraic group over K. If V is a finite-dimensional G-module, there is an induced action of $J_\infty(G)$ on $J_\infty(V)$. The quotient morphism $V \to V/\!\!/G$ induces a morphism $J_\infty(V) \to J_\infty(V/\!\!/G)$, so we have a morphism

(1.5) $$J_\infty(V)/\!\!/J_\infty(G) \to J_\infty(V/\!\!/G),$$

and the corresponding ring homomorphism

(1.6) $$K[J_\infty(V/\!\!/G)] \to K[J_\infty(V)]^{J_\infty(G)}.$$

In the case $K = \mathbb{C}$, if G is connected and $V/\!\!/G$ is smooth, it was shown in [12] that (1.6) is an isomorphism, and under some additional hypotheses this also holds when $V/\!\!/G$ is a complete intersection. In general, (1.6) is neither injective nor surjective.

Standard monomials for arc spaces. Let

(1.7) $$\mathfrak{R} = \mathfrak{R}^p = \mathbb{Z}[x_{uv}^{(k)}|\, 1 \le u,v \le p,\ k \ge 0]/(x_{uv}^{(k)} + x_{vu}^{(k)}, x_{uu}^{(k)}),$$

with a derivation ∂ characterized by $\partial x_{uv}^{(k)} = (k+1)x_{uv}^{(k+1)}$. It can be regarded as the ring of polynomial functions with integer coefficients on the arc space of $p \times p$ skew-symmetric matrices; in particular, $K[J_\infty(SM_p)] \cong \mathfrak{R} \otimes_\mathbb{Z} K$.

Let $\mathfrak{R}[h]$ be the ideal of $\mathfrak{R}$ generated by the Pfaffians of the diagonal $h \times h$ minors in the form of (1.3) and their normalized derivatives $\frac{1}{n!}\partial^n P(B)$. Let $\mathfrak{R}_h = \mathfrak{R}/\mathfrak{R}[h+2]$. Let $\mathcal{J}_r$ be the set of Pfaffians of the matrices of the form of (1.3) with $h \le r$ and their normalized derivatives $\frac{1}{n!}\partial^n P(B)$. Note that R and R_h are naturally subrings of $\mathfrak{R}$ and $\mathfrak{R}_h$, respectively. In Section 2, we will define a notion of standard monomial on $\mathcal{J}_h$ that extends the above notion on R_h, and in Section 3 we will prove the following result.

Theorem 1.2. *For an even integer h, $\mathfrak{R}_h$ has a $\mathbb{Z}$-basis given by the standard monomials of $\mathcal{J}_h$.*

The proof is based on a technical result (Lemma 2.8) whose proof is quite long and is deferred to Section 6.

Let $J_\infty(Pf_{h+2})$ be the arc space of the Pfaffian variety Pf_{h+2}. Then the affine coordinate ring $K[J_\infty(Pf_{h+2})]$ is $\mathfrak{R}_h \otimes_\mathbb{Z} K$, so we immediately have

Corollary 1.3. $K[J_\infty(Pf_{h+2})]$ *has a K-basis given by the standard monomials of $\mathcal{J}_h$.*

Application in invariant theory. Our main application of Theorem 1.2 is to prove the arc space analogue of Theorem 1.1. As above, for an even integer $h \ge 2$, let $Sp_h(K)$ be the symplectic group over K, $W = K^{\oplus h}$ its standard representation, and $V = W^{\oplus p}$ the sum of p copies of W. Then

$$K[J_\infty(V)] = K[a_{il}^{(k)}|\, 1 \le i \le p,\ 1 \le l \le h,\ k \in \mathbb{Z}_{\ge 0}],$$

which has an induced action of $J_\infty(Sp_h(K))$. The following theorem is the arc space analogue of Theorem 1.1, and is proved in Section 4.

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Theorem 1.4. *Let $X_{uv}^{(k)} = \bar{\partial}^k \sum_{i=1}^{h/2} (a_{u2i-1}^{(0)} a_{v2i}^{(0)} - a_{v2i-1}^{(0)} a_{u2i}^{(0)})$, for $1 \le u, v \le p$, where $\bar{\partial}^k = \frac{1}{k!}\partial$.*

(1) The ring of invariants $K[J_\infty(V)]^{J_\infty(Sp_h(K))}$ is generated by $X_{uv}^{(k)}$.

(2) The ideal of relations among the generators in (1) is generated by

$$(1.8) \qquad \bar{\partial}^k P \begin{pmatrix} X_{u_1u_1}^{(0)} & X_{u_1u_2}^{(0)} & \cdots & X_{u_1u_{h+2}}^{(0)} \\ X_{u_2u_1}^{(0)} & X_{u_2u_2}^{(0)} & \cdots & X_{u_2u_{h+2}}^{(0)} \\ \vdots & \vdots & \vdots & \vdots \\ X_{u_{h+2}u_1}^{(0)} & X_{u_{h+2}u_2}^{(0)} & \cdots & X_{u_{h+2}u_{h+2}}^{(0)} \end{pmatrix}, \qquad 1 \le u_i < u_{i+1} \le p.$$

(3) $K[J_\infty(V)]^{J_\infty(Sp_h(K))}$ has a K-basis given by standard monomials of $\mathcal{J}_h$.

Corollary 1.5. *For all $h \ge 1$ and $p \ge 1$, the map $K[J_\infty(V /\!\!/ Sp_h(K))] \to K[J_\infty(V)]^{J_\infty(Sp_h(K))}$ given by (1.6), is an isomorphism. In particular,*

$$J_\infty(V) /\!\!/ J_\infty(Sp_h(K)) \cong J_\infty(V /\!\!/ Sp_h(K)).$$

Corollary 1.5 generalizes Theorem 4.5 of [12], which is the case $K = \mathbb{C}$ and $p \le h + 2$. A similar result was proven in [13] for the general linear group $GL_k(K)$. The approach in this paper is similar to [13] but more involved since we need a result of Bardsley and Richardson [1] which provides a version of the Luna slice theorem in arbitrary characteristic.

Theorem 1.4 has significant applications to vertex algebras which are developed in [14, 15, 3]. First, it provides a complete description of certain cosets of affine vertex algebras inside free field algebras that are related to the classical Howe pairs. This implies the classical freeness of the simple affine vertex (super)algebras $L_k(\mathfrak{osp}_{m|2n})$ for integers $k, m, n \ge 0$ satisfying $-\frac{m}{2} + n + k + 1 > 0$. Next, for any smooth manifold X in either the algebraic, complex analytic or smooth settings, the *chiral de Rham complex* Ω_X^{ch} is a sheaf of vertex algebras on X that was introduced by Malikov, Schechtman and Vaintrob in [16]. Theorem 1.4 is essential in the description of the vertex algebra of global sections $\Gamma(X, \Omega_X^{\mathrm{ch}})$ for a d-dimensional compact Kähler manifold X with holonomy group $Sp(\frac{d}{2})$. This algebra is isomorphic to the simple small $\mathcal{N} = 4$ superconformal algebra with central charge $3d$, and is an important building block in the structure of $\Gamma(X, \Omega_X^{\mathrm{ch}})$ for an arbitrary compact Ricci-flat Kähler manifold X [14].

2. STANDARD MONOMIALS

Fix an integer $p \ge 1$ and recall the ring

$$\mathfrak{R} = \mathbb{Z}[x_{uv}^{(k)} | 1 \le u, v \le p, \ k \ge 0]/(x_{uv}^{(k)} + x_{vu}^{(k)}, x_{uu}^{(k)}),$$

with derivation ∂ given on generators by $\partial x_{uv}^{(k)} = (k+1)x_{uv}^{(k+1)}$. As above, this is an integral version of the coordinate ring of the arc space of the space SM_p of $p \times p$ skew-symmetric matrices, i.e., $K[J_\infty(SM_p)] = \mathfrak{R} \otimes_\mathbb{Z} K$, for any field K.

For $l \ge 0$, we have the lth normalized derivation $\bar{\partial}^l = \frac{1}{l!}\partial^l$ on $\mathfrak{R}$. It satisfies $\bar{\partial}^l x_{ij}^{(k)} = C_{k+l}^l x_{ij}^{(k+l)} \in \mathfrak{R}$, where for $k, n \in \mathbb{Z}_{\ge 0}$,

$$C_n^k = \begin{cases} \frac{n!}{k!(n-k)!}, & 0 \le k \le n; \\ 0, & \text{otherwise.} \end{cases}$$

The following statements are easy to verify.

Proposition 2.1. *For any $a, b \in \mathfrak{R}$,*

$$\bar{\partial}^l(ab) = \sum_{i=0}^{l} \bar{\partial}^i a\, \bar{\partial}^{l-i}b,$$

and $\bar{\partial}^l a \in \mathfrak{R}$.

Proposition 2.2. *For a skew-symmetric matrix B of the form in Equation (1.3) with $h = 2l$,*

$$(2.1) \qquad \bar{\partial}^n P(B) = \sum_{\substack{n_1+\cdots+n_l=n \\ n_i \in \mathbb{Z}_{\geq 0}}} \sum_{\sigma} \frac{\text{sign}(\sigma)}{l!2^l} x_{u_{\sigma(1)}u_{\sigma(2)}}^{(n_1)} x_{u_{\sigma(3)}u_{\sigma(4)}}^{(n_2)} \cdots x_{u_{\sigma(h-1)}u_{\sigma(h)}}^{(n_l)}$$

The second summation is over all permutations σ of $1, 2, \cdots, h$ and $\text{sign}(\sigma)$ is the sign of the permutation.

Generators. Recall that the Pfaffian $P(B)$ of the matrix B in 1.3, is represented by the ordered h-tuple $|u_h, \ldots, u_2, u_1|$ with $1 \leq u_i < u_{i+1} \leq p$ and $h \in 2\mathbb{Z}_{\geq 0}$. Similarly, let

$$(2.2) \qquad J = \bar{\partial}^n |u_h, \ldots, u_2, u_1|$$

represent $\bar{\partial}^n P(B) \in \mathfrak{R}$, the nth normalized derivative of $P(B)$. For convenience, we use the notation $\bar{\partial}^0 |u_h, \ldots, u_2, u_1|$ instead of $|u_h, \ldots, u_2, u_1|$ when $n = 0$, and we shall call such expressions $\bar{\partial}$-*lists* throughout this paper. We call $wt(J) = n$ the weight of J and call $sz(J) = h$ the size of J. Let $\mathcal{J}$ be the set of these $\bar{\partial}$-lists, and

$$\mathcal{J}_h = \{J \in \mathcal{J} | sz(J) \leq h\}$$

be the set of elements in $\mathcal{J}$ with size less than or equal to h. Let $\mathcal{E}$ be the set of ordered h-tuples of ordered pairs of the form

$$(2.3) \qquad E = |(u_h, k_h), \ldots, (u_2, k_2), (u_1, k_1)|$$

with $1 \leq u_i \leq p$, $u_i \neq u_j$ if $i \neq j$ and $k_i \in \mathbb{Z}_{\geq 0}$. Let

$$||E|| = \bar{\partial}^n |u_{\sigma(h)}, \ldots, u_{\sigma(2)}, u_{\sigma(1)}| \in \mathcal{J}.$$

Here $n = \sum k_i$, and σ is the permutation of $1, 2, \ldots, h$ such that $u_{\sigma(i)} < u_{\sigma(i+1)}$. Let

$$wt(E) = wt(||E||), \qquad sz(E) = sz(||E||).$$

Let

$$\mathcal{E}_h = \{E \in \mathcal{E} | \ sz(E) \leq h\}.$$

For $J \in \mathcal{J}$, let

$$\mathcal{E}(J) = \{E \in \mathcal{E} | \ ||E|| = J\}.$$

Note that the Pfaffians represented by $\mathcal{J}$ form a set of generators of $\mathfrak{R}$, and each $J \in \mathcal{J}$ can be represented by an element of the set $\mathcal{E}(J)$.

