Characteristic Epsilon Cycles of ℓ-Adic Sheaves on Varieties

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Abstract

Let X be a smooth variety over a finite field k. Let ℓ be a prime number invertible in k. For an ℓ-adic sheaf $\mathcal{F}$ on X, we construct a cycle supported on the singular support of $\mathcal{F}$ whose coefficients are ℓ-adic numbers modulo roots of unity. This cycle refines the characteristic cycle $CC(\mathcal{F})$ in that it satisfies a Milnor-type formula for local epsilon factors. After establishing its fundamental properties, we prove a product formula of global epsilon factors modulo roots of unity. We also extend these results to varieties over the perfections of finitely generated fields.

1 Introduction

Let k be a finite field and let ℓ be a prime number invertible in k. Let X be a smooth projective variety over k, and let $\mathcal{F}$ be an object of the triangulated category $D_c^b(X, \overline{\mathbb{Z}_\ell})$. Throughout this article, by an ℓ-adic sheaf on X we mean an object of this category.

For an object $\mathcal{F} \in D_c^b(X, \overline{\mathbb{Z}_\ell})$, the L-function $L(X, \mathcal{F}; t)$ is defined as the infinite product

$$L(X, \mathcal{F}; t) = \prod_x \frac{1}{\det(1 - \mathrm{Frob}_x t^{\deg(x/k)}, \mathcal{F}_{\bar{x}} \otimes^L_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell})} \in \overline{\mathbb{Q}_\ell}[[t]],$$

where x runs over the closed points of X, $\deg(x/k)$ denotes the degree of the residue field extension $k(x)/k$, Frob_x denotes the geometric Frobenius at x, and $\mathcal{F}_{\bar{x}}$ denotes the stalk of $\mathcal{F}$ at a geometric point $\bar{x}$ lying above x. Using his theory of étale cohomology, Grothendieck showed that this function is indeed a polynomial, rather than merely a formal power series, and that it satisfies the functional equation

$$L(X, \mathcal{F}; t) = \varepsilon(X, \mathcal{F}) t^{-\chi(X_{\bar{k}}, \mathcal{F})} L(X, \mathbb{D}_X \mathcal{F}; t^{-1}),$$

where $\mathbb{D}_X$ denotes the Verdier duality functor. Let $\mathrm{H}^i := \mathrm{H}^i(X_{\bar{k}}, \mathcal{F} \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell})$, where $\bar{k}$ is an algebraic closure of k. Then the invariants $\chi(X_{\bar{k}}, \mathcal{F})$ and $\varepsilon(X, \mathcal{F})$ can be described as

$$\chi(X_{\bar{k}}, \mathcal{F}) = \sum_i (-1)^i \dim_{\overline{\mathbb{Q}_\ell}} \mathrm{H}^i, \quad \varepsilon(X, \mathcal{F}) = \prod_i \det(-\mathrm{Frob}_k, \mathrm{H}^i)^{(-1)^{i+1}}.$$

It is a long-standing problem in arithmetic geometry to express these invariants in terms of some invariants which can be computed locally on $(X, \mathcal{F})$. For the Euler characteristic

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$\chi(X_{\bar{k}}, \mathcal{F})$, Beilinson and T. Saito made significant progress on this problem. Beilinson [1] introduced the singular support $SS(\mathcal{F})$[1], and Saito [20] constructed the characteristic cycle $CC(\mathcal{F})$ as a cycle supported on $SS(\mathcal{F})$. They proved the formula

$$\chi(X_{\bar{k}}, \mathcal{F}) = (CC(\mathcal{F}), T^*_X X)_{T^*X}, \tag{1.1}$$

which expresses $\chi(X_{\bar{k}}, \mathcal{F})$ as an intersection number (we will return to this formula after Theorem 1.3 below). When X is a curve, formula (1.1) specializes to the Grothendieck–Ogg–Shafarevich formula.

The main subject of this article is the invariant $\varepsilon(X, \mathcal{F})$, called the *global epsilon factor*. When X is a smooth projective curve, the problem of describing this invariant was settled by Laumon [15, (3.2.1.1)]. He showed that the global epsilon factor decomposes as the product of the local epsilon factors of $\mathcal{F}$ defined at the closed points of X. This result is commonly known as the product formula of global epsilon factors.

The purpose of this article is to establish a higher-dimensional analogue of this formula for $\varepsilon(X, \mathcal{F})$ modulo roots of unity, following the strategy of Beilinson and Saito in their study of the Euler characteristic $\chi(X_{\bar{k}}, \mathcal{F})$.

In this article, we introduce a cycle $\mathcal{E}(\mathcal{F})$, called the *epsilon cycle* of $\mathcal{F}$. The cycle is supported on $SS(\mathcal{F})$ and has coefficients in a group of *ℓ-adic numbers modulo roots of unity*. We establish several fundamental properties of $\mathcal{E}(\mathcal{F})$. Among them, we obtain a higher-dimensional generalization of Laumon's product formula, valid up to roots of unity.

Let us describe the contents of this article more precisely. Set

$$\Theta := \overline{\mathbb{Z}_\ell}^\times \otimes_{\mathbb{Z}} \mathbb{Q},$$

which we regard as the group of ℓ-adic numbers modulo roots of unity. Let $SS(\mathcal{F}) = \cup_a C_a$ be the decomposition into irreducible components. Since $SS(\mathcal{F})$ is a closed subset of the cotangent bundle T^*X, each C_a is an irreducible closed subset of T^*X. Moreover, as observed by Beilinson in [1], each C_a has dimension equal to that of X. Therefore, $[C_a]$ defines an element of $Z_n(T^*X)$, the group of n-cycles on T^*X, where $n = \dim X$.

The epsilon cycle $\mathcal{E}(\mathcal{F})$ is defined as an element of the group $\Theta \otimes_{\mathbb{Z}} Z_n(T^*X)$, and can be written in the form

$$\mathcal{E}(\mathcal{F}) = \sum_a \xi_a \otimes [C_a]$$

with $\xi_a \in \Theta$. We write the group law of Θ multiplicatively, while that of $\Theta \otimes_{\mathbb{Z}} Z_n(T^*X)$ is written additively. Thus,

$$\xi \otimes [C] + \xi' \otimes [C] = \xi\xi' \otimes [C].$$

In the following, we describe several fundamental properties of $\mathcal{E}(\mathcal{F})$ established in this article.

First, we explain that $\mathcal{E}(\mathcal{F})$ satisfies a Milnor-type formula for local epsilon factors. As in the case of characteristic cycles, this property uniquely characterizes $\mathcal{E}(\mathcal{F})$, and we in fact take it as the definition of the epsilon cycle.

