EILENBERG-MOORE SPECTRAL SEQUENCE AND HODGE COHOMOLOGY OF CLASSIFYING STACKS

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Abstract. Let G be a smooth connected reductive group over a field k and Γ be a central subgroup of G. We construct Eilenberg-Moore-type spectral sequences converging to the Hodge and de Rham cohomology of $B(G/\Gamma)$. As an application, building upon work of Toda and using Totaro's inequality, we show that for all $m \geq 0$ the Hodge and de Rham cohomology algebras of the classifying stacks $B\mathrm{PGL}_{4m+2}$ and $B\mathrm{PSO}_{4m+2}$ over $\mathbb{F}_2$ are isomorphic to the singular $\mathbb{F}_2$-cohomology of the classifying space of the corresponding Lie group. From this we obtain a full description of $H^{>0}(\mathrm{GL}_{4m+2}, \mathrm{Sym}^j(\mathfrak{pgl}_{4m+2}^\vee))$ and $H^{>0}(\mathrm{SO}_{4m+2}, \mathrm{Sym}^j(\mathfrak{pso}_{4m+2}^\vee))$ over $\mathbb{F}_2$.

Contents

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1. Introduction

Let p be a prime number, and let G be a split reductive group over $\mathbb{Z}$. We denote by BG the classifying stack of G, and by $BG(\mathbb{C})$ the classifying space of the topological Lie group $G(\mathbb{C})$. The computation of the mod p singular cohomology ring $H^*_{\mathrm{sing}}(BG(\mathbb{C});\mathbb{F}_p)$, or equivalently the determination of mod p characteristic classes of principal $G(\mathbb{C})$-bundles, is one of the most classical problems in algebraic topology, with contributions from a long list of illustrious authors.

Recently, in [Tot18], B. Totaro initiated the study of Hodge cohomology $H^*_{\mathrm{H}}(BG/\mathbb{F}_p)$ and de Rham cohomology $H^*_{\mathrm{dR}}(BG/\mathbb{F}_p)$ of the classifying stack $BG_{\mathbb{F}_p}$. Similarly to the topological situation, one can think of elements of these rings as Hodge and de Rham characteristic classes for $G_{\mathbb{F}_p}$-torsors. However, as Totaro showed, $H^*_{\mathrm{H}}(BG/\mathbb{F}_p)$ also has a purely representation-theoretic interpretation in terms of rational cohomology of the algebraic group $G_{\mathbb{F}_p}$ with coefficients $\mathrm{Sym}^i\,\mathfrak{g}^\vee$, where $\mathfrak{g}$ is the adjoint representation of G. In [Tot18, Theorem 9.2], he established a general result, stating that if p is not a torsion prime[1] for G then Hodge and de Rham cohomology of $BG_{\mathbb{F}_p}$ are in fact isomorphic to the mod p singular cohomology of $BG(\mathbb{C})$. The subtlety of the situation, however, is that there is no natural map between Hodge (or de Rham) and singular cohomology: the above isomorphisms are constructed by explicitly computing and comparing the two sides.

Totaro also investigated what happens at torsion primes in some particular examples. For $p=2$ and $G=\mathrm{SO}_n$ he constructed isomorphisms of graded rings

$$H^*_{\mathrm{H}}(B\mathrm{SO}_n/\mathbb{F}_2)\simeq H^*_{\mathrm{dR}}(B\mathrm{SO}_n/\mathbb{F}_2)\simeq H^*_{\mathrm{sing}}(B\mathrm{SO}_n(\mathbb{C});\mathbb{F}_2). \tag{1.1}$$

On the other hand, he computed that

$$\dim_{\mathbb{F}_2} H^{32}_{\mathrm{dR}}(B\mathrm{Spin}_{11}/\mathbb{F}_2) > \dim_{\mathbb{F}_2} H^{32}_{\mathrm{sing}}(B\mathrm{Spin}_{11}(\mathbb{C});\mathbb{F}_2),$$

showing that Hodge and de Rham cohomology of $BG_{\mathbb{F}_p}$ are not isomorphic to mod p singular cohomology of $BG(\mathbb{C})$ in general, even as graded vector spaces. Some further calculations of $H^*_{\mathrm{H}}(BG/\mathbb{F}_2)$ and $H^*_{\mathrm{dR}}(BG/\mathbb{F}_2)$ have been performed by E. Primozic [Pri19] for $G=G_2$ and $G=\mathrm{Spin}_n$ for $n\le 11$.

A general statement which holds even for torsion primes is the inequality of dimensions. First of all, the existence of the Hodge-to-de Rham spectral sequence implies the inequality

$$\dim_{\mathbb{F}_p} H^i_{\mathrm{H}}(BG/\mathbb{F}_p)\ge \dim_{\mathbb{F}_p} H^i_{\mathrm{dR}}(BG/\mathbb{F}_p).$$

[1] A prime p is called torsion if there is non-trivial p-torsion $H^*_{\mathrm{sing}}(G(\mathbb{C});\mathbb{Z})$. For any given G there are only finitely many torsion primes, and there also is a simple recipe to find them all (see [KP21a, Example 6.1.5]).

Moreover, as conjectured by Totaro and recently proved by A. Prikhodko and the first author in [KP21b], one also has an inequality

$$\dim_{\mathbb{F}_p} H^i_{\mathrm{dR}}(BG/\mathbb{F}_p) \geq \dim_{\mathbb{F}_p} H^i_{\mathrm{sing}}(BG(\mathbb{C});\mathbb{F}_p).$$

Therefore

$$\dim_{\mathbb{F}_p} H^i_{\mathrm{H}}(BG/\mathbb{F}_p) \geq \dim_{\mathbb{F}_p} H^i_{\mathrm{dR}}(BG/\mathbb{F}_p) \geq \dim_{\mathbb{F}_p} H^i_{\mathrm{sing}}(BG(\mathbb{C});\mathbb{F}_p). \tag{1.2}$$

We will refer to (1.2) as Totaro's inequality.

Main results. The computations of Totaro and Primozic are based on a version of the Hochschild–Serre spectral sequence in Hodge cohomology (see [Tot18, Proposition 9.3]). In this paper we attempt to compute the Hodge and de Rham cohomology of the classifying stacks of classical simple adjoint groups PGL_n, PSp_n, PSO_n over $\mathbb{F}_2$ by using the *Eilenberg-Moore* spectral sequence instead. The Hodge cohomology in these situations can also be reinterpreted in terms of cohomology of the classical groups GL_n, Sp_n, SO_n, but with coefficients in modules that are slightly more complicated than $\mathrm{Sym}^j\mathfrak{g}^\vee$ (see Section 9 for more details).

Let us describe our setup. Let G be a split connected reductive group over a field k, let $\Gamma \subset G$ be a central subgroup (so Γ is of multiplicative type) and consider the quotient $\overline{G} \coloneqq G/\Gamma$. For example, we could take $G = \mathrm{GL}_n$ and $\Gamma = \mathbb{G}_m$, in which case we get $\overline{G} = \mathrm{PGL}_n$. The multiplication map $\Gamma \times G \to G$ defines an action $B\Gamma \times BG \to BG$ of the group stack $B\Gamma$ and so induces a coaction of the Hopf algebra $H^*_{\mathrm{H}}(B\Gamma/k)$ on $H^*_{\mathrm{H}}(BG/k)$. We also get a similar structure for de Rham cohomology. This coaction can be used to give a first approximation to $H^*_{\mathrm{H}}(B\overline{G}/k)$, as our first general result shows.

Theorem 1.3 (Eilenberg-Moore spectral sequence)**.** *Let k be a field, and consider a short exact sequence of linear algebraic k-groups*

$$1 \to \Gamma \to G \to \overline{G} \to 1.$$

where G and $\overline{G}$ are smooth and Γ is a central subgroup of multiplicative type. Then we have two (cohomological) first-quadrant convergent spectral sequences

$$E_2^{i,j} \coloneqq \Big(\mathrm{Cotor}^i_{H^*_{\mathrm{H}}(B\Gamma/k)}(k, H^*_{\mathrm{H}}(BG/k))\Big)^j \Rightarrow H^{i+j}_{\mathrm{H}}(B\overline{G}/k),$$

$$E_2^{i,j} \coloneqq \Big(\mathrm{Cotor}^i_{H^*_{\mathrm{dR}}(B\Gamma/k)}(k, H^*_{\mathrm{dR}}(BG/k))\Big)^j \Rightarrow H^{i+j}_{\mathrm{dR}}(B\overline{G}/k).$$

Here k is the trivial comodule over $H^*_{\mathrm{H}}(BG/k)$ (and $H^*_{\mathrm{dR}}(BG/k)$) and Cotor^i are the derived functors of cotensor product (the definition is essentially dual to Tor^i in the algebra setting, see Section 2.2 for a reminder).

Remark 1.4. Recall that Hodge cohomology is in fact a bigraded algebra (see Section 2.1 for a short reminder). By our construction, in the Hodge setting, the Eilenberg-Moore spectral sequence $E_r^{i,j}$ splits as a direct sum $E_r^{i,j} \simeq \oplus_h (E_r^{i,j})^h$ where

$$(E_2^{i,j})^h \coloneqq \Big(\mathrm{Cotor}^i_{H^{*,*}_{\mathrm{H}}(B\Gamma/k)}(k, H^{*,*}_{\mathrm{H}}(BG/k))\Big)^{h,j} \Rightarrow H^{h,i+j}_{\mathrm{H}}(B\overline{G}/k).$$

A spectral sequence analogous to those of Theorem 1.3 for singular cohomology was used by H. Toda in [Tod87] to compute the $\mathbb{F}_2$-singular cohomology of $B\mathrm{PGL}_n(\mathbb{C})$, $B\mathrm{PSp}_n(\mathbb{C})$ and $B\mathrm{PSO}_n(\mathbb{C})$ when $n = 4m+2$. The main result of our paper is that the answer for Hodge and de

Rham cohomology over $\mathbb{F}_2$ stays essentially the same. (When n is odd, 2 is not a torsion prime for any of these groups, and so Hodge and de Rham cohomology are isomorphic to singular cohomology by the aforementioned result of Totaro.)

Theorem 1.5. *(1) Let $\overline{G}$ be either PSO_{4m+2} or PGL_{4m+2}. Then we have isomorphisms of graded rings*

$$H^*_{\mathrm{H}}(B\overline{G}/\mathbb{F}_2) \simeq H^*_{\mathrm{dR}}(B\overline{G}/\mathbb{F}_2) \simeq H^*_{\mathrm{sing}}(B\overline{G}(\mathbb{C});\mathbb{F}_2).$$

(2) In the case $\overline{G} = \mathrm{PSp}_{4m+2}$ we have an isomorphism of graded vector spaces

$$H^*_{\mathrm{H}}(B\overline{G}/\mathbb{F}_2) \simeq H^*_{\mathrm{dR}}(B\overline{G}/\mathbb{F}_2) \simeq H^*_{\mathrm{sing}}(B\overline{G}(\mathbb{C});\mathbb{F}_2).$$

We don't know if there exists an algebra isomorphism in Theorem 1.5(2), the main reason being that the algebra structure on $H^*_{\mathrm{sing}}(B\mathrm{PSp}_{4m+2}(\mathbb{C});\mathbb{F}_2)$ is not fully understood (see e.g. [Tod87, Proposition 4.7]). In contrast, the algebra structures of $H^*_{\mathrm{sing}}(B\mathrm{PSO}_{4m+2}(\mathbb{C});\mathbb{F}_2)$ and $H^*_{\mathrm{sing}}(B\mathrm{PGL}_{4m+2}(\mathbb{C});\mathbb{F}_2)$ can be described explicitly in terms of generators and relations; see [Tod87, Proposition 4.2, Proposition 4.5].

Let us sketch the main ideas which go into the proof of Theorem 1.5. With Theorem 1.3 at our disposal, one can try to make direct computations similar to the ones in Toda's work [Tod87]. After some extra work, this is possible to achieve for PGL_{4m+2} and PSp_{4m+2}. However, Toda's argument doesn't seem to go through directly for PSO_{4m+2} (Remark 8.14). Our key observation is that if we assume the results of [Tod87] as given, there is an easier way: some parts of Theorem 1.5 are implied by Totaro's inequality almost for free, while with a little more computational work one can also get the rest of Theorem 1.5, including the most complicated case of PSO_{4m+2}. This led us to split the proof of Theorem 1.5 in two parts.

Part 1. Isomorphisms as graded vector spaces for $B\mathrm{PGL}_{4m+2}$ and $B\mathrm{PSp}_{4m+2}$. The main step in Toda's computation of $H^*_{\mathrm{sing}}(B\overline{G}(\mathbb{C});\mathbb{F}_2)$ consists in showing that the Eilenberg-Moore spectral sequence degenerates at the second page. Assuming Toda's result, it is enough to identify the second sheets of the Eilenberg-Moore spectral sequences for Hodge and singular cohomology: indeed, by Totaro's inequality (1.2) this would immediately imply the degeneration in the Hodge setting and then also give an equality of dimensions of cohomology. This identification is done by explicitly comparing the comodule structures on $H^*_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)$ and $H^*_{\mathrm{H}}(B\mathrm{Sp}_{4m+2}/\mathbb{F}_2)$ with the ones for singular cohomology (having identified $H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_2)$ with $H^*_{\mathrm{sing}}(B\mathbb{C}^\times,\mathbb{F}_2)$ as Hopf algebras and $H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)$ with $H^*_{\mathrm{sing}}(B\mathbb{Z}/2,\mathbb{F}_2)$ as coalgebras); see Section 4.

Part 2. Isomorphisms as graded rings for $B\mathrm{PGL}_{4m+2}$ and $B\mathrm{PSO}_{4m+2}$. For $B\mathrm{PGL}_{4m+2}$, the spectral sequence argument above already produces a *ring* isomorphism between the Hodge and singular cohomology, but only after passing to the associated graded. To lift this to an isomorphism between the original rings we imitate the computation of Toda in the Hodge setting (see Sections 5 and 7). The case of $B\mathrm{PSO}_{4m+2}$ is more difficult, as the comodule structure on $H^*_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2)$ is *not* compatible with the one on singular cohomology. Nevertheless, we get around this by directly replacing some results on the structure of the comodule $H^*_{\mathrm{sing}}(B\mathrm{SO}_{4m+2}(\mathbb{C}),\mathbb{F}_2)$ with suitable Hodge cohomology analogues (see Section 5). In some ways, the Hodge context actually turns out to be easier for the computation via Toda's method; see Remark 8.9. After computing Cotor via the twisted tensor product construction (see Section 5.3) we deduce the degeneration of the Eilenberg-Moore spectral sequence in the PSO_{4m+2}-case by comparing to topological side and using Totaro's inequality. To conclude, we identify the resulting descriptions of Hodge and singular cohomology in terms of generators and relations. For more details see Section 8.

Even though the algebras $H^*_{\mathrm{H}}(B\mathrm{SO}_{4m+2}(\mathbb{C})/\mathbb{F}_2)$ and $H^*_{\mathrm{sing}}(B\mathrm{SO}_{4m+2}(\mathbb{C}),\mathbb{F}_2)$ are abstractly isomorphic, there is a rather subtle implicit distinction between topological and Hodge settings. Namely, the square

$$\begin{array}{ccc} H^*_{\mathrm{H}}(B\mathrm{PSO}_{4m+2}/\mathbb{F}_2) & \xrightarrow[\text{Section 7}]{\sim} & H^*_{\mathrm{sing}}(B\mathrm{PSO}_{4m+2}(\mathbb{C}),\mathbb{F}_2) \\ \downarrow & & \downarrow \\ H^*_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2) & \xrightarrow[\text{Totaro}]{\sim} & H^*_{\mathrm{sing}}(B\mathrm{SO}_{4m+2}(\mathbb{C}),\mathbb{F}_2), \end{array}$$

induced by pull-back with respect to the map $B\mathrm{SO}_{4m+2} \to B\mathrm{PSO}_{4m+2}$ and isomorphism (1.1) is *not* commutative.

Applications to representation theory. Recall that the Hodge cohomology of a smooth algebraic stack X over k comes with a natural bigrading: $H^n_{\mathrm{H}}(X/k) \simeq \oplus_{i+j=n} H^{i,j}(X/k)$, where $H^{i,j}(X/k) \simeq H^j(X,\Omega^i)$. In [Tot18, Theorem 2.4], Totaro showed that if G is a smooth affine k-group, one has the following representation-theoretic formula for $H^{i,j}(BG/k)$:

$$H^{i,j}(BG/k) \simeq H^{i-j}(G, \mathrm{Sym}^i\,\mathfrak{g}^\vee).$$

The right hand side denotes the cohomology of G as an algebraic group (sometimes also called "rational cohomology"), and the G-action on $\mathrm{Sym}^i\,\mathfrak{g}^\vee$ is the natural adjoint action. This gives a geometric interpretation of the cohomology of representations like $\mathrm{Sym}^i\,\mathfrak{g}^\vee$, which Totaro used to make new computations. If $\Gamma \subset G$ is a central subgroup and $\overline{G} \coloneqq G/\Gamma$, the Hodge cohomology of $B\overline{G}$ can also be interpreted in terms of rational cohomology of G, but with coefficients in more complicated modules, namely $\mathrm{Sym}^i\,\overline{\mathfrak{g}}^\vee$, where $\overline{\mathfrak{g}} \coloneqq \mathrm{Lie}(\overline{G})$.

Remark 1.6. In order to compute the groups $H^j(G, \mathrm{Sym}^i\,\mathfrak{g}^\vee)$ by the above method, it is necessary to describe Hodge cohomology of $B\overline{G}$ as a *bigraded* algebra. Given the degeneration of the Eilenberg-Moore spectral sequence, this reduces to understanding the bigraded components of the cotorsion groups Cotor^i from Theorem 1.3. In order to keep track of the bigrading, we compute Cotor^i via twisted tensor product construction with an explicit twisting cochain, imitating the original computation of Toda (see Construction 5.16 and Corollary 5.19).

For brevity, let us only discuss the result of the computation in the case $\overline{G} = \mathrm{PGL}_n$ here, and refer the reader to Section 9 for the remaining cases. For every $n \geq 1$ we have a short exact sequence of GL_n-modules $0 \to \mathbb{F}_2 \to \mathfrak{gl}_n \to \mathfrak{pgl}_n \to 0$, which is non-split if and only if n is even. From the Hochschild-Serre spectral sequence in rational cohomology one can see that $H^*(\mathrm{GL}_n, \mathrm{Sym}^i\,\mathfrak{pgl}_n^\vee) \simeq H^*(\mathrm{PGL}_n, \mathrm{Sym}^i\,\mathfrak{pgl}_n^\vee)$, and, thus

$$H^{i,j}(B\mathrm{PGL}_n/\mathbb{F}_2) \simeq H^{j-i}(\mathrm{GL}_n, \mathrm{Sym}^i\,\mathfrak{pgl}_n^\vee).$$

From Theorem 7.16, giving the description of the left hand side in the case $n = 4m+2$, we get a full computation of higher cohomology (over $\mathbb{F}_2$) of GL_{4m+2} with coefficients in $\mathrm{Sym}^i\,\mathfrak{pgl}_n^\vee$. To be more precise, there is a certain class $z \in H^1(\mathrm{GL}_{4m+2}, \mathfrak{pgl}_{4m+2}^\vee)$ and a polynomial subalgebra $A \coloneqq \mathbb{F}_2[c_1, b_i]_{1\leq i\leq 2m+1} \subset (\oplus_i \mathrm{Sym}^i\,\mathfrak{pgl}_{4m+2}^\vee)^{\mathrm{GL}_{4m+2}}$ in the GL_{4m+2}-invariants such that for any $j > 0$ one has that

$$H^j(\mathrm{GL}_{4m+2}, \oplus_i \mathrm{Sym}^i\,\mathfrak{pgl}_{4m+2}^\vee) \xrightarrow{\sim} A \cdot z^j$$

is a free A-module of rank 1 generated by $z^j \in H^j(\mathrm{GL}_{4m+2}, \mathrm{Sym}^j \mathfrak{pgl}_{4m+2}^\vee)$. Here c_1 has degree 1 and each b_i has degree $4i$. In particular, for all $i, j \geq 0$, we get a formula for the dimension of $H^j(\mathrm{GL}_{4m+2}, \mathrm{Sym}^i \mathfrak{pgl}_{4m+2}^\vee)$ as the number of ways to write $i - j$ as a sum

$$\gamma_1 + 4\beta_2 + 8\beta_3 + \cdots + (8m+4)\beta_{2m+1},$$

where γ_1 and the β_h are non-negative integers. In particular, $H^j(\mathrm{GL}_{4m+2}, \mathrm{Sym}^i \mathfrak{pgl}_{4m+2}^\vee) \neq 0$ if and only if $i \geq j$.

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2. Preliminaries

2.1. **Hodge and de Rham cohomology of stacks.** Let k be a field and X be a smooth Artin stack of finite type over k. For every $i, j \geq 0$, we denote by $H^i(X, \Omega^j)$ the i-th cohomology of the sheaf Ω^j of j-differential forms on the big étale site of X (see [Tot18, Section 2]).

We denote by $H^{i,j}_{\mathrm{H}}(X/k) \coloneqq H^j(X, \Omega^i)$ and $H^n_{\mathrm{H}}(X/k) \coloneqq \oplus_{i+j=n} H^{i,j}_{\mathrm{H}}(X/k)$ the (i,j)-th component and the total n-th Hodge cohomology group, respectively. The algebra $H^*_{\mathrm{H}}(X/k) \coloneqq \oplus_{n=0}^\infty H^n_{\mathrm{H}}(X/k) \simeq \oplus_{i,j\geq 0} H^{i,j}_{\mathrm{H}}(X/k)$ has a natural bigraded k-algebra structure.

We also denote by $H^*_{\mathrm{dR}}(X/k)$ the de Rham cohomology of X (which is a $\mathbb{Z}$-graded k-algebra): it can be defined as the hypercohomology of the (de Rham) complex of sheaves $\Omega^*_{\mathrm{dR}} \coloneqq \Omega^0 \to \Omega^1 \to \Omega^2 \to \ldots$ on big étale site of X. See [Tot18, §1] or Appendix A for more details.

The abutment filtration $F^i(\Omega^*_{\mathrm{dR}}) \coloneqq \Omega^{\geq i}_{\mathrm{dR}} \subset \Omega^*_{\mathrm{dR}}$ has the associated graded $\oplus_i \Omega^i[-i]$ and induces the Hodge-de Rham spectral sequence:

$$E_1^{i,j} \coloneqq H^{i,j}_{\mathrm{H}}(X/k) \Rightarrow H^{i+j}_{\mathrm{dR}}(X/k).$$

Remark 2.1 (Another formula for Hodge cohomology). By flat descent for the cotangent complex, for any $j \geq 0$ one has a quasi-isomorphism

$$R\Gamma(X, \Omega^j) \xrightarrow{\sim} R\Gamma(X, \wedge^j \mathbb{L}_{X/k}),$$

where $\mathbb{L}_{X/k} \in \mathrm{QCoh}(X)$ is the cotangent complex of X (see e.g. [KP21a, Proposition 1.1.4]). If $X = BG$ for some smooth k-group scheme G then, under the identification $\mathrm{QCoh}(X)^+ \simeq D(\mathrm{Rep}(G))^+$, one has an equivalence $\mathbb{L}_{X/k} \simeq \mathfrak{g}^\vee[-1]$ (see e.g. [KP21b, Example A.3.8]), where $\mathfrak{g}^\vee$ is the coadjoint representation. This leads to an equivalence $R\Gamma(BG, \Omega^j) \xrightarrow{\sim} R\Gamma(G, \mathrm{Sym}^j \mathfrak{g}^\vee[-j])$, since $\wedge^j(\mathfrak{g}^\vee[-1]) \simeq (\mathrm{Sym}^j \mathfrak{g}^\vee)[-j]$ by the decalage isomorphism (see e.g. [KP21b, Proposition A.2.49]). In particular:

$$H^{i,j}(BG/k) \simeq H^{i-j}(G, \mathrm{Sym}^i \mathfrak{g}^\vee).$$

2.2. **Cotensor product and Cotor.** Our main reference on comodules, coalgebras, cotensor products and the cobar construction is [Rav86, Appendix A]. As it is mentioned in [Rav86, Definition A1.1.1], *loc.cit.* works with coalgebras not only in the ungraded, but also in the graded or bigraded settings (or, in fact any L-graded setup with $L \coloneqq \mathbb{Z}^n$ for some $n \geq 0$), even though some of the statements in [Rav86, Appendix A] that we will refer to do not explicitly mention this.

Let k be a field and let Λ be an L-graded Hopf k-algebra. A left (resp. right) Λ-comodule is an L-graded k-vector space M together with a k-linear L-graded map $\phi_M \colon M \to \Lambda \otimes_k M$ (resp. $\phi_M \colon M \to M \otimes_k \Lambda$) which is coassociative and counital. If M is also an L-graded k-algebra and ϕ_M is a graded k-algebra homomorphism then M is called a left (right) comodule algebra. In this paper, we will only consider L-graded comodules and coalgebras for $L = \mathbb{Z}^n$ $(n = 0, 1, 2)$. When L is clear from context, we will suppress "L-graded" from the notation.

Write $\Delta \colon \Lambda \to \Lambda \otimes_k \Lambda$ for the comultiplication map of Λ. If V is an L-graded vector space, we define a comodule structure on $\Lambda \otimes_k V$ by $\Delta \otimes \mathrm{id}_V$. The functor $V \mapsto \Lambda \otimes_k V$ from L-graded vector spaces to L-graded Λ-comodules is right adjoint to the forgetful functor; see [Rav86, Definition A1.2.1].

Lemma 2.2. *(a) The category of L-graded left (resp. right) Λ-comodules is abelian.*

(b) For every L-graded k-vector space V, since V is injective, $\Lambda \otimes_k V$ is an injective L-graded Λ-comodule.

(c) The category of L-graded left (resp. right) Λ-comodules admits enough injectives.

Proof. We will only consider left Λ-comodules, the case of right Λ-comodules being entirely analogous. (a) is [Rav86, Theorem A1.1.3], and (b) and (c) are [Rav86, Lemma A1.2.2]. □

Definition 2.3. (1) Let M be a right Λ-comodule and N be a left Λ-comodule. The *cotensor product* of M and N over Λ is defined as the L-graded k-vector space

$$M \square_\Lambda N \coloneqq \mathrm{Ker}(M \otimes_k N \xrightarrow{\phi_M \otimes 1 - 1 \otimes \phi_N} M \otimes_k \Lambda \otimes_k N);$$

see [Rav86, Definition A1.1.4].

(2) Given a left Λ-comodule N, one defines the Λ-subcomodule $PN \subset N$ of *primitive elements* as

$$PN \coloneqq \{n \in N : \phi_N(n) = 1 \otimes n\}.$$

Note that the canonical isomorphism $k \otimes_k N \simeq N$ induces an isomorphism

$$k \square_\Lambda N \simeq PN.$$

It is also not hard to see that if $N = A$ is a left Λ-comodule k-algebra, then $PA \subset A$ is a k-subalgebra.

Lemma 2.4. *The functor $N \mapsto M \square_\Lambda N$ is left exact in M.*

Proof. Consider a short exact sequence of Λ-comodules

$$0 \to M' \to M \to M'' \to 0.$$

Since k is a field, the functors $M \mapsto M \otimes_k N$ and $M \mapsto M \otimes_k \Lambda \otimes_k N$ are exact in N. We obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' \otimes_k N & \longrightarrow & M \otimes_k N & \longrightarrow & M'' \otimes_k N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_{M'}\otimes 1 - 1\otimes\phi_N} & & \downarrow{\scriptstyle \phi_{M}\otimes 1 - 1\otimes\phi_N} & & \downarrow{\scriptstyle \phi_{M''}\otimes 1 - 1\otimes\phi_N} & & \\ 0 & \longrightarrow & M' \otimes_k \Lambda \otimes_k N & \longrightarrow & M \otimes_k \Lambda \otimes_k N & \longrightarrow & M'' \otimes_k \Lambda \otimes_k N & \longrightarrow & 0. \end{array}$$

The snake lemma then yields an exact sequence

$$0 \to M' \square_\Lambda N \to M \square_\Lambda N \to M'' \square_\Lambda N,$$

as desired. $\square$

By Lemma 2.4 and Lemma 2.2(c), we may give the following definition.

Definition 2.5. If $i \geq 0$ is an integer, we define the i-th *cotorsion group* $\mathrm{Cotor}^i_\Lambda(M, N)$ as the i-th right derived functor of $M \square_\Lambda N$, regarded as a left exact additive functor of M with values in L-graded vector spaces. We also let

$$\mathrm{Cotor}^*_\Lambda(M, N) \coloneqq \oplus_{i \geq 0} \mathrm{Cotor}^i_\Lambda(M, N).$$

There is a canonical isomorphism

$$\mathrm{Cotor}^0_\Lambda(M, N) \simeq M \square_\Lambda N.$$

By construction, each $\mathrm{Cotor}^i_\Lambda(M, N)$ is an L-graded vector space.

As the right derived functor, $\mathrm{Cotor}^*_\Lambda(M, N)$ can be explicitly computed by picking any injective resolution $M \to I^*$ in the category of right Λ-comodules and taking the cohomology of $I^* \square_\Lambda N$. There is a preferred such resolution given by the cobar construction.

Construction 2.6 (Cobar construction)**.** Let $\Delta \colon \Lambda \to \Lambda \otimes_k \Lambda$ and $\epsilon \colon \Lambda \to k$ denote the comultiplication and the counit maps of Λ, respectively. If M is an L-graded right Λ-comodule with the coaction $\phi_M \colon M \to M \otimes_k \Lambda$, we may construct a cosimplicial right Λ-comodule $\widetilde{\mathcal{C}}_\Lambda(M)^\bullet$ as follows. For all $s \geq 0$, set $\widetilde{\mathcal{C}}_\Lambda(M)^s \coloneqq M \otimes_k \Lambda^{\otimes s+1}$ with the coaction given by $\mathrm{id}_M \otimes \ldots \otimes \mathrm{id}_\Lambda \otimes \Delta_\Lambda$. For every $0 \leq i \leq s$, the i-th codegeneracy map $\sigma^s_i \colon \widetilde{\mathcal{C}}_\Lambda(M)^{s+1} \to \widetilde{\mathcal{C}}_\Lambda(M)^s$ is given by

$$\sigma^s_i(m \otimes \gamma_0 \otimes \cdots \otimes \gamma_{s+1}) = \epsilon(\gamma_i) m \otimes \gamma_0 \otimes \cdots \otimes \gamma_{i-1} \otimes \gamma_i \otimes \cdots \otimes \gamma_{s+1}$$

for all $\gamma_0, \ldots, \gamma_{s+1} \in \Lambda$ and $m \in M$. For every $1 \leq i \leq s$, the i-th coface map $\delta^s_i \colon \widetilde{\mathcal{C}}_\Lambda(M)^{s-1} \to \widetilde{\mathcal{C}}_\Lambda(M)^s$ is given by

$$\delta^s_i(m \otimes \gamma_0 \otimes \cdots \otimes \gamma_s) = m \otimes \gamma_0 \otimes \cdots \otimes \gamma_{i-2} \otimes \Delta(\gamma_{i-1}) \otimes \gamma_i \otimes \cdots \otimes \gamma_s$$

for all $\gamma_0, \ldots, \gamma_s \in \Lambda$ and $m \in M$, and δ^s_0 is given by

$$\delta^0_s(m \otimes \gamma_0 \otimes \cdots \otimes \gamma_s) = \phi_M(m) \otimes \gamma_0 \otimes \cdots \otimes \gamma_s.$$

By definition, the *non-normalized cobar resolution* of M is the cochain complex $\widetilde{\mathcal{C}}^*_\Lambda(M)$ such that $\widetilde{\mathcal{C}}^s_\Lambda(M) = \widetilde{\mathcal{C}}_\Lambda(M)^s$ for all $s \geq 0$ and whose differentials are given by the alternating sums of the coface maps of $\widetilde{\mathcal{C}}(M)^\bullet$.