Ordering. For any set $\mathcal{S}$, let $\mathcal{M}(\mathcal{S})$ be the set of ordered products of elements of $\mathcal{S}$. If $\mathcal{S}$ is an ordered set, we order the set $\mathcal{M}(\mathcal{S})$ lexicographically, that is

$$S_1 S_2 \cdots S_m \prec S'_1 S'_2 \cdots S'_n \quad \text{if } S_i = S'_i, i < i_0, \text{ with } S_{i_0} \prec S'_{i_0} \text{ or } i_0 = m+1, n > m.$$

We order $\mathcal{M}(\mathbb{Z})$, the set of ordered products of integers, lexicographically.

There is an ordering on the set $\mathcal{J}$:

$$\bar{\partial}^k |u_h, \ldots, u_2, u_1| \prec \bar{\partial}^{k'} |u'_{h'}, \ldots, u'_2, u'_1|$$

if

- $h' < h$;
- or $h' = h$ and $k < k'$;
- or $h' = h$, $k = k'$ and $u_h \cdots u_1 \prec u'_h \cdots u'_1$. Here we order the words of natural numbers lexicographically.

We order the pairs $(u, k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ by

$$(u, k) \leq (u', k'), \text{ if } k < k' \text{ or } k = k' \text{ and } u \leq u'.$$

There is a partial ordering on the set $\mathcal{E}$:

$$|(u_h, k_h), \ldots, (u_1, k_1)| \leq |(u'_{h'}, k'_{h'}), \ldots, (u'_1, k'_1)|$$

if $h' \leq h$ and $(u_i, k_i) \leq (u'_i, k'_i)$, for $1 \leq i \leq h'$.
Finally, there is an ordering on $\mathcal{E}$:

$$|(u_h, k_h), \ldots, (u_1, k_1)| \prec |(u'_{h'}, k'_{h'}), \ldots, (u'_1, k'_1)|$$

if

- $h > h'$;
- or $h = h'$ and $\sum k_i < \sum k'_i$;
- or $h = h'$, $\sum k_i = \sum k'_i$ and

$$(u_h, k_h) \cdots (u_1, k_1) \prec (u'_{h'}, k'_{h'}) \cdots (u'_1, k'_1)$$

Here we order the words of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ lexicographically.

Lemma 2.3. *If $E \leq E'$, then $||E|| \prec ||E'||$.*

Proof. If $sz(E') < sz(E)$ or $sz(E) = sz(E')$ and $wt(E) < wt(E')$, then $||E|| \prec ||E'||$.

If $sz(E) = sz(E')$ and $wt(E) = wt(E')$, we must have $k_i = k'_i$. So $u_i \leq u'_i$, we have $||E|| \prec ||E'||$. $\square$

Relations. In the previous notation of the $\bar{\partial}$-list $\bar{\partial}^k |u_h, \ldots, u_1|$, we require $u_i < u_{i+1}$. To describe their relations, we extend this notation $\bar{\partial}^k |u_h, \ldots, u_1|$ to any $1 \leq u_i \leq p$. $\bar{\partial}^k |u_h, \ldots, u_1|$ still represents $\bar{\partial}^k P(B)$ with B in Equation (1.3) without the requirement $u_i < u_j$. Thus we have two obvious relations:

$$\bar{\partial}^k |u_h, \ldots, u_1| = 0$$

if there is $1 \leq i < j \leq h$ such that $u_i = u_j$ and

$$\bar{\partial}^k |u_{\sigma(h)}, \ldots, u_{\sigma(2)}, u_{\sigma(1)}| = \text{sign}(\sigma) J$$

for a $\bar{\partial}$-list $J \in \mathcal{J}$ of (2.2) and a permutation σ of $1, 2, \ldots, h$.

Lemma 2.4. *For $0 \leq k < s$, and $h' \geq s + 1$, we have*

$$\sum_\sigma \frac{1}{h!s!} \text{sign}(\sigma) \bar{\partial}^0 |u_{\sigma(h+s)}, \ldots, u_{\sigma(s+1)}| \, \bar{\partial}^k |u'_{h'}, \ldots, u'_{s+1}, u_{\sigma(s)}, \ldots, u_{\sigma(1)}| \in \mathfrak{R}[h + 2].$$

Here the summation is over all permutations σ of $1, 2, \ldots, h + s$.

Proof. It is easy to see that

$$(2.4) \qquad \bar{\partial}^0 |u_h, \ldots, u_1| = \sum_{i=2}^{h} (-1)^i x_{1,i}^{(0)} \bar{\partial}^0 |u_h, \ldots, u_{i+1}, u_{i-1}, \ldots, u_2|;$$

$$(2.5) \qquad \bar{\partial}^1 |u_h, \ldots, u_1| = \sum_{1 \leq i < j \leq h} (-1)^{i+j+1} x_{i,j}^{(1)} \bar{\partial}^0 |u_h, \ldots, u_{j+1}, u_{j-1}, \ldots, u_{i+1}, u_{i-1}, \ldots, u_1|.$$

Let $l' = h'/2$. For $0 \leq k < s$,

$$\bar{\partial}^k |u'_{h'}, \ldots, u'_{s+1}, u_{\sigma(s)}, \ldots, u_{\sigma(1)}| = \sum_{k_1 + \cdots + k_{l'} = k} \sum_{\sigma'} \frac{\text{sign}(\sigma')}{l'! 2^{l'}} x_{\sigma'(u_{\sigma(1)})\sigma'(u_{\sigma(2)})}^{(k_1)} \cdots x_{\sigma'(u'_{h'-1})\sigma'(u'_{h'})}^{(k_{l'})}$$

$$(2.6) \qquad = \sum_{j=s+1}^{h'} \sum_{i=1}^{s} x_{u_{\sigma(i)} u'_j}^{(0)} f_{ij,\sigma} + \sum_{1 \leq i < j \leq s} x_{u_{\sigma(i)} u_{\sigma(j)}}^{(0)} g_{ij,\sigma} + \sum_{1 \leq i < j \leq s} x_{u_{\sigma(i)} u_{\sigma(j)}}^{(1)} h_{ij,\sigma}.$$

Here $f_{ij,\sigma}, g_{ij,\sigma}, h_{ij,\sigma} \in \mathfrak{R}$, and if σ_1 is a permutation of $1, 2, \ldots, s$ and σ_2 is a permutation of $s + 1, \ldots, h + s$, then

$$x_{u_{\sigma(i)} u}^{(0)} f_{iu,\sigma} = \text{sign}(\sigma_1) x_{u_{\sigma(i)} u}^{(0)} f_{\sigma_1^{-1}(i)u, \sigma\sigma_1\sigma_2},$$
$$x_{u_{\sigma(i)} u_{\sigma(j)}}^{(0)} g_{ij,\sigma} = \text{sign}(\sigma_1) x_{u_{\sigma(i)} u_{\sigma(j)}}^{(0)} g_{\sigma_1^{-1}(i)\sigma_1^{-1}(j), \sigma\sigma_1\sigma_2},$$
$$x_{u_{\sigma(i)} u_{\sigma(j)}}^{(1)} h_{ij,\sigma} = \text{sign}(\sigma_1) x_{u_{\sigma(i)} u_{\sigma(j)}}^{(1)} h_{\sigma_1^{-1}(i)\sigma_1^{-1}(j), \sigma\sigma_1\sigma_2}.$$

We can also require that if σ_3 is a permutation of $i, s + 1, \ldots, h + s$, $f_{iu,\sigma} = f_{iu,\sigma\sigma_3}$ and if σ_4 is a permutation of $i, j, s + 1, \ldots, h + s$, $g_{ij,\sigma} = g_{ij,\sigma\sigma_4}$, $h_{ij,\sigma} = h_{ij,\sigma\sigma_4}$. So by Equation (2.4),

(2.5) and (2.6),

$$\sum_{\sigma} \frac{1}{h!s!} \operatorname{sign}(\sigma)\bar{\partial}^0 |u_{\sigma(h+s)}, \ldots, u_{\sigma(s+1)}|\; \bar{\partial}^k |u'_{h'}, \ldots, u'_{s+1}, u_{\sigma(s)}, \ldots, u_{\sigma(1)}|$$

$$= \sum_{\sigma} \frac{\operatorname{sign}(\sigma)}{h!s!} \sum_{j=s+1}^{h'} \sum_{i=1}^{s} \frac{-1}{h+1} \bar{\partial}^0 |u_{\sigma(h+s)}, \ldots, u_{\sigma(s+1)}, u_{\sigma(i)}, u'_j|\, f_{ij,\sigma}$$

$$+ \sum_{\sigma} \frac{\operatorname{sign}(\sigma)}{h!s!} \sum_{1 \le i < j \le s} \frac{1}{(h+1)} \bar{\partial}^0 |u_{\sigma(h+s)}, \ldots, u_{\sigma(s+1)}, u_{\sigma(j)}, u_{\sigma(i)}|\, g_{ij,\sigma}$$

$$+ \sum_{\sigma} \frac{\operatorname{sign}(\sigma)}{h!s!} \sum_{1 \le i < j \le s} \frac{1}{(h+1)(h+2)} \bar{\partial}^1 |u_{\sigma(h+s)}, \ldots, u_{\sigma(s+1)}, u_{\sigma(j)}, u_{\sigma(i)}|\, h_{ij,\sigma}$$

$$= \sum_{\substack{j=s+1}}^{h'} \sum_{\substack{\sigma(h+s)>\cdots>\sigma(s) \\ \sigma(s-1)>\cdots>\sigma(1)}} -\operatorname{sign}(\sigma)\bar{\partial}^0 |u_{\sigma(h+s)}, \ldots, u_{\sigma(s+1)}, u_{\sigma(s)}, u'_j|\, f_{sj,\sigma}$$

$$+ \sum_{\substack{\sigma(h+s)>\cdots>\sigma(s) \\ \sigma(s-2)>\cdots>\sigma(1)}} \bar{\partial}^0 |u_{\sigma(h+s)}, \ldots, u_{\sigma(s+1)}, u_{\sigma(s)}, u_{\sigma(s-1)}|\, g_{ij,\sigma}$$

$$+ \sum_{\substack{\sigma(h+s)>\cdots>\sigma(s-1) \\ \sigma(s-2)>\cdots>\sigma(1)}} -\operatorname{sign}(\sigma)\bar{\partial}^0 |u_{\sigma(h+s)}, \ldots, u_{\sigma(s+1)}, u_{\sigma(s)}, u_{\sigma(s-1)}|\, h_{ij,\sigma}$$

$$\in \mathfrak{R}[h+2]$$

$\square$

Lemma 2.5. *For $i, j, h, h', k_0, m \in \mathbb{Z}_{\ge 0}$ with $h \ge h'$, $i \le h$, $j \le h'$ and $k_0 \le m$, let $l_0 = i + j - h - 1$. Given any integers a_k for $k_0 \le k \le k_0 + l_0$, there are integers a_k in the range $0 \le k < k_0$, and $k_0 + l_0 < k \le m$, such that*

(2.7) $$\sum_{k=0}^{m} a_k \sum_{\sigma} \frac{1}{i!j!} \operatorname{sign}(\sigma)$$
$$\bar{\partial}^{m-k} |u_h, \ldots, u_{i+1}, \sigma(u_i), \ldots, \sigma(u_1)|\; \bar{\partial}^k |u'_{h'}, \ldots, u'_{j+1}, \sigma(u'_j), \ldots, \sigma(u'_1)| \quad \in \mathfrak{R}[h+2].$$

Here the second summation is over all permutations σ of $u_i, \ldots, u_1, u'_j, \ldots, u'_1$ and $\operatorname{sign}(\sigma)$ is the sign of the permutation.