To state this result, we recall a basic property of $SS(\mathcal{F})$. Let $f\colon X \to \mathbb{A}^1_k$ be a function on X, and let df denote the differential of f, regarded as a section $X \to T^*X$. An

[1] More precisely, Beilinson defined the singular support for étale sheaves with finite coefficients. For an ℓ-adic sheaf $\mathcal{F}$, we define $SS(\mathcal{F})$ by reduction modulo ℓ. See Definition 2.17 for the precise construction.

important property of $SS(\mathcal{F})$ is that f is universally locally acyclic relative to $\mathcal{F}$ on the locus where df does not meet $SS(\mathcal{F}) \subset T^*X$.

Let $f\colon X \to \mathbb{A}^1_k$ be a function, and let $x \in X$ be a closed point such that df meets $SS(\mathcal{F})$ only at x. Following [20], we call such a point an *isolated* $SS(\mathcal{F})$*-characteristic point* of f. Then f is universally locally acyclic relative to $\mathcal{F}$ away from x, as recalled above. Consequently, the vanishing cycles complex of $(f, \mathcal{F})$ over the point $f(x) \in \mathbb{A}^1_k$ is supported at x; we denote it by $R\Phi_f(\mathcal{F})_x$. Let $\mathbb{A}^1_{k,(x)}$ be the unramified extension of the henselization $\mathbb{A}^1_{k,(f(x))}$ corresponding to the residue field extension $k(x)/k(f(x))$. Then $R\Phi_f(\mathcal{F})_x$ naturally defines a bounded complex of ℓ-adic representations of the fraction field of $\mathbb{A}^1_{k,(f(x))}$.

In the Milnor-type formula given below, we consider the local epsilon factor

$$\varepsilon_0(\mathbb{A}^1_{k,(x)}, R\Phi_f(\mathcal{F})_x, dt) \in \overline{\mathbb{Z}_\ell}^\times,$$

where t denotes the standard coordinate of the affine line. We follow the notation of [15, (3.1.5.6)] for local epsilon factors.

The following theorem gives the Milnor formula satisfied by $\mathcal{E}(\mathcal{F})$.

Theorem 1.1 (Theorem 4.10)**.** *Let X be a smooth variety over a finite field k. Let $\mathcal{F}$ be an object of $D^b_c(X, \overline{\mathbb{Z}_\ell})$. Write $SS(\mathcal{F}) = \bigcup_a C_a$ for the decomposition of the singular support into irreducible components. Then there exists a unique cycle*

$$\mathcal{E}(\mathcal{F}) = \sum_a \xi_a \otimes [C_a] \qquad \xi_a \in \Theta = \overline{\mathbb{Z}_\ell}^\times \otimes_{\mathbb{Z}} \mathbb{Q},$$

called the epsilon cycle, satisfying the following property. For any diagram of k-schemes

$$X \xleftarrow{j} U \xrightarrow{f} \mathbb{A}^1_k$$

with j étale, and any isolated $(SS(\mathcal{F}) \times_X U)$-characteristic point $u \in U$ of f, we have an equality

$$\varepsilon_0(\mathbb{A}^1_{k,(u)}, R\Phi_f(\mathcal{F})_u, dt)^{-1} = (\mathcal{E}(\mathcal{F}), df)_u^{\deg(u/k)} := \prod_a \xi_a^{\deg(u/k)\cdot(C_a, df)_u}$$

in Θ. Here $(C_a, df)_u$ denotes the intersection number of $j^*C_a := C_a \times_X U$ and df at u.

In particular, epsilon cycles can be computed étale locally on X.

We make the following technical remark on the definition of local epsilon factors. To define local epsilon factors, one must choose a non-trivial additive character of k. However, if a different character is chosen, the resulting local epsilon factor differs from the original one only by a root of unity. Hence the two coincide as elements of Θ. For this reason, we suppress the choice of additive character in the statement of the theorem and throughout the rest of the article.

The relationship between $CC(\mathcal{F})$ and $\mathcal{E}(\mathcal{F})$ is as follows. The characteristic cycle $CC(\mathcal{F})$ is an integral linear combination of the irreducible components C_a:

$$CC(\mathcal{F}) = \sum_a n_a [C_a],$$

where $n_a \in \mathbb{Z}$. By the compatibility of local epsilon factors with unramified twists, together with the characterizations of $\mathcal{E}(\mathcal{F})$ and $CC(\mathcal{F})$ via Milnor-type formulae, we obtain the following relation in $\Theta \otimes Z_n(T^*X)$:

$$\mathcal{E}(\mathcal{F}(1)) - \mathcal{E}(\mathcal{F}) = \sum_a q^{-n_a} \otimes [C_a], \tag{1.2}$$

where q denotes the cardinality of k and $\mathcal{F}(1)$ denotes the Tate twist (see Lemma 4.14.1 for a more general statement). Since $CC(\mathcal{F})$ can be recovered from the right-hand side of (1.2), the theory of epsilon cycles may be viewed as a refinement of the theory of characteristic cycles.

We briefly explain the proof of Theorem 1.1. We use the machinery developed in [20] for the construction of characteristic cycles. Roughly speaking, this machinery shows that, for an invariant defined on isolated characteristic points, if it varies "continuously" in families of such points, then there exists a cycle whose intersection numbers recover the invariant; see Proposition 2.13 for the precise statement.

To apply this machinery in our setting, we need a continuity result for local epsilon factors. In a separate paper [22], we prove that local epsilon factors indeed vary "continuously": more precisely, for any family of isolated characteristic points parametrized by a k-scheme S of finite type, there exists a continuous character

$$\pi_1(S)^{ab} \to \overline{\mathbb{Z}_\ell}^\times$$

whose values at geometric Frobenius elements are the corresponding local epsilon factors. For a precise statement, see [22, Theorem 4.9.2] or Theorem 4.5 below. By the finiteness theorem of Katz–Lang [13], such a character gives rise to a flat function (Definition 2.10) after passing to the quotient modulo roots of unity. This implies the existence of the cycle $\mathcal{E}(\mathcal{F})$ with the desired property.

After establishing basic properties of epsilon cycles, we prove a pullback formula for properly transversal morphisms, following an argument similar to Beilinson's treatment of $CC(\mathcal{F})$ in [20, Section 7].

Theorem 1.2 (Theorem 4.26)**.** *Let k be a finite field with q elements. Let $h\colon W \to X$ be a morphism of smooth k-schemes. Set $n = \dim X$ and $m = \dim W$. Let $\mathcal{F} \in D_c^b(X, \overline{\mathbb{Z}_\ell})$. Assume that h is properly $SS(\mathcal{F})$-transversal (Definition 2.1.2). Then we have an equality*

$$\mathcal{E}(h^*\mathcal{F}) = h^!\big(\mathcal{E}(\mathcal{F})(\frac{n-m}{2})\big)$$

in $\Theta \otimes Z_m(T^*W)$. Here, for $r \in \mathbb{Q}$, we define

$$\mathcal{E}(\mathcal{F})(r) := \mathcal{E}(\mathcal{F}) + \sum_a q^{-rn_a} \otimes [C_a] = \sum_a (\xi_a q^{-rn_a}) \otimes [C_a],$$

where the integers n_a and the elements $\xi_a \in \Theta$ are determined by

$$\mathcal{E}(\mathcal{F}) = \sum_a \xi_a \otimes [C_a], \qquad CC(\mathcal{F}) = \sum_a n_a [C_a].$$

Note that q^{-rn_a} is well-defined as an element of Θ.