Let $\overline{\Lambda} \coloneqq \mathrm{Ker}(\epsilon) \subset \Lambda$, where $\epsilon\colon \Lambda \to k$ is the counit map. The unit $\eta\colon k \to \Lambda$ defines a splitting $\Lambda \simeq k \oplus \overline{\Lambda}$ (as k-vector spaces) with an isomorphism $\overline{\Lambda} \simeq \mathrm{Coker}(\eta)$. The comultiplication $\Delta\colon \Lambda \to \Lambda \otimes_k \Lambda$ induces a map $\overline{\Delta}\colon \overline{\Lambda} \to \overline{\Lambda} \otimes_k \overline{\Lambda}$ as the composition

$$\overline{\Lambda} \hookrightarrow \Lambda \xrightarrow{\Delta} \Lambda \otimes_k \Lambda \twoheadrightarrow \overline{\Lambda} \otimes_k \overline{\Lambda},$$

which endows $\overline{\Lambda}$ with the structure of non-unital coalgebra over k. For a left (resp. right) comodule N (resp. M) with the coaction $\phi_N\colon N \to \Lambda \otimes_k N$ (resp. $\phi_M\colon M \to M \otimes_k \Lambda$) we will denote by $\overline{\phi_N}$ (resp. $\overline{\phi_M}$) the corresponding coaction of $\overline{\Lambda}$ on N (resp. M) obtained by composing with the projection $\overline{(-)}\colon \Lambda \twoheadrightarrow \overline{\Lambda}$.

The *(normalized) cobar resolution* $\mathcal{C}^*_\Lambda(M)$ of M is defined as the *normalized* cochain complex associated to $\widetilde{\mathcal{C}}_\Lambda(M)^\bullet$, that is, by

$$\mathcal{C}^s_\Lambda(M) = \bigcap_{i=0}^{s-1}(\mathrm{Ker}(\sigma_i^{s-1}\colon \mathcal{C}_\Lambda(M)^s \to \mathcal{C}_\Lambda(M)^{s-1}) = M \otimes_k \overline{\Lambda}^{\otimes s} \otimes_k \Lambda,$$

where as before $\overline{\Lambda} \simeq \mathrm{Ker}(\epsilon) \simeq \mathrm{Coker}(\eta)$, and differential $d\colon \mathcal{C}^s_\Lambda(M) \to \mathcal{C}^{s+1}_\Lambda(M)$ is induced by the one on $\widetilde{\mathcal{C}}^*_\Lambda(M)$:

$$\begin{aligned} d(m \otimes \gamma_1 \otimes \cdots \otimes \gamma_{s+1}) \coloneqq{}& \overline{\phi_M}(m) \otimes \gamma_1 \otimes \cdots \otimes \gamma_{s+1} + \\ &+ \sum_{i=1}^{s} (-1)^{i-1} m \otimes \gamma_1 \otimes \ldots \otimes \gamma_{i-1} \otimes \overline{\Delta}(\gamma_i) \otimes \gamma_{i+1} \otimes \ldots \otimes \gamma_{s+1} \end{aligned} \tag{2.7}$$

for all $\gamma_{s+1} \in \Lambda$, $\gamma_1, \ldots, \gamma_s \in \overline{\Lambda}$ and $m \in M$. See also [Rav86, Definition A1.2.11] for an analogous construction for left comodules. It is a part of the cosimplicial Dold-Kan correspondence that the natural inclusion $\mathcal{C}^*_\Lambda(M) \hookrightarrow \widetilde{\mathcal{C}}^*_\Lambda(M)$ is split and a homotopy equivalence; see e.g. [Sta, Lemma 019I, (3)] (the proof is dual to that of [Sta, Lemma 019A]).

The map $M \to M \otimes_k \Lambda^{\otimes \bullet+1} =: \widetilde{\mathcal{C}}_\Lambda(M)^\bullet$ sending $m \mapsto m \otimes 1 \otimes \ldots \otimes 1$ defines a map of cosimplicial right Λ-comodules $M^\bullet \to \widetilde{\mathcal{C}}_\Lambda(M)^\bullet$ (here $M^\bullet$ denotes the constant cosimplicial object). By passing to normalized cochain complexes we get a map

$$M \to \mathcal{C}^*_\Lambda(M)$$

of complexes of right Λ-comodules (where M is considered as a complex concentrated in degree 0). The complex $\mathcal{C}^*_\Lambda(M)$ gives an injective resolution of M; indeed, all of its terms are injective right Λ-comodules by Lemma 2.2(b). In particular, given a left Λ-comodule N one can explicitly compute $\mathrm{Cotor}^*_\Lambda(M,N)$ as the cohomology of $\mathcal{C}^*_\Lambda(M,N) \coloneqq \mathcal{C}^*_\Lambda(M)\square_\Lambda N$. Let us record that $\mathcal{C}^*_\Lambda(M,N)$ is given explicitly by $\mathcal{C}^s_\Lambda(M)\square_\Lambda N \simeq M \otimes_k \overline{\Lambda}^s \otimes_k N$ with the differential $d\colon \mathcal{C}^s_\Lambda(M,N) \to \mathcal{C}^{s+1}_\Lambda(M,N)$ given by

$$\begin{aligned} d(m \otimes \gamma_1 \otimes \ldots \otimes \gamma_s \otimes n) \coloneqq{}& \overline{\phi_M}(m) \otimes \gamma_1 \otimes \ldots \otimes \gamma_s \otimes n + (-1)^s m \otimes \gamma_1 \otimes \ldots \otimes \gamma_s \otimes \overline{\phi_N}(n) \\ &+ \sum_{i=1}^{s-1} (-1)^{i-1} m \otimes \gamma_1 \otimes \ldots \otimes \gamma_{i-1} \otimes \overline{\Delta}(\gamma_i) \otimes \gamma_{i+1} \otimes \ldots \otimes \gamma_s \otimes n. \end{aligned} \tag{2.8}$$

If Λ, M and N are L-graded, so are $\mathcal{C}^*_\Lambda(M)$ and $\mathcal{C}^*_\Lambda(M)\square_\Lambda N$; then $H^i(\mathcal{C}^*_\Lambda(M)\square_\Lambda N)$ computes $\mathrm{Cotor}^i_\Lambda(M,N)$ as an L-graded vector space.

Remark 2.9. In [Rav86, Definition A1.2.3], Ravenel defines $\mathrm{Cotor}^i(M,N)$ as the i-th right derived functor of $M\square_\Lambda N$, but considered as a functor in N. By a standard argument one can show that it doesn't matter whether to consider $M\square_\Lambda N$ as a functor in M or in N: namely, picking injective resolutions $M\to I^*_M$ and $N\to I^*_N$ as right an left Λ-comodules one has natural quasi-isomorphisms

$$I^*_M\square_\Lambda N\xrightarrow{\sim} I^*_M\square_\Lambda I^*_N\xleftarrow{\sim} M\square_\Lambda I^*_N$$

by using the fact that all rows and columns in the bicomplex defined by $I^*_M\square_\Lambda I^*_N$ are exact.

2.3. **Algebra structure on** Cotor**.** Let $m_\Lambda\colon \Lambda\otimes_k\Lambda\to\Lambda$ be the multiplication map. Given two L-graded left Λ-comodules M and N, we write $M\otimes_k N$ for the comodule tensor product of M and N. By definition, this is the usual tensor product of L-graded k-vector spaces $M\otimes_k N$, endowed with the following left coaction of Λ:

$$M\otimes_k N\xrightarrow{\phi_M\otimes\phi_N}\Lambda\otimes_k M\otimes_k\Lambda\otimes_k N\xrightarrow{\sim}\Lambda\otimes_k\Lambda\otimes_k M\otimes_k N\xrightarrow{m_\Lambda\otimes\mathrm{id}_M\otimes\mathrm{id}_N}\Lambda\otimes_k M\otimes_k N\ .$$

This definition agrees with [Rav86, Definition A1.1.2]. Let us emphasize that the left coaction of Λ on $M\otimes_k N$ crucially depends on the *algebra* structure on Λ. Similarly, one defines right Λ-comodule algebras.

Note that given right Λ-comodules M_1,M_2 and left Λ-comodules N_1,N_2 there is a natural map

$$(M_1\square_\Lambda N_1)\otimes_k(M_2\square_\Lambda N_2)\to(M_1\otimes_k M_2)\square_\Lambda(N_1\otimes_k N_2) \tag{2.10}$$

of k-vector spaces: indeed, both are subspaces of $M_1\otimes M_2\otimes N_1\otimes N_2$ and one checks easily from the definitions that the left hand side is the subspace of the right.

Construction 2.11 (External product)**.** If M_1 and M_2 are L-graded right Λ-comodules and N_1 and N_2 are L-graded left Λ-comodules, we have an external cup product map

$$\mathrm{Cotor}^{*_1}_\Lambda(M_1,N_1)\otimes_k\mathrm{Cotor}^{*_2}_\Lambda(M_2,N_2)\longrightarrow\mathrm{Cotor}^{*_1+*_2}_\Lambda(M_1\otimes_k M_2,N_1\otimes_k N_2),$$

which can be defined as follows (see also [Rav86, Definition A1.2.13]). Let $M_1\to I^*_{M_1}$ and $M_2\to I^*_{M_2}$ be injective resolutions in the category of right Λ-comodules; then $I^*_{M_1}\otimes I^*_{M_2}$ is an injective resolution for $M_1\otimes M_2$. We have a natural map

$$(I^*_{M_1}\square_\Lambda N_1)\otimes_k(I^*_{M_2}\square_\Lambda N_2)\to(I^*_{M_1}\otimes_k I^*_{M_2})\square_\Lambda(N_1\otimes_k N_2)$$

given by 2.10.

If A is a right Λ-comodule algebra and B is a left Λ-comodule algebra, then letting $M_1=M_2=A$ and $N_1=N_2=B$ and composing the external cup product with the map

$$\mathrm{Cotor}^*_\Lambda(A\otimes_k A,B\otimes_k B)\longrightarrow\mathrm{Cotor}^*_\Lambda(A,B)$$

induced by the multiplication maps $A\otimes_k A\to A$ and $B\otimes_k B\to B$ gives $\mathrm{Cotor}^*_\Lambda(A,B)$ the structure of a $(\mathbb{Z}\oplus L)$-graded k-algebra. (Here $\mathrm{Cotor}^i_\Lambda(A,B)$ has $\mathbb{Z}$-grading i and the L-grading is the one coming from Λ, M and N.)

More generally, if M (resp. N) is a left A-(resp. B-)module in the category of left (resp. right) Λ-comodules, then similarly, composing with the map

$$\mathrm{Cotor}^*_\Lambda(A\otimes_k M,B\otimes_k N)\longrightarrow\mathrm{Cotor}^*_\Lambda(M,N)$$

the external cup product endows $\mathrm{Cotor}^*_\Lambda(M,N)$ with a natural structure of $(\mathbb{Z}\oplus L)$-graded $\mathrm{Cotor}^*_\Lambda(A,B)$-algebra.

Remark 2.12. It will be useful to lift the above action of $\mathrm{Cotor}^*_\Lambda(A,B)$ on $\mathrm{Cotor}^*_\Lambda(M,N)$ to an explicit action of a DG-algebra on a DG-module.

Namely, the map $\mathrm{Cotor}^{*_1}_\Lambda(M_1,N_1)\otimes_k \mathrm{Cotor}^{*_2}_\Lambda(M_2,N_2)\to \mathrm{Cotor}^{*_1+*_2}_\Lambda(M_1\otimes_k M_2, N_1\otimes_k N_2)$ has a natural lift to the map of complexes

$$\mathrm{AW}\colon C^*_\Lambda(M_1,N_1)\otimes_k C^*_\Lambda(M_2,N_2)\longrightarrow C^*_\Lambda(M_1\otimes_k M_2, N_1\otimes_k M_2)$$

via the Alexander–Whitney product (see [Rav86, (A1.2.15)]). When $M_1=M_2=A$ and $N_1=N_2=B$ are Λ-comodule algebras the map AW endows $C^*_\Lambda(A,B)$ with the structure of DG-algebra (that lifts the algebra structure on Cotor).

We will be particularly interested in the case where the Λ-comodule structure on M_1,N_1,M_2 is trivial. Then the formula for AW simplifies significantly: given $m_1\otimes\gamma_1\otimes\ldots\otimes\gamma_s\otimes n_1\in C^s_\Lambda(M_1,N_1)$ and $m_2\otimes\gamma_{s+1}\otimes\ldots\otimes\gamma_{s+t}\otimes n_2\in C^t_\Lambda(M_2,N)$ we simply have

$$\begin{aligned}\mathrm{AW}((m_1\otimes\gamma_1\otimes\ldots\otimes\gamma_s\otimes n_1)\otimes(m_2\otimes\gamma_{s+1}\otimes\ldots\otimes\gamma_{s+t}\otimes n_2)):=\\ (m_1\otimes m_2)\otimes\gamma_1\otimes\ldots\otimes\gamma_s\otimes\gamma_{s+1}\otimes\ldots\otimes\gamma_{s+t}\otimes(n_1\otimes n_2),\end{aligned}$$

where the result is an element of $C^{s+t}_\Lambda(M_1\otimes_k M_2, N_1\otimes_k N_2)$. In the case $N_1=N_2=M_1=M_2=k$ this map induces a DG-algebra structure on $C^*_\Lambda(k,k)$ lifting the multiplication on $\mathrm{Cotor}^*_\Lambda(k,k)$. For a general left Λ-comodule N it induces a structure of DG-module over $C^*_\Lambda(k,k)$ on $C^*_\Lambda(k,N)$ which lifts the natural $\mathrm{Cotor}^*_\Lambda(k,k)$-module structure on $\mathrm{Cotor}^*_\Lambda(k,N)$.

In the case $M_1=M_2=A$, $N_1=N_2=B$ are k-algebras considered as Λ-comodule algebras with trivial actions, composing with the maps $A\otimes A\to A$, $B\otimes B\to B$ this gives a DG-algebra structure on $C^*_\Lambda(A,B)$ (with multiplication given by

$$\begin{aligned}\mathrm{AW}((a_1\otimes\gamma_1\otimes\ldots\otimes\gamma_s\otimes b_1)\otimes(a_2\otimes\gamma_{s+1}\otimes\ldots\otimes\gamma_{s+t}\otimes b_2)):=\\ a_1a_2\otimes\gamma_1\otimes\ldots\otimes\gamma_s\otimes\gamma_{s+1}\otimes\ldots\otimes\gamma_{s+t}\otimes b_1b_2),\end{aligned}$$

lifting the algebra structure on $\mathrm{Cotor}^*_\Lambda(A,B)$.

Remark 2.13. We will be working over fields of characteristic 2, so the sign conventions for multiplication will not matter for us; in particular, if M and N are commutative algebras, then so will be $\mathrm{Cotor}^*_\Lambda(M,N)$.

Example 2.14. If G is a linear algebraic group over a field k and $\Gamma\subset G$ is a central subgroup, the multiplication map $\Gamma\times G\to G$ is a k-group homomorphism and so induces a morphism of stacks $B\Gamma\times BG\to BG$. Passing to Hodge cohomology and applying Künneth's formula [Tot18, Proposition 5.1] we obtain a homomorphism of $\mathbb{Z}^2$-graded k-algebras

$$H^{*,*}_{\mathrm{H}}(BG/k)\longrightarrow H^{*,*}_{\mathrm{H}}(B\Gamma/k)\otimes_k H^{*,*}_{\mathrm{H}}(BG/k). \tag{2.15}$$

If $G=\Gamma$ is commutative, (2.15) makes $H^{*,*}_{\mathrm{H}}(BG/k)$ into a $\mathbb{Z}^2$-graded Hopf algebra over k. If G is an arbitrary linear algebraic group, (2.15) makes $H^{*,*}_{\mathrm{H}}(BG/k)$ into a left $H^{*,*}_{\mathrm{H}}(B\Gamma/k)$-comodule algebra. The same holds for de Rham cohomology, using the Künneth formula (Corollary A.5). Namely, $H^*_{\mathrm{dR}}(B\Gamma/k)$ is a $\mathbb{Z}$-graded Hopf algebra and $H^*_{\mathrm{dR}}(BG/k)$ has a natural left $H^*_{\mathrm{dR}}(B\Gamma/k)$-comodule algebra structure.

3. The Eilenberg-Moore spectral sequence

Let $K^{\bullet\bullet}$ be a first-quadrant double cochain complex, and write $\mathrm{Tot}(K^{\bullet\bullet})$ for the associated total complex. By definition, the first spectral sequence associated to $K^{\bullet\bullet}$ is the spectral sequence

$$E_1^{ij} \coloneqq H^j(K^{i\bullet}) \Rightarrow H^n(\mathrm{Tot}(K^{\bullet\bullet})) \tag{3.1}$$

associated to the decreasing filtration $F^* \mathrm{Tot}(K^{\bullet\bullet})$ of $\mathrm{Tot}(K^{\bullet\bullet})$ given by column degree, that is

$$F^i \mathrm{Tot}^n(K^{\bullet\bullet}) \coloneqq \bigoplus_{p+q=n, p\geq i}^{n} K^{p,q};$$

see [Sta, 012X].

Let $X_{\bullet\bullet}$ be a bisimplicial scheme and F be an abelian sheaf on the small étale site of $X_{\bullet\bullet}$. By [Fri82, Proposition 2.6], there exists a first-quadrant spectral sequence

$$E_1^{ij} \coloneqq H^j(X_{i\bullet}, F|_{X_{i\bullet}}) \Rightarrow H^{i+j}(X_{\bullet\bullet}, F), \tag{3.2}$$

which is functorial in F. It is defined as the first spectral sequence (3.1) associated to the double complex $K_{\bullet\bullet} = \mathrm{Hom}_{\mathrm{AbSh}(X_{\bullet\bullet})}(\mathbb{Z}_{X_{*\cdot}}, I)$, where $\mathrm{AbSh}(X_{\bullet\bullet})$ is the category of abelian sheaves on the small étale site of $X_{\bullet\bullet}$, $\mathbb{Z}_{X_{*\bullet}} \to \mathbb{Z}$ is a certain projective resolution in $\mathrm{AbSh}(X_{\bullet\bullet})$ and $F \to I$ is an injective resolution in $\mathrm{AbSh}(X_{\bullet\bullet})$. (In [Fri82, Proposition 2.4], the simplicial analogue of (3.2) is proved with more details than [Fri82, Proposition 2.6].)

Remark 3.3. The spectral sequence (3.2) is compatible with cup products in the following sense: given a homomorphism of abelian sheaves $A \otimes B \to C$ on the small étale site of $X_{\bullet\bullet}$, there is a homomorphism from the tensor product of the spectral sequence (3.2) for A and B to the spectral sequence (3.2) for C, which on the E_1-page and on the E_∞-page is the one induced by the map $A \otimes B \to C$.

Indeed, let $A \to I_A$, $B \to I_B$ and $C \to I_C$ be injective resolutions. We may construct a commutative square

$$\begin{array}{ccc} A \otimes B & \longrightarrow & I_A \otimes I_B \\ \downarrow & & \downarrow \\ C & \longrightarrow & I_C, \end{array}$$

where the map $I_A \otimes I_B \to I_C$ is uniquely determined up to homotopy. We obtain an induced homomorphism of double complexes

$$\mathrm{Hom}_{\mathrm{AbSh}(X_{\bullet\bullet})}(\mathbb{Z}_{X_{*\cdot}}, I_A) \otimes \mathrm{Hom}_{\mathrm{AbSh}(X_{\bullet\bullet})}(\mathbb{Z}_{X_{*\cdot}}, I_B) \to \mathrm{Hom}_{\mathrm{AbSh}(X_{\bullet\bullet})}(\mathbb{Z}_{X_{*\cdot}}, I_C).$$

Now an application of [CE99, XV, Exercises 2, 4] to this homomorphism yields the required cup products. It is independent of the choice of resolutions $A \to I_A$, $B \to I_B$ and $C \to I_C$ and of the map $I_A \otimes I_B \to I_C$.

For every k-scheme X, there is a simplicial scheme EX whose space of n-simplices $(EX)_n$ is X^{n+1}; see [Del74, Section 6.1.3]. If G is a linear algebraic k-group, the Čech nerve of the universal G-torsor $\mathrm{Spec}\, k \to BG$ is EG/G, the quotient of EG by the diagonal left G-action:

$$BG \longleftarrow \mathrm{Spec}\, k \leftleftarrows G \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, G \times_k G \,\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\, \cdots . \tag{3.4}$$

(We do not draw the degeneracy maps.) If H is a subgroup of G, the Čech nerve of the G-torsor $G/H \to BH$ is given by EG/H, where H acts diagonally by left multiplication:

$$BH \longleftarrow G/H \leftleftarrows G^2/H \;\overset{\longleftarrow}{\leftleftarrows}\; G^3/H \;\overset{\leftleftarrows}{\leftleftarrows}\; \cdots . \tag{3.5}$$

Proof of Theorem 1.3. We start by constructing the spectral sequence for Hodge cohomology. Since Γ is a k-group of multiplicative type, there exist a k-torus T and a k-group embedding $\Gamma \hookrightarrow T$. Define

$$\tilde{G} := (T \times G)/\Gamma.$$

The projection $\tilde{G} \to \tilde{G}/G$ is a G-torsor. The induced morphism $\tilde{G}/G \to BG$ is smooth and surjective, and the Čech nerve of $\tilde{G}/G \to BG$ is given by $E\tilde{G}/G$, that is

$$BG \longleftarrow \tilde{G}/G \leftleftarrows \tilde{G}^2/G \;\overset{\longleftarrow}{\leftleftarrows}\; \tilde{G}^3/G \;\overset{\leftleftarrows}{\leftleftarrows}\; \cdots . \tag{3.6}$$

The projection $T \to T/\Gamma$ is a Γ-torsor. The induced morphism $T/\Gamma \to B\Gamma$ is smooth and surjective, and the Čech nerve of $T/\Gamma \to B\Gamma$ is given by ET/Γ:

$$B\Gamma \longleftarrow T/\Gamma \leftleftarrows T^2/\Gamma \;\overset{\longleftarrow}{\leftleftarrows}\; T^3/\Gamma \;\overset{\leftleftarrows}{\leftleftarrows}\; \cdots . \tag{3.7}$$

For all $i \geq 1$, the projection morphism $\tilde{G}^i/G \to \overline{G}^i/\overline{G}$ is a (T^i/Γ)-torsor, hence it is smooth and surjective, and its Čech nerve is given by

$$\overline{G}^i/\overline{G} \longleftarrow T^i/\Gamma \leftleftarrows T^i/\Gamma \times \tilde{G}^i/G \;\overset{\longleftarrow}{\leftleftarrows}\; (T^i/\Gamma)^2 \times \tilde{G}^i/G \;\overset{\leftleftarrows}{\leftleftarrows}\; \cdots . \tag{3.8}$$

We obtain the following commutative diagram

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downdownarrows\downdownarrows & & \downdownarrows\downdownarrows & & \downdownarrows\downdownarrows & & \downdownarrows\downdownarrows & & \\
\overline{G}^3/\overline{G} & \longleftarrow & \tilde{G}^3/G & \leftleftarrows & T^3/\Gamma \times \tilde{G}^3/G & \overset{\longleftarrow}{\leftleftarrows} & (T^3/\Gamma)^2 \times \tilde{G}^3/G & \overset{\leftleftarrows}{\leftleftarrows} & \cdots \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \\
\overline{G}^2/\overline{G} & \longleftarrow & \tilde{G}^2/G & \leftleftarrows & T^2/\Gamma \times \tilde{G}^2/G & \overset{\longleftarrow}{\leftleftarrows} & (T^2/\Gamma)^2 \times \tilde{G}^2/G & \overset{\leftleftarrows}{\leftleftarrows} & \cdots \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows & & \\
\operatorname{Spec} k & \longleftarrow & \tilde{G}/G & \leftleftarrows & T/\Gamma \times \tilde{G}/G & \overset{\longleftarrow}{\leftleftarrows} & (T/\Gamma)^2 \times \tilde{G}/G & \overset{\leftleftarrows}{\leftleftarrows} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
B\overline{G} & \longleftarrow & BG & \leftleftarrows & B\Gamma \times BG & \overset{\longleftarrow}{\leftleftarrows} & (B\Gamma)^2 \times BG & \overset{\leftleftarrows}{\leftleftarrows} & \cdots .
\end{array} \tag{3.9}$$

In diagram (3.9), the leftmost column is $E\overline{G}/\overline{G}$, and for all the other columns are the product of an increasing number of copies of (3.7) and one copy of (3.6). The bottom row is the Čech nerve of the smooth cover of stacks $BG \to B\overline{G}$, and the other rows are given by (3.8). Let $X_{\bullet\bullet}$ be the bisimplicial scheme obtained from (3.9) by removing the left column and bottom row. The diagram shows that $X_{\bullet\bullet}$ is the Čech nerve of the morphism of simplicial schemes $E\tilde{G}/G \to E\overline{G}/\overline{G}$. Since $(E\tilde{G}/G)_i \to (E\overline{G}/\overline{G})_i$ is a (T^i/Γ)-torsor and T^i/Γ is smooth, the morphism $E\tilde{G}/G \to E\overline{G}/\overline{G}$ is a smooth cover. Therefore, for all $h \geq 0$, [Fri82, Proposition 3.7] gives an isomorphism

$$H^*(E\overline{G}/\overline{G}, \Omega^h) \xrightarrow{\sim} H^*(X_{\bullet\bullet}, \Omega^h).$$

Since $\operatorname{Spec} k \to B\overline{G}$ is smooth, we have an isomorphism

$$H^*(B\overline{G}, \Omega^h) \xrightarrow{\sim} H^*(E\overline{G}/\overline{G}, \Omega^h).$$

Moreover, for all $i \geq 0$, the simplicial scheme $X_{i\bullet}$ is the Čech nerve of the $(T^i \times \tilde{G})$-torsor $(T/\Gamma)^i \times \tilde{G}/G \to B\Gamma^i \times BG$. We obtain the isomorphisms

$$H^*(B\Gamma^i \times BG, \Omega^h) \xrightarrow{\sim} H^*(X_{i\bullet}, \Omega^h).$$

Therefore the spectral sequence (3.2) for $X_{\bullet\bullet}$ and $F = \Omega^h$ takes the form

$$E_1^{ij}(h) \coloneqq H^j((B\Gamma)^i \times BG, \Omega^h) \Rightarrow H^{i+j}(B\overline{G}, \Omega^h).$$

Letting h vary, the spectral sequences $E^{ij}(h)$ assemble into a spectral sequence of graded k-vector spaces

$$E_1^{ij} \coloneqq H^j_{\mathrm{H}}((B\Gamma)^i \times BG/k) \Rightarrow H^{i+j}_{\mathrm{H}}(B\overline{G}/k).$$

An application Remark 3.3 to the homomorphisms $\Omega^h \otimes \Omega^{h'} \to \Omega^{h+h'}$, where $h, h' \geq 0$, turns the spectral sequence into a spectral sequence of graded k-algebras.

By the Künneth formula in Hodge cohomology [Tot18, Proposition 5.1], for every $i \geq 0$,

$$E_1^{i*} = H^*_{\mathrm{H}}(B\Gamma^i \times BG/k) \simeq H^*_{\mathrm{H}}(B\Gamma/k)^{\otimes i} \otimes H^*_{\mathrm{H}}(BG/k).$$

This is the non-normalized cobar construction of the $H^*_{\mathrm{H}}(B\Gamma/k)$-comodule algebra $H^*_{\mathrm{H}}(BG/k)$. Indeed, the differentials are alternating sums of projection maps in (3.9), and using this one may check that they agree with those of the cobar construction. Therefore, by [Rav86, Corollary A1.2.12], the E_2 page of the spectral sequence computes $\operatorname{Cotor}^*_{H^*(B\Gamma)}(k, H^*_{\mathrm{H}}(BG))$.

The construction for the spectral sequence in de Rham cohomology is entirely analogous. Indeed, while [Fri82, Proposition 2.6, Proposition 3.7] are only phrased for sheaves of abelian groups, they also hold for complexes of sheaves. For the Künneth formula in de Rham cohomology, see Corollary A.5. □

Remark 3.10. If X is a smooth algebraic k-stack, U is a k-scheme, and $f\colon U \to X$ is a morphism, then the Čech nerve of f can be used to compute the Hodge or de Rham cohomology of X only when f is smooth and surjective. This explains the introduction of the auxiliary torus T in the proof of Theorem 1.3: indeed, the morphisms $G \to \overline{G}$ and $\operatorname{Spec} k \to B\Gamma$ are Γ-torsors, hence not necessarily smooth, while the morphisms $\tilde{G} \to G$ and $T/\Gamma \to B\Gamma$ are torsors under the tori T/Γ and T, respectively, and so they are smooth.

Remark 3.11. Let $K^{\bullet\bullet}$ be a first-quadrant double cochain complex, and consider the spectral sequence (3.1) for $K^{\bullet\bullet}$. Let $u \in F^iH^{i+j}(\operatorname{Tot} K^{\bullet\bullet})$ and $v \in F^{i'}H^{i'+j'}(\operatorname{Tot} K^{\bullet\bullet})$. Then u and v are represented by classes $\overline{u} \in E_\infty^{i,j}$ and $\overline{v} \in E_\infty^{i',j'}$. Suppose that $\overline{u} \cdot \overline{v} = 0$ in E_∞. This means that $uv \in F^{i+i'+1}H^{i+i'+j+j'}(\operatorname{Tot} K^{\bullet\bullet})$, and so there exists an integer $d \geq 1$ such that the representative in $\overline{uv} \in E_\infty$ of uv has bidegree $(i + j + d, i' + j')$. In other words, if $\overline{u} \cdot \overline{v} = 0$ then the column degree of $\overline{uv}$ is strictly greater than the sum of the column degrees of $\overline{u}$ and $\overline{v}$.

Since the Eilenberg-Moore spectral sequence is defined in terms of (3.1), this remark applies to it. We will make use of this observation during the proof of Theorems 7.16 and 8.17.

Remark 3.12. It follows from the construction of the Eilenberg-Moore spectral sequence in Hodge cohomology that

$$E_2^{0,*} := \mathrm{Cotor}^0_{H^*_{\mathrm{H}}(B\Gamma/k)}(k, H^*_{\mathrm{H}}(BG/k)) = PH^*_{\mathrm{H}}(BG/k),$$

where we regard $H^*_{\mathrm{H}}(BG/k)$ as a left $H^*_{\mathrm{H}}(B\Gamma/k)$-comodule algebra. The corresponding edge homomorphism

$$H^*_{\mathrm{H}}(B\overline{G}/k) \to PH^*_{\mathrm{H}}(BG/k) \subset H^*_{\mathrm{H}}(BG/k)$$

is exactly the pull-back with respect to the map $BG \to B\overline{G}$ induced by the projection $G \to \overline{G}$. A similar description holds for de Rham cohomology.

Remark 3.13. Let us emphasize that, taking into account the bigrading on Hodge cohomology and the construction of the Eilenberg-Moore spectral sequence, $E_2^{i,j}$ decomposes as a direct sum $E_2^{i,j} \simeq \oplus_h (E_2^{i,j})^h$ where

$$(E_2^{i,j})^h := \left(\mathrm{Cotor}^i_{H^*_{\mathrm{H}}(B\Gamma/k)}(k, H^*_{\mathrm{H}}(BG/k))\right)^{h,j} \Rightarrow H^{h,i+j}_{\mathrm{H}}(B\overline{G}/k).$$

Here the bigrading on $\mathrm{Cotor}^i_{H^*_{\mathrm{H}}(B\Gamma/k)}(k, H^*_{\mathrm{H}}(BG/k))$ is the one coming from $H^{*,*}_{\mathrm{H}}(B\Gamma/k)$ and $H^{*,*}_{\mathrm{H}}(BG/k)$.