For simplicity, we write Equation (2.7) in the following way,

(2.8) $$\sum \epsilon a_k \bar{\partial}^{m-k} |u_h, \ldots, u_{i+1}, \underline{u_i, \ldots, u_1}| \bar{\partial}^k |u'_{h'}, \ldots, u'_{j+1}, \underline{u'_j, \ldots, u'_1}| \in \mathfrak{R}[h+2].$$

Remark 2.6. *Since the second summation in Equation (2.7) is over all permutations, each monomial in the equation will appear $i!j!$ times, and the coefficient of each monomial will be $\pm a_k$.*

Proof of Lemma 2.5. Let $\mathcal{F}_l(u_h, \ldots, u_{i+1}; u'_{h'}, \ldots, u'_{j+1})$

$$= \sum_{\sigma} \frac{\operatorname{sign}(\sigma)}{i!j!} \bar{\partial}^0 |u_h, \ldots, u_{i+1}, \sigma(u_i), \ldots, \sigma(u_1)|\; \bar{\partial}^l |u'_{h'}, \ldots, u'_{j+1}, \sigma(u'_j), \ldots, \sigma(u'_1)|.$$

We have $\mathcal{F}_l(u_h, \ldots, u_i; u'_{h'}, \ldots, u'_{j+1})$

$$= \mathcal{F}_l(u_h, \ldots, u_{i+1}; u'_{h'}, \ldots, u'_{j+1}) \pm \mathcal{F}_l(u_h, \ldots, u_{i+1}; u'_{h'}, \ldots, u'_{j+1}, u_i).$$

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If $i = h$ and $0 \leq l \leq l_0$, by Lemma (2.4), $\mathcal{F}_l(\; ; u'_{h'}, \ldots, u'_{j+1}) \in \mathfrak{R}[h + 2]$. By induction on $h - i$, we can see that $\mathcal{F}_l(u_h, \ldots, u_{i+1}; u'_{h'}, \ldots, u'_{j+1}) \in \mathfrak{R}[h + 2]$. Thus

$$\sum_{k=0}^{m} C_k^l \sum_{\sigma} \frac{\mathrm{sign}(\sigma)}{i!j!} \bar{\partial}^{m-k} |u_h, \ldots, u_{i+1}, \sigma(u_i), \ldots, \sigma(u_1)| \; \bar{\partial}^k |u'_{h'}, \ldots, u'_{j+1}, \sigma(u'_j), \ldots, \sigma(u'_1)|$$

$$= \bar{\partial}^{m-l} \mathcal{F}_l(u_h, \ldots, u_{i+1}; u'_{h'}, \ldots, u'_{j+1}) \in \mathfrak{R}[h + 2].$$

Now the $(l_0 + 1) \times (l_0 + 1)$ integer matrix with entries $c_{ji} = C_{k_0+j}^i$, $0 \leq i, j \leq l_0$ is invertible since the determinant of this matrix is ± 1. Let $b_{ij} \in \mathbb{Z}$ be the entries of the inverse matrix. Let $a_k = \sum_{l=0}^{l_0} \sum_{j=0}^{l_0} C_k^l b_{l,j} a_{k_0+j}$. So

$$\text{left hand side of Equation (2.7)} = \sum_{l=0}^{l_0} \sum_{j=0}^{l_0} b_{l,j} a_{k_0+j} \bar{\partial}^{m-l} \mathcal{F}_l(u_h, \ldots, u_{i+1}; u'_{h'}, \ldots, u'_{j+1}) \in \mathfrak{R}[h+2].$$

$\square$

Standard monomials. Now we give the definition of standard monomials of $\mathcal{J}$.

Definition 2.7. *An ordered product $E_1 E_2 \cdots E_m$ of elements of $\mathcal{E}$ is said to be standard if*

(1) $E_a \leq E_{a+1}$, $1 \leq a < m$,
(2) E_1 is the largest in $\mathcal{E}(\|E_1\|)$ under the order $\prec$,
(3) E_{a+1} is the largest in $\mathcal{E}(\|E_{a+1}\|)$ such that $E_a \leq E_{a+1}$.

An ordered product $J_1 J_2 \cdots J_m$ of elements of $\mathcal{J}$ is said to be standard if there is a standard ordered product $E_1 E_2 \cdots E_m$ such that $E_i \in \mathcal{E}(J_i)$.

Let $\mathcal{SM}(\mathcal{J}) \subset \mathcal{M}(\mathcal{J})$ be the set of standard monomials of $\mathcal{J}$. Let $\mathcal{SM}(\mathcal{E}) \subset \mathcal{M}(\mathcal{E})$ be the set of standard monomials of $\mathcal{E}$. Let $\mathcal{SM}(\mathcal{J}_h) = \mathcal{M}(\mathcal{J}_h) \cap \mathcal{SM}(\mathcal{J})$ be the set of standard monomials of $\mathcal{J}_h$. Let $\mathcal{SM}(\mathcal{E}_h) = \mathcal{M}(\mathcal{E}_h) \cap \mathcal{SM}(\mathcal{E})$ be the set of standard monomials of $\mathcal{E}_h$.

By Definition 2.7, if $J_1 J_2 \cdots J_m$ is standard, the standard monomial $E_1 \cdots E_m \in \mathcal{SM}(\mathcal{E})$ corresponding to $J_1 \cdots J_m$ is unique and E_1 has the form

$$|(u_h, wt(E_1)), (u_{h-1}, 0), \ldots, (u_1, 0)| \in \mathcal{E}$$

with $u_i < u_{i+1}$. Therefore the map

$$\pi_h : \mathcal{SM}(\mathcal{E}_h) \to \mathcal{SM}(\mathcal{J}_h), \quad E_1 E_2 \cdots E_m \mapsto \|E_1\| \|E_2\| \cdots \|E_m\|$$

is a bijection.

We order $\mathcal{M}(\mathcal{J})$, the set of ordered products of elements of $\mathcal{J}$, lexicographically. The following lemma will be proved later in Section 6.

Lemma 2.8. *If $J = J_1 \cdots J_b \in \mathcal{M}(\mathcal{J})$ is not standard, J can be written as a linear combination of elements of $\mathcal{M}(\mathcal{J})$ preceding $J_1 \cdots J_{b-1}$, with integer coefficients.*

Recall that $\mathfrak{R}[h]$ denotes the ideal generated by $J \in \mathcal{J}$ with $sz(J) = h$, and $\mathfrak{R}_h = \mathfrak{R}/\mathfrak{R}[h + 2]$. If $h \geq p$, then $\mathcal{J}_h = \mathcal{J}$ and $\mathfrak{R}_h = \mathfrak{R}$. By the above lemma, we immediately have

Lemma 2.9. *Any element of $\mathfrak{R}_h$ can be written as a linear combination of standard monomials of $\mathcal{J}_h$ with integer coefficients.*

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Proof. We only need to show that any element of $\mathfrak{R}$ can be written as a linear combination of standard monomials of $\mathcal{J}$ with integer coefficients. If the lemma is not true, there must be a smallest element $J \in \mathcal{M}(\mathcal{J})$, which cannot be written as a linear combination of elements of $\mathcal{SM}(\mathcal{J})$ with integer coefficients. So J is not standard. By Lemma 2.8, $J = \sum_\alpha \pm J_\alpha$ with $J_\alpha \in \mathcal{M}(\mathcal{J})$ and $J_\alpha \prec J$. Since J_α can be written as a linear combination of elements of $\mathcal{SM}(\mathcal{J})$ with integer coefficients, J can also be written as such a linear combination, which is a contradiction. $\qquad\square$

3. A CANONICAL BASIS

A ring homomorphism. Let

$$\mathcal{S}_h = \{a_{il}^{(k)} \mid 1 \leq i \leq p,\ 1 \leq l \leq h,\ k \in \mathbb{Z}_{\geq 0}\},$$

and let

(3.1)
$$\mathfrak{B} = \mathbb{Z}[\mathcal{S}_h],$$

the polynomial ring generated by $\mathcal{S}_h$. Note that for a field K, if $W = K^{\oplus h}$ and $V = W^{\oplus p}$, the affine coordinate ring $K[J_\infty(V)]$ is obtained from $\mathfrak{B}$ by base change, i.e., $K[J_\infty(V)] \cong \mathfrak{B} \otimes_{\mathbb{Z}} K$.

Let ∂ be the derivation on $\mathfrak{B}$ given by $\partial a_{ij}^{(k)} = (k+1)a_{ij}^{(k+1)}$. We have a ring homomorphism

$$\tilde{Q}_h : \mathfrak{R} \to \mathfrak{B}, \qquad x_{uv}^{(k)} \mapsto \bar{\partial}^k \sum_{i=1}^{h/2} (a_{u2i-1}^{(0)} a_{v2i}^{(0)} - a_{v2i-1}^{(0)} a_{u2i}^{(0)}).$$

For any $J \in \mathcal{J}$ with $sz(J) > h$, we have $\tilde{Q}_h(J) = 0$, so $\tilde{Q}_h$ induces a ring homomorphism

(3.2)
$$Q_h : \mathfrak{R}_h \to \mathfrak{B}.$$

Tableaux. Let $\tilde{\mathcal{S}}_h = \mathcal{S}_h \cup \{*\}$. We define an ordering on the set $\tilde{\mathcal{S}}_h$:
for $a_{ij}^{(k)}, a_{i'j'}^{(k')} \in \mathcal{S}_h$, $a_{ij}^{(k)} < *$ and $a_{ij}^{(k)} \leq a_{i'j'}^{(k')}$ if $kij \prec k'i'j'$.

We use tableaux to represent the monomials of $\mathfrak{B}$. Let $\mathcal{T}$ be the set of the following tableaux:

(3.3)
$$\left| \begin{array}{c} y_{1,h}, \cdots, y_{1,2}, y_{1,1} \\ \vdots \\ y_{m,h}, \cdots, y_{m,2}, y_{m,1} \end{array} \right| .$$

Here $y_{s,l}$ are some $a_{il}^{(k)}$ or $*$, every row of the tableau has elements in $\mathcal{S}_h$ and $y_{s,j} \leq y_{s+1,j}$. We use the tableau (3.3) to represent a monomial in $\mathfrak{B}$, which is the product of $a_{ij}^{(k)'}s$ in the tableau. It is easy to see that the representation is a one-to-one correspondence between $\mathcal{T}$ and the set of monomials of $\mathfrak{B}$. We associate to the tableau (3.3) the word:

$$y_{1,h} \cdots y_{1,1} y_{2,h} \cdots y_{2,1} \cdots y_{m,h} \cdots y_{m,1}$$

and order these words lexicographically. For a polynomial $f \in \mathfrak{B}$, let $Ld(f)$ be its leading monomial in f under the order we defined on $\mathcal{T}$.