The symbol $h^!$ denotes the pullback map for cycles supported on the singular support; see Definition 4.17 for the precise definition.

Finally, we state and prove a product formula for global epsilon factors.

Theorem 1.3 (Theorem 5.9)**.** *Let X be a smooth projective variety of dimension n over a finite field k. Let $\mathcal{F} \in D^b_c(X, \overline{\mathbb{Z}_\ell})$, and let $\mathcal{E}(\mathcal{F}) = \sum_a \xi_a \otimes [C_a]$ be its epsilon cycle. Then*

$$\prod_i \det\bigl(\mathrm{Frob}_k, \mathrm{H}^i(X_{\bar{k}}, \mathcal{F} \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell})\bigr)^{(-1)^i} = (\mathcal{E}(\mathcal{F}), T^*_X X)_{T^*X} := \prod_a \xi_a^{(C_a, T^*_X X)_{T^*X}}$$

as elements of $\overline{\mathbb{Z}_\ell}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$. Here $(C_a, T^*_X X)_{T^*X}$ denotes the degree of the 0-cycle class

$$[C_a] \cap [T^*_X X] \in \mathrm{CH}_0(T^*_X X) = \mathrm{CH}_0(X).$$

When X is a curve, the above formula recovers Laumon's product formula modulo roots of unity. This follows from the description of $\mathcal{E}(\mathcal{F})$ in the curve case given in Example 4.20.

Note that

$$\prod_i \det\bigl(\mathrm{Frob}_k, \mathrm{H}^i(X_{\bar{k}}, \mathcal{F}(1) \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell})\bigr)^{(-1)^i} = q^{-\chi(X_{\bar{k}}, \mathcal{F})} \cdot \prod_i \det\bigl(\mathrm{Frob}_k, \mathrm{H}^i(X_{\bar{k}}, \mathcal{F} \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell})\bigr)^{(-1)^i}.$$

Therefore, Theorem 1.3 is compatible with (1.1) and (1.2).

As a consequence of Theorem 1.3, we obtain a formula expressing the p-adic valuations of global epsilon factors in terms of those of local epsilon factors (Example 5.11). Let

$$\iota\colon \overline{\mathbb{Q}_\ell} \hookrightarrow \overline{\mathbb{Q}_p}$$

be a field embedding, where p is the characteristic of k. Then Theorem 1.3 yields the formula

$$\bigl|\iota\bigl(\det(\mathrm{Frob}_k, R\Gamma(X_{\bar{k}}, \mathcal{F}))\bigr)\bigr|_p = \prod_a |\iota(\xi_a)|_p^{\deg(C_a, T^*_X X)_{T^*X}},$$

where $|\cdot|_p$ denotes the p-adic absolute value on $\overline{\mathbb{Q}_p}$.

In [23], N. Umezaki, E. Yang, and Y. Zhao proved a twist formula for global epsilon factors [23, Theorem 5.23]. A weaker version modulo roots of unity can also be deduced from the theorem above together with Lemma 4.14.1.

More generally, we develop a theory of epsilon cycles when the base field k is the perfection of a finitely generated field over its prime field (in particular, the characteristic may be 0). In this setting, the coefficient group $\overline{\mathbb{Z}_\ell}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ is replaced by

$$\mathrm{Hom}(G_k, \overline{\mathbb{Z}_\ell}^\times) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where G_k denotes the absolute Galois group of k, and Hom denotes the group of continuous homomorphisms. When k is finite, we identify $\mathrm{Hom}(G_k, \overline{\mathbb{Z}_\ell}^\times)$ with $\overline{\mathbb{Z}_\ell}^\times$ via $\chi \mapsto \chi(\mathrm{Frob}_k)$. Under this identification, the results described above are recovered as the special case where k is finite.

To formulate the Milnor-type formula when k is the perfection of a finitely generated field, we need a theory of local epsilon factor in this general setting. When k is a perfect field of positive characteristic, we use the theory of local epsilon factors over general perfect fields, developed independently by Yasuda [25, 26] and Guignard [7]. Yasuda defines local epsilon factors for representations with torsion coefficients, whereas Guignard defines them for $\overline{\mathbb{Q}_\ell}$-representations. Since we work primarily with $\overline{\mathbb{Z}_\ell}$-representations, which can be realized as inverse limits of representations with torsion coefficients, we mainly follow Yasuda's approach. Guignard's theory could also be used, because the two constructions give the same local epsilon factor for a $\overline{\mathbb{Z}_\ell}$-representation V:

$$\varepsilon_0(T, V, \omega) = \varepsilon_0(T, V \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell}, \omega),$$

where the ε_0 on the left is defined using Yasuda's theory, whereas the one on the right is defined using Guignard's theory.

When the characteristic is 0, we define local epsilon factors for representations V with unramified determinant using the Jacobi sum characters constructed in [18]. For general V, we take a direct sum of copies of V so that its determinant becomes unramified. Consequently, in characteristic 0, we can define local epsilon factors only modulo roots of unity.

In [8], Guignard provides another method for computing global epsilon factors in higher dimensions, different in nature from the theory of epsilon cycles. It would be interesting to clarify the relationship between his results and ours.

We explain the organization of this paper. In Section 2, we recall the theory of relative singular support developed by Hu and Yang [9], which is a relative analogue of Beilinson's theory. We also review the machinery that will be used to establish the existence of epsilon cycles. In Section 3, we collect and prove the results on local epsilon factors needed for the construction of epsilon cycles. Section 4 is devoted to the construction of epsilon cycles and the proof of their basic properties.

In Section 5, we introduce the notion of epsilon classes, which are analogues of the characteristic classes defined in [20]. Using epsilon classes together with the formalism of Radon transforms, we prove the product formula for $\det(R\Gamma(X_{\bar{k}}, \mathcal{F}))$ in Theorem 5.9. We also give an axiomatic description of epsilon cycles in Theorem 5.12.

We fix notation and conventions that will be used throughout the paper.

- For a field k, we denote by G_k its absolute Galois group.
- We denote by $\chi_{\mathrm{cyc}}\colon G_k \to \mathbb{Z}_\ell^\times$ the ℓ-adic cyclotomic character.
- For a finite separable extension k'/k of fields, we denote by $\mathrm{tr}_{k'/k}\colon G_k^{ab} \to G_{k'}^{ab}$ the transfer morphism induced by the inclusion $G_{k'} \hookrightarrow G_k$. We denote by $\delta_{k'/k}$ the determinant character of the induced representation $\mathrm{Ind}_{G_{k'}}^{G_k} \mathbb{Q}_\ell$ of the trivial representation.
- For a scheme X and a point $x \in X$, we denote by $k(x)$ the residue field of X at x.
- For a finite extension x'/x of spectra of fields, we denote by $\deg(x'/x)$ the degree of the extension. When $x = \mathrm{Spec}(k)$ and $x' = \mathrm{Spec}(k')$, we also write $\deg(k'/k)$ for $\deg(x'/x)$.
- Let X be a scheme, and let $x \to X$ be a morphism from the spectrum of a field. We define the scheme $X_{(x)}$ as follows. Let $\mathcal{I}$ be the category of diagrams

$$x \to U \to X,$$

where U is an étale X-scheme and $x \to U$ is a lift of the given morphism $x \to X$. Define

$$X_{(x)} := \varprojlim_{U \in \mathcal{I}} U.$$

For example, if x is a geometric point of X, then $X_{(x)}$ is the strict henselization of X at x. If x is a point of X, then $X_{(x)}$ is the henselization of X at x.