4. Additive part of Theorem 1.5 for projective linear and symplectic groups

The goal of this section is to identify the E_2 page of the Eilenberg-Moore spectral sequence for the Hodge and de Rham cohomology with their topological counterparts in the case of PGL_n and PSp_n. This is done by explicitly comparing the coalgebras and comodules that take part in the Cotor description on both sides. In Section 4.3 we then record how to deduce the degeneration of the above spectral sequences in the case $n = 4m + 2$ from Totaro's inequality and the results of Toda [Tod87] for the singular cohomology.

4.1. **Projective linear group.** Let $n \geq 1$ be an integer and p be a prime number. We have isomorphisms

$$H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p) \simeq \mathbb{F}_p[x_2], \qquad H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_p) \simeq \mathbb{F}_p[c_1, \ldots, c_n], \tag{4.1}$$

where $x_2 \in H^{1,1}_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p)$ and $c_i \in H^{i,i}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_p)$. If $\iota\colon T_n \hookrightarrow \mathrm{GL}_n$ is the diagonal maximal torus, then by the Künneth formula $H^*_{\mathrm{H}}(BT_n/\mathbb{F}_p)$ is a polynomial ring in n generators $t_1, \ldots, t_n$ of bidegree $(1, 1)$, and the pullback map

$$B\iota^*\colon H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_p) \longrightarrow H^*_{\mathrm{H}}(BT_n/\mathbb{F}_p) \simeq \mathbb{F}_p[t_1, \ldots, t_n]$$

is injective and identifies c_i with the i-th symmetric function on variables t_i; see [Tot18, End of proof of Theorem 9.2].

If we identify $\mathbb{G}_{\mathrm{m}}$ with the center of GL_n, the multiplication map $\mathbb{G}_{\mathrm{m}} \times \mathrm{GL}_n \to \mathrm{GL}_n$ is a group homomorphism, and so induces a morphism of stacks $B\mathbb{G}_{\mathrm{m}} \times B\mathrm{GL}_n \to B\mathrm{GL}_n$. By the Künneth formula, we obtain a ring homomorphism

$$\phi\colon H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_p) \to H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p) \otimes H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_p), \tag{4.2}$$

which endows $H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_p)$ with the structure of an $H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p)$-comodule algebra (as discussed in Example 2.14).

Let us explicitly describe the coalgebra structure on $H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p)$. The restriction of the product map $\mathbb{G}_{\mathrm{m}}\times\mathbb{G}_{\mathrm{m}}\to\mathbb{G}_{\mathrm{m}}$ to $\mathbb{G}_{\mathrm{m}}\times\{1\}$ and $\{1\}\times\mathbb{G}_{\mathrm{m}}$ is the identity, hence the comultiplication map

$$\Delta\colon H^2_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p)\to H^2_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}\times B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p)\xrightarrow[\sim]{\text{Künneth}} H^2_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p)\oplus H^2_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p)$$

is the diagonal embedding. It then follows that the Hopf algebra structure on $H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p)\simeq\mathbb{F}_p[x_2]$ is the unique one given by $\Delta(x_2)=x_2\otimes 1+1\otimes x_2$, namely

$$\Delta(x_2^n)=\sum_{i=0}^{n}\binom{n}{i}x_2^{n-i}\otimes x_2^i.$$

Since $H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_p)$ is concentrated in bidegrees (i,i) and is a polynomial algebra, the Hodge-de Rham spectral sequence for $B\mathrm{GL}_n$ degenerates and induces an isomorphism

$$H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_p)\simeq H^*_{\mathrm{dR}}(B\mathrm{GL}_n/\mathbb{F}_p).$$

Thus the previous discussion also applies to $H^*_{\mathrm{dR}}(B\mathrm{GL}_n/\mathbb{F}_p)$. Below, in this section we will be giving proofs in the case of Hodge cohomology, but the same arguments then apply to de Rham context.

We now write

$$H^*_{\mathrm{sing}}(B\mathbb{C}^\times;\mathbb{F}_p)\simeq\mathbb{F}_p[x_2^{\mathrm{top}}],\qquad H^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_p)\simeq\mathbb{F}_p[c_1^{\mathrm{top}},\dots,c_n^{\mathrm{top}}],$$

where $|x_2^{\mathrm{top}}|=2$ and $|c_i^{\mathrm{top}}|=2i$. We naturally regard $H^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_p)$ as a $H^*_{\mathrm{sing}}(B\mathbb{C}^\times;\mathbb{F}_p)$-comodule algebra, and let ϕ^{top} be the coaction map. We note that the coalgebra structure on $H^*_{\mathrm{sing}}(B\mathbb{C}^\times;\mathbb{F}_p)$ is the unique one given by $\Delta(x_2^{\mathrm{top}})=x_2^{\mathrm{top}}\otimes 1+1\otimes x_2^{\mathrm{top}}$.

It is easy to see that there is an isomorphism of Hopf algebras

$$H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p)\simeq H^*_{\mathrm{sing}}(B\mathbb{C}^\times;\mathbb{F}_p)$$

defined by sending x_2 to x_2^{top}. The next lemma shows that it can be extended to an isomorphism between the comodule-algebras $H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_p)$ and $H^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_p)$, inducing in particular an isomorphism of cotorsion groups of our interest.

Lemma 4.3. *Consider the $H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p)$-comodule algebra $H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_p)=\mathbb{F}_p[c_1,\dots,c_n]$, with the coaction ϕ of (4.2).*

(a) For all $i=1,\dots,n$, we have

$$\phi(c_i)=\sum_{i_1+i_2=i}\binom{n-i_2}{i_1}x_2^{i_1}\otimes c_{i_2},$$

where we use the convention that $c_0:=1$.

(b) The isomorphisms

$$H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p)\xrightarrow{\sim}H^*_{\mathrm{sing}}(B\mathbb{C}^\times;\mathbb{F}_p),\qquad H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_p)\xrightarrow{\sim}H^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_p)$$

given by sending $x_2\mapsto x_2^{\mathrm{top}}$ and $c_i\mapsto c_i^{\mathrm{top}}$ induce an isomorphism of bigraded algebras

$$\mathrm{Cotor}^*_{H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p)}(\mathbb{F}_p,H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_p))\simeq\mathrm{Cotor}^*_{H^*_{\mathrm{sing}}(B\mathbb{C}^\times;\mathbb{F}_p)}(\mathbb{F}_p,H^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_p)).$$

Entirely analogous statements hold with Hodge cohomology replaced by de Rham cohomology.

Proof. (a) The proof is analogous to that of [Tod87, Proposition 3.2]. We have a commutative diagram

$$\begin{array}{ccc} H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_p) & \xrightarrow{\phi} & H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p)\otimes H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_p) \\ \big\downarrow{\scriptstyle B\iota^*} & & \big\downarrow{\scriptstyle 1\otimes B\iota^*} \\ H^*_{\mathrm{H}}(BT_n/\mathbb{F}_p) & \xrightarrow{\phi'} & H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p)\otimes H^*_{\mathrm{H}}(BT_n/\mathbb{F}_p), \end{array}$$

where ϕ' is the coaction map for T_n. We have $H^*_{\mathrm{H}}(BT_n/\mathbb{F}_p) = \mathbb{F}_p[t_1,\dots,t_n]$, where t_i has degree 2 for all i. Recall that $\Delta\colon H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p)\to H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p)\otimes H^*_{\mathrm{H}}(BT_n/\mathbb{F}_p)$ sends x_2 to $x_2\otimes 1 + 1\otimes x_2$. Thus, for all $i=1,\dots,n$, the commutativity of the square

$$\begin{array}{ccc} \mathbb{G}_{\mathrm{m}}\times T_i & \longrightarrow & T_i \\ \big\downarrow{\scriptstyle \mathrm{id}\times\mathrm{pr}_i} & & \big\downarrow{\scriptstyle \mathrm{pr}_i} \\ \mathbb{G}_{\mathrm{m}}\times\mathbb{G}_{\mathrm{m}} & \longrightarrow & \mathbb{G}_{\mathrm{m}} \end{array}$$

implies that

$$\phi'(t_i) = 1\otimes t_i + x_2\otimes 1.$$

Moreover

$$B\iota^*(\sum_{i=1}^n c_i) = \prod_{i=1}^n (1+t_i).$$

Elementary calculations show that

$$(1\otimes B\iota)^*\phi(\sum c_i) = (1\otimes B\iota)^*\sum_{j=0}^n\sum_{i=0}^j \binom{n-j}{i} x_2^i\otimes c_j.$$

Now (a) follows from the injectivity of $(1\otimes B\iota)^*$.

(b) By (a) and [Tod87, Proposition 3.2], the stated isomorphisms induce a commutative square

$$\begin{array}{ccc} H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_p) & \xrightarrow{\phi} & H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_p)\otimes H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_p) \\ \big\downarrow{\scriptstyle \wr} & & \big\downarrow{\scriptstyle \wr} \\ H^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_p) & \xrightarrow{\phi^{\mathrm{top}}} & H^*_{\mathrm{sing}}(B\mathbb{C}^\times;\mathbb{F}_p)\otimes H^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_p), \end{array}$$

where vertical maps are the ones in the statement of the proposition. This shows that the corresponding comodule-algebras are equivalent, and, consequently, one also has an isomorphism of the corresponding Cotor-algebras. □

4.2. **Projective symplectic group.** Let $n\geq 1$ be an integer. By [Tot18, Theorem 9.2], we have an isomorphism

$$H^*_{\mathrm{H}}(B\mathrm{Sp}_{2n}/\mathbb{F}_2) = \mathbb{F}_2[q_1,\dots,q_n],$$

where q_i has bidegree $(2i,2i)$ for $i=1,\dots,n$. By [Tot18, Proposition 10.1], we have an isomorphism of Hopf algebras

$$H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)\simeq\mathbb{F}_2[x_1,x_2]/(x_1^2),$$

where $x_1 \in H^{1,0}_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)$ and $x_2 \in H^{1,1}_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)$. We identify μ_2 with the center of Sp_{2n}. Then the multiplication map $\mu_2 \times \mathrm{Sp}_{2n} \to \mathrm{Sp}_{2n}$ induces a ring homomorphism

$$\phi\colon H^*_{\mathrm{H}}(B\mathrm{Sp}_{2n}/\mathbb{F}_2) \longrightarrow H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2) \otimes H^*_{\mathrm{H}}(B\mathrm{Sp}_{2n}/\mathbb{F}_2). \tag{4.4}$$

We can view $H^*_{\mathrm{H}}(B\mathrm{Sp}_{2n}/\mathbb{F}_2)$ as an $H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)$-comodule algebra, with coaction map ϕ.

Lemma 4.5. *Let $j\colon \mathrm{Sp}_{2n} \hookrightarrow \mathrm{GL}_{2n}$ be the tautological inclusion. Then*

$$Bj^*(c_{2h}) = q_h, \qquad Bj^*(c_{2h-1}) = 0$$

for all $h = 1, \dots, n$. In particular, Bj^* is injective.

Proof. Let $T_{2n} \subset \mathrm{GL}_{2n}$ be the diagonal maximal torus, and let $T'_{2n} := j^{-1}(T_{2n})$. Then T'_{2n} is a maximal torus of Sp_{2n}, and j induces a commutative square

$$\begin{array}{ccc} H^*_{\mathrm{H}}(B\mathrm{GL}_{2n}/\mathbb{F}_2) & \xrightarrow{Bj^*} & H^*_{\mathrm{H}}(B\mathrm{Sp}_{2n}/\mathbb{F}_2) \\ \downarrow & & \downarrow \\ H^*_{\mathrm{H}}(BT_{2n}/\mathbb{F}_2) & \longrightarrow & H^*_{\mathrm{H}}(BT'_{2n}/\mathbb{F}_2). \end{array}$$

The vertical maps are injective. Indeed, by a theorem of Borel [Bor61] (the computation of torsion primes for all connected compact Lie groups), the classifying spaces $B\mathrm{SU}(2n)$ and $B\mathrm{Sp}(2n)$ have no torsion primes, hence the injectivity of the two maps follows from [Tot18, End of proof of Theorem 9.2].

For $i = 1, \dots, 2n$, let $t_i \in H^2(BT_{2n}/\mathbb{F}_2)$ be the generator corresponding to the i-th coordinate of T_{2n}. Then the bottom horizontal arrow gives an identification

$$H^*_{\mathrm{H}}(BT'_{2n}/\mathbb{F}_2) \simeq H^*_{\mathrm{H}}(BT_{2n}/\mathbb{F}_2)/(t_1 + t_{n+1}, t_2 + t_{n+2}, \dots, t_n + t_{2n}).$$

The conclusion follows from the fact that c_i is the i-th symmetric function of $t_1, \dots, t_{2n}$, and that under the previous identification q_i is the i-th symmetric function of $t_1^2, \dots, t_n^2$; see the last line of the proof of [CR10, Proof of Theorem 6.6]. □

Let us also identify the coalgebra structure on $H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)$. Similarly to the $\mathbb{G}_m$-case one can see that the comultiplication map

$$\Delta\colon H^1_{\mathrm{H}}(B\mu_2/\mathbb{F}_p) \longrightarrow H^1_{\mathrm{H}}(B\mu_2 \times B\mu_2/\mathbb{F}_p) \xrightarrow[\sim]{\text{Künneth}} H^1_{\mathrm{H}}(B\mu_2/\mathbb{F}_p) \oplus H^1_{\mathrm{H}}(B\mu_2/\mathbb{F}_p)$$

is the diagonal embedding, so $\Delta(x_1) = x_1 \otimes 1 + 1 \otimes x_1$. The class $x_2 \in H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)$ is the pullback of the generator of $H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_2)$ (see the proof of [Tot18, Proposition 10.1]) and so we also have $\Delta(x_2) = x_2 \otimes 1 + 1 \otimes x_2$. This defines a Hopf-algebra structure on $H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2) \simeq \mathbb{F}_2[x_1, x_2]/x_1^2$ uniquely.

Recall that $H^*_{\mathrm{sing}}(B\mathbb{Z}/2; \mathbb{F}_2) \simeq \mathbb{F}_2[z]$, where $|z| = 1$. The natural Hopf algebra structure here is uniquely defined by $\Delta(z) = z \otimes 1 + 1 \otimes z$.

Remark 4.6. Note that $H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)$ is *not* isomorphic to $H^*_{\mathrm{sing}}(B\mathbb{Z}/2; \mathbb{F}_2)$ as an algebra. Nevertheless, the unique $\mathbb{F}_2$-linear map $\psi\colon H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2) \to H^*_{\mathrm{sing}}(B\mathbb{Z}/2; \mathbb{F}_2)$ which sends a x_2^i to z^{2i} and $x_1 x_2^i$ to z^{2i+1} is easily checked to be an isomorphism of *coalgebras*.

Recall that

$$H^*_{\mathrm{sing}}(B\mathrm{Sp}_{2n};\mathbb{F}_2) = \mathbb{F}_2[q_1^{\mathrm{top}},\dots,q_n^{\mathrm{top}}],$$

where q_i^{top} has degree $4i$ for $i = 1,\dots,n$.

Lemma 4.7. *Consider $H^*_{\mathrm{H}}(B\mathrm{Sp}_{2n}/\mathbb{F}_2) = \mathbb{F}_2[q_1,\dots,q_n]$, with the coaction ϕ of (4.4).*

(a) For all $i = 1,\dots,n$, we have

$$\phi(q_i) = \sum_{i_1+i_2=i} \binom{n-i_2}{i_i} x_2^{2i_1} \otimes q_{i_2}.$$

(b) The isomorphism of coalgebras $\psi\colon H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2) \to H^*_{\mathrm{sing}}(B\mathbb{Z}/2;\mathbb{F}_2)$ together with the map of comodules $H^*_{\mathrm{H}}(B\mathrm{Sp}_{2n}/\mathbb{F}_2) \to H^*_{\mathrm{sing}}(B\mathrm{Sp}_{2n};\mathbb{F}_2)$ sending q_i to q_i^{top} induces an isomorphism of bigraded $\mathbb{F}_2$-vector spaces

$$\mathrm{Cotor}^*_{H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)}(\mathbb{F}_2, H^*_{\mathrm{H}}(B\mathrm{Sp}_{2n}/\mathbb{F}_2)) = \mathrm{Cotor}^*_{H^*_{\mathrm{sing}}(B(\mathbb{Z}/2\mathbb{Z});\mathbb{F}_2)}(\mathbb{F}_2, H^*_{\mathrm{sing}}(B\mathrm{Sp}_{2n};\mathbb{F}_2)).$$

Proof. (a) Since ϕ respects the gradings and $H^*_{\mathrm{H}}(B\mathrm{Sp}_{2n}/\mathbb{F}_2)$ is concentrated in even degrees, the $x_2^i x_1$-component of $\phi(q_h)$ must be zero for all h. To compute the x_2^i-components, observe that we have the following commutative square

$$\begin{array}{ccc} H^*_{\mathrm{H}}(B\mathrm{Sp}_{2n}/\mathbb{F}_2) & \xrightarrow{Bj^*} & H^*_{\mathrm{H}}(B\mathrm{GL}_{2n}/\mathbb{F}_2) \\ \downarrow & & \downarrow \\ H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2) & \longrightarrow & H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_2), \end{array}$$

where $\mathbb{G}_{\mathrm{m}}$ is viewed as the center of GL_{2n} and μ_2 as the center of Sp_{2n}. The conclusion follows from Lemma 4.3(a) and Lemma 4.5.

(b) We have

$$H^*_{\mathrm{sing}}(B\mathrm{Sp}_{2n};\mathbb{F}_2) = \mathbb{F}_2[q_1^{\mathrm{top}},\dots,q_n^{\mathrm{top}}],$$

where q_i^{top} has degree $4i$ for $i = 1,\dots,n$. The coaction map

$$\phi^{\mathrm{top}}\colon H^*_{\mathrm{sing}}(B\mathrm{Sp}_{2n};\mathbb{F}_2) \to H^*_{\mathrm{sing}}(B(\mathbb{Z}/2\mathbb{Z});\mathbb{F}_2) \otimes H^*_{\mathrm{sing}}(B\mathrm{Sp}_{2n}/\mathbb{F}_2)$$

has been computed in [Tod87, Proposition 3.4] and agrees with the formula that we have proved in part (a). Recall (Remark 4.6) that we have an isomorphism of coalgebras

$$\psi\colon H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2) \xrightarrow{\sim} H^*_{\mathrm{sing}}(B(\mathbb{Z}/2\mathbb{Z});\mathbb{F}_2).$$

Comparison of (a) with Toda's formula for ϕ^{top} yields the commutativity of the following diagram of graded linear maps

$$\begin{array}{ccc} H^*_{\mathrm{H}}(B\mathrm{Sp}_{2n}/\mathbb{F}_2) & \xrightarrow{\phi} & H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2) \otimes H^*_{\mathrm{H}}(B\mathrm{Sp}_{2n}/\mathbb{F}_2) \\ \downarrow \wr & & \downarrow \wr \\ H^*_{\mathrm{sing}}(B\mathrm{Sp}_{2n};\mathbb{F}_2) & \xrightarrow{\phi^{\mathrm{top}}} & H^*_{\mathrm{sing}}(B(\mathbb{Z}/2\mathbb{Z});\mathbb{F}_2) \otimes H^*_{\mathrm{sing}}(B\mathrm{Sp}_{2n};\mathbb{F}_2), \end{array}$$

where the vertical maps are induced by ψ and the graded algebra isomorphism

$$H^*_{\mathrm{H}}(B\mathrm{Sp}_{2n}/\mathbb{F}_2) \xrightarrow{\sim} H^*_{\mathrm{sing}}(B\mathrm{Sp}_{2n};\mathbb{F}_2), \qquad q_h \mapsto q_h^{\mathrm{top}}.$$

This induces the desired isomorphism between Cotor* (as graded vector spaces). □

4.3. **Proof of the additive part of Theorem 1.5 for general linear and symplectic groups.** Let G be one of the split reductive $\mathbb{Z}$-groups GL_{4m+2} and Sp_{4m+2}, and let Γ be the center of G. We now prove that we have isomorphisms of $\mathbb{Z}$-graded $\mathbb{F}_2$-vector spaces

$$H^*_{\mathrm{H}}(B\overline{G}/\mathbb{F}_2) \simeq H^*_{\mathrm{dR}}(B\overline{G}/\mathbb{F}_2) \simeq H^*_{\mathrm{sing}}(B\overline{G}(\mathbb{C});\mathbb{F}_2).$$

In particular, this establishes Theorem 1.5(2).

Consider the Eilenberg-Moore spectral sequence associated by Theorem 1.3 to the central short exact sequence

$$1 \to \Gamma_{\mathbb{F}_2} \to G_{\mathbb{F}_2} \to \overline{G}_{\mathbb{F}_2} \to 1. \tag{4.8}$$

By Lemmas 4.3(b) and 4.7(b), its E_2 page is isomorphic to the E_2 page of the topological Eilenberg-Moore spectral sequence of [Tod87, (4.1)] for the fibration $\Gamma(\mathbb{C}) \to G(\mathbb{C}) \to \overline{G}(\mathbb{C})$. Toda showed that the latter spectral sequence degenerates on the second page; see [Tod87, p. 99, line 8] for $G = \mathrm{GL}_{4m+2}$ and [Tod87, p. 102, line 9] for Sp_{4m+2}. Therefore,

$$\dim_{\mathbb{F}_2} H^i_{\mathrm{H}}(B\overline{G}/\mathbb{F}_2) \leq \dim_{\mathbb{F}_2} H^i_{\mathrm{sing}}(B\overline{G}(\mathbb{C});\mathbb{F}_2)$$

for all $i \geq 0$ (where the equality holds for all i if and only if the spectral sequence for Hodge cohomology also degenerates). On the other hand, by [KP21b, Theorem 5.3.6] and the existence of the Hodge-de Rham spectral sequence, we have

$$\dim_{\mathbb{F}_2} H^i_{\mathrm{H}}(B\overline{G}/\mathbb{F}_2) \geq \dim_{\mathbb{F}_2} H^i_{\mathrm{dR}}(B\overline{G}/\mathbb{F}_2) \geq \dim_{\mathbb{F}_2} H^i_{\mathrm{sing}}(B\overline{G}(\mathbb{C});\mathbb{F}_2).$$

We conclude that the algebraic Eilenberg-Moore spectral sequence associated to (4.8) degenerates and that we have equality of dimensions, as desired.

Remark 4.9. The same argument would also apply for $n \neq 4m+2$, if we knew that the Eilenberg-Moore spectral sequence degenerates on the topological side in this case.

5. Computation of Cotor

In this section we set up the main tool for an explicit computation of Cotor: namely, the twisted tensor product construction (Construction 5.5).

5.1. **Totalizations.** Below, we will need the following notation. Let k be a field and let $V^{*_1,*_2}$ be a $\mathbb{Z}\oplus\mathbb{Z}$-graded vector space over k. Given a bihomogeneous element $v \in V^{i_1,i_2}$ we denote $|v| := (\deg_1(v), \deg_2(v))$ where $\deg_1(v) := i_1$ and $\deg_2(v) := i_2$. Given a bigraded vector space $V^{*_1,*_2}$, we can associate to it a $\mathbb{Z}$-graded vector space V^{tot} by putting $(V^{\mathrm{tot}})^i := \oplus_{i_1+i_2=i} V^{i_1,i_2}$. For all $v \in V^{i_1,i_2}$, we denote by $|v|_{\mathrm{tot}} := \deg_1(v)+\deg_2(v) = i_1+i_2$ the total degree of v. The *totalization* functor $V^{*_1,*_2} \mapsto V^{\mathrm{tot}}$ from $\mathbb{Z}^2$-graded vector spaces to $\mathbb{Z}$-graded vector spaces is symmetric monoidal with respect to monoidal structures given by the Day convolution[2]. In particular, given a $\mathbb{Z}^2$-graded coalgebra $C^{*_1,*_2}$ or a $\mathbb{Z}^2$-graded algebra $A^{*_1,*_2}$, the corresponding totalizations C^{tot} and A^{tot} are naturally a $\mathbb{Z}$-graded coalgebra and a $\mathbb{Z}$-graded algebra, respectively. Note that the underlying (ungraded) coalgebra C and algebra A stay the same:

$$C := \oplus_{i_1,i_2} C^{i_1,i_2} \simeq \oplus_i (C^{\mathrm{tot}})^i \quad \text{and} \quad A := \oplus_{i_1,i_2} A^{i_1,i_2} \simeq \oplus_i (A^{\mathrm{tot}})^i.$$

[2]Namely $(V^{*_1,*_2} \otimes W^{*_1,*_2})^{i,j} := \bigoplus_{\substack{n_1+n_2=i \\ m_1+m_2=j}} V^{n_1,n_2} \otimes W^{m_1,m_2}$ and $(V^* \otimes W^*)^i := \bigoplus_{n+m=i} V^n \otimes W^m$.

5.2. **Twisting cochains.** In this section we briefly remind the concept of twisting cochains following [Pos11]. All the results hold true in the presence of extra L-grading for any abelian group L (see [Pos11, Remark in Section (1.1)]). We will assume that k is a field of characteristic 2, so we can ignore all the signs. Given a $\mathbb{Z}$-graded vector space $V = \oplus_{p\in\mathbb{Z}} V^p$ we denote by $V[i]$ the i-th shift to the left, so that $(V[i])^p \coloneqq V^{p-i}$.

Let A a DG-algebra over k. Thus, $A = \oplus_{p\in\mathbb{Z}} A^p$ is a $\mathbb{Z}$-graded[3] associative algebra over k, endowed with a k-linear map $d_A\colon A \to A$ of degree 1 (in other words a $\mathbb{Z}$-graded map $A \to A[1]$), such that $d^2 = 0$ and satisfies the Leibnitz rule $d_A(ab) = d_A(a)\cdot b + a \cdot d_A(b)$. We will assume A is augmented, in other words that there is a preferred DG-algebra homomorphism $\epsilon\colon A \to k$.

Dually, let C be a coaugmented DG-coalgebra and let $\eta\colon k \to C$ be the coaugmentation. One can consider a DG-algebra $\mathrm{Hom}_k(C,A)^*$ defined as follows: the n-th graded component $\mathrm{Hom}_k(C,A)^n \subset \mathrm{Hom}_k(C,A)$ is given by the space of graded maps $C \to A[n]$, the differential $d\colon \mathrm{Hom}_k(C,A) \to \mathrm{Hom}_k(C,A)$ is given by the formula

$$d(f) = d_C \circ f + f \circ d_A,$$

and multiplication is defined by

$$f * g \coloneqq m_A \circ (f \otimes g) \circ \Delta_C.$$

Here m_A and Δ_C denote the multiplication and comultiplication maps on A and C correspondingly.

Definition 5.1. A k-linear map $\theta\colon C \to A$ is called a *twisting cochain* if

(1) θ is homogeneous of degree 1: $\theta \in \mathrm{Hom}_k(C,A)^1$;
(2) $\epsilon \circ \theta = \theta \circ \eta = 0$;
(3) $d_A \circ \theta + \theta * \theta = 0$.

Example 5.2. The zero map $0\colon C \to A[1]$ is a twisting cochain.

Given a coalgebra C with cougmentation η there is a canonical example of twisting cochain provided by the Cobar-construction. For simplicity, let us only consider the case when C is classical: in other words, $C = C^0$ and $d_C = 0$. Let us also denote $\overline{C} \coloneqq \ker(\eta)$; non-canonically one has $C \simeq k \oplus \overline{C}$. The counital comultiplication $\Delta_C\colon C \to C \otimes C$ induces a well-defined map $\overline{\Delta}_C\colon \overline{C} \to \overline{C} \otimes \overline{C}$.

First, we recall the definition of Cobar-construction.

Construction 5.3 (DG-algebra $\mathrm{Cobar}_\eta(C)$)**.** To a coaugmented coalgebra (C,η) one can associate a natural DG-algebra $\mathrm{Cobar}_\eta(C)$ called its *Cobar construction.* By definition, the underlying $\mathbb{Z}$-graded algebra

$$\mathrm{Cobar}_\eta(C) \coloneqq \mathrm{T}_k(\overline{C}[-1]) \simeq \oplus_{n\geq 0} \overline{C}[-1]^{\otimes n}$$

is the free algebra on $\overline{C}[-1]$. Note that various DG-algebra structures lifting the free algebra structure on $\mathrm{Cobar}_\eta(C)$ are in bijection with the graded k-linear maps $\overline{C}[-1] \to \mathrm{Cobar}_\eta(C)$; indeed, the differential extends uniquely from $\overline{C}[-1]$ via the Leibnitz rule. We put the corresponding map $\overline{C}[-1] \subset \mathrm{Cobar}_\eta(C)$ to be $\overline{\Delta}_C\colon \overline{C} \to \overline{C} \otimes \overline{C}$, which defines a $\mathbb{Z}$-graded map

[3]In the L-graded setting we assume each A^p has an extra L-grading and all linear maps in consideration preserve this grading.

$\overline{C}[-1] \to (\overline{C}[-1] \otimes \overline{C}[-1])[1] \subset \mathrm{Cobar}_\eta(C)[1]$. The augmentation $\mathrm{Cobar}_\eta(C)$ is given by projection to the 0-th summand (given by k).

Explicitly, the underlying complex of $\mathrm{Cobar}_\eta(C)$ is given by

$$0 \to k \to \overline{C} \to \overline{C} \otimes \overline{C} \to \overline{C} \otimes \overline{C} \otimes \overline{C} \to \dots$$

with the differential given by

$$d_{\mathrm{Cobar}_\eta(C)}(c_1 \otimes \ldots \otimes c_n) = \sum_{i=1}^{n} c_1 \otimes \ldots \otimes \overline{\Delta}_C(c_i) \otimes \ldots c_n \in \overline{C}^{\otimes(n+1)},$$

while the multiplication is given by stacking the tensors together:

$$(c_1 \otimes \ldots \otimes c_k) \cdot (c'_1 \otimes \ldots \otimes c'_\ell) = c_1 \otimes \ldots \otimes c_k \otimes c'_1 \otimes \ldots \otimes c'_\ell \in \overline{C}^{k+\ell}.$$

In the case $C = \Lambda$ has a compatible structure of Hopf algebra, this agrees with the DG-algebra $\mathcal{C}^*_\Lambda(k,k)$ (see Remark 2.12). In particular, upon taking the cohomology we get an isomorphism of $\mathbb{Z}$-graded algebras

$$H^*(\mathrm{Cobar}_\eta(\Lambda)) \simeq \mathrm{Cotor}^*_\Lambda(k,k).$$

Example 5.4 (Universal twisting cochain)**.** There is a natural twisting cochain,

$$\theta_{\mathrm{can}} \colon C \to \mathrm{Cobar}_\eta(C)[1]$$

simply given by the composition of the projection $C \twoheadrightarrow \overline{C}$ and the embedding of the summand $\overline{C}[-1][1] \simeq C \hookrightarrow \mathrm{Cobar}_\eta(\Lambda)[1]$. One sees that $\epsilon \circ \theta = \theta \circ \eta = 0$, and also for $c \in C$

$$(d \circ \theta_{\mathrm{can}} + \theta_{\mathrm{can}} * \theta_{\mathrm{can}})(c) = \overline{\Delta}(c) + \overline{\Delta}(c) = 2\overline{\Delta}(c) = 0,$$

where $\overline{\Delta}(c) \in \overline{C} \otimes \overline{C}$ is considered as an element of $\mathrm{Cobar}_\eta(C)$.