For $E_i = |(u^i_{h_i}, k^i_{h_i}), \ldots, (u^i_2, k^i_2), (u^i_1, k^i_1)| \in \mathcal{E}$, $1 \le i \le m$, we use a tableau to represent $E_1 \cdots E_m \in \mathcal{SM}(\mathcal{E})$,

(3.4)
$$
\begin{vmatrix}
(u^1_{h_1}, k^1_{h_1}), & \cdots, & (u^1_2, k^1_2), (u^1_1, k^1_1) \\
(u^2_{h_2}, k^2_{h_2}), & \cdots, & (u^2_2, k^2_2), (u^2_1, k^2_1) \\
& \vdots & \\
(u^m_{h_m}, k^m_{h_m}), & \cdots, & (u^m_2, k^m_2), (u^m_1, k^m_1)
\end{vmatrix}.
$$

Let $T : \mathcal{SM}(\mathcal{J}_h) \to \mathcal{T}$ with

$$
T(E_1 \cdots E_m) =
\begin{vmatrix}
*, \cdots, *, a^{(k^1_{h_1})}_{u^1_{h_1} h_1}, & \cdots, & a^{(k^1_1)}_{u^1_1 1} \\
*, \cdots, *, a^{(k^2_{h_2})}_{u^2_{h_2} h_2}, & \cdots, & a^{(k^2_1)}_{u^2_1 1} \\
& \vdots & \\
*, \cdots, *, a^{(k^m_{h_m})}_{u^m_{h_m} h_m}, & \cdots, & a^{(k^m_1)}_{u^m_1 1}
\end{vmatrix}.
$$

Obviously, T is an injective map and $T(E) \prec T(E')$ if $E \prec E'$.

Lemma 3.1. *Let $J_1 \cdots J_m \in \mathcal{SM}(\mathcal{J}_h)$ and $E_1 \cdots E_m \in \mathcal{SM}(\mathcal{E}_h)$ be its associated standard monomial. Assume the tableau representing $E_1 \cdots E_m$ is (3.4). Then the leading monomial of $Q_h(J_1 \cdots J_m)$ is represented by the tableau $T(E_1 E_2 \cdots E_m)$. Thus*

$$
Ld \circ Q_h = T \circ \pi_h^{-1} : \mathcal{SM}(\mathcal{J}_h) \to \mathcal{T}
$$

is injective. The coefficient of the leading monomial of $Q_h(J_1 \cdots J_m)$ is ± 1.

Proof. Let W_m be the monomial corresponding to the tableau $T(E_1 \cdots E_m)$. Let

$$
M_m = a^{(k^m_{h_m})}_{u^m_{h_m} h_m} \cdots a^{(k^m_1)}_{u^m_1 1}
$$

be the monomial corresponding to the tableau $T(E_m)$. Then $W_m = W_{m-1} M_m$. Let $l_m = \frac{h_m}{2}$, by a direct calculation,

$$
Q_h(J_m) = \sum \pm a^{(k_1)}_{u^m_{\sigma(1)} 2s_1 - 1} a^{(k_2)}_{u^m_{\sigma(2)} 2s_1} a^{(k_3)}_{u^m_{\sigma(3)} 2s_2 - 1} a^{(k_4)}_{u^m_{\sigma(4)} 2s_2} \cdots a^{(k_{h_m-1})}_{u^m_{\sigma(h_m-1)} 2s_{l_m} - 1} a^{(k_{h_m})}_{u^m_{\sigma(h_m)} 2s_{l_m}}.
$$

The summation is over all $k_i \ge 0$ with $\sum k_i = wt(E_m)$, all s_i with $1 \le s_1 < s_2 < \cdots < s_{l_m} \le h/2$ and all permutations σ of $1, 2, \ldots, h_m$.

We prove the lemma by induction on m. If $m = 1$, the lemma is obviously true. Assume it is true for $J_1 \cdots J_{m-1}$. Then $Ld(Q_h(J_1 \cdots J_{m-1})) = W_{m-1}$, the monomial corresponding to $T(E_1 \cdots E_{m-1})$, and the coefficient of W_{m-1} in $Q_h(J_1 \cdots J_{m-1})$ is ± 1. M_m is one of the monomials in $Q_h(J_m)$ with coefficient ± 1. All of the monomials in $Q_h(J_1 \cdots J_{m-1})$ except W_{n-1} are less than W_{n-1}, so any monomial in $Q_h(J_1 \cdots J_{m-1})$ except W_{n-1} times any monomial in $Q_h(J_m)$ is less than W_{m-1}. Since $W_{m-1} \prec W_m$, the coefficient of W_m in $Q_h(J_1 \cdots J_m)$ is not zero. Now

$$
W_{m-1} \prec W_m \prec Ld(Q_h(J_1 \cdots J_m)).
$$

The leading monomial $Ld(Q_h(J_1 \cdots J_m))$ must have the form

$$
W = W_{m-1} a^{(k_1)}_{u^m_{\sigma(1)} 2s_1 - 1} a^{(k_2)}_{u^m_{\sigma(2)} 2s_1} a^{(k_3)}_{u^m_{\sigma(3)} 2s_2 - 1} a^{(k_4)}_{u^m_{\sigma(4)} 2s_2} \cdots a^{(k_{h_m-1})}_{u^m_{\sigma(h_m-1)} 2s_{l_m} - 1} a^{(k_{h_m})}_{u^m_{\sigma(h_m)} 2s_{l_m}}.
$$

If some s_i is greater than $h_{m-1}/2$, then $W \prec W_{n-1}$. So $s_i \le h_{m-1}/2$. We must have $a^{(k_i)}_{u^m_i s_i} \ge a^{(k^{m-1}_{s_i})}_{u^{m-1}_{s_i} s_i}$; otherwise, $W \prec W_{m-1}$. If there is some $h_{m-1}/2 \ge s_i > l_m$, then $W \prec W_m$.

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So we can assume $s_i = i$. Such monomials in $Q_h(J_m)$ are in one-to-one correspondence with $E'_m \in \mathcal{E}(J_m)$ such that $E_{m-1} \leq E'_m$. E_m is the largest in $\mathcal{E}(J_m)$ with $E_{m-1} \leq E_m$ since E is standard, so W_m is the leading term of $Q_h(J_1 \cdots J_m)$. The coefficient of W_m in $Q_h(J_1 \cdots J_m)$ is ± 1 since the coefficients of W_{m-1} in $Q_h(J_1 \cdots J_{m-1})$ and M_m in $Q_h(J_m)$ are ± 1. $\qquad \square$

Proof of Theorem 1.2. By Lemma 3.1, $Ld(Q_h(\mathcal{SM}(\mathcal{J}_h)))$ are linearly independent, so $\mathcal{SM}(\mathcal{J}_h)$ are linearly independent. By Lemma 2.9, $\mathcal{SM}(\mathcal{J}_h)$ generates $\mathfrak{R}_h$. So $\mathcal{SM}(\mathcal{J}_h)$ is a $\mathbb{Z}$-basis of $\mathfrak{R}_h$. $\qquad \square$

Theorem 3.2. $Q_h : \mathfrak{R}_h \to \mathfrak{B}$ *is injective. So we may identify $\mathfrak{R}_h$ with its image $Im(Q_h)$, which is the subring of $\mathfrak{B}$ generated by $\bar{\partial}^k \sum_i^{h/2}(a_{u2i-1}^{(0)} a_{v2i}^{(0)} - a_{v2i-1}^{(0)} a_{u2i}^{(0)})$. In particular, $Q_h(\mathcal{SM}(\mathcal{J}_h))$ is a $\mathbb{Z}$-basis of $Im(Q_h)$.*

Proof. By Lemma 3.1, $Ld(Q_h(\mathcal{SM}(\mathcal{J}_h)))$ are linearly independent. Since $\mathcal{SM}(\mathcal{J}_h)$ is a $\mathbb{Z}$-basis of $\mathfrak{R}_h$, $Q_h : \mathfrak{R}_h \to \mathfrak{B}$ is injective. $\qquad \square$

Since Q_h is injective and $\mathfrak{B}$ is an integral domain,

Corollary 3.3. $\mathfrak{R}_h$ *is an integral domain.*

4. APPLICATION

In this section, we give the main application of our standard monomial basis, which is the arc space analogue of Theorem 1.1.

Arc spaces. Suppose that X is a scheme of finite type over K. Its arc space (cf. [5]) $J_\infty(X)$ is determined by its functor of points. For every K-algebra A, we have a bijection

$$\mathrm{Hom}(\mathrm{Spec}\, A, J_\infty(X)) \cong \mathrm{Hom}(\mathrm{Spec}\, A[[t]], X).$$

If $i : X \to Y$ is a morphism of schemes, we get a morphism of schemes $i_\infty : J_\infty(X) \to J_\infty(Y)$. If i is a closed immersion, then i_∞ is a closed immersion. If $i : X \to Y$ is an étale morphism, then we have a Cartesian diagram

$$\begin{array}{ccc} J_\infty(X) & \to & J_\infty(Y) \\ \downarrow & & \downarrow \\ X & \to & Y \end{array}$$

If $X = \mathrm{Spec}\, K[x_1, \ldots, x_n]$, then $J_\infty(Y) = \mathrm{Spec}\, K[x_i^{(k)} | 1 \leq i \leq n, k \in \mathbb{Z}_{\geq 0}]$. The identification is made as follows: for a K-algebra A, a morphism $\phi : K[x_1, \ldots, x_n] \to A[[t]]$ determined by $\phi(x_i) = \sum_{k=0}^\infty a_i^{(k)} t^k$ corresponds to a morphism $K[x_i^{(k)}] \to A$ determined by $x_i^{(k)} \to a_i^{(k)}$. Note that $K[x_1, \ldots, x_n]$ can be identified with the subalgebra $K[x_1^{(0)}, \ldots, x_n^{(0)}] \subset K[x_i^{(k)}]$, and from now on we use $x_i^{(0)}$ instead of x_i.

The polynomial ring $K[x_i^{(k)}]$ has a derivation ∂ defined on generators by

$$(4.1) \qquad \qquad \partial x_i^{(k)} = (k+1)x_i^{(k+1)}.$$

It is more convenient to work with the normalized k-derivation $\bar{\partial}^k = \frac{1}{k!}\partial^k$, but this is a priori not well-defined on $K[x_i^{(k)}]$ if char K is positive. But ∂ is well defined on $\mathbb{Z}[x_i^{(k)}]$,

and $\bar{\partial}^k$ maps $\mathbb{Z}[x_i^{(k)}]$ to itself, so for any K, there is an induced K-linear map

(4.2) $$\bar{\partial}^k : K[x_i^{(k)}] \to K[x_i^{(k)}],$$

obtained by tensoring with K.

If X is the affine scheme $\operatorname{Spec} K[x_1^{(0)}, \ldots, x_n^{(0)}]/(f_1, \ldots, f_r)$, then $J_\infty(X)$ is the affine scheme
$$\operatorname{Spec} K[x_i^{(k)} \mid i = 1, \ldots, k \in \mathbb{Z}_{\geq 0}]/(\bar{\partial}^l f_j \mid j = 1, \ldots, r, \ l \geq 0).$$
Indeed, for every $f \in K[x_1^{(0)}, \ldots, x_n^{(0)}]$, we have

$$\phi(f) = \sum_{k=0}^{\infty} (\bar{\partial}^k f)(a_1^{(0)}, \ldots, a_n^{(k)}) \, t^k.$$

It follows that ϕ induces a morphism $K[x_1^{(0)}, \ldots, x_n^{(0)}]/(f_1, \ldots, f_r) \to A[[t]]$ if and only if

$$(\bar{\partial}^k f_i)(a_1^{(0)}, \ldots, a_n^{(k)}) = 0, \ \text{for all } i = 1, \ldots, r, \ k \geq 0.$$

If Y is the affine scheme $\operatorname{Spec} K[y_1^{(0)}, \ldots, y_m^{(0)}]/(g_1, \ldots, g_s)$, a morphism $P : X \to Y$ gives a ring homomorphism $P^* : K[Y] \to K[X]$. Then the induced homomorphism of arc spaces $P_\infty : J_\infty(X) \to J_\infty(Y)$ is given by

$$P_\infty^*(y_i^{(k)}) = \bar{\partial}^k P^*(y_i^{(0)}).$$

In particular, P_∞^* commutes with $\bar{\partial}^k$ for all $k \geq 0$.