- For a scheme X of finite type over S, we say that X is of relative dimension n over S if every fiber of $X \to S$ is equidimensional of dimension n.
- We fix an algebraic closure $\overline{\mathbb{Q}_\ell}$ of $\mathbb{Q}_\ell$. Let μ denote the group of roots of unity in $\overline{\mathbb{Q}_\ell}$. For a finite extension $E/\mathbb{Q}_\ell$, we denote by $\mathcal{O}_E$ the ring of integers of E.
- For the ℓ-adic formalism of sheaves on a noetherian topos T, we refer to [4]. We review the necessary material in the appendix. The derived category of constructible complexes of $\mathcal{O}_E$-sheaves on T is denoted by $D_c^b(T, \mathcal{O}_E)$.

Contents

2 Relative Singular Supports and Characteristic Cycles ([1], [9], [20])

In this section, we recall the theory of singular supports over general bases and that of characteristic cycles.

Let S be a noetherian scheme, and let Λ be a finite local ring whose characteristic is invertible in S. We denote by Λ_0 its residue field. Let X be an S-scheme of finite type. A complex K of étale sheaves of Λ-modules on X is said to be *constructible* if each cohomology sheaf $\mathcal{H}^i(K)$ is constructible and vanishes for all but finitely many integers i. We write $D_{\mathrm{ctf}}(X, \Lambda)$ for the full subcategory of $D(X, \Lambda)$ consisting of constructible complexes of finite tor-dimension.

2.1 Relative singular support

Singular support was first defined by Beilinson in [1] for smooth varieties over a field. Later, Hu and Yang [9] generalized these results to smooth schemes over a general noetherian scheme. In what follows, we briefly recall their theory; for details, we refer the reader to [9].

Let X be a smooth scheme over S. We write $T^*(X/S)$ for the relative cotangent bundle of X over S, and $T^*_X(X/S)$ for its zero section. When S is the spectrum of a field, we often simply write T^*X and $T^*_X X$, respectively.

For a morphism $x \to X$ from the spectrum of a field, $T^*_x(X/S)$ denotes the base change $T^*(X/S) \times_X x$. We say that a closed subset C of $T^*(X/S)$ is *conical* if C is stable under the natural action of $\mathbb{G}_m$ on the vector bundle $T^*(X/S)$.

First we recall the notions of C-transversality.

Definition 2.1. *Let X be a smooth scheme over S, and let C be a closed conical subset of $T^*(X/S)$.*

1. *([9, 2.4]) We say that an S-morphism $h\colon W \to X$ from a smooth S-scheme W is* C-transversal *if, for every geometric point $w \to W$, every non-zero element of $C \times_X w \subset T^*_{h(w)}(X/S) := T^*(X/S) \times_X w$ maps to a non-zero element of $T^*_w(W/S)$ under $dh_w\colon T^*_{h(w)}(X/S) \to T^*_w(W/S)$.*

2. *Assume that X and C are of relative dimension n over S. Let W be a smooth S-scheme of relative dimension m. We say that an S-morphism $h\colon W \to X$ is* properly C-transversal *if h is C-transversal in the sense of* 1 *and $W \times_X C$ is of relative dimension m over S.*

3. *([9, 2.4]) We say that an S-morphism $f\colon X \to Y$ to a smooth S-scheme Y is* C-transversal *if, for every geometric point $x \to X$, every non-zero element of $T^*_{f(x)}(Y/S) := T^*(Y/S) \times_Y x$ maps to an element of $T^*_x(X/S)$ outside $C \times_X x$ under $df_x\colon T^*_{f(x)}(Y/S) \to T^*_x(X/S)$.*

Lemma 2.2 (cf. [1, 1.2]). *Let $h\colon W \to X$ be a morphism of smooth S-schemes, and let C be a closed conical subset of $T^*(X/S)$. If h is C-transversal, then the map $dh\colon C \times_X W \to T^*(W/S)$ is finite.*

Proof. Replacing W by an open covering, we may assume that W is affine. Then the same argument as in the proof of [1, §1.2, Lemma (ii)] works in our setting. □

Definition 2.3. *Let X and C be as in Definition 2.1. Let W and Y be smooth S-schemes.*

1. *([9, 2.4]) Let $h\colon W \to X$ be a morphism of S-schemes. When h is C-transversal, we define $h^\circ C$ to be the image of $dh\colon C \times_X W \to T^*(W/S)$. By Lemma 2.2, this is a closed conical subset of $T^*(W/S)$.*
2. *([9, 2.4]) Let $f\colon X \to Y$ be a morphism of S-schemes. Assume that f is proper on the base $C \cap T^*_X(X/S)$ of C. We define*
$$f_\circ C = \mathrm{pr}_1(df^{-1}(C)),$$
where $df\colon T^*(Y/S) \times_Y X \to T^*(X/S)$ and $\mathrm{pr}_1\colon T^*(Y/S) \times_Y X \to T^*(Y/S)$. This is a closed conical subset of $T^*(Y/S)$.
3. *([9, 4.1]) Let (h, f) be a pair of S-morphisms*
$$X \xleftarrow{h} W \xrightarrow{f} Y.$$
*We say that (h, f) is C-*transversal *if h is C-transversal and f is $h^\circ C$-transversal, where $h^\circ C$ is as in* 1.

Definition 2.4 ([9, 4.2]). *Let X and C be as in Definition 2.1. Let $K \in D_{\mathrm{ctf}}(X, \Lambda)$. We say that K is micro-supported on C if, for every C-transversal pair (h, f) in the sense of Definition 2.3.3, f is locally acyclic relative to h^*K.*

Let $\Lambda \to \Lambda'$ be a local ring homomorphism between finite local rings. The following lemma shows that local acyclicity for $K \in D_{\mathrm{ctf}}(X, \Lambda)$ is equivalent to that for $K \otimes^L_\Lambda \Lambda'$.