In fact this twisting cochain is universal: given any other twisting cochain $\theta \colon C \to A[1]$ one gets a map $\theta[-1] \colon C[-1] \to A$, which factors through $\overline{C}[-1]$, and which then extends to an algebra map

$$\mathrm{T}(\theta[-1]) \colon \mathrm{Cobar}_\eta(C) \dashrightarrow A$$

by multiplicativity. To see that it commutes with the differential it is enough to do so when restricted to $\overline{C}[-1] \subset \mathrm{Cobar}_\eta(C)$, where this reduces exactly to Definition 5.1(3):

$$\mathrm{T}(\theta[-1])(d_{\mathrm{Cobar}_\eta(C)}(c)) = \mathrm{T}(\theta[-1])(\overline{\Delta}(c)) = (m_A \circ \theta \otimes \theta \circ \overline{\Delta})(c))) = (\theta * \theta)(c) = d_A(\theta(c)).$$

Moreover, by construction, one has $\theta = \mathrm{T}(\theta[-1]) \circ \theta_{\mathrm{can}}$. The other way around, given a DG-algebra homomorphism $\tau \colon \mathrm{Cobar}_\eta(C) \to A$ one sees from the equation above that the restriction

$$\tau[1]|_{\overline{C}} \colon \overline{C} \to A[1]$$

(or rather its composition with $C \twoheadrightarrow \overline{C}$) defines a twisting cochain between C and A.

5.3. **Twisted tensor product.** The choice of a twisting cochain $\theta\colon C \to A[1]$ (Definition 5.1) allows to define natural functors between the categories of left DG-comodules over C and left DG-modules over A given by the *twisted tensor product*.

Construction 5.5 ([Pos11, Section 6.2])**.** Let (C, η) be a coaugmented DG-coalgebra and let M be a DG-comodule over C with the coaction map $\phi_M\colon M \to C \otimes M$. Let (A, ϵ) be an augmented DG-algebra and let $\theta\colon C \to A[1]$ be a twisting cochain. The *twisted tensor product* $A \otimes_\theta M$ is the left A-DG-module given by $A \otimes M$ endowed with the natural action of A from the left and the differential

$$d_\theta(a \otimes m) = d_A(a) \otimes m + a \otimes d_M(m) + a \cdot (\theta \otimes \mathrm{id}_M)(\phi_C(m)).$$

Remark 5.6. Similarly, if N is a DG-module over A, one defines the C-DG-comodule $C \otimes_\theta N$ by taking $C \otimes N$ with the left coaction of C induced by the one on itself, and with the differential d_θ defined as

$$d_\theta(c \otimes n) = d_C(c) \otimes n + c \otimes d_M(n) + c \otimes (\theta(c) \cdot n).$$

By [Pos11, Theorem in (6.5)], in the case C is conilpotent, the functor $A \otimes_\theta -$ is left adjoint to $C \otimes_\theta -$, when considered as functors between the homotopy categories of DG-categories of A-DG-modules and C-DG-comodules.

Example 5.7. Let C be classical and let $\theta_{\mathrm{can}}\colon C \to A$ be the universal twisting cochain from Example 5.4 for $A = \mathrm{Cobar}_\eta(C)$. Let M be a classical C-DG-comodule, meaning that $M = M^0$ (with $d_M \equiv 0$) and let $\overline{\phi}_M\colon M \to \overline{C} \otimes M$ be the composition of the coaction $\phi_M\colon M \to C \otimes M$ and the projection $C \otimes M \to \overline{C} \otimes M$. Unwinding the definitions, the twisted tensor product $A \otimes_{\theta_{\mathrm{can}}} M$ is identified with the complex

$$0 \to M \to \overline{C} \otimes M \to \overline{C} \otimes \overline{C} \otimes M \to \dots$$

with the differential d_θ given by

$$d_\theta(c_1 \otimes \ldots \otimes c_n \otimes m) = d_{\mathrm{Cobar}_\eta(C)}(c_1 \otimes \ldots \otimes c_n) \otimes m \;+\; c_1 \otimes \ldots \otimes c_n \otimes \phi_M(m) =$$
$$= \sum_{i=1}^{n} c_1 \otimes \ldots \otimes \overline{\Delta}_C(c_i) \otimes \ldots c_n \otimes m \;+\; c_1 \otimes \ldots \otimes c_n \otimes \overline{\phi}_M(m),$$

which agrees with $\mathcal{C}^*_\Lambda(M, N)$ from Construction 2.6 for $\Lambda = C$, $N = k$ and $M = M$. In particular, $H^*(A \otimes_{\theta_{\mathrm{can}}} M)$ computes $\mathrm{Cotor}^*_C(k, M)$, and the natural A-DG-module structure on $A \otimes_{\theta_{\mathrm{can}}} M$ agrees with the natural action of $\mathrm{Cotor}^*_C(k, k)$ on $\mathrm{Cotor}^*_C(k, M)$ by passing to cohomology (using Remark 2.12).

Sometimes, one can find a smaller DG-model for $\mathrm{Cobar}_\eta(C)$ that would still allow to compute $\mathrm{Cotor}^*_C(k, -)$. Usually, for that some assumptions on the coalgebra C are also necessary.

Definition 5.8. A twisting cochain $\theta\colon C \to A[1]$ is called *acyclic* if the induced map of DG-algebras

$$\mathrm{T}(\theta[-1])\colon \mathrm{Cobar}_\eta(C) \to A$$

from Example 5.4 is a quasi-isomorphism.

Definition 5.9. A classical coalgebra C is called *conilpotent* if any classical C-comodule M has an exhaustive filtration indexed by $\mathbb{N}$ with the associated graded given by the trivial comodules.[4]

[4]Namely, comodules M where the reduced coaction $\overline{\phi_M}\colon M \to M \otimes \overline{C}$ is the zero map.

Remark 5.10. The underlying coalgebra of the Hopf algebra defining a unipotent group k-scheme is conilpotent (see [Jan03, I.2.14(8)]).

Lemma 5.11. *Let C be a classical conilpotent coalgebra, and let M be a classical C-comodule. If $\theta\colon C \to A[1]$ is an acyclic twisting cochain, then*

(1) There is a natural isomorphism

$$\mathrm{Cotor}^*_C(k,k) \simeq H^*(A)$$

of $\mathbb{Z}$-graded algebras;

(2) There is a natural isomorphism

$$\mathrm{Cotor}^*_C(k,M) \simeq H^*(A \otimes_\theta M).$$

Moreover, the left action of A on $A \otimes_\theta M$ agrees with the natural action of $\mathrm{Cotor}^*_C(k,k)$ on $\mathrm{Cotor}^*_C(k,M)$ by passing to cohomology.

Proof. The isomorphism in (1) is induced by the DG-algebra map

$$\mathrm{T}(\theta[-1])\colon \mathrm{Cobar}_\eta(C) \to A$$

after passing to cohomology. Indeed, by the definition of acyclic twisting cochain that map is an equivalence. Moreover, by functoriality of the twisted tensor product we also get a map of complexes

$$\mathrm{Cobar}_\eta(C) \otimes_{\theta_{\mathrm{can}}} M \to A \otimes_\theta M.$$

Since C is unipotent, $M = \mathrm{colim}_{n\geq 0} M_n$ where M_n/M_{n-1} is given by some vector space V with the trivial coaction. The map above is the colimit of analogous maps for M replaced by M_n, and since the filtration on M is exhaustive it is enough to show that the map is a quasi-isomorphism on the associated graded pieces. Since the coaction is trivial we can identify the map $\mathrm{Cobar}_\eta(C) \otimes_{\theta_{\mathrm{can}}} \mathrm{gr}_n M \to A \otimes_\theta \mathrm{gr}_n M$ with the tensor product

$$\mathrm{T}(\theta[-1]) \otimes \mathrm{id}_{\mathrm{gr}_n M}\colon \mathrm{Cobar}_\eta(C) \otimes \mathrm{gr}_n M \to A \otimes \mathrm{gr}_n M,$$

which is a quasi-isomorphism by (1). Finally, the compatibility of the actions follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Cobar}_\eta(C) \otimes \mathrm{Cobar}_\eta(C) \otimes_{\theta_{\mathrm{can}}} M & \longrightarrow & \mathrm{Cobar}_\eta(C) \otimes_{\theta_{\mathrm{can}}} M \\ \downarrow & & \downarrow \\ A \otimes A \otimes_\theta M & \longrightarrow & A \otimes_\theta M \end{array}$$

(with horizontal arrow given by the left action, and vertical one induced by $\mathrm{T}(\theta[-1])$) and the discussion in Example 5.7 by applying $H^*(-)$. □

5.4. **Twisting cochains for Λ_1 and Λ_2.** For simplicity[5] further we will assume that $k = \mathbb{F}_2$. We regard $\mathbb{F}_2$ as a trivial left and right comodule over any Hopf algebra over $\mathbb{F}_2$ via the counit map.

[5] This way any coefficient in any k-linear expression is either 0 or 1.

Construction 5.12. Consider the primitively generated $\mathbb{Z}^2$-graded Hopf algebras

$$\Lambda_1 \coloneqq \mathbb{F}_2[x_2], \qquad \Lambda_2 \coloneqq \mathbb{F}_2[x_1, x_2]/(x_1^2), \tag{5.13}$$

where the bigradings of the generators are $|x_1| \coloneqq (1,0)$ and $|x_2| \coloneqq (1,1)$. Their total gradings are $|x_i|_{\mathrm{tot}} = i$. As before, by "primitively generated," we mean that x_1 and x_2 are primitive elements, that is, the comultiplication map sends x_i to $x_i \otimes 1 + 1 \otimes x_i$ for $i = 1, 2$.

Remark 5.14. We note that $\Lambda_1 \simeq H_H^{*,*}(B\mathbb{G}_m/\mathbb{F}_2)$ and $\Lambda_1^{\mathrm{tot}} \simeq H^*_{\mathrm{H}}(B\mathbb{G}_m/\mathbb{F}_2) \simeq H^*_{\mathrm{dR}}(B\mathbb{G}_m/\mathbb{F}_2)$, while $\Lambda_2 \simeq H_H^{*,*}(B\mu_2/\mathbb{F}_2)$ and $\Lambda_1^{\mathrm{tot}} \simeq H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2) \simeq H^*_{\mathrm{dR}}(B\mu_2/\mathbb{F}_2)$.

Remark 5.15. The affine k-group scheme $\operatorname{Spec}\Lambda_1$ can be identified with $\mathbb{G}_a$, while $\operatorname{Spec}\Lambda_2$ is naturally isomorphic to $\mathbb{G}_a \times \alpha_2$. Both group schemes are unipotent, so coalgebras underlying Λ_1 and Λ_2 are conilpotent.

We will now define some explicit twisting cochains for Λ_1 and Λ_2, essentially following [Tod87]. Below we consider Λ_i (resp. Λ_i^{tot}) as a classical DG-coalgebra in $\mathbb{Z}^2$-(resp. $\mathbb{Z}$)-vector spaces.

Construction 5.16 (Twisting cochain for Λ_i)**.**

(1) Consider a $\mathbb{Z}^2$-graded polynomial algebra

$$R_1 \coloneqq \mathbb{F}_2[z_3, z_5, \dots, z_{2^h+1}, \dots]$$

with the bigrading of the variables given by $|z_{2^h+1}| \coloneqq (2^{h-1}, 2^{h-1})$. The corresponding total grading is $|z_i|_{\mathrm{tot}} = i-1$. We endow R_1 with augmentation $\eta_1\colon R_1 \to \mathbb{F}_2$ by putting $\eta_1(z_i) = 0$ for all i.

Let us consider R_1 as a DG-algebra in $\mathbb{Z}^2$-graded vector spaces by assigning to each z_i a DG-grading 1 (which we will also denote $\deg_z$ further) and letting the differential d to be 0.

Consider an $\mathbb{F}_2$-linear graded[6] map $\theta_1\colon \Lambda_1 \to R_1[1]$ by setting:

- $\theta_1(x_2^{2^h}) = z_{2^{h+1}+1}$;
- $\theta_1(x_2^j) = 0$ if j is not a power of 2.

It induces a map $\theta_1^{\mathrm{tot}}\colon \Lambda_1^{\mathrm{tot}} \to R_1^{\mathrm{tot}}[1]$ as well.

(2) Similarly, consider a $\mathbb{Z}^2$-graded polynomial algebra

$$R_2 \coloneqq \mathbb{F}_2[z_2, z_3, z_5, \dots, z_{2^h+1}, \dots]$$

with bigrading $|z_2| = (1,0)$ and $|z_{2^h+1}| = (2^{h-1}, 2^{h-1})$ for $h \geq 1$. The total gradings again are given by $|z_i|_{\mathrm{tot}} = i-1$. We endow R_2 with augmentation $\eta_2\colon R_2 \to \mathbb{F}_2$ by putting $\eta_2(z_i) = 0$ for all i.

As above, we consider R_2 as a DG-algebra in $\mathbb{Z}^2$-graded vector spaces by assigning to each z_i grading 1 and putting the differential d to be 0. Define a graded map $\theta_2\colon \Lambda_2 \to R_2[1]$ by:

- $\theta_2(x_1) = z_2$,
- $\theta_2(x_2^{2^h}) = z_{2^{h+1}+1}$,
- $\theta_2(x_2^j) = 0$ if j is not a power of 2,
- $\theta_2(x_1 x_2^l) = 0$ for all $l \geq 1$.

It also induces a map $\theta_2^{\mathrm{tot}}\colon \Lambda_2^{\mathrm{tot}} \to R_2^{\mathrm{tot}}[1]$.

[6] Here $-[1]$ denotes the shift with respect to the DG-degree.

Lemma 5.17. *The maps $\theta_i\colon \Lambda_i \to R_i[1]$ define twisting cochains that are acyclic.*

In particular, the corresponding DG-algebra maps $\mathrm{T}(\theta_i[-1])\colon \mathrm{Cobar}(\Lambda_i) \to R_i$ are quasi-isomorphisms and induce an algebra isomorphism

$$\mathrm{Cotor}^*_{\Lambda_i}(\mathbb{F}_2, \mathbb{F}_2) \simeq R_i$$

by passing to cohomology.

Proof. Properties (1) and (2) of Definition 5.1 are obvious by construction. Let $\mu_{R_i}\colon R_i \otimes R_i \to R_i$ and $\Delta_{\Lambda_i}\colon \Lambda_i \to \Lambda_i \otimes \Lambda_i$ denote the multiplication and comultiplication maps, respectively. Since $d_{R_i} \equiv 0$, to prove (3) we need to show that

$$m_{R_i} \circ (\theta_i \otimes \theta_i) \circ \Delta_{\Lambda_i} = 0.$$

Suppose first that $i = 1$. If $x = \sum_j x_2^{a_j}$, then

$$\Delta_{\Lambda_1}(x) = \sum_j (x_2 \otimes 1 + 1 \otimes x_2)^{a_j} = \sum_{j, 0 \le h \le a_j} \binom{a_j}{h} x_2^h \otimes x_2^{a_j - h}.$$

We have $\theta_1(x_2^h) \otimes \theta_1(x_2^{a_j-h}) = 0$ unless $h = 2^d$ and $a_j - h = 2^e$ for some $d, e \ge 0$. In that case

$$(\mu_{R_1} \circ (\theta_1 \otimes \theta_1))(x_2^{2^d} \otimes x_2^{2^e}) = z_{2^{d+1}+1} z_{2^{e+1}+1} = (\mu_{R_1} \circ (\theta_1 \otimes \theta_1))(x_2^{2^e} \otimes x_2^{2^d}).$$

Since $\binom{2^d+2^e}{2^d} = \binom{2^d+2^e}{2^e}$, the contributions of $\binom{2^d+2^e}{2^d} x_2^{2^d} \otimes x_2^{2^e}$ and $\binom{2^d+2^e}{2^e} x_2^{2^e} \otimes x_2^{2^d}$ add to zero, hence $(\mu_{R_1} \circ (\theta_1 \otimes \theta_1) \circ \Delta_\Lambda)(x) = 0$, as desired.

Suppose now that $i = 2$. We can write $x \in \Lambda_2$ as $x_1 \sum_{i \ge 0} x_2^{a_i} + \sum_{j \ge 0} x_2^{b_j}$. Then

$$\Delta_{\Lambda_2}(x) = (x_1 \otimes 1 + 1 \otimes x_1) \sum_i (x_2 \otimes 1 + 1 \otimes x_2)^{a_i} + \sum_j (x_2 \otimes 1 + 1 \otimes x_2)^{b_j} =$$

$$= \left(\sum_{i, 0 \le h \le a_i} \binom{a_i}{h} \left(x_1 x_2^h \otimes x_2^{a_i - h} + x_2^h \otimes x_1 x_2^{a_i - h} \right) \right) + \sum_{j, 0 \le h \le b_j} \binom{b_j}{h} x_2^h \otimes x_2^{b_j - h}.$$

Again we have that $\theta_2(x_2^h) \otimes \theta_2(x_2^{b_j - h}) = 0$ unless $h = 2^d$ and $b_j - h = 2^e$ for some $d, e \ge 0$, and so, as above, contributions to $m_{R_2} \circ (\theta_2 \otimes \theta_2) \circ \Delta_{\Lambda_2}(x)$ coming from $\binom{2^d+2^e}{2^d} x_2^{2^d} \otimes x_2^{2^e}$ and $\binom{2^d+2^e}{2^e} x_2^{2^e} \otimes x_2^{2^d}$ cancel each other. Also, $\theta_2(x_1 x_2^h) \otimes \theta_2(x_2^{a_j - h}) = 0$ unless $h = 0$ and $a_j = 2^n$ for some $n \ge 0$ and, similarly, $\theta_2(x_2^h) \otimes \theta_2(x_1 x_2^{a_j - h}) = 0$ unless $h = a_j$ and $a_j = 2^n$.

It remains to show that the twisting cochains θ_i are acyclic. By Remark 5.15, $\mathrm{Cotor}^*_{\Lambda_1}(\mathbb{F}_2, \mathbb{F}_2)$ is computed by the group cohomology $R\Gamma(\mathbb{G}_a, \mathbb{F}_2)$. By [Jan03, Proposition in I.4.27] the latter is given by the polynomial ring $\mathbb{F}_2[t_1, t_2, \ldots]$ in infinitely many generators $t_i \in H^1(\mathbb{G}_a, \mathbb{F}_2)$. The underlying complex of $\mathrm{Cobar}(\Lambda_1)$ is given by

$$0 \to \mathbb{F}_2 \xrightarrow{d_0} x_2\mathbb{F}_2[x_2] \xrightarrow{d_1} (x_2\mathbb{F}_2[x_2]) \otimes (x_2\mathbb{F}_2[x_2]) \to \ldots$$

with d_1 sending x_2^n to $(x_2 \otimes 1 + 1 \otimes x_2)^n - x_2^n \otimes 1 - 1 \otimes x_2^n \in x_2\mathbb{F}_2[x_2]) \otimes (x_2\mathbb{F}_2[x_2])$. By Lukas's theorem for binomial coefficients we get that $d_1(x_2^n) = 0$ if and only if n is a power of 2. Recalling the definition of θ_1 we see that the map

$$\mathrm{T}(\theta_1[-1])\colon \mathrm{Cobar}(\Lambda_1) \to R_1 \simeq \mathbb{F}_2[z_3, z_5, \ldots]$$

induces an isomorphism on H^1, and, since the cohomology of both sides are polynomial rings in H^1, is a quasi-isomorphism.

Similarly, by Remark 5.15 $\mathrm{Cotor}^*_{\Lambda_2}(\mathbb{F}_2,\mathbb{F}_2)$ is computed by the group cohomology $R\Gamma(\mathbb{G}_a \times \alpha_2,\mathbb{F}_2) \simeq R\Gamma(\mathbb{G}_a,\mathbb{F}_2)\otimes R\Gamma(\alpha_2,\mathbb{F}_2)$. By [Jan03, I.4.26(2)] cohomology $H^*(\alpha_2,\mathbb{F}_2)$ is given by the polynomial ring $\mathbb{F}_2[t_0]$ on a class in degree 1. Consequently, $H^*(\mathbb{G}_a\times\alpha_2,\mathbb{F}_2)\simeq\mathbb{F}_2[t_0,t_1,t_2,\ldots]$ is again a polynomial ring in infinitely many generators. $\mathrm{Cobar}(\Lambda_2)$ is given by

$$0\to\mathbb{F}_2\xrightarrow{d_0}\overline{\Lambda}_2\xrightarrow{d_1}\overline{\Lambda}_2\otimes\overline{\Lambda}_2\to\ldots$$

with d_1 sending x_2^n to $(x_2\otimes 1+1\otimes x_2)^n\equiv(x_2\otimes 1+1\otimes x_2)^n - x_2^n\otimes 1 - 1\otimes x_2^n\in\overline{\Lambda}_2\otimes\overline{\Lambda}_2$, which is 0 only if n is a power of 2, and $x_1x_2^n$ to $(x_1\otimes 1+1\otimes x_1)(x_2\otimes 1+1\otimes x_2)^n$ which is 0 in $\overline{\Lambda}_2\otimes\overline{\Lambda}_2$ if an only if $n=0$. So we get that H^1 is generated by x_1 and $x_2^{2^h}$ for $h\geq 0$. Looking at the definition of θ_2 and arguing as for θ_1 we get that the map

$$\mathrm{T}(\theta_2[-1])\colon \mathrm{Cobar}(\Lambda_2)\to R_2\simeq\mathbb{F}_2[z_2,z_3,z_5,\ldots]$$

is a quasi-isomorphism. □

Remark 5.18. Lemma 5.17 shows that the DG-algebras $\mathrm{Cobar}(\Lambda_i)$ are formal.

Corollary 5.19. *Let M be a graded Λ_i-comodule. Then there is a natural isomorphism*

$$\mathrm{Cotor}^*_{\Lambda_i}(\mathbb{F}_2,M)\simeq H^*(R_i\otimes_{\theta_i}M,d_{\theta_i})$$

where θ_i is the twisting cochain from Construction 5.16. Moreover, $\mathrm{Cotor}^*_{\Lambda_i}(\mathbb{F}_2,\mathbb{F}_2)$-module structure on $\mathrm{Cotor}^*_{\Lambda_i}(\mathbb{F}_2,M)$ agrees with left action of R_i on the twisted tensor product $R_i\otimes_{\theta_i}M$.

Proof. This follows from Lemmas 5.11, 5.17 and Remark 5.15. □

Remark 5.20. For the reader's convenience, let us describe the resulting complex $(R_i\otimes_{\theta_i}M,d_{\theta_i})$ more explicitly. Let M be a $\mathbb{Z}^2$-graded left Λ_i-comodule. Consider the tensor product $R_i\otimes M$; cohomological grading on $R_i\otimes_{\theta_i}M$ corresponds to $\deg_z$-grading on R_i and zero grading on M. The resulting complex $(R_i\otimes_{\theta_i}M,d_{\theta_i})$ looks as

$$\ldots\to 0\to(R_i\otimes M)^{\deg_z=0}\xrightarrow{d_{\theta_i}}(R_i\otimes M)^{\deg_z=1}\xrightarrow{d_{\theta_i}}(R_i\otimes M)^{\deg_z=2}\xrightarrow{d_{\theta_i}}\ldots,$$

or, to write down explicitly the first few terms:

$$\ldots\to 0\to M\xrightarrow{d_{\theta_i}}M\otimes(\oplus_j\mathbb{F}_2\cdot z_j)\xrightarrow{d_{\theta_i}}M\otimes(\oplus_{j_1\leq j_2}\mathbb{F}_2\cdot z_{j_1}z_{j_2})\xrightarrow{d_{\theta_i}}\ldots.$$

Here, every term has an additional $\mathbb{Z}^2$-grading (coming from (non $\deg_z$) $\mathbb{Z}^2$-grading on R_i and the $\mathbb{Z}^2$-grading on M).

The complex $(R_i\otimes_{\theta_i}M,d_{\theta_i})$ is a DG-module over R_i (with differential on the latter being 0), so d_{θ_i} is linear with respect to all z_i's. On M (in degree 0) d_{θ_i} is given by the composition

$$M\xrightarrow{\overline{\phi_M}}\overline{\Lambda_i}\otimes M\xrightarrow{\theta_i\otimes 1}R_i[1]\otimes M,$$

where $\phi_M\colon M\to\Lambda_i\otimes M$ is the coaction on M and θ_i is the twisting cochain $\theta_i\colon\Lambda_i\to R_i[1]$. This defines d_{θ_i} uniquely.

5.5. **Controlling multiplication.** Let A be a ($\mathbb{Z}^2$-graded) comodule Λ_i-algebra. Given Corollary 5.19 it is natural to ask if one can enhance the isomorphism in Corollary 5.19 for $M = A$ to an algebra isomorphism. Unfortunately, the twisted tensor product construction doesn't interact well with multiplication, and typically there is no natural DG-algebra structure on $R_i \otimes_{\theta_i} A$ compatible with the multiplication on A.

Nevertheless, here is an observation which partially remedies this problem. Let $i \in \{1, 2\}$ and A be a ($\mathbb{Z}^2$-graded) comodule Λ_i-algebra. Recall the subalgebra $PA \subset A$ of primitive elements in A; by definition the coaction of Λ_i on PA is trivial. We have a natural map of complexes given by twisted tensor products

$$R_i \otimes_{\theta_i} PA \longrightarrow R_i \otimes_{\theta_i} A.$$

Since the coaction $\overline{\phi_{PA}}\colon PA \to PA \otimes \overline{\Lambda_i}$ is trivial the differential on $R_i \otimes_{\theta_i} PA$ is zero, and so $R_i \otimes_{\theta_i} PA \simeq H^*(R_i \otimes_{\theta_i} PA, d_{\theta_i})$. We thus get a natural graded map

$$R_i \otimes PA \longrightarrow R_i \otimes_{\theta_i} A \xrightarrow{\sim} \mathrm{Cotor}^*_{\Lambda_i}(\mathbb{F}_2, A).$$

Let us endow $R_i \otimes PA$ with the algebra structure induced by algebra structures on R_i and PA.

Lemma 5.21. *The composite map $R_i \otimes PA \to \mathrm{Cotor}^*_{\Lambda_i}(\mathbb{F}_2, A)$ is an algebra homomorphism.*

Proof. There are many ways to see this, but let us justify the claim using Cobar construction. Namely we have a roof

$$R_i \otimes PA \xleftarrow[\sim]{\text{q.i.}} \mathrm{Cobar}_\eta(\Lambda_i) \otimes_{\theta_{\mathrm{can}}} PA \longrightarrow \mathrm{Cobar}_\eta(\Lambda_i) \otimes_{\theta_{\mathrm{can}}} A$$

with the left arrow induced by the universality of θ_{can} (Example 5.4), and where the right arrow can be identified with the DG-algebra map $C^*_{\Lambda_i}(\mathbb{F}_2, PA) \to C^*_{\Lambda_i}(\mathbb{F}_2, A)$ (endowed with Alexander-Whitney product, see Remark 2.12 and Example 5.7). We can rewrite the roof above as

$$R_i \otimes PA \xleftarrow[\sim]{\text{q.i.}} C^*_{\Lambda_i}(\mathbb{F}_2, PA) \longrightarrow C^*_{\Lambda_i}(\mathbb{F}_2, A)$$

Since the coaction on PA is trivial, there is in fact a quasi-isomorphism of DG-algebras

$$C^*_{\Lambda_i}(\mathbb{F}_2, PA) \simeq C^*_{\Lambda_i}(\mathbb{F}_2, \mathbb{F}_2) \otimes PA$$

(as follows e.g. from Remark 2.12). Thus the left arrow above is also a quasi-isomorphism of DG-algebras by Lemma 5.17. The map in question is given by the map induced on cohomology by the roof above; since the right map is also a map of DG-algebras (with DG-algebra structures lifting the multiplication on Cotor) we get that the map $R_i \otimes PA \to \mathrm{Cotor}^*_{\Lambda_i}(\mathbb{F}_2, A)$ is an algebra homomorphism. ☐

6. Decomposing comodule algebras

In this section we extend a key computational lemma of Toda that allows to decompose a graded Λ_1-comodule algebra A as a tensor product of two simplier things, to graded Λ_2-comodule algebras. This will be used in Sections 7 and 8 to compute the bigrading on the E_2 page of the Eilenberg-Moore spectral sequence for PGL_{4m+2} and PSO_{4m+2}.

Let $i \in \{1, 2\}$. Let A be a $\mathbb{Z}^2$-graded left Λ_i-comodule algebra and $\phi\colon A \to \Lambda_i \otimes A$ be the corresponding coaction map.

Assume first that $i = 1$. Recall that $\Lambda_1 \simeq \mathbb{F}_2[x_2]$, where x_2 is primitive and $|x_2| = (1, 1)$.

Construction 6.1. For all $j \geq 0$, define linear maps $d_j \colon A \to A$ of bidegree $(-j, -j)$ by

$$\phi(a) = \sum_{j \geq 0} x_2^j \otimes d_j(a).$$

For all $a, b \in A$, we have

$$d_0(a) = a, \qquad d_h(ab) = \sum_{i+j=h} d_i(a) d_j(b), \qquad d_i d_j(a) = \binom{i+j}{i} d_{i+j}(a) \tag{6.2}$$

Example 6.3. Let us view Λ_1 as $H_{\mathrm{H}}^{*,*}(B\mathbb{G}_m/\mathbb{F}_2)$ and consider the comodule algebra given by $H_{\mathrm{H}}^{*,*}(B\mathrm{GL}_n/\mathbb{F}_2) \simeq \mathbb{F}_2[c_1, \dots, c_n]$, as in Section 4.1. Then from Lemma 4.3(a) one sees that $d_i(c_j) = \binom{n-j+i}{i} c_{j-i}$, where we put $c_0 \coloneqq 1$.

One has the following lemma by Toda, which allows to decompose A into two simpler parts, provided that there is an element $a_\sharp \in A$ satisfying a special condition.

Lemma 6.4. *Let q be a power of 2, and let $a_\sharp \in A$ be a bihomogeneous element such that*

$$d_q(a_\sharp) = 1, \qquad d_h(a_\sharp) = 0 \textit{ for all } h > q.$$

Consider the linear subspace

$$P_q A \coloneqq \{a \in A : d_i(a) \textit{ for all } i \geq q\} \subset A.$$

Then the map $\mathbb{F}_2[x] \otimes P_q A \to A$ with $|x| \coloneqq |a_\sharp| = 2q$ given by $f(x) \otimes b \mapsto f(a_\sharp) b$ is a bigraded linear isomorphism. Consequently, the projection $A \twoheadrightarrow A/(a_\sharp)$ induces an additive isomorphism $P_q A \simeq A/(a_\sharp)$.

Proof. This is [Tod87, Lemma 3.7]. Even though A is $\mathbb{Z}$-graded in *loc.cit.*, the same proof works in the bigraded setting (this is because $a_\sharp$ is bigraded, and so the d_i are bigraded). Our $P_q A$ is denoted there by B. ☐

Remark 6.5. The main point of Lemma 6.4 is that it provides a unique lift of an element in $A/(a_\sharp)$ to an element in $P_q A \subset A$ (in other words, to an element of A on which d_i's act by 0 for $i \geq q$).

Construction 6.6. The additive isomorphism $P_q A \simeq A/(a_\sharp)$ induces a new multiplication $*$ on $P_q A$ by setting

$$a * a' \coloneqq a'' \text{ if } aa' \equiv a'' \pmod{a_\sharp}. \tag{6.7}$$

Remark 6.8. In the case $q = 2$, following [Tod87, p. 92] there is also an explicit formula for the $*$-product:

$$a * a' = a \cdot a' + d_1(a) \cdot d_1(a') \cdot a_\sharp. \tag{6.9}$$

Indeed, $a * a' \equiv aa' \pmod{a_\sharp}$, so it is just enough to check that the right hand side lies in $P_2 A$. From the relations in (6.2) it is clear that $d_i(a * a') = 0$ for $i > 2$, while

$$d_2(a * a') = d_2(a \cdot a') + d_2(d_1(a) \cdot d_1(a') \cdot a_\sharp) = d_1(a) \cdot d_1(a') + d_1(a) \cdot d_1(a') \cdot 1 = 0.$$

It also follows from the formula above that if one of the a or a' belongs to $PA \subset P_2 A$, then $a * a' = a \cdot a'$.