Arc space of the Pfaffian variety. Recall that the space SM_p of skew-symmetric $p \times p$ matrices over K has affine coordinate ring

$$K[SM_p] = K[x_{ij}^{(0)} \mid 1 \leq i, \ j \leq p]/(x_{ij}^{(0)} + x_{ji}^{(0)}, x_{ii}^{(0)}) = R \otimes_{\mathbb{Z}} K,$$

where R is given by (1.2). The Pfaffian variety Pf_h is the subvariety of SM_p determined by the ideal $K[SM_p][h]$ generated by the Pfaffians of all diagonal h-minors, so

$$K[Pf_h] = K[SM_p]/K[SM_p][h] = R_{h-2} \otimes_{\mathbb{Z}} K,$$

where R_{h-2} is given by (1.4). Similarly, recall that

$$K[J_\infty(SM_p)] = K[x_{ij}^{(k)} \mid 1 \leq i, \ j \leq p]/(x_{ij}^{(k)} + x_{ji}^{(k)}, x_{ii}^{(0)}) = \mathfrak{R} \otimes_{\mathbb{Z}} K.$$

Then

$$K[J_\infty(Pf_h)] = K[J_\infty(SM_p)]/K[J_\infty(SM_p)][h-2] \cong \mathfrak{R}_{h-2} \otimes_{\mathbb{Z}} K.$$

Proof of Corollary 1.3. By Theorem 1.2, $\mathcal{SM}(\mathcal{J}_{h-2})$ is a $\mathbb{Z}$-basis of $\mathfrak{R}_{h-2}$. So it is a K-basis of $K[J_\infty(Pf_h)]$. $\qquad\square$

Recall the map $Q_h : \mathfrak{R}_h \to \mathfrak{B}$ given by (3.2), which extends to a map

(4.3) $$Q_h^K : K[J_\infty(Pf_{h+2})] \to K[J_\infty(V)],$$

where $K[J_\infty(Pf_{h+2})]$ and $K[J_\infty(V)]$ are identified with $\mathfrak{R}_h \otimes_{\mathbb{Z}} K$ and $\mathfrak{B} \otimes_{\mathbb{Z}} K$, respectively, and $Q_h^K = Q_h \otimes \operatorname{Id}$.

Theorem 4.1. *Q_h^K is injective, so we may identify $K[J_\infty(Pf_{h+2})]$ with the subring $Im(Q_h^k)$ of $K[J_\infty(V)]$. In particular, $K[J_\infty(Pf_{h+2})]$ is integral.*

Proof. By Lemma 3.1, $Ld(Q_h(\mathcal{SM}(\mathcal{J}_h)))$ are linearly independent. By Corollary 1.3, $\mathcal{SM}(\mathcal{J}_h)$ is a K-basis of $\mathfrak{R}_h$, so Q_h^K is injective. Since $K[J_\infty(V)]$ is integral, so as $K[J_\infty(Pf_{h+2})]$. $\quad\square$

In general, if char $K = 0$, the arc space of an integral scheme is irreducible [7] but it may not be reduced. $Pf_h(K)$ is an example whose arc space is integral.

Principle G-bundles. Let G be an algebraic group over K. If G acts morphically on an algebraic variety X, then we say that X is a G-variety. An affine G-variety X is a principal G-bundle in the the étale topology if, for every $x \in X /\!\!/ G$, there is an étale neighborhood $V \to X$ of x such that $V \times_{X/G} X \cong V \times G$ as G-varieties. The following result is from [1] by P. Bardsley and R. W. Richardson,

Proposition 4.2. *Let X be an affine G-variety. Then X is a principal G-bundle in the étale topology if and only if for every x in X, the orbit $G \cdot x$ is separable and the stabilizer G_x is trivial.*

The group structure $G \times G \to G$ induces the group structure on its arc space

$$J_\infty(G) \times J_\infty(G) \to J_\infty(G).$$

So $J_\infty(G)$ is an algebraic group. For a G-variety X, the action $G \times X \to X$ induces the action of $J_\infty(G)$ on $J_\infty(X)$,

$$J_\infty(G) \times J_\infty(X) \to J_\infty(X).$$

The quotient map $X \to X /\!\!/ G$ induces morphisms $J_\infty(X) \to J_\infty(X /\!\!/ G)$ and

$$\pi_X : J_\infty(X) /\!\!/ J_\infty(G) \to J_\infty(X /\!\!/ G).$$

Proposition 4.3. *If X is a principal G-bundle in the étale topology and $X /\!\!/ G$ is smooth, then π_X is an isomorphism.*

Proof. For any étale morphism $V \to X /\!\!/ G$ with $V \times_{X /\!\!/ G} X \cong V \times G$ as G-varieties, we have Cartesian diagrams

$$
\begin{array}{ccc}
J_\infty(V) & \to & J_\infty(X /\!\!/ G) \\
\downarrow & & \downarrow \\
V & \to & X /\!\!/ G
\end{array}
\quad \text{and} \quad
\begin{array}{ccccc}
J_\infty(V) \times J_\infty(G) & \to & J_\infty(X) \\
\downarrow & & \downarrow \\
V \times G & \to & X \\
\downarrow & & \downarrow \\
V & \to & X /\!\!/ G
\end{array} \; .
$$

So $J_\infty(V) \to J_\infty(X /\!\!/ G)$ is an étale morphism and

$$
\begin{aligned}
J_\infty(V) \times_{J_\infty(X /\!\!/ G)} J_\infty(X) /\!\!/ J_\infty(G) &\cong (J_\infty(V) \times_{J_\infty(X /\!\!/ G)} J_\infty(X)) /\!\!/ J_\infty(G) \\
&\cong (V \times_{X /\!\!/ G} J_\infty(X)) /\!\!/ J_\infty(G) \\
&\cong J_\infty(V) \times J_\infty(G) /\!\!/ J_\infty(G) \\
&\cong J_\infty(V).
\end{aligned}
$$

If π_X is not an isomorphism, there is an étale morphism $V \to X /\!\!/ G$ such that $V \times_{X /\!\!/ G} X \cong V \times G$ as G-varieties and $\pi_V : J_\infty(V) \times_{J_\infty(X /\!\!/ G)} J_\infty(X) /\!\!/ J_\infty(G) \to J_\infty(V)$ is not an isomorphism. But $V \times_{X /\!\!/ G} J_\infty(X /\!\!/ G) \cong J_\infty(V)$, which is a contradiction. $\quad\square$

Invariants for the arc space of the symplectic group action. Let $G = Sp_h(K)$ be the symplectic group over K, $W = K^{\oplus h}$ its standard representation, and $V = W^{\oplus p}$. Recall that V has affine coordinate ring

$$K[V] = K[a_{il}^{(0)} | 0 \le i, j \le p, 1 \le l \le h].$$

The action of G on V induces an action of $J_\infty(G)$ on the affine coordinate ring

$$K[J_\infty(V)] = K[a_{il}^{(k)} | 0 \le i, j \le p, \ 1 \le l \le h, \ k \in \mathbb{Z}_{\ge 0}],$$

which is identified with $\mathfrak{B} \otimes_{\mathbb{Z}} K$, where $\mathfrak{B}$ is given by (3.1).

If $p \ge h$, let $\Delta = Q_h^K(|h, \ldots, 1|)$, and let $K[J_\infty(V)]_\Delta$ and $\mathrm{Im}(Q_h^K)_\Delta$ be the localization of $K[J_\infty(V)]$ and $\mathrm{Im}(Q_h^K)$ at Δ.

Lemma 4.4. *If $p \ge h$,*

$$K[J_\infty(V)]_\Delta^{J_\infty(Sp_h(K))} = \mathrm{Im}(Q_h^K)_\Delta.$$

Proof. Let $K[V]_\Delta$ be the localization of $K[V]$ at Δ and $V_\Delta = \mathrm{Spec}\, K[V]_\Delta$. By Theorem 1.1, the ring of invariants $K[V]^{Sp_h}$ is generated by $Q_h^K(x_{uv}^{(0)})$, so the affine coordinate ring of $J_\infty(V_\Delta /\!/ G)$ is isomorphic to $\mathrm{Im}(Q_h^K)_\Delta$. To prove the lemma, we only need to show that $\pi_{V_\Delta} : J_\infty(V_\Delta) /\!/ J_\infty(G) \cong J_\infty(V_\Delta /\!/ G)$. By Proposition 4.2, V_Δ is a principal bundle in the étale topology, and by Proposition 4.3, π_{V_Δ} is an isomorphism. $\qquad\square$

Theorem 4.5. $K[J_\infty(V)]^{J_\infty(Sp_h(K))} = \mathrm{Im}(Q_h^K)$.

Proof. If $p \ge h$, we regard $K[J_\infty(V)]$ and $\mathrm{Im}(Q_h^K)_\Delta$ as subrings of $K[J_\infty(V)]_\Delta$. By Lemma 4.4, we have

$$K[J_\infty(V)]_\Delta^{J_\infty(Sp_h(K))} = K[J_\infty(V)] \cap \mathrm{Im}(Q_h^K)_\Delta.$$

Now for any $f \in K[J_\infty(V)] \cap \mathrm{Im}(Q_h^K)_\Delta$, $f = \frac{g}{\Delta^n}$, with $\Delta^n f = g \in \mathrm{Im}(Q_h^K)$. The leading monomial of g is

$$Ld(g) = (a_{11}^{(0)} \cdots a_{hh}^{(0)})^n Ld(f),$$

with coefficient $C_0 \ne 0$. Since $g \in \mathrm{Im}(Q_h^K)$, there is a standard monomial $J \in \mathcal{SM}(\mathcal{J}_h)$, with $Ld(Q_h(J)) = Ld(g)$. Since J has the factor $|h, \ldots, 1|^n$, $Q_h(J)$ has the factor Δ^n. Thus $f - C_0 \frac{Q_h^K(J)}{\Delta^n} \in K[J_\infty(V)] \cap \mathrm{Im}(Q_h^K)_\Delta$ with a lower leading monomial and $\frac{Q_h^K(J)}{\Delta^n} \in \mathrm{Im}(Q_h^K)$. By induction on the leading monomial of f, $f \in \mathrm{Im}(Q_h^K)$, so $K[J_\infty(V)] \cap \mathrm{Im}(Q_h^K)_\Delta = \mathrm{Im}(Q_h^K)$, and $K[J_\infty(V)]^{J_\infty(Sp_h(K))} = \mathrm{Im}(Q_h^K)$.