Lemma 2.5. *Let $f\colon X \to Y$ be a morphism of schemes of finite type. Suppose that the characteristic of Λ is invertible in Y. Let $K \in D_{\mathrm{ctf}}(X, \Lambda)$. Then f is (resp. universally) locally acyclic relative to K if and only if it is (resp. universally) locally acyclic relative to $K \otimes^L_\Lambda \Lambda'$.*

Proof. Let Λ_0 be the residue field of Λ. Let x be a geometric point of X, and let y be a geometric point of Y that is a generalization of $f(x)$. Since the functor $\Gamma(X_{(x)} \times_{Y_{(f(x))}} y, -)$ is of finite cohomological dimension, we have
$$R\Gamma(X_{(x)} \times_{Y_{(f(x))}} y, K) \otimes^L_\Lambda \Lambda_0 \cong R\Gamma(X_{(x)} \times_{Y_{(f(x))}} y, K \otimes^L_\Lambda \Lambda_0).$$
On the other hand, Λ has a finite filtration whose graded quotients are finite free Λ_0-modules. It follows that local acyclicity for K is equivalent to local acyclicity for $K \otimes^L_\Lambda \Lambda_0$. Therefore, we may assume that Λ and Λ' are finite fields. Then the assertion is clear since Λ' is a finite free Λ-module. □

Lemma 2.6. *Let X and C be as in Definition 2.1. Let $K \in D_{\mathrm{ctf}}(X, \Lambda)$. Assume that K is micro-supported on C.*

1. *([9, Lemma 4.7(ii)],[1, §2.1, Lemma(ii)]) Let (h, f) be a C-transversal pair in the sense of Definition 2.3.3. Then f is universally locally acyclic relative to h^*K.*

2. *Let $h\colon X' \to X$ be an S-morphism where X' is a smooth S-scheme. If h is C-transversal, then h^*K is micro-supported on $h^\circ C$.*

Proof. 1. The case where Λ is a field is proved in [9, Lemma 4.7(ii)]. The general case follows from this and Lemma 2.5.

2. Let $X' \xleftarrow{h'} W \xrightarrow{f} Y$ be an $h^\circ C$-transversal pair. Then the pair $(h \circ h', f)$ is C-transversal. Thus, f is locally acyclic relative to h'^*h^*K. □

We recall the theorem on the existence of relative singular supports.

Theorem 2.7. *Let X be a smooth S-scheme of finite type. Let $K \in D_{\mathrm{ctf}}(X, \Lambda)$. After replacing S by a dense open subscheme, the following statements hold.*

1. *([9, Theorem 5.2(2)]) There exists a smallest closed conical subset of $T^*(X/S)$ on which K is micro-supported. This smallest closed conical subset is called* the relative singular support *and denoted by $SS(K, X/S)$.*

2. *([9, Theorem 5.3]) For a morphism $s \to S$ from the spectrum of a field, we have*

$$SS(K|_{X_s}) = SS(K, X/S) \times_S s.$$

   Here $SS(K|_{X_s}) = SS(K|_{X_s}, X_s/s)$ is the singular support in the case over a field, as defined in [1].

3. *Let $\Lambda \to \Lambda'$ be a local homomorphism of finite local rings. Then $SS(K, X/S)$ exists if and only if $SS(K \otimes^L_\Lambda \Lambda', X/S)$ exists. If these conditions are satisfied, then*

$$SS(K, X/S) = SS(K \otimes^L_\Lambda \Lambda', X/S).$$

When S is the spectrum of a field, we simply write $SS(K)$ for $SS(K, X/S)$, following [1].

Proof. Assertion 3 follows from Lemma 2.5. When Λ is a field, assertions 1 and 2 are proved in loc. cit. The general case then follows from 3. □

Remark 2.8. *If the relative singular support $SS(K, X/S)$ exists, then the structure morphism $X \to S$ is universally locally acyclic relative to K, since $X \xleftarrow{\mathrm{id}} X \to S$ is $SS(K, X/S)$-transversal.*

Moreover, if X is projective over S, then the existence of $SS(K, X/S)$ is equivalent to the universal local acyclicity of $X \to S$ relative to K by [9, Theorem 5.2].

We give some examples of singular supports.

Proposition 2.9. *Assume that $S = \mathrm{Spec}(k)$ is the spectrum of a field k. We write $SS(-)$ for $SS(-, X/S)$.*

1. *([20, Lemma 4.3.2]) Let X be a smooth irreducible curve over k, and let $K \in D_{\mathrm{ctf}}(X, \Lambda)$. Then*

$$SS(K) \subset T^*_X X \cup \bigcup_x T^*_x X,$$

   where x runs through the closed points at which K is not locally constant. Moreover, the inclusion is an equality if and only if the restriction of K to the generic point of X is not acyclic.

2. *([17, Theorem 2.2.3]) Let X_1 and X_2 be smooth schemes over k. Let $K_i \in D_{\mathrm{ctf}}(X_i, \Lambda)$ for $i = 1, 2$. Then*

$$SS(K_1 \boxtimes^L K_2) = SS(K_1) \times_k SS(K_2) \subset T^*X_1 \times_k T^*X_2 \cong T^*(X_1 \times_k X_2).$$

Proof. The assertions are proved in the references cited above. □

2.2 Flat function and characteristic cycle

In this subsection, we recall the key ingredients in the construction of characteristic cycles in [20], since they also play an essential role in the construction of epsilon cycles.

Let k be a perfect field. We generalize some notions and results on isolated C-characteristic points introduced in [20], where k is assumed to be algebraically closed, to the case where k is not necessarily algebraically closed.

For a scheme X, we write $|X|$ for the set of closed points of X. We fix an abelian group A.

Definition 2.10 (cf. [20, Definition 5.5])**.** *Let Z be a scheme locally of finite type over k. Let $\varphi\colon |Z| \to A$ be a function.*

1. *For a morphism $f\colon Z' \to Z$ of schemes of finite type over k, we define a function*

$$f^*\varphi\colon |Z'| \to A, \qquad z' \mapsto \deg(z'/f(z'))\varphi(f(z')).$$

   If no confusion arises, we simply write $\varphi|_{Z'} = f^*\varphi$.

2. *The function φ is said to be* constant *if there exists a function $\psi\colon |\operatorname{Spec}(k)| \to A$ such that $\psi|_Z = \varphi$.*

3. *Let $g\colon Z \to S$ be a quasi-finite morphism of schemes locally of finite type over k. We say that φ is* flat *over S if, for every closed point $z \in Z$, there exists a commutative diagram*

(2.1)
$$\begin{array}{ccccc} U & \hookrightarrow & V \times_S Z & \longrightarrow & Z \\ & \overset{\tilde{g}}{\searrow} & \downarrow & & \downarrow g \\ & & V & \longrightarrow & S \end{array}$$

   of k-schemes satisfying the following properties:

   (a) *The morphism $V \to S$ is étale, and there exists a closed point $v \in V$ whose image in S is $g(z)$. The induced homomorphism $k(g(z)) \to k(v)$ of the residue fields is an isomorphism.*

(b) *U is an open neighborhood of (v, z) in $V \times_S Z$.*

(c) *U is finite over V, and the fiber of $\tilde{g}$ over v consists only of (v, z).*

(d) *The function*

$$\tilde{g}_*(\varphi|_U) \colon |V| \to A, \qquad x \mapsto \sum_{y \in \tilde{g}^{-1}(x)} \varphi|_U(y)$$

is constant in the sense of 2.