Example 6.10. Continuing Example 6.3, let $n = qk$ where q is a power of 2 and k is odd. Then

$$d_q(c_q) = \binom{qk}{q} \cdot c_0 = 1.$$

In particular Lemma 6.4 applies to $A = H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2) \simeq \mathbb{F}_2[c_1, \dots, c_n]$ where we take $a_\sharp$ to be c_q.

Remark 6.11. It is immediate from the formula for $d_i d_j$ in Construction 6.1 that $d_1 \colon A \to A$ preserves $P_q A$ and that $d_1^2 = 0$. Thus d_1 can be considered as a differential on A as well as any of the subalgebras $P_q A \subset A$.

Note, however, that d_1 doesn't satisfy the Leibnitz rule with respect to the $*$-multiplication on $P_q A$. Instead, one has the formula

$$d_1(a * b) = d_1(a) * b + a * d_1(b) + d_1(a) \cdot d_1(b) \cdot a_1.$$

It follows that d_1 is at least PA-linear.

Remark 6.12. Lemma 6.4 also works in the $\mathbb{Z}$-graded setting, namely when we consider Λ_1^{tot} and a $\mathbb{Z}$-graded Λ_1^{tot}-comodule algebra A.

Assume now that $i = 2$. Recall that $\Lambda_2 \simeq \mathbb{F}_2[x_1, x_2]/(x_1^2)$, where both x_i are primitive and $|x_1| = (1, 0)$, $|x_2| = (1, 1)$.

Construction 6.13. For all $j \geq 0$, define maps $d_{2j}, d_{2j+1} \colon A \to A$ of bidegrees $(-j, -j)$ and $(-j-1, -j)$ by

$$\phi(a) = \sum_{j \geq 0} x_2^j \otimes d_{2j}(a) + \sum_{j \geq 0} x_1 x_2^j \otimes d_{2j+1}(a).$$

Again, we have

$$d_0(a) = a \qquad \text{and} \qquad d_i d_j(a) = \binom{i+j}{i} d_{i+j}(a). \tag{6.14}$$

The formula for $d_i(ab)$ is, however, different from Construction 6.1, due to the fact that $x_1^2 = 0 \in \Lambda_2$: namely, for all $a, b \in A$, we still have

$$d_{2h+1}(ab) = \sum_{i+j=2h+1} d_i(a) d_j(b),$$

but

$$d_{2h}(ab) = \sum_{i+j=h} d_{2i}(a) d_{2j}(b), \tag{6.15}$$

with the terms in the sum only having even indices.

Here is a version of Toda's lemma for Λ_2-comodule algebras.

Lemma 6.16. *Let $q > 1$ be a power of 2, and let $a_\sharp \in A$ be a bihomogeneous element such that*

$$d_q(a_\sharp) = 1, \qquad d_h(a_\sharp) = 0 \text{ for all } h > q.$$

(a) Consider the linear subspace

$$P_q A \coloneqq \{a \in A : d_i(a) = 0 \text{ for all } i \geq q\}.$$

Consider the polynomial ring $\mathbb{F}_2[x]$, *where* $|x| := |a_\sharp| = (q/2, q/2)$. *Then the map* $\mathbb{F}_2[x] \otimes P_q A \to A$ *given by* $f(x) \otimes b \mapsto f(a_\sharp)b$ *is an* $\mathbb{F}_2$*-linear isomorphism.*

(b) If $q = 2$, *then* $P_2 A$ *is a subring of* A *and the map* $\mathbb{F}_2[x] \otimes P_2 A \to A$ *in (a) is a ring isomorphism.*

Remark 6.17. Let us point out that there is no analogue of Lemma 6.16(b) in the setting of Lemma 6.4.

Remark 6.18. As with Lemma 6.4, Lemma 6.16 also works in the $\mathbb{Z}$-graded setting, when we consider Λ_2^{tot} and a $\mathbb{Z}$-graded Λ_2^{tot}-comodule algebra A.

Proof. (a) The proof is analogous to that of [Tod87, Lemma 3.6]. To prove injectivity, it suffices to show the following: if $b_1, \dots, b_r \in P_q A$ are bihomogeneous and $\sum_{i=0}^r a_\sharp^i b_i = 0$, then $b_1 = \cdots = b_r = 0$. To see this, suppose by contradiction that $b_r \neq 0$, and let $h \geq 0$ be such that $d_h(b_r) \neq 0$ and $d_{h'}(b_r) = 0$ for all $h' > h$. (Such h exists, because $d_0(b_r) = b_r$.) Note that by our assumptions on $a_\sharp$ and multiplicativity (6.15) we have $d_{rq}(a_\sharp^r) = d_q(a_\sharp)^r = 1$. We then also have

$$0 = d_{rq+h}(\sum_{i=0}^r a_\sharp^i b_i) = d_{rq}(a_\sharp^r) d_h(b_r) = d_h(b_r) \neq 0,$$

a contradiction.

We now prove surjectivity. It suffices to show that every bihomogeneous element $a \neq 0$ of A is in the image. Let h be the maximal integer such that $d_h(a) \neq 0$, and write $h = qj + h'$, where $0 \leq h' < q$. For all $s > h'$, $d_s d_{qj}(a)$ is a multiple of $d_{s+qj}(a)$, hence zero. Note that this means that $d_{qj}(a) \in P_q A$. Since q is a power of 2 and $h' < q$, e.g. by Lucas's formula we have $\binom{qj+h'}{h'} = 1$, and so by (6.14) we have $d_{h'} d_{qj}(a) = d_h(a)$. Setting $b := d_{qj}(a) \in P_q A$ and $a' := a - a_\sharp^j b$, then for all $i \geq 0$ we have

$$d_i(a') = d_i(a) - \sum_{s=0}^{h'} d_{i-h'}(a_\sharp^j) d_s(b).$$

We now show that $d_i(a') = 0$ for all $i \geq h$. Indeed, if $i > h$, then $i - s > qj$ for all $0 \leq s \leq h'$ and so, due to the multiplicativity of ϕ, we have $d_{i-s}(a_\sharp^j) = 0$. It follows that $d_i(a') = d_i(a) = 0$ for all $i > h$. On the other hand, when $i = h$, then

$$d_h(a') = d_h(a) - d_{qj}(a_\sharp^j) d_{h'}(b) = d_h(a) - (d_q(a_\sharp))^j d_h(a) = d_h(a) - d_h(a) = 0.$$

Thus $a = a_\sharp^j b + a'$, where $d_i(a') = 0$ for all $i \geq h$. Since a, $a_\sharp$ and b are bihomogeneous, $a_\sharp^j b$ and a' are homogeneous of degree $|a|$. Therefore, applying the same reasoning to a' and iterating, we eventually write a as $\sum_{i=0}^r a_\sharp^i b_i$ for some $b_i \in B$.

(b) Let $a, b \in P_2 A$. Then $\phi(a) = a + d_1(a)x_1$ and $\phi(b) = b + d_1(b)x_1$. Since $x_1^2 = 0$, we obtain $\phi(ab) = \phi(a)\phi(b) = ab + (d_1(a) + d_1(b))x_1$, that is, $ab \in P_2 A$. It follows that $P_2 A$ is a subring of A. It remains to note that the explicit formula for isomorphism in (a) is obviously multiplicative. ∎

Remark 6.19. It follows from Equation (6.14) that $d_1^2 = 0$ and that d_1 preserves $P_q A$. Thus, as in Remark 6.11, d_1 defines a differential on both A and $P_q A \subset A$.

Remark 6.20. We also get that the differential d_1 satisfies the following cyclic equation:

$$d_1(ab)d_1(c) + d_1(bc)d_1(a) + d_1(ac)d_1(b) = 0. \tag{6.21}$$

Indeed, the latter expression is nothing but $d_1(d_1(abc))$, which is equal to 0.

7. Hodge cohomology of $B\mathrm{PGL}_{4m+2}$

The goal of this section is to prove Theorem 7.16, which gives an explicit description of $H^{*,*}_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2)$ and $H^*_{\mathrm{dR}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2)$ as bigraded and graded algebras, respectively. It also implies Theorem 1.5(1) for $\overline{G} = \mathrm{PGL}_{4m+2}$. For the proof we use the Eilenberg-Moore spectral sequence (Theorem 1.3) and the strategy devised by Toda [Tod87] in the topological setting.

Consider $H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)$ as a $H^{*,*}_{\mathrm{H}}(B\mathbb{G}_m/\mathbb{F}_2)$-comodule algebra. By Example 6.10, if we take $a_\sharp = c_2 \in H^{1,1}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)$, then Lemma 6.4 applies.

We will start by identifying the subalgebras

$$PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2) \subset P_2H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2) \subset H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2).$$

Recall that the operation d_1 (defined in Construction 6.1) induces a $PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)$-linear differential on $P_2H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)$; see Remark 6.11.

Lemma 7.1. *Assume that $n = 4m+2$.*

(a) There exists a unique sequence

$$\overline{c}_1, \dots, \overline{c}_{4m+2} \in H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)$$

such that $\overline{c}_i$ has bidegree (i,i) and

(1) $\overline{c}_i$ are polynomial generators: $H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2) = \mathbb{F}_2[\overline{c}_1, \overline{c}_2 \dots, \overline{c}_{4m+2}]$;

(2) $\overline{c}_1 \coloneqq c_1$, $\overline{c}_2 \coloneqq c_2$;

(3) for all $j > 1$, we have

$$\overline{c}_{2j} \equiv c_{2j} \pmod{c_2} \quad \textit{and} \quad \overline{c}_{2j} \in P_2H^{j,j}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)$$

(so $d_i(\overline{c}_{2j}) = 0$ for $i \geq 2$).

(4) $\overline{c}_{2j-1} = d_1(\overline{c}_{2j})$;
$\overline{c}_{2j-1}$ is primitive: $\overline{c}_{2j-1} \in PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)$.

(b) (1) The subalgebra $P_2H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2) \subset H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)$ is freely generated by $\overline{c}_1, \overline{c}_3, \overline{c}_4, \overline{c}_5, \dots, \overline{c}_{4m+2}$ under the $*$-multiplication (see Construction 6.6);

(2) For all $1 < h \leq 2m+1$ define elements

$$b_h \coloneqq \overline{c}_{2h} * \overline{c}_{2h} + c_1\overline{c}_{2h}\overline{c}_{2h-1} \in H^{4h,4h}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2).$$

Then $b_h \in PH^{*,*}_{\mathrm{H}}$ (in particular, $b_h \in P_2H^{*,*}_{\mathrm{H}}$ and $d_1(b_h) = 0$) and the natural map

$$\mathbb{F}_2[c_1, b_2, \dots, b_{2m+1}] \longrightarrow H^*(P_2H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2), d_1)$$

is an isomorphism.

(c) For any (unordered) tuple of integers $I = \{i_1, \dots, i_r\}$ write $l(I) \coloneqq r$ and $d(I) \coloneqq \sum_{k=1}^r i_k$. Define

$$y_I \coloneqq d_1(\overline{c}_{2i_1} * \dots * \overline{c}_{2i_r}) \in PH^{2d(I)-1,2d(I)-1}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2).$$

In particular, $y_{\{i\}} = \overline{c}_{2i-1}$.

(1) *The subalgebra* $PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2) \subset H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2)$ *of primitive elements is generated by* c_1*,* b_h *and* y_I *for* $I = \{1 < i_1 < \ldots < i_r \le 2m+1\}$*, and the relations given by*

$$y_I y_J = \sum_{\emptyset \neq K \subset I} y_{(I-K)\cup J} y_{\{k_1\}} \cdots y_{\{k_s\}} c_1^{l(K)-1},$$

for all subsets $I, J \subset \{2, \ldots, 2m+1\}$ *and where we put*

$$y_{\{h,h,j_1,\ldots,j_s\}} := y_{\{j_1,\ldots,j_s\}} b_h + y_{\{h,j_1,\ldots,j_s\}} y_{\{h\}} c_1.$$

(2) $PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2)$ *is a finitely generated module over the polynomial subalgebra*

$$\mathbb{F}_2[c_1, \overline{c}_3, \overline{c}_5, \ldots, \overline{c}_{4m-1}, b_2, b_3, \ldots, b_{2m+1}] \subset PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2).$$

Analogous statements hold with Hodge cohomology replaced by de Rham cohomology. (In this case $\overline{c}_i, b_h, y_I \in H^*_{\mathrm{dR}}$ *have degrees* $2i$*,* $8h$ *and* $4d(I)-2$*, respectively.)*

Proof. We first show the statement for the $\mathbb{Z}$-graded algebra $H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2)$, that is, we only keep track of the total grading instead of the bigrading. By Lemma 4.3(b), it suffices to prove the analogous assertions in the topological setting. Thus (a) and (b1) follow from [Tod87, Proposition 3.7], (b2) is given by [Tod87, Lemma 3.10(ii)], and (c) follows from [Tod87, Proposition 3.11].

It remains to explain the bidegrees of the elements. The $\overline{c}_i$ are constructed from the c_i by applying Lemma 6.4. In particular, since the isomorphism of Lemma 6.4 respects the bigrading, $\overline{c}_i$ and c_i have the same bidegree (i,i). The $*$-product preserves bigrading and so we get b_h is homogeneous of bidegree $(4h, 4h)$. Similarly d_1 reduces the bigrading by $(1,1)$ and so we get $|y_I| = (2d(I)-1, 2d(I)-1)$. Everything works similarly in the de Rham setting where the degrees of $\overline{c}_i$, b_h and y_I are given by $2i$, $8h$ and $4d(I)-2$, respectively. □

Remark 7.2. Let us comment upon the logic behind the statements of Lemma 7.1 (and how the proof could go without appealing to [Tod87]). Having Lemma 6.4, it is more or less immediate that $P_2H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2)$ is isomorphic to the polynomial ring over $\overline{c}_1, \overline{c}_3, \overline{c}_4, \ldots, \overline{c}_{4m+2}$ via the $*$-multiplication. The subalgebra $PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2) \subset P_2H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2)$ then is identified with the kernel of the differential d_1 on $P_2H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2)$. One can understand this kernel in two steps: first, by finding a subalgebra in $\mathrm{Ker}(d_1)$ that maps isomorphically to the cohomology $H^*(P_2H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2), d_1)$ of d_1 — this is given by $\mathbb{F}_2[c_1, b_2, \ldots, b_{2m+1}]$; second, by describing the image of d_1 — the latter contains elements $\overline{c}_{2i-1}$, and the whole image is spanned by the remaining y_I's (with $l(I) \ge 2$) over the polynomial algebra $\mathbb{F}_2[c_1, \overline{c}_3, \overline{c}_5, \ldots, \overline{c}_{4k+1}, b_2, \ldots, b_{2m+1}]$.

The reader can also look at the proof of the analogous statement for $B\mathrm{SO}_{4m+2}$ (Lemma 8.5), which is slightly more natural due to the fact that the $*$-multiplication on $P_2H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2)$ coincides with the usual one.

Example 7.3. Formulas for the elements $\overline{c}_i \in P_2H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2) \subset H^*_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2) \simeq \mathbb{F}_2[c_1, \ldots, c_{4m+2}]$ get complicated pretty fast as i grows. Here are the first few of them:

$$\overline{c}_1 = c_1 \qquad \overline{c}_2 = c_2 \qquad \overline{c}_3 = c_3 + mc_1^3 \qquad \overline{c}_4 = c_4 + m(c_2 + c_1^2)c_2$$
$$\overline{c}_5 = c_5 + c_4c_1 + c_3(c_2 + c_1^2) \qquad \overline{c}_6 = c_6 + (c_4 + c_3c_1)c_2.$$

Let us also record that

$$b_h = \overline{c}_{2h} * \overline{c}_{2h} + c_1\overline{c}_{2h}\overline{c}_{2h-1} = \overline{c}_{2h}^2 + \overline{c}_{2h-1}^2 c_2 + c_1\overline{c}_{2h}\overline{c}_{2h-1}$$

(here we simply use the explicit formula for $*$-product from Remark 6.8).

Remark 7.4. Note that since all alements $c_1, b_2, \dots, b_{2m+1}$ are primitive (so lie in $PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2)$) by Lemma 7.1(b) we get that the natural map

$$PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2) \longrightarrow H^*(P_2H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2), d_1)$$

induced by embedding $PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2) \subset P_2H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)$ is a surjection.

We can now compute the E_2-page of the Eilenberg-Moore spectral sequence.

Lemma 7.5. *Assume that $n = 4m+2$, for some integer $m \geq 0$. Then*

$$\mathrm{Cotor}^*_{H^{*,*}_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_2)}(\mathbb{F}_2, H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2))$$

is isomorphic, as a $\mathbb{Z}\oplus\mathbb{Z}^2$-graded[7] *algebra, to*

$$(1 \otimes PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)) \oplus (z_3\mathbb{F}_2[z_3] \otimes \mathbb{F}_2[c_1, b_h]_{h=2}^{2m+1}),$$

where the gradings of the elements are $|c_1| = (0,1,1)$, $|b_h| = (0,4h,4h)$, $|z_3| = (1,1,1)$.

Analogous assertion holds with Hodge cohomology replaced by de Rham cohomology (with the corresponding $\mathbb{Z}\oplus\mathbb{Z}$-gradings $|c_1| = (0,2)$, $|b_h| = (0,8h)$, $|z_3| = (1,2)$).

Remark 7.6. Here the algebra structure on the direct sum above is induced by the surjective homomorphism

$$\phi\colon PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)) \twoheadrightarrow H^*(P_2H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2), d_1) \simeq \mathbb{F}_2[c_1, b_h]_{h=2}^{2m+1}, \tag{7.7}$$

(the last isomorphism is Lemma 7.1(b)).

More precisely, let $\varphi\colon A \to B$ be a homomorphism of L-graded algebras and z a formal variable; then we can define a $(\mathbb{Z}\oplus L)$-graded algebra $A_{\phi,z}$ with the underlying vector space $A \oplus (\oplus_{j\geq 1} Bz^j)$ as the pull-back

$$A_{\phi,z} := A \times_B B[z],$$

where the map $A \to B$ is given by ϕ, the map $B[z] \to B$ sends z to 0 and z has grading $(1, 0_L)$. One can think of $A_{\phi,z}$ as the ring of polynomials $\{a + b_1z + \ldots + b_nz^n\}$ where $a \in A$, $b_i \in B$ and $a \cdot z^n := \phi(a)z^n$.

The lemma then claims that $\mathrm{Cotor}^*_{H^{*,*}_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_2)}(\mathbb{F}_2, H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2))$ is isomorphic to $A_{\phi,z}$ for $z := z_3$,

$$A = PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)), \qquad B = H^*(P_2H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2), d_1) \simeq \mathbb{F}_2[c_1, b_h]_{h=2}^{2m+1},$$

and φ given by (7.7).

Proof. Consider the natural map $PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2)\otimes R_1 \to \mathrm{Cotor}^*_{H^*_{\mathrm{H}}(B\mathbb{G}_{\mathrm{m}}/\mathbb{F}_2)}(\mathbb{F}_2, H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2))$ as in Lemma 5.21. We will show that this map induces the required isomorphism. We have a

[7] Here, the $\mathbb{Z}^2$-component of the grading is coming from the bigrading on Hodge cohomology, and $\mathbb{Z}$-component is the Cotor-grading.

commutative diagram of graded vector spaces

$$\begin{array}{ccc} PH^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2)\otimes R_1 & \xrightarrow{\sim} & PH^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_2)\otimes R_1 \\ \downarrow & & \downarrow \\ H^*(H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2)\otimes_{\theta_1} R_1) & \xrightarrow{\sim} & H^*(H^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_2)\otimes_{\theta_1} R_1) \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{Cotor}^*_{H^*_{\mathrm{H}}(B\mathbb{G}_m/\mathbb{F}_2)}(\mathbb{F}_2, H^*_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2)) & \xrightarrow{\sim} & \mathrm{Cotor}^*_{H^*_{\mathrm{sing}}(B\mathbb{C}^\times;\mathbb{F}_2)}(\mathbb{F}_2, H^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_2)), \end{array}$$

where the top square is induced by the embeddings $PH^*_H \hookrightarrow H^*_H$ and $PH^*_{\mathrm{sing}} \hookrightarrow H^*_{\mathrm{sing}}$ and Lemma 4.3(b), while the vertical identifications in the bottom square come from Corollary 5.19. Here, the elements $z_i \in R_1$, $\overline{c}_i$, b_h and y_I from Lemma 7.1 are getting mapped to the analogous elements (z_i, $\overline{a}_i$, b_h and y_I) in the notation of [Tod87, Section 3 and (4.7)]. The statement of the lemma then follows from the analogous description in [Tod87, Section 3 and (4.7)]. To see that the algebra structure above is the correct one, note that the maps in the outer rectangle in the above diagram are homomorphisms of algebras by Lemma 5.21, and, by the description in [Tod87, (4.7)], the composition of the vertical maps on the right is surjective, hence the same is true for the composition of the two vertical maps on the left. □

Remark 7.8. For the reader's convenience, let us also sketch the idea behind Toda's computation on the topological side. First of all, [Tod87, Theorem 4.1] identifies the cohomology $H^*(H^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_2)\otimes_{\theta_1} R_1)$ with the cohomology of a subcomplex

$$C := (P_2H^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_2)\otimes \mathbb{F}_2[z_3], d_{\theta_1}|_{P_2H^*_{\mathrm{sing}}\otimes\mathbb{F}_2[z_3]}) \subset (H^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_2)\otimes_{\theta_1} R_1, d_{\theta_1}).$$

By the definition of d_{θ_1}, its restriction to C can be identified with $d_1\otimes z_3$. The cohomology of C then can be identified with $H^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_2)$ plus direct sum of infinitely many copies of $H^*(P_2H^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_2), d_1)$ multiplied by powers of z_3. Using the topological analogue of Lemma 7.1(b) to describe the latter one arrives at the description of $H^*(H^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_2)\otimes_{\theta_1} R_1)$ as in Lemma 7.5. Moreover, since $H^*(P_2H^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_2), d_1)$ is generated by classes in $PH^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_2)$ one gets that the map

$$PH^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_2)\otimes R_1 \longrightarrow \mathrm{Cotor}^*_{H^*_{\mathrm{sing}}(B\mathbb{C}^\times;\mathbb{F}_2)}(\mathbb{F}_2, H^*_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_2))$$

is a surjective homomorphism of algebras and one recovers the algebra structure on Cotor as well.

Remark 7.9. Note that elements y_I from Lemma 7.1(c) are defined as images under d_1 of $\overline{c}_{i_1} * \dots * \overline{c}_{i_r} \in P_2H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_n/\mathbb{F}_2)$, and so they map to 0 under the map (7.7). Consequently, (by Lemma 7.5) in $\mathrm{Cotor}^*_{H^{*,*}_{\mathrm{H}}(B\mathbb{G}_m/\mathbb{F}_2)}(\mathbb{F}_2, H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2))$ we have $y_Iz_3 = 0$ for any I.

Let us now compute Hodge and de Rham cohomology of $B\mathrm{PGL}_n$ for all n in degrees ≤ 3.

Lemma 7.10. *We have isomorphisms*

(1) $H^0_{\mathrm{H}}(B\mathrm{PGL}_n/\mathbb{F}_2)\simeq \mathbb{F}_2$,
(2) $H^1_{\mathrm{H}}(B\mathrm{PGL}_n/\mathbb{F}_2)\simeq 0$,
(3) $H^2_{\mathrm{H}}(B\mathrm{PGL}_n/\mathbb{F}_2)\simeq H^{1,1}_{\mathrm{H}}(B\mathrm{PGL}_n/\mathbb{F}_2)\simeq \mathbb{F}_2$ *if n is even and* $H^2_{\mathrm{H}}(B\mathrm{PGL}_n/\mathbb{F}_2)\simeq 0$ *if n is odd,*

(4) $H^3_{\mathrm{H}}(B\mathrm{PGL}_n/\mathbb{F}_2) \simeq H^{1,2}_{\mathrm{H}}(B\mathrm{PGL}_n/\mathbb{F}_2) \simeq \mathbb{F}_2$ *if* n *is even and* $H^3_{\mathrm{H}}(B\mathrm{PGL}_n/\mathbb{F}_2) \simeq 0$ *if* n *is odd.*

Entirely analogous assertions hold for de Rham cohomology. We will denote by x_2 *and* x_3 *the unique non-zero elements in* H^2_{H} *and* H^3_{H} *in the case* n *is even.*

Proof. By [Tot18, Theorem 2.4] and [CR10, Theorem 1.1] (cf. [Tot18, Theorem 8.1]) for every split simple k-group G not of type C we have

$$\begin{aligned} H^2_{\mathrm{H}}(BG/k) =& H^2(BG,\mathcal{O}) \oplus H^1(BG,\Omega^1) \oplus H^0(BG,\Omega^2) \\ \simeq& H^2(G,k) \oplus (\mathfrak{g}^*)^G \oplus H^{-2}(G,S^2(\mathfrak{g}^*)) \\ \simeq& H^2(G,k) \oplus (\mathfrak{t}^*)^W, \end{aligned}$$

where $\mathfrak{g}$ and $\mathfrak{t}$ are the Lie algebras of G and a maximal torus $T \subset G$, respectively, and where W is the Weyl group of G. By [Jan03, II, Corollary 4.11], we have $H^0(G,k) = k$ and $H^i(G,k) = 0$ for $i > 0$. Setting $k = \mathbb{F}_2$ and $G = \mathrm{PGL}_n$ yields

$$H^2_{\mathrm{H}}(B\mathrm{PGL}_n/\mathbb{F}_2) = (\mathfrak{t}^*)^W. \tag{7.11}$$

If $G = \mathrm{PGL}_n$, then $W = S_n$ and the S_n-representation $\mathfrak{t}^*$ fits into a short exact sequence

$$0 \to \mathfrak{t}^* \to \mathbb{F}_2^{\oplus n} \xrightarrow{\Sigma} \mathbb{F}_2 \to 0,$$

where S_n acts on $\mathbb{F}_2^{\oplus n}$ by permutation and the map Σ is given by the summation of coordinates. The invariants $(\mathbb{F}_2^{\oplus n})^{S_n}$ are spanned by the vector $(1,1,\dots,1)$ which lies in $\mathfrak{t}^*$ if and only if n is even. This gives (3); in particular, if n is even there is a unique $x_2 \in (\mathfrak{t}^*)^{S_n} \simeq H^1(B\mathrm{PGL}_n, \Omega^1)$ such that

$$H^2_{\mathrm{H}}(B\mathrm{PGL}_n/\mathbb{F}_2) = \mathbb{F}_2 \cdot x_2.$$

To compute the other cohomology groups we will use a Leray-Serre-type spectral sequence. Namely, let PGL_n act on $\mathbb{P}^{n-1}$ as its automorphism group, and let $P \subset \mathrm{PGL}_n$ be the stabilizer of $(1:0:\cdots:0) \in \mathbb{P}^{n-1}(\mathbb{F}_2)$. We have $H^*_{\mathrm{H}}(\mathbb{P}^{n-1}/\mathbb{F}_2) = \mathbb{F}_2[h]/(h^n)$, where h has Hodge bidegree $(1,1)$ (meaning $h \in H^1(\mathbb{P}^{n-1},\Omega^1) \simeq H^2_{\mathrm{H}}(\mathbb{P}^{n-1}/\mathbb{F}_2)$). The Levi subgroup corresponding to the parabolic P is isomorphic to GL_{n-1}. Therefore, by [Tot18, Proposition 9.3] we have a spectral sequence

$$E_2^{i,j} \coloneqq H^i_{\mathrm{H}}(B\mathrm{PGL}_n/\mathbb{F}_2) \otimes H^j_{\mathrm{H}}(\mathbb{P}^{n-1}/\mathbb{F}_2) \Rightarrow H^*_{\mathrm{H}}(B\mathrm{GL}_{n-1}/\mathbb{F}_2). \tag{7.12}$$

We know $H^2_{\mathrm{H}}(B\mathrm{GL}_{n-1}/\mathbb{F}_2) = \mathbb{F}_2$ and $H^3_{\mathrm{H}}(B\mathrm{GL}_{n-1}/\mathbb{F}_2) = 0$. From this it is easy to see that the E_2 page of the spectral sequence (7.12) is given in low degrees by

	0	1	2	3
2	h	$\cdot$	$\cdot$	
1	0	0	$\cdot$	
0	1	0	x_2	x_3

n even

	0	1	2	3
2	h	$\cdot$	$\cdot$	
1	0	0	$\cdot$	
0	1	0	0	0

n odd

where $x_3 \in H^2(B\mathrm{PGL}_{4m+2}, \Omega^1)$ is defined by $d_3(h) = x_3$. Thus for all n we get

(7.13) $$H^1_{\mathrm{H}}(B\mathrm{PGL}_n/\mathbb{F}_2) \simeq 0.$$

We also get that

$$H^3_{\mathrm{H}}(B\mathrm{PGL}_n/\mathbb{F}_2) \simeq H^{1,2}_{\mathrm{H}}(B\mathrm{PGL}_n/\mathbb{F}_2) \simeq \mathbb{F}_2 \cdot x_3$$

when n is even and

$$H^3_{\mathrm{H}}(B\mathrm{PGL}_n/\mathbb{F}_2) \simeq 0$$

if n is odd. This gives us the groups $H^i_{\mathrm{H}}(B\mathrm{PGL}_n/\mathbb{F}_2)$ for $i \le 3$. By looking at the bigrading of generators it is also easy to see that the Hodge-de-Rham spectral sequence necessarily degenerates in degrees ≤ 3, and so

(7.14) $$H^i_{\mathrm{H}}(\mathrm{PGL}_n/\mathbb{F}_2) \simeq H^i_{\mathrm{dR}}(\mathrm{PGL}_n/\mathbb{F}_2) \text{ for all } i \le 3.$$ □

Now let us compute the rest of the Hodge cohomology ring in the case of $B\mathrm{PGL}_{4m+2}$.

Remark 7.15. If n is odd, then 2 is not a torsion prime for PGL_n and by the general result of Totaro [Tot18, Theorem 9.2] one has $H^*_{\mathrm{H}}(B\mathrm{PGL}_n/\mathbb{F}_2) \simeq H^*_{\mathrm{dR}}(B\mathrm{PGL}_n/\mathbb{F}_2) \simeq \mathbb{F}_2[c_2, \dots, c_n]$ where $|c_i| = 2i$. Thus understanding the case $n = 4k+2$ is the next natural step.

Theorem 7.16. *The bigraded ring $H^{*,*}_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2)$ is generated by elements*

$$x_2 \in H^{1,1}_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2), \quad x_3 \in H^{1,2}_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2),$$

$$b_h \in H^{4h,4h}_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2), \quad y_I \in H^{2d(I)-1,2d(I)-1}_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2).$$

Here $1 < h \le 2m+1$, $I = (i_1, \dots, i_r) \in \{1 < i_1 < \dots < i_r \le 2m+1\}$, and $d(I) \coloneqq i_1 + \dots + i_r$. The relations are generated by

(7.17) $$y_I y_J = \sum_{\emptyset \neq K \subset I} y_{(I-K)\cup J} y_{\{k_1\}} \cdots y_{\{k_s\}} x_2^{l(K)-1},$$

(7.18) $$y_{\{h,h,j_1,\dots,j_s\}} = y_{\{j_1,\dots,j_s\}} b_h + y_{\{h,j_1,\dots,j_s\}} y_{\{h\}} x_2 = 0,$$

$$x_3 y_I = 0 \text{ for all } I. \tag{7.19}$$

Similarly, the graded ring $H^*_{\mathrm{dR}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2)$ has generators $x_2 \in H^2_{\mathrm{dR}}$, $x_3 \in H^3_{\mathrm{dR}}$, $b_h \in H^{8h}_{\mathrm{dR}}$, $y_I \in H^{4d(I)-2}_{\mathrm{dR}}$, where h, I and $d(I)$ are as above, and relations generated by (7.17), (7.18) and (7.19). In particular, we have an isomorphism of graded rings $H^*_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2) \simeq H^*_{\mathrm{dR}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2)$.