More generally, let $V' = W^{\oplus p+h}$ where $W = K^{\oplus h}$ as before. Its arc space has affine coordinate ring

$$K[J_\infty(V')] = K[a_{il}^{(k)} | 0 \le i, j \le p+h, \ 1 \le l \le h, \ k \in \mathbb{Z}_{\ge 0}],$$

which contains $K[J_\infty(V)]$ as a subalgebra, and has an action of $J_\infty(G)$. By the above argument, $K[J_\infty(V')]^{J_\infty(G)}$ is generated by $X_{uv}^{(k)} = \bar{\partial}^k \sum_{i=1}^{h/2} (a_{u2i-1}^{(0)} a_{v2i}^{(0)} - a_{v2i-1}^{(0)} a_{u2i}^{(0)})$. Let $\mathcal{I}$ be the ideal of $K[J_\infty(V')]$ generated by $a_{il}^{(k)}$ with $i > p$. Then

$$K[J_\infty(V')] = K[J_\infty(V)] \oplus \mathcal{I}.$$

Note that $K[J_\infty(V)]$ and $\mathcal{I}$ are $J_\infty(G)$-invariant subspaces of $K[J_\infty(V')]$, and

$$K[J_\infty(V')]^{J_\infty(G)} = K[J_\infty(V)]^{J_\infty(G)} \oplus \mathcal{I}^{J_\infty(G)}.$$

If $i > p$ or $j > p$, $X_{ij}^{(k)} \in \mathcal{I}^{J_\infty(G)}$, so

$$K[J_\infty(V)]^{J_\infty(G)} \cong K[J_\infty(V')]^{J_\infty(G)}/\mathcal{I}^{J_\infty(G)}$$

is generated by $X_{ij}^{(k)}$, $1 \le i, j \le p$. Therefore $K[J_\infty(V)]^{J_\infty(G)} = \mathrm{Im}(Q_h^K)$, as claimed. $\qquad\square$

Proof of Theorem 1.4. By Theorem 4.5 and Theorem 4.1, $K[J_\infty(V)^{J_\infty(Sp_h(K))} = \mathrm{Im}(Q_h^K) \cong K[J_\infty(Pf_{h+2})]$. $\qquad\square$

Proof of Corollary 1.5. This is immediate from Theorem 1.4 because $V/\!/ Sp_h(K)$ is isomorphic to the Pfaffian variety Pf_{h+2}. $\qquad\square$

5. SOME PROPERTIES OF STANDARD MONOMIALS

By the definition of standard monomials, if $E_1 E_2 \cdots E_n \in \mathcal{SM}(\mathcal{E})$, then E_{i+1} is the largest element in $\|\mathcal{E}(E_{i+1})\|$ such that $E_i \le E_{i+1}$. In this section, we study the properties of $\|\mathcal{E}(E_{i+1})\|$ and E_{i+1} that need to be satisfied to make $E_1 E_2 \cdots E_n$ a standard monomial.

Let

$$E = |(u_h, k_h), \ldots, (u_1, k_1)| \in \mathcal{E},$$
$$J' = \bar{\partial}^{n'} |u'_{h'}, \ldots, u'_1| \in \mathcal{J}.$$

$L(E, J')$. For $h' \le h$, let σ be the permutation of $\{1, 2, \ldots, h'\}$ such that $u_{\sigma_L(i)} < u_{\sigma_L(i+1)}$. Let $L(E, J')$ be the smallest non-negative integer i_0 such that $u'_i \ge u_{\sigma_L(i-i_0)}$, $i_0 < i \le h'$. Let

$$E(h') = |(u_{h'}, k_{h'}), \ldots, (u_1, k_1)|.$$

Then $L(E, J') = L(E(h'), J')$.

The following lemma is obvious.

Lemma 5.1. *For $J'' = \bar{\partial}^k |u''_{h'}, \ldots, u''_1| \in \mathcal{J}$, if there are at least s elements in $\{u''_{h'}, \ldots, u''_1\}$ from the set $\{u'_{h'}, \ldots, u'_1\}$, then $L(E, J'') \ge L(E, J') - h' + s$;*

A criterion for J' to be greater than E. We say J' is greater than E if there is an element $E' \in \mathcal{E}(J')$ with $E \le E'$. Then J' is greater than E if and only if J' is greater than $E(h')$. The following lemma is a criterion for J' to be greater than E.

Lemma 5.2. *J' is greater than E if and only if $wt(E(h')) - wt(J') \ge L(E, J')$.*

Proof. Let $i_0 = L(E, J')$ and σ be the permutation of $\{1, 2, \ldots, h'\}$ such that $u_{\sigma_L(i)} < u_{\sigma_L(i+1)}$.

If $wt(E(h')) - wt(J') \ge L(E, J')$, let

$$\tilde{u}'_{\sigma(i)} = \begin{cases} u'_{i+i_0}, & \sigma(i) + i_0 \le h' \\ u'_{i+i_0-h_b}, & i + i_0 > h' \end{cases}, \qquad k'_{\sigma(i)} = \begin{cases} k_{\sigma(i)}, & i + i_0 \le h', i \ne h' \\ k_{\sigma(i)} + 1, & i + i_0 > h', i \ne h' \end{cases},$$

$$k'_{\sigma(h')} = wt(J') - \sum_{i=1}^{h'-1} k'_i.$$

Then

$$k'_{\sigma(h')} = wt(J') - wt(E(h')) - i_0 + k_{\sigma(h')} + 1 - \delta_{i_0}^0 \ge k_{\sigma(h')} + 1 - \delta_{i_0}^0.$$
$$(\tilde{u}'_{\sigma(i)}, k'_{\sigma(i)}) \ge (u_{\sigma(i)}, k_{\sigma(i)}).$$

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So
$$\tilde{E}' = |(\tilde{u}'_{h'}, k'_{h'}), \ldots, (\tilde{u}'_2, k'_2), (\tilde{u}'_1, k'_1)|$$
is an element in $\mathcal{E}(J')$ with $\tilde{E}' \geq E$.

On the other hand, suppose $\tilde{E}' \in \mathcal{E}(J')$ with $\tilde{E}' \geq E$. Assume
$$\tilde{E}' = |(\tilde{u}'_{h'}, k'_{h'}), \ldots, (\tilde{u}'_2, k'_2), (\tilde{u}'_1, k'_1)|.$$
We have $(\tilde{u}'_i, k'_i) \geq (u_i, k_i)$ i.e., $k'_i > k_i$ or $k'_i = k_i, \tilde{u}'_i \geq u_i$. So
$$\sum_{i=1}^{h'}(k'_i - k_i) + \sharp\{\tilde{u}'_i \geq u_i, i | 1 \leq i \leq h'\} \geq h',$$
Let $i'_0 = h' - \sharp\{\tilde{u}'_i \geq u_i, i | 1 \leq i \leq h'\}$. Then
$$i'_0 \leq \sum_{i=1}^{h'}(k'_i - k_i) = wt(J') - wt(E(h')).$$
Here $\tilde{u}'_1, \ldots, \tilde{u}'_{h'}$ is a permutation of $u'_1, \ldots, u'_{h'}$. By the definition of i'_0, it is easy to see that $u'_i \geq u_{\sigma(i-i'_0)}, i'_0 < i \leq h'$. So $i'_0 \geq L(E, J')$. Thus
$$wt(E(h')) - wt(J') \geq i'_0 \geq L(E, J').$$

$\square$

Corollary 5.3. J' is greater than E if and only if $||E(h')||J'$ is standard.

Proof. By Lemma 5.2, J' is greater than E if and only if $wt(E(h')) - wt(J') \geq L(E, J')$ and $||E(h')||J'$ is standard if and only if $wt(E(h')) - wt(J') \geq L(E(h'), J') = L(E, J')$. $\square$

The property "largest". Let
$$\mathcal{W}_s(E, J') = \{J = \bar{\partial}^k|u'_{i_s} \ldots u'_{i_1}| \,|\, 1 \leq i_l \leq h', \ J \text{ is greater than } E\}.$$

Lemma 5.4. *If E' is the largest element in $\mathcal{E}(J')$ such that $E \leq E'$, then for $s < h'$, $||E'(s)||$ is the smallest element in $\mathcal{W}_s(E, J')$.*

Proof. Assume
$$E' = |(u'_{h'}, k_{h'}), \ldots, (u'_2, k_2), (u'_1, k_1)|.$$
For $s < h'$, let J_s be the smallest element in $\mathcal{W}_s(E, J')$. Let
$$E_s = |(u'_{i_s}, \tilde{k}_s), \ldots, (u'_{i_2}, \tilde{k}_2), (u'_{i_1}, \tilde{k}_1)|$$
be the largest element in $\mathcal{E}(J_s)$ such that $E(s) \leq E_s$.

Assume l is the largest number such that $(u'_j, k'_j) = (u'_{i_j}, \tilde{k}_j)$ for $j < l \leq s+1$. If $l \leq s$, then $i_l \geq l$ and $(u'_{i_l}, \tilde{k}_l) \neq (u'_l, k'_l)$. If $i_l = l$, by the maximality of E' and the minimality of J_s, we must have $(u'_{i_l}, \tilde{k}_l) = (u'_l, k'_l)$, a contradiction. So $i_l > l$.

If $(u'_{i_l}, \tilde{k}_l) < (u'_l, k'_l)$, then $(u'_l, k'_l + k'_{i_l} - \tilde{k}_l) > (u'_{i_l}, k'_{i_l})$. Let E'' be the element in $\mathcal{E}(J')$ obtained by replacing (u'_l, k'_l) and (u'_{i_l}, k'_{i_l}) in E' by $(u'_{i_l}, \tilde{k}_l)$ and $(u'_l, k'_l + k'_{i_l} - \tilde{k}_l)$, respectively. We have $E' \prec E''$ and $E(h') \leq E''$. But $E' \neq E''$ is the largest element in $\mathcal{E}(||E'^L||)$ such that $E \leq E'$, which is a contradiction.

Assume $(u'_{i_l}, \tilde{k}_l) > (u'_l, k'_l)$. If $l \notin \{i_1, \ldots, i_s\}$, replacing $(u'_{i_l}, \tilde{k}_l)$ in E_s by (u'_l, k'_l), we get E'_s with $E(s) \leq E'_s$ and $||E'_s|| \prec J_s$. This is impossible since $J_s \neq ||E'_s||$ is the smallest element

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in $\mathcal{W}_s(E, E')$. If $l = i_j \in \{i_1, \ldots, i_s\}$, $(u'_{i_l}, \tilde{k}_l + \tilde{k}_j - k'_l) > (u'_{i_j}, \tilde{k}_j)$. Let E'_s be the element in $\mathcal{E}(J_s)$ obtained by replacing $(u'_{i_l}, \tilde{k}_l)$ and $(u'_{i_j}, \tilde{k}_j)$ in E_s by (u'_l, k'_l) and $(u'_{i_l}, \tilde{k}_l + \tilde{k}_j - k'_l)$, respectively. We have $E_s \prec E'_s$ and $E \leq E'_s$. But $E'_s \neq E_s$ is the largest element in $\mathcal{E}(J_s)$ such that $E \leq E'_s$, a contradiction. Therefore $E_s = E'(s)$, and $||E'(s)||$ is the smallest element in $\mathcal{W}_s(E, J')$. $\square$

Corollary 5.5. *If E' is the largest element in $\mathcal{E}(||E'||)$ such that $E \leq E'$, then for $s < h'$,*
$$L(E, ||E'(s)||) = wt(E'(s)) - wt(E(s)).$$

Proof. Since $E \leq E'$, $E \leq E'(s)$. By Lemma 5.4, $||E'(s)||$ is the smallest element in $\mathcal{W}_s(E, J')$. By Lemma 5.2, $L(E, ||E'(s)||) = wt(E'(s)) - wt(E(s))$. $\square$

Corollary 5.6. *If E' is the largest element in $\mathcal{E}(||E'||)$ such that $E \leq E'$, then for $s < h'$ and any $J \in \mathcal{W}_s(E, E')$,*
$$L(E, ||E'(s)||) \leq L(E, J).$$

Proof. Assume $J = \bar{\partial}^k|u_s, \ldots, u_1|$ and $||E'(s)|| = \bar{\partial}^l|u'_s, \ldots, u'_1|$. If $m = L(E, J) - L(E, ||E'(s)||) < 0$, let $J'' = \bar{\partial}^{l+m}|u_s, \ldots, u_1|$. By Lemma 5.2, $J'' \in W_s(E, J')$. By Lemma 5.4, $||E'(s)||$ is the smallest element in $\mathcal{W}_s(E, ||E'||)$. But $wt(||E'(s)||) > wt(J'')$, a contradiction. $\square$

Lemma 5.7. *Let $E_i = |(u^i_{h_i}, k^i_{h_i}), \ldots, (u^i_1, k^i_1)|$, $i = a, b$. Suppose that $E_b \leq E_a$, and that E_a is the largest element in $\mathcal{E}(||E_a||)$ such that $E_b \leq E_a$. Let $1 \leq h < h_a$ and σ_i be permutations of $\{1, \ldots, h\}$, such that $u^i_{\sigma_i(1)} < u^i_{\sigma_i(2)} < \cdots < u^i_{\sigma_i(h)}$. Let $u'_1, \ldots, u'_{h_a}$ be a permutation of $u^a_1, \ldots, u^a_{h_a}$ such that $u'_1 < u'_2 < \cdots < u'_{h_a}$. Assume $u'_{i_2} = u^a_{\sigma(i_1)}$ with $i_2 > i_1$. Then for any*
$$K = \bar{\partial}^k|u_h, \ldots, u_{s+1}, u'_{t_s}, \ldots u'_{t_1}|,$$
with $t_1 < t_2 < \cdots < t_s < i_2$, $L(E_b, K) > L(E_b, ||E_a(h)||) + s - i_1$.