Let X be a smooth k-scheme and C be a closed conical subset of T^*X. Let $f \colon X \to Y$ be a k-morphism to a smooth curve Y over k. Let $x \in X$ be a closed point.

Definition 2.11. *Let the notation be as above.*

1. *We say that x is* an at most isolated C-characteristic point *of f if there exists an open neighborhood U of x such that the restriction $f|_{U \setminus \{x\}}$ is C-transversal.*

2. *Assume that X is purely of dimension n and that every irreducible component C_a of C has dimension n. Let $x \in X$ be an at most isolated C-characteristic point of f, and let ω be a basis of T^*Y on a neighborhood of $f(x) \in Y$.*

   For each irreducible component C_a, we define $(C_a, df)_{T^*X,x}$, or simply $(C_a, df)_x$, to be the intersection number of C_a with the section $f^*\omega$ of T^*X. Equivalently, $(C_a, df)_{T^*X,x}$ is the integer determined by

   $$[C_a] \cap [f^*\omega] = (C_a, df)_{T^*X,x}[x] \in Z_0(x).$$

   More generally, for every element

   $$\alpha = \sum_a \beta_a \otimes [C_a] \in A \otimes Z_n(T^*X),$$

   we define the intersection number $(\alpha, df)_{T^*X,x} \in A$, or simply $(\alpha, df)_x$, by

   $$(\alpha, df)_{T^*X,x} = \sum_a (C_a, df)_{T^*X,x} \cdot \beta_a.$$

Definition 2.12. *Let X and C be as above.*

1. *An* A-valued function on isolated C-characteristic points *is an assignment*

   $$(f, u) \longmapsto \varphi(f, u) \in A,$$

   where

   (2.2) $$\begin{array}{ccc} U & \xrightarrow{f} & Y \\ {\scriptstyle j}\downarrow & & \\ X & & \end{array}$$

   is a diagram of k-schemes with Y a smooth curve over k and j étale, and $u \in |U|$ is an at most isolated C-characteristic point of f, satisfying the following conditions:

(a) *If u is not an isolated C-characteristic point, then $\varphi(f,u)=0$.*

(b) *Let*

(2.3)
$$\begin{array}{ccc} U' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & Y \\ \downarrow & & \\ X & & \end{array}$$

be a commutative diagram of k-schemes where the vertical arrows are étale, and Y and Y' are smooth curves over k. Let $u' \in U'$ be an at most isolated $C \times_X U'$-characteristic point of f'. Then

$$\varphi(f',u') = \deg(u'/u) \cdot \varphi(f,u),$$

where u denotes the image of u' in U.

2. *Let φ be an A-valued function on isolated C-characteristic points. We say that φ is* flat *if, for every commutative diagram*

(2.4)
$$\begin{array}{ccccc} Z & \hookrightarrow & U & \xrightarrow{f} & Y \\ & & {\scriptstyle \mathrm{pr}_1}\downarrow & \searrow & \swarrow \\ & & X & & S \end{array}$$

of k-schemes such that

- *S is a smooth scheme over k,*
- *$Y \to S$ is a relative smooth curve,*
- *the induced morphism $U \to X \times_k S$ is étale,*
- *Z is a closed subscheme of U that is quasi-finite over S, and*
- *the pair (pr_1, f) is C-transversal (in the sense of Definition 2.3.3) outside Z,*

the function

$$\varphi_f\colon |Z| \to A, \qquad z \mapsto \varphi(f_s, z),$$

where s denotes the image of z under $Z \to S$ and $f_s\colon U_s \to Y_s$ is the base change of f along $s \to S$, is flat over S in the sense of Definition 2.10.3.

Proposition 2.13 ([20, Proposition 5.8])**.** *Assume that A is uniquely divisible (i.e., the canonical map $A \to A \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism). Let X be a smooth scheme of pure dimension n over k, and let C be a closed conical subset of T^*X. Assume that every irreducible component C_a of C has dimension n. Let φ be an A-valued function on isolated C-characteristic points.*

Then the following conditions are equivalent.

1. *The function φ is flat.*

2. There exists a cycle

$$\alpha = \sum_a \beta_a \otimes [C_a] \in A \otimes_{\mathbb{Z}} Z_n(T^*X),$$

where Z_n denotes the group of n-cycles, supported on C such that

$$\varphi(f,u) = \deg(u/k)(j^*\alpha, df)_{T^*U,u} \tag{2.5}$$

for every diagram (2.2) and every at most isolated C-characteristic point $u \in U$ of f.

Moreover, if these conditions hold, then the cycle α in 2 *is unique.*

Proof. Since the proof is completely similar to [20, Proposition 5.8], and since we only use the implication $1 \Rightarrow 2$ in the sequel, we sketch the proof of $1 \Rightarrow 2$.

First, assume that k is algebraically closed. By a similar argument to that in [20, Proposition 5.8], there exists a unique cycle $\alpha_X \in A \otimes Z_n(T^*X)$ such that

$$\varphi(f,u) = (\alpha_X, df)_{T^*X,u}$$

for every diagram (2.2) in which $U \to X$ is an open immersion. Let $j\colon W \to X$ be an étale morphism. By restricting φ to W, we obtain an A-valued function on isolated j^*C-characteristic points. Since this function is also flat, there exists a unique cycle $\alpha_W \in A \otimes Z_n(T^*W)$ satisfying $\varphi(f,u) = (\alpha_W, df)_{T^*W,u}$ for every diagram (2.2) with X replaced by W and in which the morphism $U \to W$ is an open immersion.

We claim that

$$\alpha_W = j^*\alpha_X.$$

This is a consequence of [20, Proposition 5.8.2]. This proves the case where k is algebraically closed.

Next, we consider the general case. Let $\bar{k}$ be an algebraic closure of k. For any k-scheme Y, we write $Y_{\bar{k}} := Y \times_k \bar{k}$. The function φ induces an A-valued function $\varphi_{\bar{k}}$ on isolated $C_{\bar{k}}$-characteristic points, defined as follows. Let

$$\begin{array}{ccc} U & \xrightarrow{f} & Y \\ \downarrow & & \\ X_{\bar{k}} & & \end{array} \tag{2.6}$$

be a diagram as in (2.2), and let $u \in U$ be an at most isolated $C_{\bar{k}}$-characteristic point of f. Assume that U and Y are quasi-compact. Then there exists a finite subextension k'/k of $\bar{k}/k$ and a diagram of k'-schemes

$$\begin{array}{ccc} U' & \xrightarrow{f'} & Y' \\ \downarrow & & \\ X_{k'} & & \end{array}$$

whose base change along $k' \to \bar{k}$ is isomorphic to (2.6). Let $u' \in U'$ be the image of u. Define

$$\varphi_{\bar{k}}(f,u) := \frac{1}{\deg(u'/k)}\varphi(f',u'),$$

where the valule $\varphi(f', u')$ is computed with respect to the diagram

$$\begin{array}{ccc} U' & \xrightarrow{f'} & Y' \\ \downarrow & & \\ X_{k'} & & \\ \downarrow & & \\ X. & & \end{array}$$

One checks that this definition is independent of the choice of (k', f'). Hence it defines an A-valued function on isolated $C_{\bar{k}}$-characteristic points.