Remark 7.20. From Lemma 7.1(c) and the conclusion of Theorem 7.16 one can see that there is a slightly more compact expression for $H^*_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2)$ as the middle term in the short exact sequence

$$0 \to \mathbb{F}_2[x_2, x_3, b_2, \dots, b_{2m+1}] \xrightarrow{\cdot x_3} H^*_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2) \xrightarrow{Bp^*} PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2) \to 0.$$

Proof of Theorem 7.16. Consider the Eilenberg-Moore spectral sequence of Theorem 1.3 associated to the short exact sequence

$$1 \to \mathbb{G}_{\mathrm{m}} \to \mathrm{GL}_{4m+2} \xrightarrow{p} \mathrm{PGL}_{4m+2} \to 1.$$

In Section 4.3 we proved that it degenerates on the E_2 page, which (by the computation in Lemma 7.5) is given by

$$(1 \otimes PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)) \oplus (z_3\mathbb{F}_2[z_3] \otimes \mathbb{F}_2[c_1, b'_h]_{h=2}^{2m+1}),$$

with $E_2^{0,*} \simeq E_\infty^{0,*} \simeq (1 \otimes PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2))$ and $E_2^{>0,*} \simeq E_\infty^{>0,*} \simeq (z_3\mathbb{F}_2[z_3] \otimes \mathbb{F}_2[c_1, b'_h]_{h=2}^{2m+1})$. Below let us denote by $b'_h \in PH^{4h,4h}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)$ the elements that we called b_h in Lemma 7.5. Let $b_h \in H^{4h,4h}_{\mathrm{H}}(B\mathrm{PGL}_{4\mathrm{m}+2}/\mathbb{F}_2)$ be a choice liftings of b'_h, meaning that $Bp^*(b_h) = 1 \otimes b'_h$ (such a lifting exists since the spectral sequence degenerates). Note that since Bp^* preserves the Hodge bigrading we can pick b_h to be bihomogeneous (explicitly, lying in $H^{4h,4h}_{\mathrm{H}}(B\mathrm{PGL}_{4\mathrm{m}+2}/\mathbb{F}_2)$).

Recall that (by Remark 3.13) a homogeneous element

$$x \in (\mathrm{Cotor}^i_{H^{*,*}_{\mathrm{H}}(B\mathbb{G}_m)}(\mathbb{F}_2, H^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)))^{h,j} \simeq (E_\infty^{i,j})^h$$

gives a class in $\mathrm{gr}_i(H^{h,i+j}_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2))$. We let $i+j+h$ be the "total degree". For elements in $(1 \otimes PH^{s,t}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2))$ the $\mathbb{Z}^3$-grading (i,j,k) is $(0,s,t)$, while for z_3 it is $(1,1,1)$.

Note that by Lemma 7.5 we have an embedding of the subalgebra $\mathbb{F}_2[z_3]\otimes\mathbb{F}_2[c_1, b'_2, \dots, b'_{2m+1}] \subset E_\infty^{*,*}$. Since $H^2_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2) \simeq \mathbb{F}_2 \cdot x_2$ and $H^3_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2) \simeq \mathbb{F}_2 \cdot x_3$ and the elements c_1 and z_3 are the only non-zero elements of total degree 2 and 3, respectively, they must be the "images" of elements x_2 and x_3 from Lemma 7.10 in the infinity page $E_\infty^{*,*}$. Here we identify $E_\infty^{*,*}$ with the associated graded for the spectral sequence filtration, and by "images" we mean the images in this associated graded. More generally, by definition, the image of each b_h is $b_{h'}$, and so the subalgebra $\mathbb{F}_2[z_3] \otimes \mathbb{F}_2[c_1, b'_2, \dots, b'_{2m+1}]$ is the image in the associated graded of the subalgebra in $H^*_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2)$ generated by x_2, x_3 and b_h's. In particular, we see that there are no non-trivial relations between x_2, x_3 and b_h's (otherwise there would be some in the associated graded as well) and we get an embedding $\mathbb{F}_2[x_2, x_3, b_2, \dots, b_{2m+1}] \subset H^*_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2)$. Moreover, looking at the description in Lemma 7.5 once again, we see that $x_3\mathbb{F}_2[x_2, x_3, b_2, \dots, b_{2m+1}]$ maps isomorphically to $\mathrm{Ker}\, Bp^*$ (indeed, this follows from the isomorphism $z_3\mathbb{F}_2[c_1, z_3, b_2, \dots, b_{2m+1}] \simeq E_\infty^{>0,*}$ for the associated graded). We thus have isomorphisms

$$\mathrm{Im}\, Bp^* = PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2), \qquad \mathrm{Ker}\, Bp^* = x_3\mathbb{F}_2[x_2, x_3, b_2, \dots, b_{2m+1}]. \tag{7.21}$$

To complete the proof of Theorem 7.16, it remains to construct elements $y_I \in H^*_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2)$ and show that all relations are generated by (7.17), (7.18), (7.19).

By Lemma 7.1, the subalgebra $PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)$ is generated by

$$y'_I = y'(i_1,\dots,i_r) \coloneqq d_1(\overline{c}_{2i_1} * \dots * \overline{c}_{2i_r}), \qquad I = \{i_1,\dots,i_r\}, \quad 1 < i_j \le 2m+1 \text{ for all } j.$$

as an $\mathbb{F}_2[c_1, b'_2, \dots, b'_{2m+1}]$-module (here again we call by y'_I the elements that were called y_I in Lemma 7.1).

By the degeneration of the spectral sequence, we may pick $y_I \in H^{4d(I)-2,4d(I)-2}_{\mathrm{H}}(B\mathrm{PGL}_{4m+2})$ such that $Bp^*(y_I) = y'_I$. Recall from Remark 3.11 that the spectral sequence filtration on $H^*_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2)$ is given by column degree.

Note that $E^{>k,*}_\infty \simeq z_3^k\mathbb{F}_2[z_3] \otimes \mathbb{F}_2[c_1, b'_2, \dots, b'_h]$ and x_3 maps to z_3 in the associated graded. By Lemma 7.5 we have $(z_3 \otimes 1)\cdot(1 \otimes y'_I) = 0$ in E_2, hence $x_3 y_I = 0$ in the associated graded in the E_∞-page. By Remark 3.11, this means that there exists $f \in H^*_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2)$ such that $x_3 y_I = x_3^2 f$ in $H^*_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2)$. Replacing y_I by $y_I - x_3 f$, we now have $Bp^*(y_I) = y_I$ and $x_3 y_I = 0$, that is, (7.19) holds for such y_I.

To check the relations (7.17) and (7.18) we proceed as follows. By Lemma 7.1(c), the relations (7.17) and (7.18) hold after we apply Bp^*, or, in other words, the difference of the left and right hand sides lies in $\operatorname{Ker} Bp^*$. Since $x_3 y_I = 0$ and every term in relations (7.17) and (7.18) contains at least one y_I, they are killed by multiplication by x_3. However, by (7.21), no element in $\operatorname{Ker} Bp^*$ is killed by x_3, hence relations (7.17) and (7.18) hold on the nose.

It remains to show that there are no further relations. Since we proved that the relations in the proposition hold, we have the map from the ring in the statement of the theorem (call it A^*) to $H^*_{\mathrm{H}}(B\mathrm{PGL}_{4m+2}/\mathbb{F}_2)$. The associated graded of A^* by the powers of x_3 is isomorphic to A^* again and also coincides with the description of Cotor in Lemma 7.5 (via the description of $PH^*_{\mathrm{H}}(B\mathrm{GL}_{4k+2}, \mathbb{F}_2)$ from Lemma 7.1) by an easy direct inspection. It follows that the map to E_∞-page is an isomorphism. Since both filtrations are complete, this implies the result for Hodge cohomology. The proof in de Rham cohomology context is entirely analogous, using (7.14) as the starting input in degrees ≤ 3. □

Proof of Theorem 1.5(1) for PGL_{4m+2}. The conclusion follows by comparing the explicit descriptions in Theorem 7.16 and [Tod87, Proposition 4.2]. □

8. Hodge cohomology of $B\mathrm{PSO}_{4m+2}$

In this section we compute the Hodge cohomology ring of $B\mathrm{PSO}_{4m+2}$ (Theorem 8.17). This then implies Theorem 1.5(1) for $B\mathrm{PSO}_{4m+2}$ by explicitly comparing the answers in the singular, Hodge and de Rham settings. For the computation we again follow Toda's strategy, but, contrary to the PGL_{4m+2} case, the details will be quite different.

8.1. **Cohomology of $B\mathrm{O}_n$ and $B\mathrm{SO}_n$.** By the orthogonal group O_n over $\mathbb{F}_2$ we mean the corresponding Chevalley model, namely the group scheme of linear transformations of $(\mathbb{F}_2^{\oplus n}, q)$ that preserve the non-singular quadratic form $q = x_1x_2 + x_3x_4 + \dots + x_{n-2}x_{n-1} + x_n^2$ in the case n is odd and $q = x_1x_2 + \dots + x_{n-1}x_n$ in the case n is even. The correct definition of special orthogonal group SO_n in characteristic 2 is somewhat tricky: namely, if n is odd, SO_n is defined in the usual way as the kernel of the determinant map $\det\colon \mathrm{O}_n \to \mu_2$, but if n is even one considers the "Dickson determinant" $D\colon \mathrm{O_n} \to \mathbb{Z}/2$ (see [CR10, Section 4.1.2]) instead, and defines $\mathrm{SO}_n \coloneqq \ker(D) \subset \mathrm{O}_n$.

We briefly recall the structure of Hodge cohomology rings of the corresponding classifying stacks $B\mathrm{O}_n$ and $B\mathrm{SO}_n$ established in [Tot18, Section 11]. If $n = 2r$ is even then

$$H^{*,*}_{\mathrm{H}}(B\mathrm{O}_{2r}/\mathbb{F}_2) \simeq \mathbb{F}_2[u_1, \dots, u_{2r}] \quad \text{and} \quad H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{2r}/\mathbb{F}_2) \simeq \mathbb{F}_2[u_2, \dots, u_{2r}]$$

with $|u_{2a}| = (a, a)$ and $|u_{2a+1}| = (a, a+1)$. The natural restriction map $H^{*,*}_{\mathrm{H}}(B\mathrm{O}_{2r}/\mathbb{F}_2) \to H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{2r}/\mathbb{F}_2)$ induced by the embedding $\mathrm{SO}_{2r} \to \mathrm{O}_{2r}$ simply sends u_1 to 0 (and u_i to u_i for $i \geq 2$).

It is often convenient to pull-back cohomology of $B\mathrm{O}_{2r}$ (and $B\mathrm{SO}_{2r}$) to the classifying stack of a product of several copies of $B\mathrm{O}_2$ inside. Namely, we have an embedding $\mathrm{O}_2^r \to \mathrm{O}_{2r}$ which induces a cover $(B\mathrm{O}_2)^r \to B\mathrm{O}_{2r}$. Let $s_i, t_i \in H^{*,*}_{\mathrm{H}}(B\mathrm{O}_2^r/\mathbb{F}_2)$ to be the pull-back of $u_1, u_2 \in H^{*,*}_{\mathrm{H}}(B\mathrm{O}_2/\mathbb{F}_2)$ under the i-th projection. By [Tot18, Lemma 11.3], the pull-back map

$$H^{*,*}_{\mathrm{H}}(B\mathrm{O}_{2r}/\mathbb{F}_2) \longrightarrow H^{*,*}_{\mathrm{H}}(B\mathrm{O}_2^r/\mathbb{F}_2) \simeq \mathbb{F}_2[s_1, t_1, s_2, t_2, \dots, s_r, t_r]$$

is an embedding and sends

$$u_{2a} \mapsto \sum_{1 \leq i_1 < \dots < i_a \leq r} t_{i_1} \cdots t_{i_a}, \quad u_{2a+1} \mapsto \sum_{j=1}^{r} \left(s_j \cdot \sum_{\substack{1 \leq i_1 < \dots < i_a \leq r \\ i_h \ \underline{\text{not}} \text{ equal to } j}} t_{i_1} \cdots t_{i_a} \right). \tag{8.1}$$

One can also get similar formulas in the case $n = 2r+1$ is odd, but we won't need them further, so let us just refer the reader to [Tot18, Section 11].

Finally, let us note that by [Tot18, Theorem 10.1 and Theorem 11.1], the Hodge-de Rham spectral sequences for $B\mathrm{O}_n$, $B\mathrm{SO}_n$ and $B\mu_2$ degenerate and induce natural isomorphisms of graded rings $H^*_{\mathrm{H}}(B\mathrm{O}_n/\mathbb{F}_2) \simeq H^*_{\mathrm{dR}}(B\mathrm{O}_n/\mathbb{F}_2)$ and $H^*_{\mathrm{H}}(B\mathrm{SO}_n/\mathbb{F}_2) \simeq H^*_{\mathrm{dR}}(B\mathrm{SO}_n/\mathbb{F}_2)$. This way the above discussion also applies to the de Rham cohomology rings $H^*_{\mathrm{dR}}(B\mathrm{O}_n/\mathbb{F}_2)$ and $H^*_{\mathrm{dR}}(B\mathrm{SO}_n/\mathbb{F}_2)$.

8.2. **The coaction of $H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)$.** Let $n = 2r$ be an even integer. We have

$$H^{*,*}_{\mathrm{H}}(B\mu_2/\mathbb{F}_2) \simeq \Lambda_2 \coloneqq \mathbb{F}_2[x_1, x_2]/(x_1^2), \qquad H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{2r}/\mathbb{F}_2) \simeq \mathbb{F}_2[u_2, \dots, u_{2r}],$$

where

$$x_1 \in H^{1,0}_{\mathrm{H}}(B\mu_2/\mathbb{F}_2),\ x_2 \in H^{1,1}_{\mathrm{H}}(B\mu_2/\mathbb{F}_2),\ u_{2a} \in H^{a,a}_{\mathrm{H}}(B\mathrm{SO}_n/\mathbb{F}_2),\ u_{2a+1} \in H^{a,a+1}_{\mathrm{H}}(B\mathrm{SO}_n/\mathbb{F}_2).$$

For even n, the center of SO_n is non-trivial and isomorphic to μ_2. As in Example 2.14, we can consider the multiplication map $\mu_2 \times \mathrm{SO}_{2r} \to \mathrm{SO}_{2r}$ that induces a ring map

$$\phi\colon H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{2r}/\mathbb{F}_2) \longrightarrow H^{*,*}_{\mathrm{H}}(B\mu_2/\mathbb{F}_2) \otimes H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{2r}/\mathbb{F}_2). \tag{8.2}$$

We view $H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{2r}/\mathbb{F}_2)$ as a $H^{*,*}_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)$-comodule algebra, with the coaction map ϕ. Similarly, we can consider de Rham cohomology instead of Hodge.

We will first describe the $H^{*,*}_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)$-comodule structure on $H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{2r}/\mathbb{F}_2)$.

Lemma 8.3. *Let $n = 2r$ be an even integer, and consider the $H^{*,*}_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)$-comodule algebra $H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{2r}/\mathbb{F}_2)$, with coaction ϕ as in (8.2). We have*

$$\phi(u_{2a}) = \sum_{i+j=a} \binom{r-j}{i} x_2^i \otimes u_{2j} + \sum_{i+j=a} \binom{r-j}{i} x_2^i x_1 \otimes u_{2j-1}$$

and

$$\phi(u_{2a+1}) = \sum_{i+j=a} \binom{r-j}{i} x_2^i \otimes u_{2j+1},$$

where we put $u_1 := 0$ *and* $u_0 := 1$.

Proof. Let $V \simeq \mathbb{F}_2^{\oplus 2}$ be the tautological 2-dimensional k-linear representation of O_2, and let $H \subset \mathrm{O}_2$ be the subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mu_2$, where $\mathbb{Z}/2\mathbb{Z}$ permutes the coordinates of V and μ_2 acts by scaling. We have

$H^{*,*}_{\mathrm{H}}(B(\mathbb{Z}/2\mathbb{Z})/\mathbb{F}_2) \simeq \mathbb{F}_2[z]$, $H^{*,*}_{\mathrm{H}}(B\mu_2/\mathbb{F}_2) \simeq \mathbb{F}_2[x_1, x_2]/(x_1^2)$, $H^{*,*}_{\mathrm{H}}(BH/\mathbb{F}_2) \simeq \mathbb{F}_2[z, x_1, x_2]/(x_1^2)$,

where z, x_2 and x_1 have bidegrees $(0, 1)$, $(1, 1)$ and $(1, 0)$, respectively.

The coaction $H^{*,*}_{\mathrm{H}}(BH/\mathbb{F}_2) \to H^{*,*}_{\mathrm{H}}(B\mu_2/\mathbb{F}_2) \otimes H^{*,*}_{\mathrm{H}}(BH/\mathbb{F}_2)$ induced by $\mu_2 \times H \to H$ sends

$$z \mapsto 1 \otimes z, \quad t \mapsto 1 \otimes x_2 + x_2 \otimes 1, \quad x_1 \mapsto 1 \otimes x_1 + x_1 \otimes 1.$$

By [Tot18, Discussion above Lemma 11.3],[8] the pullback map $H^{*,*}_{\mathrm{H}}(B\mathrm{O}_2/\mathbb{F}_2) \to H^{*,*}_{\mathrm{H}}(BH/\mathbb{F}_2)$ sends $u_1 \mapsto z$, $u_2 \mapsto x_2 + x_1 z$. This allows to compute the coaction $\phi\colon H^{*,*}_{\mathrm{H}}(B\mathrm{O}_2/\mathbb{F}_2) \to H^{*,*}_{\mathrm{H}}(B\mu_2/\mathbb{F}_2) \otimes H^{*,*}_{\mathrm{H}}(B\mathrm{O}_2/\mathbb{F}_2)$: it sends

$$u_1 \mapsto 1 \otimes u_1, \quad u_2 \mapsto 1 \otimes u_2 + x_2 \otimes 1 + x_1 \otimes u_1.$$

Write

$$H^*_{\mathrm{H}}(B\mathrm{O}_2^m/\mathbb{F}_2) = \mathbb{F}_2[s_1, \ldots, s_m, t_1, \ldots, t_m],$$

where s_i, t_i are pullbacks of $u_1, u_2 \in H^{*,*}_{\mathrm{H}}(B\mathrm{O}_2/\mathbb{F}_2)$ along the i-th projection. By the above computation the coaction $H^{*,*}_{\mathrm{H}}(B\mathrm{O}_2^m/\mathbb{F}_2) \to H^{*,*}_{\mathrm{H}}(B\mu_2/\mathbb{F}_2) \otimes H^{*,*}_{\mathrm{H}}(B\mathrm{O}_2^m/\mathbb{F}_2)$ sends

$$s_i \mapsto 1 \otimes s_i, \quad t_i \mapsto 1 \otimes t_i + t \otimes 1 + v \otimes s_i.$$

Finally, following Equation (8.1), the pullback $H^{*,*}_{\mathrm{H}}(B\mathrm{O}_{2m}/\mathbb{F}_2) \to H^{*,*}_{\mathrm{H}}(B\mathrm{O}_2^m/\mathbb{F}_2)$ sends

$$u_{2a} \mapsto \sum_{1 \le i_1 < \cdots < i_a \le m} t_{i_1} \cdots t_{i_a}, \quad u_{2a+1} \mapsto \sum_{j=1}^{r} s_j \cdot \left(\sum_{1 \le i_1 < \cdots < i_a \le m, i_h \ne j} t_{i_1} \cdots t_{i_a} \right).$$

One then checks formulas in statement of the lemma by plugging $\phi(t_i)$ and $\phi(s_i)$, and opening the brackets (this is a direct computation that we leave to the reader).

Finally, following the discussion in Section 8.1, to pass from O_{2m} to $\mathrm{SO}_{2m} \subset \mathrm{O}_{2m}$ we just need to set $u_1 = 0$ in all the formulas. □

Remark 8.4. On the topological side we have isomorphisms

$$H^*_{\mathrm{sing}}(B(\mathbb{Z}/2\mathbb{Z}); \mathbb{F}_2) \simeq \mathbb{F}_2[z_1], \qquad H^*_{\mathrm{sing}}(B\mathrm{SO}_n(\mathbb{C}); \mathbb{F}_2) = \mathbb{F}_2[w_2, \ldots, w_n],$$

where $|z_1| = 1$, and w_i, with $|w_i| = i$ are the Stiefel-Whitney classes. We immediately see that $H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)$ is *not* isomorphic to $H^*_{\mathrm{sing}}(B(\mathbb{Z}/2\mathbb{Z}); \mathbb{F}_2)$ as an algebra. In fact the situation is even worse: if we take the (unique) isomorphism of $H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)$ and $H^*_{\mathrm{sing}}(B(\mathbb{Z}/2\mathbb{Z}); \mathbb{F}_2)$ as coalgebras (Remark 4.6), it can't be extended to an isomorphism of the comodule algebras $H^*_{\mathrm{H}}(B\mathrm{SO}_n/\mathbb{F}_2)$ and $H^*_{\mathrm{sing}}(B\mathrm{SO}_n(\mathbb{C}); \mathbb{F}_2)$. This is the reason why the argument from Section 4 doesn't apply to PSO_n as well. In particular we don't know yet that the Eilenberg-Moore spectral sequence collapses on second page (but we will show this in Corollary 8.13).

[8] In *loc.cit.* z, x_1, x_2 are denoted by s, v, t, respectively.

8.3. **Computation of** Cotor**.** Note that $A = H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2)$ with $a_\sharp \coloneqq u_2$ satisfies the assumptions of Lemma 6.16 with $q = 2$. Indeed, $d_2(\overline{u}_2) = 1$, and $d_i(\overline{u}_2) = 0$ for $i > 2$ by degree reasons. We now establish an analogue of Lemma 7.1 in the setting of the special orthogonal group. Namely, we will describe more or less explicitly the subalgebras

$$PH^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2) \subset P_2H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2) \subset H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2),$$

as well as the cohomology of $P_2H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2)$ with respect to the differential induced by d_1.

Lemma 8.5. *Assume that $n = 4m + 2$.*

(a) There exists a unique sequence

$$\overline{u}_2, \dots, \overline{u}_{4m+2} \in H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2)$$

with bidegrees $|\overline{u}_{2a}| = (a, a)$ and $|\overline{u}_{2a+1}| = (a, a + 1)$, such that

(1) $\overline{u}_2 = u_2$,

(2) For all $j \geq 2$, $\overline{u}_j \equiv u_j \pmod{u_2}$ and $\overline{u}_j \in P_2H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2)$,

(3) For all $j \geq 2$, $\overline{u}_{2j-1} = d_1(\overline{u}_{2j})$,

(4) $H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2) = \mathbb{F}_2[\overline{u}_2, \overline{u}_3, \dots, \overline{u}_{4m+2}]$,

(5) $P_2H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2) = \mathbb{F}_2[\overline{u}_3, \overline{u}_4, \dots, \overline{u}_{4m+2}]$.

(b) (1) The elements $\overline{u}_3, \overline{u}_5, \dots, \overline{u}_{4m+1}$ and $b_h \coloneqq \overline{u}^2_{2h}$ for $h > 1$ are primitive (equivalently, they lie in $P_2H^{*,*}_{\mathrm{H}}$ and are killed by d_1).

(2) The natural map

$$\mathbb{F}_2[b_2, \dots, b_{2m+1}] \longrightarrow H^*(P_2H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2), d_1)$$

is an isomorphism.

(c) For any (unordered) tuple of integers $I = \{i_1, \dots, i_r\}$, set $d(I) \coloneqq i_1 + \ldots + i_r$,

$$\overline{u}_I \coloneqq \overline{u}_{2i_1} \cdot \ldots \cdot \overline{u}_{2i_r} \in P_2H^{d(I),d(I)}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2),$$

and

$$y_I = y(i_1, \dots, i_r) \coloneqq d_1(u_{\overline{I}}) \in PH^{d(I)-1,d(I)}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2).$$

In particular, $y_{\{i\}} = \overline{u}_{2i-1}$.

(1) The subalgebra $PH^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2) \subset H^*_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2)$ of primitive elements is generated by the b_h and the y_I for $I = \{1 < i_1 < \cdots < i_r \leq 2m + 1\}$, with the relations generated by

$$y_{\{i_1,\dots,i_r\}} \cdot y_{\{j_1,\dots j_s\}} = \sum_{h=1}^{r} y_{\{i_1,\dots,i_{h-1},i_{h+1},\dots,i_r,j_1,\dots j_s\}} \cdot y_{\{i_h\}}, \tag{8.6}$$

where we have $y_{\{h,h,i_1,\dots,i_r\}} = b_h \cdot y_I$.

(2) $PH^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2)$ is a finitely generated module over the polynomial subalgebra

$$\mathbb{F}_2[\overline{u}_3, \overline{u}_5, \dots, \overline{u}_{4m+1}, b_2, b_3, \dots, b_{2m+1}] \subset PH^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2).$$

Entirely analogous statements hold with Hodge cohomology replaced by de Rham cohomology.

Proof. We write A for $H^*_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2)$.

(a) By Lemma 6.16(b) applied to A and $a_\sharp = u_2$, the composite map

$$P_2A \hookrightarrow A \to A/(u_2)$$

is a ring isomorphism. Thus we can let $\overline{u}_2 \coloneqq u_2$ and $\overline{u}_j \in P_2A$ be the inverse image of u_j (mod u_2) for all $j \geq 2$ under this isomorphism. Elements $\overline{u}_j$ satisfy all the properties except, possibly, 3). For that one, note that by Lemma 8.3 we have $d_1(u_2) = 0$ and $d_1(u_{2j}) = u_{2j-1}$ for all $j > 1$. We have $d_1(\overline{u}_{2j}) \in P_2A$ and it follows from the above that $d_1(\overline{u}_{2j}) \equiv d_1(u_{2j}) \equiv u_{2j-1}$ (mod u_2) and so $d_1(\overline{u}_{2j}) = \overline{u}_{2j-1}$.

(b) Recall that we put $b_h \coloneqq \overline{u}_{2h}^2$. We have $d_1(b_h) = 2 \cdot \overline{u}_{2h} \cdot d_1(\overline{u}_{2h}) = 0$, while $d_1(\overline{u}_{2i-1}) = d_1(d_1(\overline{u}_{2i})) = 0$, so (1) follows. For (2) let us describe the differential d_1 on $P_2H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2) \simeq \mathbb{F}_2[\overline{u}_3, \overline{u}_4, \ldots, \overline{u}_{4m+2}]$:

$$d_1(\overline{u}_{2i-1}) = 0 \quad \text{and} \quad d_1(\overline{u}_{2i}) = \overline{u}_{2i-1}.$$

Let $C \coloneqq \mathbb{F}_2[\overline{u}_3, \overline{u}_5, \ldots, \overline{u}_{4m+1}, b_2, b_3, \ldots, b_{2m+1}]$. Note that $C \subset \operatorname{Ker} d_1$, and so d_1 is C-linear. Moreover, elements $\overline{u}_I \coloneqq u_{2i_1} \cdot \ldots \cdot u_{2i_r}$ for all $I = \{1 < i_1 < \ldots < i_r \leq 2m+1\}$ form a basis of $P_2H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2)$ over C. One can then identify $(P_2H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2), d_1)$ with the Koszul complex over C for the regular sequence $\overline{u}_3, \overline{u}_5, \ldots, \overline{u}_{4m+1}$. This way we get that the cohomology of $(P_2H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2), d_1)$ is given by $C/(\overline{u}_3, \overline{u}_5, \ldots, \overline{u}_{4m+1}) \simeq \mathbb{F}_2[b_2, b_3, \ldots, b_{2m+1}]$.

(c) Recall that $PH^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2)$ is identified with $\mathrm{Ker}(d_1) \subset P_2H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2)$. In (b) we showed that $\mathbb{F}_2[b_2, b_3, \ldots, b_{2m+1}] \subset \mathrm{Ker}(d_1)$ maps isomorphically to the cohomology. From this we get that the whole $\mathrm{Ker}(d_1)$ is a direct sum of $\mathbb{F}_2[b_2, b_3, \ldots, b_{2m+1}]$ and the image $\mathrm{Im}(d_1)$. By the discussion in b) we have that $\mathrm{Im}(d_1)$ is spanned over C by $y_I \coloneqq d_1(\overline{u}_I)$ for all $I = \{1 < i_1 < \ldots < i_r \leq 2m+1\}$. Moreover $y_{\{i\}} = d_1(\overline{u_{2i}}) = \overline{u}_{2i-1} \in C$, so we get that $PH^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2)$ is spanned over C by y_I's with $l(I) \geq 2$. It remains to understand the relations between y_I.

First of all, indeed

$$y_{\{h,h,i_1,\ldots,i_r\}} = d_1(\overline{u}_h^2 \cdot \overline{u}_I) = \overline{u}_h^2 \cdot d_1(\overline{u}_I) = b_h \cdot y_I.$$

Then, applying (6.21) to $a = \overline{u}_{I \setminus \{i_r\}}$, $b = \overline{u}_{i_r}$, $c = \overline{u}_J$, for any $I = \{i_1, \ldots, i_r\}$ and $J = \{j_1, \ldots j_s\}$ we get

$$y_{\{i_1,\ldots,i_{r-1},i_r\}} \cdot y_{\{j_1,\ldots j_s\}} = y_{\{i_1,\ldots,i_{r-1}\}} \cdot y_{\{i_r,j_1,\ldots j_s\}} + y_{\{i_1,\ldots,i_{r-1},j_1,\ldots j_s\}} \cdot y_{\{i_r\}}.$$

Applying the same equation again to the first term on the right and continuing, we end up with (8.6). Taking $J = \{j\}$ be a 1-element set, we get

$$y_{\{i_1,\ldots,i_r\}} y_{\{j\}} = \sum_{s=1}^{r} y_{\{i_1,\ldots,i_{s-1},i_{s+1},\ldots,i_r,j\}} \cdot y_{\{i_s\}}. \tag{8.7}$$

Note that this is exactly the relation which is obtained from $d_1^2(\overline{u}_{I \cup \{j\}}) = 0$. Indeed, for any $J = \{j_1, \ldots, j_r\}$

$$d_1(\overline{u}_I) = \sum_{s=1}^{r} \overline{u}_{J \setminus \{j_s\}} \cdot d_1(\overline{u}_{2j_s}) = \sum_{s=1}^{r} \overline{u}_{J \setminus \{j_s\}} \cdot y_{j_s}. \tag{8.8}$$

Putting $J = I \cup \{j\}$ and applying again we exactly get the sum of the left and right hand sides in Equation (8.7).