Proof. Let $n = wt(E_a(h))$.
Let $i_0 = R(E_b, K)$, then $u'_{t_i} \geq u^b_{\sigma_b(i-i_0)}$, for $s \geq i > i_0$.
Let $i'_0 = R(E_b, ||E_a(h)||)$, then $u^a_{\sigma_a(i)} \geq u^b_{\sigma_b(i-i'_0)}$, for $h \geq i > i'_0$.

We have $i_1 > i'_0$. Otherwise, $i_1 \leq i'_0$. Replacing $u^a_{\sigma_a(i_1)}$ in $||E^a(h)||$ by some $u^a_i < u^a_{\sigma_a(i_1)}$ with $i > h$, (such u^a_i exists since $i_2 > i_1$), we get
$$J = \bar{\partial}^n|u^a_{\sigma'_a(h)}, \ldots, u^a_{\sigma'_a(i_1+1)} u^a_i, u^a_{\sigma'_a(i_1-1)}, \ldots, u^a_{\sigma'_a(1)}|$$
with $R(E_b, J) \leq i'_0$. By Lemma 5.2, J is greater than E_b. It is impossible by Lemma 5.4 since $J \prec ||E_a(h)||$ and E_a is the largest element in $\mathcal{E}(||E_a||)$ such that $E_b \leq E_a$.

If $s \geq i_1$, let
$$J' = \bar{\partial}^n|u^a_{\sigma_a(h)}, \ldots, u^a_{\sigma_a(i_1+1)}, u'_{t_s}, \ldots, u'_{t_{s-i_1+1}}|.$$
If $L(E_b, K) \leq L(E_b, ||E_a(h)||) + s - i_1$,
$$u'_{t_i} \geq u^b_{\sigma_b(i-i_0)} \geq u^b_{\sigma_b(i-i'_0-s+i_1)}, \text{ for } i \leq s.$$
We have $R(E_b, J') \leq i'_0$. By Lemma 5.2, J' is greater than E_b. By Lemma 5.4, it is impossible since $J' \prec ||E_a(h)||$ and E_a is the largest element in $\mathcal{E}(||E_a||)$ such that $E_b \leq E_a$. So when $s \geq i_1$, $L(E_b, K) \geq L(E_b, ||E_a||) + s - i_1$.

If $s < i_1$, let $t'_1 < t'_2 < \cdots < t'_{i_1} < i_2$ with $\{t_1, \ldots, t_s\} \subset \{t'_1, \ldots, t'_{i_1}\}$. Let
$$K' = \bar{\partial}^k|u'_{t'_1}, \ldots, u'_{t'_{i_1}}, u_{i_1+1}, \ldots, u_h|.$$

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By Lemma 5.1, we have

$$L(E_b, K) \geq L(E_b, K') + s - i_1 \geq L(E_b, ||E_a||) + s - i_1.$$

So for any $s > 0$, we have $L(E_b, K) > L(E_b, ||E_a||) + s - i_1$. $\square$

The following lemmas are obvious.

Lemma 5.8. *If $J_1 \prec J_2 \prec \cdots \prec J_n$, σ is a permutation of $\{1, \ldots, n\}$, then*

$$J_1 J_2 \cdots J_n \prec J_{\sigma(1)} \cdots J_{\sigma(n)}.$$

Lemma 5.9. *If $K_1 K_2 \cdots K_k \prec J_1 \cdots J_l$, then*

$$K_1 K_2 \cdots K_{s-1} J K_s \cdots K_k \prec J_1 \cdots J_{s-1} J J_s \cdots J_l.$$

6. PROOF OF LEMMA 2.8

In this section we prove Lemma 2.8. By Lemma 5.8, we can assume the monomials are expressed as ordered products $J_1 J_2 \cdots J_b$ with $J_a \prec J_{a+1}$. For $\alpha \in \mathcal{M}(\mathcal{J})$, let

$$\mathfrak{R}(\alpha) = \{\sum c_i \beta_i \in \mathfrak{R} | c_i \in \mathbb{Z}, \beta_i \in \mathcal{M}(\mathcal{J}), \beta_i \prec \alpha, \beta_i \neq \alpha\},$$

the space of linear combinations of elements preceding α in $\mathcal{M}(\mathcal{J})$ with integer coefficients.

Lemma 6.1. *If $J_1 J_2$ is not standard, $J_1 J_2 \in \mathfrak{R}(J_1)$.*

Proof. Assume $J_i = \bar{\partial}^{n_i} |u_{h_i}^i, \ldots, u_2^i, u_1^i|$, for $i = 1, 2$.
Let $E_1 = |(u_{h_1}^1, n_1), \ldots, (u_1^1, 0)|$. Let $i_0 = L(E_1, J_2)$. If $i_0 \neq 0$, there is $i_0 \leq i_1 \leq h_2$, such that $u_{i_1}^2 < u_{i_1 - i_0 + 1}^1$. If $i_0 = 0$, let $i_1 = 0$. Let $m = n_1 + n_2$ and $a_{n_2 - l} = \delta_l^0$, $0 \leq l \leq l_0 - 1$. By Lemma 2.5, there are integers a_k such that

$$(6.1) \quad \sum \epsilon a_k \bar{\partial}^{m-k} |\underline{u_{h_1}^1, \ldots, u_{i_1 - i_0 + 1}^1}, u_{i_1 - i_0}^1, \ldots, u_1^1 | \bar{\partial}^k |\underline{u_{h_2}^2, \ldots, u_{i_1 + 1}^2}, u_{i_1}^2, \ldots, u_1^2| \in \mathfrak{R}[h_1 + 1].$$

(1) If $h_1 = h_2$, then $n_1 \leq n_2$. Since $J_1 J_2$ is not standard, J_2 is not greater than E_1. By Lemma 5.2, $i_0 > n_2 - n_1 \geq 0$. $J_1 J_2 \in \mathfrak{R}(J_1)$ since in Equation (6.1):
   - All the terms with $k = n_2$ precede J_1 except $J_1 J_2$ itself;
   - All the terms with $k = n_2 - 1, \ldots, n_1$ vanish since $a_k = 0$;
   - All the terms with $k = n_2 + 1, \ldots, m$ precede J_1 since the weight of the upper $\bar{\partial}$-list is $m - k < n_1$;
   - All the terms with $k = 0, \ldots, n_1 - 1$ precede J_1 after exchanging the upper $\bar{\partial}$-list and the lower $\bar{\partial}$-list since the weight of the lower $\bar{\partial}$-list is $k < n_1$;
   - The terms in $\mathfrak{R}[h_1 + 2]$ precede J_1 since they have bigger sizes.
(2) If $h_1 > h_2$. Since $J_1 J_2$ is not standard, by Lemma 5.2, $i_0 > n_2$. $J_1 J_2 \in \mathfrak{R}(J_1)$ since in Equation (6.1):
   - All the terms with $k = n_2$ precede J_1 in the lexicographic order except $J_1 J_2$ itself;
   - All the terms with $k = n_2 - 1, \ldots, 0$ vanish since $a_k = 0$;
   - All the terms with $k = n_2, \ldots, m$ precede J_1 since the weight of the upper $\bar{\partial}$-list is $m - k < n_1$;
   - The terms in $\mathfrak{R}[h_1 + 2]$ precede J_1 since they have bigger sizes.

$\square$

Proof of Lemma 2.8. We prove the lemma by induction on b.

If $b = 1$, J_1 is standard.

If $b = 2$, by Lemma 6.1, the lemma is true.

For $b \geq 3$, assume the lemma is true for $b - 1$. We can assume $J_1 \cdots J_{b-1}$ is standard by induction and Lemma 5.9. Let $E_1 \cdots E_{b-1} \in \mathcal{SM}(\mathcal{E})$ be the standard ordered product of elements of $\mathcal{E}$ corresponding to $J_1 \cdots J_{b-1}$. If $J_1 \cdots J_b$ is not standard, then J_b is not greater than E_{b-1}. By Lemma 6.2 (below), $J_{b-1}J_b = \sum K_i f_i$ with $K_i \in \mathcal{J}$, $f_i \in \mathfrak{R}$ such that K_i is either smaller than J_{b-1} or K_i is not greater than E_{b-2}. If K_i is smaller than J_{b-1}, then $J_1 \cdots J_{b-1} K_i f \in \mathfrak{R}(J_1 \cdots J_{b-1})$. If K_i is not greater than E_{b-2}, $J_1 \cdots J_{b-2} K_i$ is not standard, so it is in $\mathfrak{R}(J_1 \cdots J_{b-2})$ by induction. Then $J_1 \cdots J_{b-2} K_i f \in \mathfrak{R}(J_1 \cdots J_{b-1})$. So $J_1 \cdots J_b = \sum J_1 \cdots J_{b-2} K_i f_i \in \mathfrak{R}(J_1 \cdots J_{b-1})$. $\qquad\square$

Lemma 6.2. *Let $E \in \mathcal{E}$, J_a and J_b in $\mathcal{J}$ with $J_a \prec J_b$, and suppose that E_a is the largest element in $\mathcal{E}(J_a)$ such that $E \leq E_a$. If J_b is not greater than E_a, then $J_a J_b = \sum K_i f_i$ with $K_i \in \mathcal{J}$, $f_i \in \mathfrak{R}$ such that K_i is either smaller than J_a or K_i is not greater than E.*

Proof. Assume $J_a = \bar{\partial}^{n_a}|u'_{h_a}, \ldots, u'_1|$ and $J_b = \bar{\partial}^{n_b}|u^b_{h_b}, \ldots, u^b_1|$. $J_a \prec J_b$, so $h_a \geq h_b$. Assume $\|E_a(h_b)\| = \bar{\partial}^{m_a}|u^a_{h_b}, \ldots, u^a_1|$. Let $m = n_b + n_a$. Let $i_0 = L(E_a, J_b)$. If $i_0 \neq 0$, there is $i_0 \leq i_1 \leq h_b$, such that $u^b_{i_1} < u^a_{i_1 - i_0 + 1}$. If $i_0 = 0$, let $i_1 = 0$. Since J_b is not greater than E_a, by Lemma 5.2,

(6.2) $$i_0 > n_b - m_a.$$

By definition, $\{u^a_{h_b}, \ldots, u^a_1\}$ is a subset of $\{u'_{h_a}, \ldots, u'_1\}$ with $u'_i < u'_{i+1}$ and $u^a_i < u^a_{i+1}$. If we assume $u'_{i_2} = u^a_{i_1 - i_0 + 1}$, we have $i_2 \geq i_1 - i_0 + 1$.