Since $\varphi_{\bar{k}}$ is flat, there exists a unique cycle $\alpha_{\bar{k}} \in A \otimes_{\mathbb{Z}} Z_n(T^*X_{\bar{k}})$ satisfying (2.5). By the construction of $\varphi_{\bar{k}}$ and the uniqueness of $\alpha_{\bar{k}}$, the cycle $\alpha_{\bar{k}}$ is invariant under the action of $\mathrm{Gal}(\bar{k}/k)$. By Galois descent, we obtain a cycle α satisfying the required condition. □

We now introduce notation for the vanishing cycles complex. Let X be a smooth scheme over k, and let $K \in D_{\mathrm{ctf}}(X, \Lambda)$, where Λ is a finite local ring as before.

Let $X \overset{j}{\leftarrow} U \overset{f}{\rightarrow} Y$ be a diagram as in (2.2), and let $u \in U$ be a closed point. Assume that u is an at most isolated $j^*SS(K)$-characteristic point of f. Let $U_{(u)}$ and $Y_{(f(u))}$ denote the henselizations of U and Y at u and $f(u)$, respectively. Let $Y_{(u)}$ denote the finite étale cover of $Y_{(f(u))}$ corresponding to the separable extension $k(u)/k(f(u))$ of residue fields.

Definition 2.14. *Let the notation be as above. Let $f_{(u)} \colon U_{(u)} \to Y_{(u)}$ denote the induced morphism. We write $R\Phi_f(K)_u$ for the vanishing cycles complex of $K|_{U_{(u)}}$ with respect to $f_{(u)}$, and call it* the vanishing cycles complex of K supported at u.

By definition, $R\Phi_f(K)_u$ is a constructible complex on $U_{(u)} \times_{Y_{(u)}} \bar{u}$ supported at $\bar{u}$, where $\bar{u}$ denotes a geometric point lying above u. Moreover, it is endowed with an equivariant action of the absolute Galois group G_{η_u} of η_u, where η_u denotes the generic point of $Y_{(u)}$ ([2]).

Since $R\Phi_f(K)_u$ is supported at the single point $\bar{u}$, we shall usually regard it as a complex of Λ-modules equipped with a continuous action of G_{η_u}, or equivalently as an object of $D_{\mathrm{ctf}}(\eta_u, \Lambda)$.

For an object $M \in D_{\mathrm{ctf}}(\eta_u, \Lambda)$, define the total dimension of M by

$$\mathrm{dimtot}\, M := \mathrm{rk}\, M + \mathrm{Sw}\, M.$$

We recall the definition of characteristic cycles.

Theorem 2.15 ([20, Theorems 5.9, 5.18])**.** *Let X be a smooth scheme over k, and let $K \in D_{\mathrm{ctf}}(X, \Lambda)$. Let C be a closed conical subset of T^*X on which K is micro-supported. Assume that all irreducible components of X and C have dimension n.*

1. *There exists a cycle $CC(K)$ in $\mathbb{Q} \otimes Z_n(T^*X)$, supported on C, satisfying the following property:*

   For every diagram as in (2.2) and every at most isolated C-characteristic point $u \in U$ of f, we have
   $$-\,\mathrm{dimtot}\, R\Phi_f(K)_u = (CC(K), df)_u.$$
   Moreover, $CC(K)$ is uniquely determined by this property and is independent of the choice of C on which K is micro-supported. Furthermore, $CC(K)$ has coefficients in $\mathbb{Z}$.

2. Let $\Lambda \to \Lambda'$ be a local ring homomorphism of finite local rings. Then

$$CC(K) = CC(K \otimes^L_\Lambda \Lambda').$$

Proof. 1. The existence and uniqueness follow immediately from Proposition 2.13, once one knows that the $\mathbb{Q}$-valued function on isolated C-characteristic points defined by $\varphi(f,u) = -\deg(u/k) \cdot \operatorname{dimtot} R\Phi_f(K)_u$ is flat. This flatness is proved in [20, Proposition 2.16]. The integrality of $CC(K)$ is proved in [20, Theorem 5.18].

2. Let $X \overset{j}{\leftarrow} U \overset{f}{\to} Y$ be as (2.2). The assertion follows from

$$R\Phi_f(K) \otimes^L_\Lambda \Lambda' \cong R\Phi_f(K \otimes^L_\Lambda \Lambda'),$$

and the equality $\operatorname{dimtot} M = \operatorname{dimtot}(M \otimes^L_\Lambda \Lambda')$ for $M \in D_{\mathrm{ctf}}(\eta_u, \Lambda)$. □

Let ℓ be a prime number invertible on all schemes under consideration,a and let $\overline{\mathbb{Q}_\ell}$ be an algebraic closure of $\mathbb{Q}_\ell$. We define local acyclicity, singular supports, and characteristic cycles for $\overline{\mathbb{Z}_\ell}$-sheaves. For the ℓ-adic formalism, we follow Ekedahl's approach, which we review in Section 6. In the sequel, E is a finite subextension in $\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell$, and $\mathcal{O}_E$ denotes its ring of integers.

Definition 2.16. *Let $f\colon X \to Y$ be a morphism of finite type between noetherian schemes. Let $\mathcal{O}$ be either $\mathcal{O}_E$ or $\overline{\mathbb{Z}_\ell}$. Let $\mathcal{F}$ be an object in $D^b_c(X, \mathcal{O})$.*

1. *Assume that $\mathcal{O} = \mathcal{O}_E$. We say that f is (resp. universally) locally acyclic relative to $\mathcal{F}$ if, for some (equivalently, every) $n \geq 0$, the morphism f is (resp. universally) locally acyclic relative to $\mathcal{F} \otimes^L_{\mathcal{O}} \mathcal{O}/\ell^{n+1}$.*
2. *Assume that $\mathcal{O} = \overline{\mathbb{Z}_\ell}$. Choose a finite subextension E and an object $\mathcal{F}_E \in D^b_c(X, \mathcal{O}_E)$ such that $\mathcal{F}_E \otimes_{\mathcal{O}_E} \overline{\mathbb{Z}_\ell} \cong \mathcal{F}$. We say that f is (resp. universally) locally acyclic relative to $\mathcal{F}$ if f is (resp. universally) locally acyclic relative to $\mathcal{F}_E$ in the sense of* 1.

By Lemma 2.5, this definition is independent of the choices of E and $\mathcal{F}_E$.