Now, to show that Equations (8.6) generate all the relations assume $\sum f_I y_I = 0$ for some $f_I \in C$. Since $d_1(f_I) = 0$ for all I, we get that $\sum f_I \overline{u}_I \in \operatorname{Ker} d_1$. From the proof of b) we know that we can write

$$\sum f_I \overline{u}_I = d_1(\sum g_J \overline{u}_J) + z,$$

for some $g_J \in C$ and $z \in \mathbb{F}_2[b_2, b_3, \dots, b_{2m+1}]$. Note that $d_1(g_J) = 0$ and so we get that

$$\sum f_I \overline{u}_I = \sum g_J d_1(\overline{u}_J) + z.$$

By the discussion above (8.8) the expression for $d_1(\overline{u}_I)$ as a sum of u_J's with coefficients in C gives (8.7) after plugging y_J's in place of u_J's. Since $\overline{u}_I$ (with $I = \{1 < i_1 < \cdots < i_r \le 2m+1\}$) form a basis of $P_2 H^{*,*}$ over the algebra C we get that $\sum f_I y_I = 0$ in fact is a linear combination of relations in (8.8) which are a particular case of (8.6). □

Remark 8.9. Note the difference in the formulas for elements b_h as defined in Lemma 8.5(b) and those defined in the topological setting [Tod87, p. 92]. Again, this is due to the fact that $H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)$ and $H^*_{\mathrm{sing}}(B(\mathbb{Z}/2\mathbb{Z}); \mathbb{F}_2)$ are not isomorphic as algebras: Lemma 6.16(b), which is available only for $H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)$, makes the formula for b_h very simple.

We are ready to compute $\operatorname{Cotor}^*_{H^{*,*}_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)}(\mathbb{F}_2, H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2))$.

Lemma 8.10. *For every $m \ge 0$, we have an isomorphism of $\mathbb{Z} \oplus \mathbb{Z}^2$-graded*[9] *algebras*

$$\operatorname{Cotor}^*_{H^{*,*}_{\mathrm{H}}(B\mu_2)}(\mathbb{F}_2, H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2})) \simeq (1 \otimes PH^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2})) \oplus (z_2\mathbb{F}_2[z_2] \otimes \mathbb{F}_2[b_2, \dots, b_{2m+1}]),$$

with the grading of elements given by $|z_2| = (1,1,0)$, $|b_h| = (0, 2h, 2h)$.

Analogous assertion holds with Hodge cohomology replaced by de Rham cohomology (with the corresponding $\mathbb{Z} \oplus \mathbb{Z}$-gradings given by $|z_2| = (1,1)$, $|b_h| = (0, 4h)$).

Remark 8.11. Analogously to Remark 7.6, the multiplication on the direct sum above is prescribed by the surjective homomorphism

$$\varphi \colon PH^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}) \twoheadrightarrow H^*(P_2 H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2})), d_1) \simeq \mathbb{F}_2[b_2, \dots, b_{2m+1}]. \tag{8.12}$$

More precisely, (following the terminology in Remark 7.6) Lemma 8.10 claims that the algebra $\operatorname{Cotor}^*_{H^{*,*}_{\mathrm{H}}(B\mu_2)}(\mathbb{F}_2, H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}))$ is isomorphic to $A_{\phi,z}$ for $z := z_2$,

$$A = PH^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}), \qquad B = H^*(P_2 H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2})), d_1) \simeq \mathbb{F}_2[c_1, b_h]_{h=2}^{2m+1},$$

and φ is given by (8.12).

Proof. Let $A := H^*_{\mathrm{H}}(B\mathrm{SO}_n/\mathbb{F}_2)$. By Corollary 5.19, Cotor groups can be computed as the cohomology $H^*(R_2 \otimes_{\theta_2} A, d_{\theta_2})$ of the twisted tensor product.

Let us compute the differential $d_{\theta_2} \colon R_2 \otimes A \to R_2 \otimes A$ explicitly. The differential d_{θ_2} is $R_2 \otimes 1$-linear and so it is enough to understand d_{θ_2} on $1 \otimes A$. Recall that $A \simeq P_2 A[u_2]$ by Lemma 6.16(b). We have $\phi(u_2) = 1 \otimes u_2 + x_2 \otimes 1$ and $\phi(u_2^{2^i}) = \phi(u_2)^{2^i} = 1 \otimes u_2^{2^i} + x_2^{2^i} \otimes 1$. By definition of d_{θ_2} (see Construction 5.5) and our choice of the twisted cochain (Construction 5.16)

[9] Where the $\mathbb{Z}$-component in $\mathbb{Z} \oplus \mathbb{Z}^2$ is the Cotor-grading.

this shows that $d_{\theta_2}(1 \otimes u_2^{2^i}) = z_{2^{i+1}+1} \otimes 1$ for $i \geq 0$. More generally, if we take u_2^n and take the 2-adic expansion $n = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_r}$ with all $i_1 < i_2 < \ldots < i_r$, then we have

$$\phi(u_2^n) = \phi(u_2)^{2^{i_1}} \cdot \ldots \cdot \phi(u_2)^{2^{i_r}} = (1 \otimes u_2^{2^{i_1}} + x_2^{2^{i_1}} \otimes 1) \cdot \ldots \cdot (1 \otimes u_2^{2^{i_r}} + x_2^{2^{i_r}} \otimes 1).$$

Opening the brackets, one gets a formula for d_{θ_2}:

$$d_{\theta_2}(1 \otimes u_2^n) = \sum_{j=1}^{r} z_{2^{i_j+1}+1} \otimes u_2^{n-2^{i_j}}.$$

Moreover, if $a \in P_2A$, then $\phi(a) = 1 \otimes a + x_1 \otimes d_1(a)$ and so $d_{\theta_2}(a) = z_2 \otimes d_1(a)$. Finally, $\phi(au_2^n) = \phi(a)\phi(u_2^n)$ and by opening the brackets in this product one sees that

$$d(1 \otimes au_2^n) = d(1 \otimes a) \cdot 1 \otimes u_2^n + 1 \otimes a \cdot d(u_2^n).$$

Note that by the above formulas $\mathbb{F}_2[z_2] \otimes P_2A \subset R_2 \otimes A$ is closed under the differential, and so is $\mathbb{F}_2[z_3, z_5, z_9, \ldots] \otimes \mathbb{F}_2[u_2] \subset R_2 \otimes A$. Moreover, the partial Leibnitz rule above shows that there is a decomposition as a tensor product

$$(R_2 \otimes_{\theta_2} A, d_{\theta_2}) \simeq (\mathbb{F}_2[z_2] \otimes P_2A, d_{\theta_2}) \bigotimes_{\mathbb{F}_2} (\mathbb{F}_2[z_3, z_5, z_9, \ldots] \otimes \mathbb{F}_2[u_2], d_{\theta_2}).$$

We claim that the right term in the tensor product is quasi-isomorphic to $\mathbb{F}_2$ in (cohomological) degree 0. Indeed, consider a tensor product product $C = \wedge_{\mathbb{F}_2}(\xi_3, \xi_5, \xi_9, \ldots) \otimes \mathbb{F}_2[z_3, z_5, z_9, \ldots]$ where $\wedge_{\mathbb{F}_2}(\xi_3, \xi_5, \xi_9, \ldots)$ is the exterior algebra in variables ξ_i (with same indices as for z_i before). Endow K with the unique $\mathbb{F}_2[z_3, z_5, \ldots]$–linear differential d_K sending ξ_i to z_i and satisfying Leibnitz rule. The complex (K, d_K) is nothing but the Koszul complex (a free resolution of trivial module $\mathbb{F}_2$ over $\mathbb{F}_2[z_3, z_5, z_9, \ldots]$) and is quasi-isomorphic to $\mathbb{F}_2$ in degree 0. There is a $\mathbb{F}_2[z_3, z_5, z_9, \ldots]$–linear map of complexes

$$(K, d_K) \to (\mathbb{F}_2[z_3, z_5, z_9, \ldots] \otimes \mathbb{F}_2[u_2], d_{\theta_2})$$

sending $\xi_{2^{i+1}+1}$ to $1 \otimes u_2^{2^i}$ and extended by multiplicativity (meaning $\xi_{2^{i_1+1}+1}\xi_{2^{i_2+1}+1} \cdots \xi_{2^{i_k+1}+1}$ is sent to $1 \otimes (u_2^{2^{i_1}+2^{i_2} \ldots + 2^{i_k}})$). Since for any $n \geq 0$ $u_2^n = u_2^{2^{i_1}+2^{i_2}+\ldots+2^{i_k}}$ for a unique distinct set of natural numbers $\{i_1, \ldots, i_k\}$ (given by powers in the 2-adic expansion for n), this map is an isomorphism of complexes.

Consequently, we get a quasi-isomorphism

$$(R_2 \otimes_{\theta_2} A, d_{\theta_2}) \simeq (\mathbb{F}_2[z_2] \otimes P_2A, d_{\theta_2}|_{\mathbb{F}_2[z_2] \otimes P_2A}).$$

Recall that $d_{\theta_2}|_{\mathbb{F}_2[z_2] \otimes P_2A}$ is sending $z_2^k \otimes a \mapsto z_2^{k+1} \otimes d_1(a)$. Thus $(\mathbb{F}_2[z_2] \otimes P_2A, d_{\theta_2})$ looks like

$$1 \otimes P_2A \xrightarrow{z_2 \otimes d_1(-)} z_2 \otimes P_2A \xrightarrow{z_2 \otimes d_1(-)} z_2^2 \otimes P_2A \xrightarrow{z_2 \otimes d_1(-)} \ldots,$$

from which we get that $H^0(\mathbb{F}_2[z_2] \otimes P_2A, d_{\theta_2}) \simeq 1 \otimes PA \simeq \ker(1 \otimes d_1) \subset 1 \otimes P_2A$, while $H^i(\mathbb{F}_2[z_2] \otimes P_2A, d_{\theta_2}) \simeq z_2^i \otimes H^*(P_2A, d_1)$, which by Lemma 8.5(b2) is isomorphic to $z_2^i \otimes \mathbb{F}_2[b_1, \ldots, b_h]$. Therefore, the algebra structure can be understood via Lemma 5.21: namely, the map $\mathbb{F}_2[z_2] \otimes PA \to H^*(\mathbb{F}_2[z_2] \otimes P_2A, d_{\theta_2})$ is an algebra homomorphism. □

Corollary 8.13. *The Eilenberg-Moore spectral sequence (from Theorem 1.3) for*

$$1 \to \mu_2 \to \mathrm{SO}_{4k+2} \to \mathrm{PSO}_{4k+2} \to 1$$

over $\mathbb{F}_2$ degenerates on the E_2 page.

Proof. Similarly to Section 4.3 we need to show that the dimensions of terms in the second page for the Eilenberg-Moore spectral sequence in Hodge and singular cohomology are the same. Indeed, by Toda's result [Tod87, Section 4.4] the spectral sequence degenerates on the singular cohomology side, so by Totaro's inequality we would get that it should also degenerate for Hodge and de Rham cohomology. The comparison is established by direct inspection. First, comparing Lemma 8.5(c1) and [Tod87, Proposition 3.11] one sees that $PH^*_{\mathrm{H}}(B\mathrm{SO}_{4k+2}/\mathbb{F}_2)$ and $PH^*_{\mathrm{sing}}(B\mathrm{SO}_{4k+2}(\mathbb{C}),\mathbb{F}_2)$ are given by the same generators and relations. This gives an isomorphism between $E_2^{0,*}$ in Hodge and singular cohomology. Finally, one observes $E_2^{>0,*}$ in both settings is given by a free module over a polynomial ring with generators in same degrees, see Lemma 8.10 and [Tod87, Section 4.4], so the dimensions are also the same. □

Remark 8.14. Let us point out that Toda's strategy of proving degeneration (using pull-back with respect to the "tensor product" map $\mathrm{O}_2 \times \mathrm{SO}_{2m+1} \to \mathrm{SO}_{4m+2}$) doesn't work in the Hodge setting: the reason is that the corresponding pull-back map

$$H^*_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2) \to H^*_{\mathrm{H}}(B\mathrm{O}_2/\mathbb{F}_2) \otimes H^*_{\mathrm{H}}(B\mathrm{SO}_{2m+1}/\mathbb{F}_2)$$

is no longer an embedding.

8.4. **Computation of Hodge cohomology of $B\mathrm{PSO}_{4m+2}$.** We begin by understanding the Hodge cohomology of $B\mathrm{PSO}_{2r}$ in low degrees.

Lemma 8.15. *We have isomorphisms*

(1) $H^0_{\mathrm{H}}(B\mathrm{PSO}_{2r}/\mathbb{F}_2) \simeq \mathbb{F}_2$
(2) $H^1_{\mathrm{H}}(B\mathrm{PSO}_{2r}/\mathbb{F}_2) \simeq 0,$
(3) $H^2_{\mathrm{H}}(B\mathrm{PSO}_{2r}/\mathbb{F}_2) \simeq H^{1,1}_{\mathrm{H}}(B\mathrm{PSO}_{2r}/\mathbb{F}_2) \simeq \begin{cases} \mathbb{F}_2, & \text{if } r = 2k+1 \\ \mathbb{F}_2 \oplus \mathbb{F}_2, & \text{if } r = 2k. \end{cases}$

In the case $r = 2k+1$ we let x_2 be the (unique) generator of $H^2_{\mathrm{H}}(B\mathrm{PSO}_{4k+2}/\mathbb{F}_2)$.

Proof. As in the proof of Lemma 7.10, using that SO_n (and so PSO_{2r}) is smooth and connected, we have isomorphisms

$$H^i_{\mathrm{H}}(B\mathrm{PSO}_{2r}/\mathbb{F}_2) \simeq \begin{cases} \mathbb{F}_2 & i = 0 \\ 0 & i = 1 \\ (\mathfrak{t}^\vee)^W & i = 2, \end{cases}$$

where $\mathfrak{t}$ is the Lie algebra of the maximal torus of PSO_{2r} and W is the Weyl group. Thus it remains to show that $(\mathfrak{t}^\vee)^W \simeq \mathbb{F}_2$. The maximal torus $T' \simeq \mathbb{G}_m^r \subset \mathrm{SO}_{2r}$ is given by $\{(t_1, t_1^{-1}, \dots, t_r, t_r^{-1})\} \subset \mathrm{SO}_{2r}$ (in the basis where the quadratic form q_{2r} is given by $x_1x_2 + \dots + x_{2r-1}x_{2r}$). The maximal torus $T \subset \mathrm{PSO}_{2r}$ is obtained as the quotient of T' by diagonal copy of μ_2. Note that both tori are split and so there is a W-equivariant identification of the Lie algebras $\mathfrak{t}$ and $\mathfrak{t}'$ and the mod 2 reductions $X_*(T)_{\mathbb{F}_2}$ and $X_*(T')_{\mathbb{F}_2}$ of the cocharacter lattices. Let $\chi_i \colon \mathbb{G}_m \to T'$ be cocharacter corresponding to t_i; it is not hard to see that $X_*(T')$ is embedded into $X_*(T)$ as the lattice $\mathbb{Z}\cdot\chi_1 \oplus \dots \oplus \mathbb{Z}\cdot\chi_r$ inside the lattice generated by $\frac{1}{2}(\chi_1 + \dots + \chi_r)$ and χ_i. The group W is isomorphic to $S_r \ltimes (\mathbb{Z}/2\mathbb{Z})^{r-1}$ where $(\mathbb{Z}/2\mathbb{Z})^{r-1}$ acts trivially on $X_*(T')$ (and $\mathfrak{t}'$) and S_r acts by transpositions on χ_i's. For $X_*(T')$ we then have a short exact sequence

$$0 \to \mathbb{Z}\cdot(\chi_1 + \dots + \chi_r) \to X_*(T') \to L \to 0$$

for a W-module L. This gives a short exact sequence

$$0 \to \mathbb{F}_2 \to \mathfrak{t}' \to L/2 \to 0. \tag{8.16}$$

Note however that the image of $\mathfrak{t}' \to \mathfrak{t}$ is exactly given by $L/2$ and so gives a W-equivariant splitting $\mathfrak{t} \simeq \mathbb{F}_2 \oplus L/2$. In particular, $(\mathfrak{t}^\vee)^W \simeq ((L/2)^\vee)^W \oplus \mathbb{F}_2$.

From (8.16) we have a short exact sequence

$$0 \to (L/2)^\vee \to \mathfrak{t}'^\vee \to \mathbb{F}_2 \to 0,$$

inducing a left-exact sequence

$$0 \to ((L/2)^\vee)^W \to (\mathfrak{t}'^\vee)^W \to \mathbb{F}_2 \to \dots$$

By direct inspection $\mathfrak{t}'^\vee \simeq \mathbb{F}_2^{\oplus r}$ is the "permutation module"[10] for W and maps to $\mathbb{F}_2$ by $(x_1, \dots, x_n) \mapsto \sum x_i$, while $(\mathfrak{t}'^\vee)^W$ is spanned by the vector $(1, 1, \dots, 1)$. So the map $(\mathfrak{t}'^\vee)^W \simeq \mathbb{F}_2$ is given by multiplication by r and we get that $((L/2)^\vee)^W$ is $\mathbb{F}_2$ or 0 depending on whether r is even or odd. □

Theorem 8.17. *The bigraded ring $H^*_{\mathrm{H}}(B\mathrm{PSO}_{4m+2}/\mathbb{F}_2)$ is generated by*

$$x_2 \in H^{1,1}_{\mathrm{H}}(B\mathrm{PSO}_{4m+2}/\mathbb{F}_2), \qquad b_h \in H^{2h,2h}_{\mathrm{H}}(B\mathrm{PSO}_{4m+2}/F_2),$$
$$y_I = y_{\{i_1,\dots,i_r\}} \in H^{d(I),d(I)-1}_{\mathrm{H}}(B\mathrm{PSO}_{4m+2}/\mathbb{F}_2).$$

Here $1 < h \le 2m+1$, $I = (i_1, \dots, i_r)$, where $1 < i_1 < \dots < i_r \le 2m+1$, and $d(I) := i_1 + \dots + i_r$. The relations are generated by

$$x_2 \cdot y_I = 0, \tag{8.18}$$

$$y_{\{i_1,\dots,i_r\}} y_{\{i_{r+1},\dots,i_s\}} = \sum_{j=1}^{r} y_{\{i_1,\dots,i_{j-1},i_{j+1},\dots,i_s\}} y_{\{i_j\}} \text{ for } s > r \ge 2, \tag{8.19}$$

with the convention that

$$y_{\{h,h,j_1,\dots,j_s\}} = y_{\{j_1,\dots,j_s\}} b_h. \tag{8.20}$$

Similarly, the graded ring $H^*_{\mathrm{dR}}(B\mathrm{PSO}_{4m+2}/\mathbb{F}_2)$ has generators $x_2 \in H^2$, $b_h \in H^{4h}$, $y_I \in H^{2d(I)-1}$, where h, I and $d(I)$ are as above, and relations generated by (8.18), (8.20) and (8.19). In particular, the Hodge-to-de Rham spectral sequence for $B\mathrm{PSO}_{4m+2}$ degenerates and we have an isomorphism of graded rings $H^*_{\mathrm{H}}(B\mathrm{PSO}_{4m+2}/\mathbb{F}_2) \simeq H^*_{\mathrm{dR}}(B\mathrm{PSO}_{4m+2}/\mathbb{F}_2)$.

Proof. The proof is similar to that of Theorem 7.16. As there, we only prove the theorem for Hodge cohomology, but a similar argument applies to de Rham cohomology. By Corollary 8.13, Eilenberg-Moore spectral sequence for

$$1 \to \mu_2 \to \mathrm{SO}_{4k+2} \xrightarrow{p} \mathrm{PSO}_{4k+2} \to 1$$

degenerates on the E_2 page, which is isomorphic to

$$1 \otimes PH^*_{\mathrm{H}}(B\mathrm{SO}_{4m+2}) \oplus z_2\mathbb{F}_2[z_2] \otimes \mathbb{F}_2[b'_2, \dots, b'_{2m+1}]$$

[10] Meaning $(\mathbb{Z}/2\mathbb{Z})^{r-1}$ acts trivially, while there is a choice of r vectors that form a basis, such that S_r acts on them by permutations.

by an explicit computation which we made in Lemma 8.10 (and where we now call $b'_h \in H^{2h,2h}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2)$ what we called b_h in Lemma 8.10(b)). From degeneration, we see that the pull-back map $Bp^*\colon H^*_{\mathrm{H}}(B\mathrm{PSO}_{4m+2}) \twoheadrightarrow 1 \otimes PH^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2})$ is a surjection. We let $b_h \in H^{2h,2h}_{\mathrm{H}}(B\mathrm{PSO}_{4m+2}/\mathbb{F}_2)$ be any fixed lift of $1 \otimes b'_h$.

The only element of total degree 2 in $E^{*,*}_\infty \simeq E^{*,*}_2$ is z_2. Thus it has to be the image of the class $x_2 \in H^{1,1}_{\mathrm{H}}(B\mathrm{PSO}_{4m+2}/\mathbb{F}_2)$ from Lemma 8.15 in $E^{*,*}_\infty$. Since $\mathbb{F}_2[z_2] \otimes \mathbb{F}_2[b'_2, \dots, b'_{2m+1}]$ embeds into $E^{*,*}_2$ we get that the natural map $\mathbb{F}_2[x_2] \otimes \mathbb{F}_2[b_2, \dots, b_{2m+1}] \to H^*_{\mathrm{H}}(B\mathrm{PSO}_{4m+2}/\mathbb{F}_2)$ is an embedding. We also get isomorphisms

$$\mathrm{Im}\, Bp^* = PH^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2), \qquad \mathrm{Ker}\, Bp^* = x_2\mathbb{F}_2[x_2, b_2, \dots, b_{2m+1}]. \tag{8.21}$$

By Lemma 8.5, the subalgebra $PH^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2)$ is generated by

$$y'_I = y'(i_1, \dots, i_r) \coloneqq d_1(\overline{u}_{2i_1} * \dots * \overline{u}_{2i_r}), \qquad I = \{i_1, \dots, i_r\}, \quad 1 < i_j \leq 2m+1 \text{ for all } j.$$

as an $\mathbb{F}_2[b'_2, \dots, b'_{2m+1}]$-module (here again we call by y'_I the elements that were called y_I in Lemma 7.1). Then, as in Theorem 7.16, taking $y_I \in H^{4d(I),4d(I)-1}_{\mathrm{H}}(B\mathrm{PSO}_{4m+2}/\mathbb{F}_2)$ such that $Bp^*(y_I) = y'_I$ and possibly replacing them by $y_I - x_2 f$ one makes them satisify the relations Equation (8.18) and Equation (8.19). The same argument as in Theorem 7.16 also shows that these are the only relations. □

Proof of Theorem 1.5 for PSO_{4m+2}. The conclusion follows by comparing the descriptions given by Theorem 8.17 and [Tod87, Proposition 4.5]. □

9. Applications to representation theory

In this section, we reinterpret our computations in terms of representation theory. Let k be a field, G be a connected reductive k-group, Γ be a central subgroup of G, and $\overline{G} \coloneqq G/\Gamma$ be the adjoint quotient. Recall that by Totaro's work [Tot18, Corollary 2.2] one has the following interpretation of the Hodge cohomology of $\overline{G}$: for all $i, j \geq 0$ we have

$$H^{i,j}(B\overline{G}/k) \xrightarrow{\sim} H^{j-i}(\overline{G}, \mathrm{Sym}^i \overline{\mathfrak{g}}^\vee),$$

where $\overline{\mathfrak{g}}$ is the Lie algebra of $\overline{G}$. Note that $\overline{\mathfrak{g}}$ is also a G-module via the projection $G \twoheadrightarrow \overline{G}$. Since Γ is of multiplicative type, the functor of Γ-invariants is exact, hence the Hochschild-Serre spectral sequence for $1 \to \Gamma \to G \to \overline{G} \to 1$ provides isomorphisms

$$H^*(G, \mathrm{Sym}^i \overline{\mathfrak{g}}^\vee) \xrightarrow{\sim} H^*(\overline{G}, \mathrm{Sym}^i \overline{\mathfrak{g}}^\vee).$$

Altogether, this shows the following: $H^{i,j}(B\overline{G}/k) \simeq 0$ if $i > j$, the "pure" part $\oplus_i H^{i,i}(B\overline{G}/k)$ of Hodge cohomology is isomorphic to the algebra

$$H^0(G, \mathrm{Sym}^* \overline{\mathfrak{g}}^\vee) \simeq (\mathrm{Sym}^* \overline{\mathfrak{g}}^\vee)^G,$$

and the "non-pure" part $\oplus_{i \neq j} H^{i,j}(B\overline{G}/k)$ is given by the higher cohomology $H^{>0}(G, \mathrm{Sym}^* \overline{\mathfrak{g}}^\vee)$.

Using the computations of Hodge cohomology that we made in previous sections we will analyze this picture in the case when $k = \mathbb{F}_2$, G is GL_{4m+2}, SO_{4m+2} or Sp_{4m+2}, and Γ is the center of G. Recall that for each of the G under consideration, the Eilenberg-Moore spectral sequence in Hodge cohomology for $1 \to \Gamma \to G \to \overline{G} \to 1$ degenerates at the E_2 page. Thus, in order to calculate the dimensions of $H^{i,j}(B\overline{G}/k)$, it will be enough to compute the dimension of the corresponding (see Remark 9.1) bigraded component on the E_2 page. Following the proofs of Theorems 7.16 and 8.17 for PGL_{4m+2} and PSO_{4m+2}, the E_2 page is in fact isomorphic to Hodge cohomology as a bigraded algebra (so we can also understand the multiplicative structure on $H^*(G, \mathrm{Sym}^* \overline{\mathfrak{g}}^\vee)$ this way).

Remark 9.1. Recall that (assuming the degeneration of the Eilenber-Moore spectral sequence) a homogeneous class

$$x \in (\mathrm{Cotor}^i_{H^{*,*}_{\mathrm{H}}(B\Gamma)}(\mathbb{F}_2, H^{*,*}_{\mathrm{H}}(BG/\mathbb{F}_2)))^{h,j} \simeq (E^{i,j}_\infty)^h$$

gives a class in $\mathrm{gr}_i(H^{h,i+j}_{\mathrm{H}}(B\overline{G}/\mathbb{F}_2))$. Thus the bigrading we are interested in is given by $(h, i+j)$. We will call it *Hodge bigrading* from now on and will denote it by $|x|_{\mathrm{H}} \in \mathbb{Z}^2$.

9.1. **Projective linear group.** When $\overline{G} = \mathrm{PGL}_n$, the representation in question is $\mathfrak{pgl}_n^\vee$. We have a short exact sequence of GL_n-modules

$$0 \to \mathbb{F}_2 \to \mathfrak{gl}_n \to \mathfrak{pgl}_n \to 0,$$

giving a short exact sequence

$$0 \to \mathfrak{pgl}_n^\vee \to \mathfrak{gl}_n^\vee \to \mathbb{F}_2 \to 0.$$

Remark 9.2. When n is even this short exact sequence is non-split. Indeed, any such splitting would induce a Lie algebra direct sum decomposition of $\mathfrak{gl}_n$ as $\mathbb{F}_2 \oplus [\mathfrak{gl}_n, \mathfrak{gl}_n]$. However, when n is even one has $\mathbb{F}_2 \subset [\mathfrak{gl}_n, \mathfrak{gl}_n]$. Indeed, if $n = 2$ then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right],$$

and the general $n = 2r$ case reduces to this one by considering the analogous block-diagonal matrices (with r blocks of size 2).

Remark 9.3. We also note that if n is even the representations $\mathfrak{pgl}_n$ and $\mathfrak{pgl}_n^\vee$ are not irreducible. Indeed, the trace function $\mathrm{tr}\colon \mathfrak{gl}_n \to \mathbb{F}_2$ is GL_n-invariant and is 0 on scalars $\mathbb{F}_2 \subset \mathfrak{gl}_n$ and so defines a (non-zero) map $\mathfrak{pgl}_n \to \mathbb{F}_2$. Its kernel, however, is an irreducible GL_n-module.

By [Tot18, Theorem 9.1], the higher cohomology of GL_n with coefficients in $\mathrm{Sym}^j \mathfrak{gl}_n^\vee$ is 0. In contrast, for even n, due to the non-splitness of the above short exact sequence, the higher cohomology of $\mathrm{Sym}^j \mathfrak{pgl}_n^\vee$ become quite complicated. Our computation (Theorem 8.17) of Hodge cohomology of $B\mathrm{PGL}_{4m+2}$ allows to describe it fully in the case $n = 4m+2$.

Recall that the E_2 page for PGL_{4m+2} has been computed in Lemma 7.5 as

$$PH^*_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2) \;\oplus\; z_3\mathbb{F}_2[z_3] \otimes \mathbb{F}_2[c_1, b_2, \ldots, b_{2m+1}],$$

where $PH^*_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)$ is the 0-th column $E_2^{0,*}$ and $z_3\mathbb{F}_2[z_3] \otimes \mathbb{F}_2[c_1, b_2, \ldots, b_{2m+1}]$ gives the rest. The Hodge-bidegrees here are given as follows: $PH^{*,*}_{\mathrm{H}}(B\mathrm{GL}_{4m+2}/\mathbb{F}_2)$ is pure, $|z_3|_{\mathrm{H}} = (1,2)$, $|c_1|_{\mathrm{H}} = (1,1)$ and $|b_h|_{\mathrm{H}} = (4h, 4h)$. Therefore, we have an isomorphism

$$H^{>0}(\mathrm{GL}_{4m+2}, \mathrm{Sym}^* \mathfrak{pgl}_{4m+2}^\vee) \simeq z_3\mathbb{F}_2[z_3] \otimes \mathbb{F}_2[c_1, b_2, \ldots, b_{2m+1}].$$

where $c_1, b_2, \ldots, b_{2m+1} \in H^0(\mathrm{GL}_{4m+2}, \mathfrak{pgl}_{4m+2}^\vee) \simeq (\mathrm{Sym}^* \mathfrak{pgl}_{4m+2}^\vee)^{\mathrm{GL}_{4m+2}}$ are certain invariant polynomials of degrees $1, 4, 8, \ldots, 8m+4$ (and which can be explicitly understood via Lemma 7.1) and[11] $z_3 \in H^1(\mathrm{GL}_{4m+2}, \mathfrak{pgl}_{4m+2}^\vee)$. This way we get that $H^i(\mathrm{GL}_{4m+2}, \mathrm{Sym}^j \mathfrak{pgl}_{4m+2}^\vee)$ has a basis

[11] This class is exactly the one that classifies the non-split extension $0 \to \mathfrak{pgl}_n^\vee \to \mathfrak{gl}_n^\vee \to \mathbb{F}_2 \to 0$..

consisting of monomials of the form $z_3^{i-j}f$, where f is a monomial in $c_1, b_2, \dots, b_{2m+1}$ of total degree $j-i$. Therefore $\dim_{\mathbb{F}_2} H^i(\mathrm{GL}_{4m+2}, \mathrm{Sym}^j\, \mathfrak{pgl}_{4m+2}^\vee)$ equals to the number of ways to write $j-i$ as a sum

$$\gamma_1 + 4\beta_2 + 8\beta_3 + \dots + (8m+4)\beta_{2m+1},$$

where γ_1 and the β_h are non-negative integers.

9.2. **Projective orthogonal group.** When $\overline{G} = \mathrm{PSO}_{4m+2}$, the representation in question is $\mathfrak{pso}_{4m+2}^\vee$. Since μ_2 is not smooth, the "Lie algebra" of μ_2 is in fact a complex, namely the dual $\mathbb{L}^\vee_{\mu_2/\mathbb{F}_2}$ to the cotangent complex $\mathbb{L}_{\mu_2/\mathbb{F}_2}$. We have $H^0(\mathbb{L}^\vee_{\mu_2/\mathbb{F}_2}) = H^1(\mathbb{L}^\vee_{\mu_2/\mathbb{F}_2}) = \mathbb{F}_2$. We have a fiber sequence $\mathbb{L}^\vee_{\mu_2/\mathbb{F}_2} \to \mathfrak{g} \to \overline{\mathfrak{g}}$ in the derived category of G-modules (where G acts trivially on $\mathbb{L}^\vee_{\mu_2/\mathbb{F}_2}$) which gives an exact sequence of SO_{4m+2}-modules

$$0 \to \mathbb{F}_2 \to \mathfrak{so}_{2r} \to \mathfrak{pso}_{2r} \to \mathbb{F}_2 \to 0 \tag{9.4}$$

as the long exact sequence of cohomology. However, for odd r the first map $\mathbb{F}_2 \to \mathfrak{so}_{4m+2}$ is in fact split, as the next lemma shows.