Now we prove the lemma. The proof is quite long and it is divided into three cases.

Case 1: $h_a = h_b$. Let $a_{n_b - l} = \delta^0_l$ for $0 \leq l \leq i_0 - 1$. By Lemma 2.5, there are integers a_k, such that

(6.3) $$\sum \epsilon a_k \bar{\partial}^{m-k} |\underline{u'_{h_a}, \ldots, u'_{i_1 - i_0 + 1}}, u'_{i_1 - i_0}, \ldots, u'_1| \, \bar{\partial}^k |u^b_{h_b}, \ldots, u^b_{i_1 + 1}, \underline{u^b_{i_1}, \ldots, u^b_1}| \in \mathfrak{R}[h_a + 2].$$

$J_a J_b \in \mathfrak{R}(J_a)$ since in the above equation,

- All the terms with $k = n_b$ precede J_a in the lexicographic order except $J_a J_b$ itself;
- All the terms with $k = n_b - 1, \ldots, n_a$ vanish since $n_a = m_a$, $a_k = 0$;
- All the terms with $k = n_b + 1, \ldots, m$ precede J_a since the weight of the upper $\bar{\partial}$-list is $m - k < n_a$;
- All the terms with $k = 0, \ldots, n_a - 1$ precede J_a after exchanging the upper $\bar{\partial}$-list and the lower $\bar{\partial}$-list since the weight of the lower $\bar{\partial}$-list is $k < n_a$;
- The terms in $\mathfrak{R}[h_a + 2]$ precede J_a since they have bigger sizes.

Case 2: $h_b < h_a$ and $n_b < m_a$.

By Lemma 2.5,

(6.4) $$\sum_{0 \leq i < h_b} \sum_{\sigma} \frac{(-1)^i \operatorname{sign}(\sigma)}{i!(h_b - i)} \sum \epsilon a^i_k$$

$$\bar{\partial}^{m-k} |\underline{u'_{h_a}, \ldots, u'_{i+1}}, u^b_{\sigma(i)}, \ldots, u^b_{\sigma(2)}, u^b_{\sigma(1)}| \, \bar{\partial}^k |\underline{u^b_{\sigma(h_b)}, \ldots, u^b_{\sigma(i+1)}, u'_i, \ldots, u'_1}| \in \mathfrak{R}[h_a + 2].$$

Here a^i_k are integers and $a^i_{n_b - l} = \delta_{0,l}$ for $0 \leq l < h_b - i$. The second summation is over all permutations σ of $\{1, \ldots, h_b\}$. In the above equation:

- The terms in $\mathfrak{R}[h_a + 2]$ precede J_a since they have bigger sizes.
- All the terms with $k = n_b, \ldots, m$ precede J_a since the weight of the upper $\bar{\partial}$-list is $m - k < n_a$.
- The terms with $k = n_b$ are $J_a J_b$ and the terms with the lower $\bar{\partial}$-lists

$$K_0 = \bar{\partial}^{n_b} |u'_{i_{h_b}}, \ldots, u'_{i_2}, u'_{i_1}| \in \mathcal{J}.$$

And all the other terms are cancelled. By Corollary 5.5 and 5.6,

$$L(E_{b-2}, K_0) \geq L(E_{b-2}, E_a(h_b)) = m_a - m_{b-2} > n_b - m_{b-2}.$$

By Lemma 5.2, K_0 is not greater than E_{b-2}.
- The terms with $k < n_b$ vanish unless $h_b - i \leq n_b - k$. In this case the lower $\bar{\partial}$-lists of the terms are

$$K_1 = \bar{\partial}^k |u^b_{\sigma(h_b)}, \ldots, u^b_{\sigma(i+1)}, u'_{s_i}, \ldots, u'_{s_1}|.$$

By Lemma 5.1,

$$L(E_{b-2}, K_1) \geq L(E_{b-2}, K_0) - (h_b - i).$$

So

$$L(E_{b-2}, K_1) > n_b - m_{b-2} - (h_b - i) \geq k - m_{b-2}.$$

By Lemma 5.2, K_1 is not greater than E_{b-2}.

Case 3: $h_a > h_b$ and $n_b \geq m_a$. If $i_0 = 0$, then J_b is greater than E_a and $J_1 \cdots J_b$ is standard. So $i_0 > 0$. By Lemma 2.5, we have

(6.5)
$$\sum_{\min\{i_1, i_2\} > s} \sum_{\sigma} \frac{(-1)^s \operatorname{sign}(\sigma)}{s!(i_2 - 1 - s)!} \sum_{k=0}^{m} \epsilon a^s_k$$

$$\bar{\partial}^{m-k} |u'_h, \ldots, u'_{i_2}, u'_{\sigma(i_2 - 1)}, \ldots, u'_{\sigma(s+1)}, u^b_s, \ldots, u^b_1| \; \bar{\partial}^k |u^b_{h_b}, \ldots, u^b_{i_1+1}, \underline{u^b_{i_1}, \ldots, u^b_{s+1}}, u'_{\sigma(s)}, \ldots, u'_{\sigma(1)}|$$

$$\in \mathfrak{R}[h_a + 1].$$

Here a^s_k are integers, $a^s_{n_b - l} = \delta_{0,l}$ for $0 \leq l < i_1 - s$, and σ are permutations of $\{1, \ldots, i_2 - 1\}$.

(6.6)
$$\sum_{\substack{i \geq i_1 > s \\ i_2 > s}} \sum_{\sigma, \sigma_1} \frac{(-1)^{i+s+t} \operatorname{sign}(\sigma) \operatorname{sign}(\sigma_1)}{(i - i_1)!(h_b - i)!s!(i_2 - 1 - s)!} \sum_{k=0}^{m} \epsilon a^{i,s}_k$$

$$\bar{\partial}^{n-k} |u^b_{\sigma(h_b)} \ldots u^b_{\sigma(i)} \ldots u^b_{\sigma(i_1+1)}, u^b_{i_1} \ldots u^b_1, u'_{h_a} \ldots u'_{h_b+1}| \; \bar{\partial}^k |\underline{u'_{h_b+1}, \ldots u'_{i_2}}, u'_{\sigma_1(i_2-1)}, \ldots u'_{\sigma_1(s+1)}, u'_{\sigma_1(s)}, \ldots u'_{\sigma_1(1)}|$$

$$\in \mathfrak{R}[h_a + 1].$$

Here $a^{i,s}_k$ are integers, $a^{i,s}_{n_b - l} = \delta_{0,l}$ for $0 \leq l < (i + j_1 - s)$, σ are permutations of $\{h_b, \ldots, i_1 + 1\}$, and σ_1 are permutations of $\{1, \ldots, i_2 - 1\}$.

- We use Equation (6.5) when $i_2 > i_1 - i_0 + 1$;
- We use Equation (6.6) when $i_2 = i_1 - i_0 + 1$.

In the above relations:

(1) The terms in $\mathfrak{R}[h_a + 2]$ precede J_a since they have bigger sizes.
(2) All the terms with $k = n_b + 1, \ldots, n$ precede J_a since the weight of the upper $\bar{\partial}$-list is $n - k < n_a$.

(3) The terms with $k = n_b$ are $J_a J_b$, the terms with the upper $\bar{\partial}$-list preceding J_a (the upper $\bar{\partial}$-lists are the $\bar{\partial}$-lists given by replacing some u'_i, $i \geq i_2$ in J_a by some u^b_k) and the terms with the lower $\bar{\partial}$-lists

$$K_0 = \bar{\partial}^{n_b} |u^b_{h_b}, \ldots, u^b_{i_1+1}, u'_{\sigma_1(i_1)}, \ldots u'_{\sigma_1(1)}|.$$

And all of the other terms cancel. By Lemma 5.7,

$$L(E_{b-2}, K_0) > L(E_{b-2}, ||E_a(h)||) + i_1 - (i_1 - i_0 + 1);$$

By the above inequality,

$$\begin{aligned} L(E_{b-2}, K_0) &\geq& L(E_{b-2}, ||E_a(h_b)||) + i_0 \\ \text{(by corollary 5.5)} &=& wt(E_a(h_b)) - wt(E_{b-2}) + i_0 \\ \text{(by Equation (6.2)} &>& wt(E_b) - wt(E_{b-2}(h_b)). \end{aligned}$$

By Lemma 5.2, K_0 is not greater than E_{b-2}.

(4) When $i_2 > i_1 - i_0 + 1$, the terms with $k < n_b$ in Equation (6.5) vanishes unless $i_1 - s \leq n_b - k$. In this case the lower $\bar{\partial}$-lists of the terms are

$$K_1 = \bar{\partial}^k |u^b_{h_b}, \ldots, u^b_{i_1+1}, \underline{u^b_{i_1}, \ldots, u^b_{s+1}}, u'_{\sigma(s)}, \ldots, u'_{\sigma(1)}|$$

The underlined u in K_1 can be any underlined u in Equation (6.5). By Lemma 5.7

$$L(E_{b-2}, K_1) > L(E_{b-2}, E_a(h_b)) + s - (i_1 - i_0 + 1);$$

$$\begin{aligned} L(E_{b-2}, K_1) &\geq& L(E_{b-2}, E_a(h_b)) + k - n_b + i_0 \\ \text{(by Corollary 5.5)} &=& wt(E_a(h_b)) - wt(E_{b-2}(h_b)) + k - n_b + i_0 \\ \text{(by Equation (6.2)} &>& k - wt(E_{b-2}(h_b)). \end{aligned}$$

By Lemma 5.2, K_1 is not greater than E_{b-2}.

(5) When $i_2 = i_1 - i_0 + 1$, the terms with $k < n_b$ are the terms in the Equation (6.6), such that $i - s \leq n_b - k$. In this case, the lower $\bar{\partial}$-lists of the terms are

$$K_1 = \bar{\partial}^k |u'_{h_b+1}, \ldots, u'_{i_2}, \underline{u'_{\sigma_1(i_2-1)}, \ldots, u'_{\sigma_1(s+1)}}, u'_{\sigma_1(s)}, \ldots, u'_{\sigma_1(1)}|.$$

The underlined u in K_1 can be any underlined u in Equation (6.6). Let

$$K'_1 = \bar{\partial}^k |u'_{k_{h_b}}, \ldots, u'_{k_{i_2}}, u'_{\sigma(i_2-1)}, \ldots, u'_{\sigma(1)}|.$$

Here $i_2 \leq k_{i_2} < k_{i_2+1} < \cdots < k_{h_b} \leq h_a$. By Lemma 5.1, there is some K'_1 such that

(6.7) $$L(E_{b-2}, K_1) \geq L(E_{b-2}, K'_1) - (i_2 - 1 - s) - (i - i_1),$$

since in K_1 the number of u^b_l with $u^b_l > u'_{i_2}$ is at most $i - i_1$. By Corollary 5.6,

(6.8) $$L(E_{b-2}, K'_1) \geq L(E_{b-2}, ||E_a(h_b)||).$$

By Equation (6.7, 6.8),

$$\begin{aligned} L(E_{b-2}, K_1) &\geq& L(E_{b-2}, ||E_a(h_b)||) + s - (i - i_0) \\ \text{(by Corollary 5.5)} &\geq& wt(E_a(h_b)) - wt(E_{b-2}(h_b)) + k - n_b + i_0 \\ \text{(by Equation 6.2)} &>& k - wt(E_{b-2}(h_b)). \end{aligned}$$

By Lemma 5.2, K_1 is not greater than E_{b-2}.

$\square$

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