Lemma 9.5. *The Lie algebra $\mathfrak{so}_{4m+2}$ over $\mathbb{F}_2$ splits as $\mathbb{F}_2 \oplus \mathfrak{l}$ where $\mathfrak{l} \simeq [\mathfrak{so}_{4m+2}, \mathfrak{so}_{4m+2}]$. This splitting is preserved by the SO_{4m+2}-action and gives the decomposition of $\mathfrak{so}_{4m+2}$ into a sum of simple representations.*

Proof. By [Hog82, Table 1][12] we have that the center $\mathfrak{z}(\mathfrak{so}_{4m+2})$ and $\mathfrak{l} := [\mathfrak{so}_{4m+2}, \mathfrak{so}_{4m+2}]$ are the only non-trivial Lie ideals in $\mathfrak{so}_{4m+2}$. In particular, $\mathfrak{z}(\mathfrak{so}_{4m+2})$ is 1-dimensional and is exactly given by the image of $\mathbb{F}_2$ under the above map. Thus we only need to check that $\mathfrak{z}(\mathfrak{so}_{4m+2})$ doesn't belong to $\mathfrak{l}$. Let's identify a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{so}_{4m+2}$ with $X_*(T) \otimes \mathbb{F}_2$ (where $X_*(T)$ are cocharacters of the maximal torus corresponding to $\mathfrak{h}$). E.g. by [Hog82, Section 1] the intersection of $[\mathfrak{so}_{4m+2}, \mathfrak{so}_{4m+2}]$ with the Cartan subalgebra $\mathfrak{h}$ is given by the image of coroot lattice $R^\vee \otimes \mathbb{F}_2 \to X_*(T) \otimes \mathbb{F}_2$. In the standard basis for $X_*(T)$ (dual to what is usually denoted denoted $\varepsilon_1, \dots, \varepsilon_{2m+1} \in X^*(T)$) the center $\mathfrak{z}(\mathfrak{so}_{4m+2})$ is spanned by the vector $(1, 1, \dots, 1, 1)$, while the image of $R^\vee \otimes \mathbb{F}_2$ is described as the kernel of the sum-of-coordinates map $\mathbb{F}_2^{\oplus 2m+1} \xrightarrow{\Sigma} \mathbb{F}_2$. Since $2m+1$ is odd we see that $(1, 1, \dots, 1, 1)$ doesn't belong to $\ker(\Sigma)$, and so $\mathfrak{so}_{4m+2} \simeq \mathbb{F}_2 \oplus \mathfrak{l}$. The adjoint action of SO_{4m+2} preserves both the center and the commutator and so respects this decomposition. It remains to show that $\mathfrak{l}$ is irreducible. But, any SO_{4m+2}-invariant subspace would in particular give a Lie ideal in $\mathfrak{l}$, of which there aren't any by [Hog82, Table 1] again. □

Remark 9.6. Essentially by the definition of roots, the highest weight for $\mathfrak{l}$ is the longest root $\theta \in X_*(T)$ of SO_{4m+2}. Since $\mathfrak{l}$ is irreducible we have $\mathfrak{l} = L(\theta)$. Recall that the highest weight of the dual $L(\lambda)^\vee$ is $-w_0\lambda$ where w_0 is the longest element of the Weyl group, and so $L(\lambda)^\vee \simeq L(-w_0\lambda)$. Since $w_0\theta = -\theta$ we get that $L(\theta)^\vee \simeq L(\theta)$ and so $\mathfrak{l}$ is self-dual.

Note that by Totaro's computation $H^{1,2}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2) \simeq \mathbb{F}_2 \cdot u_3$ and that

$$H^{1,2}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2) \simeq H^1(\mathrm{SO}_{4m+2}, \mathfrak{so}_{4m+2}^\vee) \simeq H^1(\mathrm{SO}_{4m+2}, \mathfrak{l}^\vee) \oplus H^1(\mathrm{SO}_{4m+2}, \mathbb{F}_2).$$

[12] Note that we are in the D_ℓ-type ℓ odd, intermediate case, in the notations of *loc.cit.*

Since $H^1(\mathrm{SO}_{4m+2}, \mathbb{F}_2) \simeq 0$ (either by Kempf vanishing or comparing with $H^{0,1}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2)$) we get

$$H^1(\mathrm{SO}_{4m+2}, \mathfrak{l}) \simeq H^1(\mathrm{SO}_{4m+2}, \mathfrak{l}^\vee) \simeq \mathbb{F}_2.$$

Now, from Lemma 9.5 and (9.4) we get a short exact sequence of SO_{4m+2}-modules:

$$0 \to \mathfrak{l} \to \mathfrak{pso}_{4m+2} \to \mathbb{F}_2 \to 0.$$

Lemma 9.7. *This extension is non-split.*

Proof. Due to SO_{4m+2}-equivariance, the map $\mathfrak{pso}_{4m+2} \to \mathbb{F}_2$ is in fact a map of Lie algebras and a splitting would also give a decomposition of the Lie algebra $\mathfrak{pso}_{4m+2}$ as $\mathbb{F}_2 \oplus \mathfrak{l}$. However, the center $\mathfrak{z}(\mathfrak{pso}_{4m+2})$ is trivial (again, see [Hog82, Table 1]), and so this is impossible. □

As a consequence, we get that the SO_{4m+2}-representation $\mathfrak{pso}_{4m+2}$ is the unique non-zero class in $\mathrm{Ext}^1_{\mathrm{SO}_{4m+2}}(\mathbb{F}_2, \mathfrak{l}) \simeq H^1(\mathrm{SO}_{4m+2}, \mathfrak{l})$. This class necessarily corresponds to the (unique non-zero) class $u_3 \in H^{1,2}_{\mathrm{H}}(B\mathrm{SO}_{4m+2}/\mathbb{F}_2)$.

Passing to linear duals, we get a short (non-split) exact sequence

$$0 \to \mathbb{F}_2 \to \mathfrak{pso}^\vee_{4m+2} \to \mathfrak{l} \to 0.$$

Since $\mathfrak{l}$ is self-dual, by the above discussion we also know that such a non-split extension is unique.

Remark 9.8. Using [Hog82, Table 1] one can see that the Lie algebra $\mathfrak{spin}_{4m+2}$ is also a non-split extension of $\mathfrak{l}$ by $\mathbb{F}_2$. By uniqueness, we get that the SO_{4m+2}-representations $\mathfrak{pso}^\vee_{4m+2}$ and $\mathfrak{spin}_{4m+2}$ are isomorphic.

Let us now compute the cohomology of $\mathrm{Sym}^* \mathfrak{pso}^\vee_{4m+2}$. By Lemma 8.10 the E_2 page of the Eilenberg-Moore spectral sequence is given by

$$PH^*_{\mathrm{H}}(B\mathrm{SO}_{4m+2}) \oplus z_2\mathbb{F}_2[z_2] \otimes \mathbb{F}_2[b_2, \dots, b_{2m+1}],$$

with $PH^*_{\mathrm{H}}(B\mathrm{SO}_{4m+2})$ being the 0-th column $E_2^{0,*}$ and $z_2\mathbb{F}_2[z_2] \otimes \mathbb{F}_2[b_2, \dots, b_{2m+1}] \simeq E_2^{>0,*}$ being the rest of E_2. The Hodge bigradings here are given by $|z_2|_{\mathrm{H}} = (1,1)$ and $|b_h|_{\mathrm{H}} = (2h, 2h)$. It follows that the non-pure part of the Hodge cohomology in fact lies in $E_2^{0,*}$ and embeds in the non-pure part of Hodge cohomology of $B\mathrm{SO}_{4m+2}$.

On the level of representations we get the following: let $q^\vee \colon \mathrm{Sym}^* \mathfrak{pso}^\vee_{4m+2} \to \mathrm{Sym}^* \mathfrak{so}^\vee_{4m+2}$ be the natural map induced by $\mathfrak{pso}^\vee_{4m+2} \to \mathfrak{so}^\vee_{4m+2}$. Then $q^\vee$ induces an embedding

$$H^{>0}(\mathrm{SO}_{4m+2}, \mathrm{Sym}^* \mathfrak{pso}^\vee_{4m+2}) \hookrightarrow H^{>0}(\mathrm{SO}_{4m+2}, \mathrm{Sym}^* \mathfrak{so}^\vee_{4m+2}).$$

Moreover, the image can be described fairly explicitly. Namely, in the notations of Lemma 8.5, $H^{>0}(\mathrm{SO}_{4m+2}, \mathrm{Sym}^* \mathfrak{pso}^\vee_{4m+2})$ can be identified with the ideal generated by non-pure elements $\overline{u}_{2k+1}$ and y_I inside $PH^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2})$. It can be also seen as the intersection of the ideal $(u_3, u_5, \dots, u_{4m+1}) \subset H^{*,*}_{\mathrm{H}}(\mathrm{SO}_{4m+2})$ (which is isomorphic to $H^{>0}(B\mathrm{SO}_{4m+2}, \mathrm{Sym}^*(\mathfrak{so}^\vee_{4m+2}))$ by Totaro's computation [Tot18, Theorem 11.1]) and the subalgebra $PH^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2})$ of primitive elements. This reduces the problem of computing $H^i(B\mathrm{SO}_{4m+2}, \mathrm{Sym}^j(\mathfrak{pso}^\vee_{4m+2}))$ to a much simpler linear algebra computation: namely one just needs to take the corresponding bigraded component in $(u_3, u_5, \dots, u_{4m+1}) \subset H^{*,*}_{\mathrm{H}}(B\mathrm{SO}_{4m+2})$ and compute the intersection of kernels of d_i's (from Construction 6.13) for all $i > 0$.

9.3. **Projective symplectic group.** Assume now that $\overline{G} = \mathrm{PSp}_{4m+2}$. Similarly to the case of PSO_n we have an exact sequence of PSp_{4m+2}-modules

$$0 \to \mathbb{F}_2 \to \mathfrak{sp}_{4m+2} \to \mathfrak{psp}_{4m+2} \to \mathbb{F}_2 \to 0.$$

which then gives an exact sequence

$$0 \to \mathbb{F}_2 \to \mathfrak{psp}_{4m+2}^\vee \to \mathfrak{sp}_{4m+2}^\vee \to \mathbb{F}_2 \to 0 \tag{9.9}$$

by passing to duals.

Remark 9.10. In contrast to the SO_{4m+2}-case, the map $\mathbb{F}_2 \to \mathfrak{sp}_{4m+2}$ is non-split. Indeed, coroots of Sp_{4m+2} span the coweight lattice $X^*(T)$, which forces the center $\mathbb{F}_2$ to lie in the commutator $[\mathfrak{sp}_{4m+2}, \mathfrak{sp}_{4m+2}]$. Also, the quotient $\mathfrak{sp}_{4m+2}/\mathbb{F}_2$ is not irreducible.

For brevity, we only sketch how to compute the second sheet of the Eilenberg-Moore spectral sequence in this case. Using Lemma 4.7 we can explicitly identify the coaction of $H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)$ on $H^*_{\mathrm{H}}(B\mathrm{Sp}_{4m+2}/\mathbb{F}_2)$ with the coaction of $H^*_{\mathrm{sing}}(B\mathbb{Z}/2, \mathbb{F}_2)$ on $H^*_{\mathrm{sing}}(B\mathrm{Sp}_{4m+2}(\mathbb{C}), \mathbb{F}_2)$. This identification then also induces an isomorphism of the primitive parts $PH^*_{\mathrm{H}}(B\mathrm{Sp}_{4m+2}/\mathbb{F}_2) \simeq PH^*_{\mathrm{sing}}(B\mathrm{Sp}_{4m+2}(\mathbb{C}), \mathbb{F}_2)$. Moreover the computations of Cotor using the twisted cochains (as in Corollary 5.19) are also compatible, which allows to identify the algebra structures on

$$\mathrm{Cotor}^*_{H^*_{\mathrm{H}}(B\mu_2/\mathbb{F}_2)}(\mathbb{F}_2, H^*_{\mathrm{H}}(B\mathrm{Sp}_{4m+2}/\mathbb{F}_2)) \quad \text{and} \quad \mathrm{Cotor}^*_{H^*_{\mathrm{sing}}(B\mathbb{Z}/2, \mathbb{F}_2)}(\mathbb{F}_2, H^*_{\mathrm{sing}}(B\mathrm{Sp}_{4m+2}, \mathbb{F}_2))$$

via Lemma 5.21 and Lemma 5.21.

This gives an identification

$$E_2^{*,*} \simeq (\mathbb{F}_2[z_2, z_3] \otimes PH^*_{\mathrm{H}}(B\mathrm{Sp}_{4m+2}/\mathbb{F}_2)) + \mathbb{F}_2[z_2, z_3, z_5, b_2, b_3, \dots, b_{2m+1}],$$

where $b_h \in PH^{16h}_{\mathrm{H}}(B\mathrm{Sp}_{4m+2}/\mathbb{F}_2)$ are certain elements defined similarly to Lemma 7.1 (or [Tod87, Lemma 3.10]) using the $*$-product of Construction 6.6 for $a_\sharp \coloneqq q_2$. However, the twisted tensor product construction of (5.5) is naturally bigraded, which allows to compute the Hodge bidegrees (see Remark 9.1) for $E_2^{*,*}$. Namely, $|z_2|_{\mathrm{H}} = (1,1)$, $|z_3|_{\mathrm{H}} = (1,2)$, $|z_5|_{\mathrm{H}} = (2,3)$ and $|b_h|_{\mathrm{H}} = (8h, 8h)$ (more generally, all elements in $PH^*_{\mathrm{H}}(B\mathrm{Sp}_{4m+2}/\mathbb{F}_2)$ are pure).

Remark 9.11. There is another way to understand the bigradings of z_i, by computing the Hodge cohomology of $B\mathrm{PSp}_{4m+2}$ in low degrees directly. Namely, let E be the Lagrangian Grassmannian, that is, $E \coloneqq \mathrm{PSp}_{4m+2}/P$ where $P \subset \mathrm{PSp}_{4m+2}$ is the standard right-end root parabolic subgroup with Levi subgroup isomorphic to GL_{2m+1}; see [Pra91, Proposition 6.1]. By [Tot18, Proposition 9.3] we have a spectral sequence

$$E_2^{i,j} \coloneqq H^i_{\mathrm{H}}(B\mathrm{PSp}_{4m+2}/\mathbb{F}_2) \otimes H^j_{\mathrm{H}}(E/\mathbb{F}_2) \Rightarrow H^*_{\mathrm{H}}(B\mathrm{GL}_{2m+1}/\mathbb{F}_2). \tag{9.12}$$

The bigraded ring $H^{*,*}_{\mathrm{H}}(E/\mathbb{F}_2)$ is well understood, as we now explain. By [Tot18, Proposition 7.1], the cycle class map

$$CH^*(E) \otimes_{\mathbb{Z}} \mathbb{F}_2 \longrightarrow H^*_{\mathrm{H}}(E/\mathbb{F}_2)$$

is an isomorphism. In particular, $H^{i,j}_{\mathrm{H}}(E/\mathbb{F}_2) = 0$ unless $i = j$. The Chow group $CH^*(E)$ is torsion-free and can be computed using the cell decomposition; see [Pra91, Corollary 6.3].

Using (9.12) and the above description of $H^*_{\mathrm{H}}(E/\mathbb{F}_2)$, low-degree computations analogous to those in Lemma 7.10 show that

$$H^i_{\mathrm{H}}(B\mathrm{PSp}_{4m+2}/\mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{if } i=0 \\ 0 & \text{if } i=1 \\ \mathbb{F}_2\langle x_2\rangle & \text{if } i=2 \\ \mathbb{F}_2\langle x_3\rangle & \text{if } i=3 \\ \mathbb{F}_2\langle x_2^2\rangle & \text{if } i=4 \\ \mathbb{F}_2\langle x_2x_3, x_5\rangle & \text{if } i=5. \end{cases} \tag{9.13}$$

Here x_2 has bidegree $(1,1)$, x_3 has bidegree $(1,2)$ and x_5 has bidegree $(2,3)$.

Returning to the computation of dimensions of Hodge cohomology, we have that the E_∞ page of the Eilenberg-Moore spectral sequence is isomorphic to

$$(\mathbb{F}_2[z_2,z_3]\otimes PH^{*,*}_{\mathrm{H}}(B\mathrm{Sp}_{4m+2}/\mathbb{F}_2))\oplus\mathbb{F}_2[z_2,z_3,z_5,b_2,b_3,\dots,b_{2m+1}].$$

From the representation-theoretic point of view, we get the following: there are two classes $x_3\in H^1(\mathrm{Sp}_{4m+2},\mathfrak{psp}^\vee_{4m+2})$, $x_5\in H^1(\mathrm{Sp}_{4m+2},\mathrm{Sym}^2\,\mathfrak{psp}^\vee_{4m+2})$, such that the higher cohomology $H^{>0}(\mathrm{Sp}_{4m+2},\mathrm{Sym}^*\,\mathfrak{psp}^\vee_{4m+2})$ is generated by the ideal

$$(x_3,x_5)\subset\mathbb{F}_2[x_3,x_5]\subset H^{>0}(\mathrm{Sp}_{4m+2},\mathrm{Sym}^*\,\mathfrak{psp}^\vee_{4m+2})$$

as a module over the invariants $(\mathrm{Sym}^*\,\mathfrak{psp}^\vee_{4m+2})^{\mathrm{Sp}_{4m+2}}$. Moreover,

- $(\mathrm{Sym}^*\,\mathfrak{psp}^\vee_{4m+2})^{\mathrm{Sp}_{4m+2}}\otimes x_3\cdot\mathbb{F}_2[x_3]$ embeds into $H^{>0}(\mathrm{Sp}_{4m+2},\mathrm{Sym}^*\,\mathfrak{psp}^\vee_{4m+2})$ via the action map;
- the cokernel of the above map can be described as $x_5\cdot\mathbb{F}_2[x_2,x_3,x_5,b_2,\dots,b_{2m+1}]$, where $x_2\in(\mathfrak{psp}^\vee_{4m+2})^{\mathrm{Sp}_{4m+2}}$ and $b_h\in(\mathrm{Sym}^{8h}\,\mathfrak{psp}^\vee_{4m+2})^{\mathrm{Sp}_{4m+2}}$ are fairly explicit invariant polynomials.[13]

Remark 9.14. The extension class given by

$$x_3\in\mathrm{Ext}^1_{\mathrm{Sp}_{4m+2}}(\mathbb{F}_2,\mathfrak{psp}^\vee_{4m+2})\simeq H^1(\mathrm{Sp}_{4m+2},\mathfrak{psp}^\vee_{4m+2})\simeq H^{1,2}_{\mathrm{H}}(B\mathrm{PSp}_{4m+2}/\mathbb{F}_2)$$

can be described explicitly. Indeed, the exact sequence (9.9) gives a class in $\mathrm{Ext}^2_{\mathrm{Sp}_{4m+2}}(\mathbb{F}_2,\mathbb{F}_2)$, which is necessarily 0, since $\mathrm{Ext}^2_{\mathrm{Sp}_{4m+2}}(\mathbb{F}_2,\mathbb{F}_2)\simeq H^2(\mathrm{Sp}_{4m+2},\mathbb{F}_2)\simeq H^{2,0}_{\mathrm{H}}(B\mathrm{Sp}_{4m+2}/\mathbb{F}_2)\simeq 0$. Thus, (9.9) comes from some Sp_{4m+2}-representation V with a two-step filtration $0\subset V_1\subset V_2\subset V$ and such that $V_1\simeq\mathbb{F}_2$, $V_2\simeq\mathfrak{psp}^\vee_{4m+2}$, $V/V_1\simeq\mathfrak{sp}_{4m+2}$ and $V/V_2\simeq\mathbb{F}_2$. In particular, V fits into a short exact sequence

$$0\to\mathfrak{psp}^\vee_{4m+2}\to V\to\mathbb{F}_2\to 0,$$

giving a class $[V]\in\mathrm{Ext}^1_{\mathrm{Sp}_{4m+2}}(\mathbb{F}_2,\mathfrak{psp}^\vee_{4m+2})$. Moreover, one sees from Remark 9.10 that this extension is non-split and thus $[V]\neq 0$, which forces it to be equal to x_3 since

$$\mathrm{Ext}^1_{\mathrm{Sp}_{4m+2}}(\mathbb{F}_2,\mathfrak{psp}^\vee_{4m+2})\simeq H^{1,2}_{\mathrm{H}}(B\mathrm{PSp}_{4m+2}/\mathbb{F}_2)\simeq\mathbb{F}_2.$$

[13] In particular, $x_2\in(\mathfrak{psp}^\vee_{4m+2})^{\mathrm{Sp}_{4m+2}}$ is exactly the image of 1 under the map $\mathbb{F}_2\to\mathfrak{psp}^\vee_{4m+2}$ from Equation (9.9).

Appendix A. Künneth formula for de Rham cohomology

In this section we give a proof of the Künneth formula for de Rham cohomology in the context of Artin stacks. The generality we consider is bigger than what is necessary for the applications in the body of the paper: this doesn't really affect the proof and might be useful for a future use.

Let R be a base ring. We will work in the setting of higher Artin stacks (in the sense of [TV08, Section 1.3.3], see also [KP21b, Appendix A.1]) these are sheaves of spaces in étale topology on the site Aff_R of affine R-schemes, that admit a smooth $(n-1)$-representable atlas for some $n \neq 0$ (the latter being an inductively defined notion, see loc. cit. for more details).

Remark A.1. A classical stack $\mathcal{X}\colon \mathrm{Aff}_R \to \mathrm{Grpd}$ can be considered as a higher stack via composing with the nerve functor $N\colon \mathrm{Grpd} \to \mathrm{Spcs}$. The image of this functor can be identified with the subcategory spanned by higher stacks that take values in 1-truncated spaces $\mathrm{Spcs}_{\le 1} \hookrightarrow \mathrm{Spcs}$ (a space $X \in \mathrm{Spcs}$ is called 1-truncated if $\pi_i(X,x) = 0$ for $i > 1$ and any base point $x \in X$). See e.g. [Hol08].

Construction A.2. A useful fact ([Pri15, Theorem 4.7]) is that for any n-Artin stack $\mathcal{X}$ there exists an $(n-1)$-coskeletal smooth hypercover $X_\bullet \to X$, such that each X_i is a (possibly infinite) union of affine schemes. If we assume that $\mathcal{X}$ is smooth itself and, moreover, is quasi-compact and quasi-separated, the schemes X_i can be chosen to be smooth affine schemes.

Example A.3. The classical quotient stack $\mathcal{X} = [X/G]$ with X and G being smooth affine schemes over R is a smooth qcqs 1-Artin stack. In this case a hypercover as in Construction A.2 can be taken to be the Čech nerve of the smooth cover $X \to [X/G]$. We have $X_\bullet \simeq X \times G^{\times \bullet}$ with the standard maps.

Given a smooth higher Artin stack $\mathcal{X}$, one can consider its (relative) de Rham cohomology ([KP21a, Defintion 1.1.3]), defined as the homotopy limit

$$R\Gamma_{\mathrm{dR}}(\mathcal{X}/R) := \lim_{(S\to\mathcal{X})\in(\mathrm{Aff}^{\mathrm{sm}}_{/\mathcal{X}})^{\mathrm{op}}} R\Gamma_{\mathrm{dR}}(S/R).$$

This functor satisfies étale descent (which follows from the analogous statement for Hodge cohomology). Since smooth maps are étale surjections, for a smooth qcqs Artin stack $\mathcal{X}$ one gets a more economical formula in terms of a hypercover $|X_\bullet| \to \mathcal{X}$ as in Construction A.2:

$$R\Gamma_{\mathrm{dR}}(\mathcal{X}/R) \xrightarrow{\sim} \mathrm{Tot}\, R\Gamma_{\mathrm{dR}}(X_\bullet/R) \in D(\mathrm{Mod}_R)$$

the totalization[14] of the cosimplicial complex $R\Gamma_{\mathrm{dR}}(X_\bullet/R)$. Here, $R\Gamma_{\mathrm{dR}}(X_n/R) \in D(\mathrm{Mod}_R)$ is given by the usual de Rham complex

$$\Omega^0_{X_n/R} \to \Omega^1_{X_n/R} \to \Omega^2_{X_n/R} \to \dots,$$

[14] Or, in other words, $\lim_{[\bullet]\in\Delta}$.

and the totalization above can be computed by the means of the corresponding double-complex:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\Omega^2_{X_0/R} & \longrightarrow & \Omega^2_{X_1/R} & \longrightarrow & \Omega^2_{X_2/R} & \longrightarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
\Omega^1_{X_0/R} & \longrightarrow & \Omega^1_{X_1/R} & \longrightarrow & \Omega^1_{X_2/R} & \longrightarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
\Omega^0_{X_0/R} & \longrightarrow & \Omega^0_{X_1/R} & \longrightarrow & \Omega^0_{X_2/R} & \longrightarrow & \dots
\end{array}$$

In particular, in the case of Example A.3, $R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)$ agrees with the definition given by Totaro in [Tot18]. We also note that $R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)$ lies in[15] $D(\mathrm{Mod}_R)^{\geq 0}$ for any (smooth) $\mathcal{X}$.

We are now ready to prove the Künneth formula. We say that a ring R is of finite Tor-dimension if there exists $k \geq 0$ such that for any two (classical) modules $M, N \in \mathrm{Mod}_R$ their derived tensor product $M \otimes^{\mathbb{L}}_R N$ lies in cohomological degrees $\geq -k$.

Proposition A.4 (Künneth formula for de Rham cohomology)**.** *Let $\mathcal{X}$, $\mathcal{Y}$ be smooth qcqs higher Artin stacks over a base ring R that is of finite Tor-dimension. Then multiplication induces a natural equivalence*

$$R\Gamma_{\mathrm{dR}}(\mathcal{X}/R) \otimes^{\mathbb{L}}_R R\Gamma_{\mathrm{dR}}(\mathcal{Y}/R) \xrightarrow{\sim} R\Gamma_{\mathrm{dR}}(\mathcal{X} \times_R \mathcal{Y}/R).$$

Proof. The proof is analogous to [KP21b, Proposition 2.2.15], the main idea being to reduce to the case of affine schemes. Let $X_\bullet \to \mathcal{X}$ and $Y_\bullet \to \mathcal{Y}$ be hypercovers as in Construction A.2; this provides a hypercover $X_\bullet \times_R Y_\bullet \to X \times_R Y$ as well. We have

$$R\Gamma_{\mathrm{dR}}(\mathcal{X}/R) \otimes^{\mathbb{L}}_R R\Gamma_{\mathrm{dR}}(\mathcal{Y}/R) \xrightarrow{\sim} \mathrm{Tot}\,(R\Gamma_{\mathrm{dR}}(X_\bullet/R)) \otimes^{\mathbb{L}}_R R\Gamma_{\mathrm{dR}}(\mathcal{Y}/R).$$

Under the Tor-finiteness assumption on R, the derived tensor product $- \otimes^{\mathbb{L}}_R R\Gamma_{\mathrm{dR}}(\mathcal{Y}/R)$ is left t-exact up to a shift, and thus by [KP21b, Corollary 3.1.13] we can move $- \otimes^{\mathbb{L}}_R R\Gamma_{\mathrm{dR}}(\mathcal{Y}/R)$ inside the totalization:

$$\mathrm{Tot}\,(R\Gamma_{\mathrm{dR}}(X_\bullet/R)) \otimes^{\mathbb{L}}_R R\Gamma_{\mathrm{dR}}(\mathcal{Y}/R) \xrightarrow{\sim} \mathrm{Tot}\left(R\Gamma_{\mathrm{dR}}(X_\bullet/R) \otimes^{\mathbb{L}}_R R\Gamma_{\mathrm{dR}}(\mathcal{Y}/R)\right).$$

Now, using the hypercover $Y_\bullet \to \mathcal{Y}$ in the same way, we get

$$\mathrm{Tot}\left(R\Gamma_{\mathrm{dR}}(X_\bullet/R) \otimes^{\mathbb{L}}_R R\Gamma_{\mathrm{dR}}(\mathcal{Y}/R)\right) \xrightarrow{\sim} \lim_{[\bullet_1,\bullet_2]\in\Delta\times\Delta} R\Gamma_{\mathrm{dR}}(X_{\bullet_1}/R) \otimes^{\mathbb{L}}_R R\Gamma_{\mathrm{dR}}(Y_{\bullet_2}/R).$$

Since Δ is sifted, the limit over $\Delta \times \Delta$ can be computed after restriction to the diagonal $\Delta \xrightarrow{\mathrm{diag}} \Delta \times \Delta$, and we get an equivalence

$$\lim_{[\bullet_1,\bullet_2]\in\Delta\times\Delta} R\Gamma_{\mathrm{dR}}(X_{\bullet_1}/R) \otimes^{\mathbb{L}}_R R\Gamma_{\mathrm{dR}}(Y_{\bullet_2}/R) \xrightarrow{\sim} \lim_{[\bullet]\in\Delta} R\Gamma_{\mathrm{dR}}(X_\bullet/R) \otimes^{\mathbb{L}}_R R\Gamma_{\mathrm{dR}}(Y_\bullet/R).$$

[15] Here, $D(\mathrm{Mod}_R)^{\geq 0} \subset D(\mathrm{Mod}_R)$ is the full subcategory spanned by complexes M with $H^{-i}(M) = 0$ for any $i > 0$.

We then have a commutative diagram

$$\begin{array}{ccc} R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)\otimes^{\mathbb{L}}_R R\Gamma_{\mathrm{dR}}(\mathcal{Y}/R) & \longrightarrow & R\Gamma_{\mathrm{dR}}(\mathcal{X}\times_R\mathcal{Y}/R) \\ \downarrow\wr & & \downarrow\wr \\ \lim_{[\bullet]\in\Delta} R\Gamma_{\mathrm{dR}}(X_\bullet/R)\otimes^{\mathbb{L}}_R R\Gamma_{\mathrm{dR}}(Y_\bullet/R) & \longrightarrow & \lim_{[\bullet]\in\Delta} R\Gamma_{\mathrm{dR}}(X_\bullet\times_R Y_\bullet/R) \end{array}$$

where the horizontal arrows are induced by multiplication, while the vertical ones are induced by pull-backs. The left vertical map is an equivalence by the above discussion, while the right one is an equivalence by descent. By [Sta, Tag 0FMB] the maps

$$R\Gamma_{\mathrm{dR}}(X_\bullet/R)\otimes^{\mathbb{L}}_R R\Gamma_{\mathrm{dR}}(Y_\bullet/R) \longrightarrow R\Gamma_{\mathrm{dR}}(X_\bullet\times_R Y_\bullet/R)$$

are equivalences, thus so is the map between the limits. From the commutative diagram we then deduce the same for the upper horizontal map. ☐

Corollary A.5. *Let $R = k$ be a field and let $\mathcal{X}$, $\mathcal{Y}$ be smooth qcqs higher Artin stacks over k. Then multiplication induces a natural isomorphism of graded k-algebras*

$$H^*_{\mathrm{dR}}(\mathcal{X}/k)\otimes_k H^*_{\mathrm{dR}}(\mathcal{Y}/k) \xrightarrow{\sim} H^*_{\mathrm{dR}}(\mathcal{X}\times_k\mathcal{Y}/k).$$

Proof. This follows from Proposition A.4 by passing to cohomology. ☐

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