Definition 2.17. *Let X be a smooth scheme of finite type over a noetherian scheme S. Let $\mathcal{O}$ be either $\mathcal{O}_E$ or $\overline{\mathbb{Z}_\ell}$. Let $\mathcal{F} \in D^b_c(X, \mathcal{O})$.*

1. *Assume that $\mathcal{O} = \mathcal{O}_E$. We define $SS(\mathcal{F}, X/S) := SS(\mathcal{F} \otimes^L_{\mathcal{O}_E} \mathcal{O}_E/\ell^{n+1}, X/S)$ for some (equivalently, every) $n \geq 0$. If S is the spectrum of a perfect field, we also define $CC(\mathcal{F}) := CC(\mathcal{F} \otimes^L_{\mathcal{O}_E} \mathcal{O}_E/\ell^{n+1})$ for some (equivalently, every) $n \geq 0$.*
2. *Assume that $\mathcal{O} = \overline{\mathbb{Z}_\ell}$. Choose a finite subextension E and an object $\mathcal{F}_E \in D^b_c(X, \mathcal{O}_E)$ such that $\mathcal{F}_E \otimes^L_{\mathcal{O}_E} \overline{\mathbb{Z}_\ell} \cong \mathcal{F}$. We define $SS(\mathcal{F}, X/S) := SS(\mathcal{F}_E, X/S)$. If S is the spectrum of a perfect field, we define $CC(\mathcal{F}) := CC(\mathcal{F}_E)$, where the right-hand side is defined by* 1.

By Theorems 2.7.3 and 2.15.2, the above definitions are independent of the auxiliary choices.

Remark 2.18. *As explained in [23, 5.3], the characteristic cycle is defined for objects of $D^b_c(X, \overline{\mathbb{Q}_\ell})$.*

2.3 Reminder on good pencils

We recall the theory of the universal hyperplane sections, following [20, Section 3.2], and the notion of good pencils, following [21].

Let X be a smooth quasi-projective scheme over a field k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let V be a finite-dimensional k-vector space, and let $V \to \Gamma(X, \mathcal{L})$ be a k-linear map inducing a surjection $V \otimes_k \mathcal{O}_X \to \mathcal{L}$. Assume that the induced morphism $h\colon X \to \mathbb{P} = \mathbb{P}(V^\vee)$ is an immersion. Here, we adopt the convention opposite to Grothendieck's for projective bundles: namely, $\mathbb{P}(V^\vee)$ parametrizes line subbundles of $V^\vee$.

Let $\mathbb{P}^\vee := \mathbb{P}(V)$ be the dual projective space. The universal hyperplane $Q \subset \mathbb{P} \times \mathbb{P}^\vee$ parametrizes pairs (x, H) consisting of a point $x \in \mathbb{P}$ and a hyperplane $H \in \mathbb{P}^\vee$ containing x. Since the kernel of the tautological surjection $V \otimes_k \mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{P}}$ is canonically isomorphic to the cotangent bundle $\Omega^1_{\mathbb{P}}$, the scheme Q is identified with the projective space bundle $\mathbb{P}(T^*\mathbb{P})$. The composite morphism

$$T^*_Q(\mathbb{P} \times \mathbb{P}^\vee) \hookrightarrow Q \times_{\mathbb{P} \times \mathbb{P}^\vee} T^*(\mathbb{P} \times \mathbb{P}^\vee) \to Q \times_{\mathbb{P}} T^*\mathbb{P}$$

identifies $T^*_Q(\mathbb{P} \times \mathbb{P}^\vee)$ with the universal line subbundle on $Q = \mathbb{P}(T^*\mathbb{P})$.

Consider the commutative diagram

(2.7)
$$\begin{array}{ccccc} X & \xleftarrow{\;p\;} & X \times_{\mathbb{P}} Q & \xrightarrow{\;p^\vee\;} & \mathbb{P}^\vee \\ \downarrow{\scriptstyle h} & & \downarrow & \nearrow{\scriptstyle \boldsymbol{p}^\vee} & \\ \mathbb{P} & \xleftarrow[\boldsymbol{p}]{} & Q, & & \end{array}$$

where the square is cartesian. We have $X \times_{\mathbb{P}} Q = \mathbb{P}(X \times_{\mathbb{P}} T^*\mathbb{P})$. Let $L \subset \mathbb{P}^\vee$ be a line in $\mathbb{P}^\vee$. We have a commutative diagram

(2.8)
$$\begin{array}{ccccc} & & X_L & \xrightarrow{\;f\;} & L \\ & \swarrow{\scriptstyle \pi} & \downarrow & & \downarrow \\ X & \xleftarrow[p]{} & X \times_{\mathbb{P}} Q & \xrightarrow[p^\vee]{} & \mathbb{P}^\vee, \end{array}$$

where the right square is cartesian. Let A_L denote the axis of L in $\mathbb{P}$. Then A_L is a linear subspace of $\mathbb{P}$ of codimension 2. The $\mathbb{P}$-scheme $\mathbb{P}_L = Q \times_{\mathbb{P}^\vee} L$ is isomorphic to the blow-up of $\mathbb{P}$ along A_L. Consequently, if X and A_L meet transversally in $\mathbb{P}$, then X_L is the blow-up of X along the smooth subvariety $X \cap A_L$.

Definition 2.19. *Let $X \subset \mathbb{P}$ be a smooth closed subvariety of pure dimension n over k. Let C be a closed conical subset of T^*X whose irreducible components have dimension n. We say that the pair (π, f) as in (2.8) is a* good pencil *with respect to C if the following conditions are satisfied.*

1. *The subvarieties X and A_L meet transversally (hence X_L is smooth).*
2. *The morphism π is properly C-transversal.*
3. *The morphism f has at most isolated $\pi^\circ C$-characteristic points.*

4. *For every closed point $y \in L$, there is at most one $\pi^\circ C$-characteristic point on the fiber $f^{-1}(y)$.*

5. *No isolated $\pi^\circ C$-characteristic point of f is contained in the exceptional locus of π.*

6. *For every irreducible component C_a of C, there exists an isolated $\pi^\circ C$-characteristic point $x \in X_L$ such that df meets C_a at x and does not meet any other component of C at x. Moreover, for every isolated $\pi^\circ C$-characteristic point x, the section df meets exactly one irreducible component at x.*

7. *For every isolated $\pi^\circ C$-characteristic point $x \in X_L$ of f, the induced morphism $x \to f(x)$ of spectra of fields is purely inseparable.*

The existence of good pencils over a finite extension of the base field is proved in [23, Lemma 4.2.7] using [21, Lemma 2.3].

Lemma 2.20 ([23, Lemma 4.2.7], [21, Lemma 2.3])**.** *Let X and C be as in Definition 2.19. Assume that k is an infinite field. Let $\mathrm{Gr}(1, \mathbb{P}^\vee)$ be the Grassmannian variety parametrizing lines in $\mathbb{P}^\vee$.*

After replacing the immersion $X \hookrightarrow \mathbb{P}$ by its composition with a Veronese embedding $\mathbb{P} \hookrightarrow \mathbb{P}'$ of degree ≥ 3 if necessary, there exists a dense open subset $U \subset \mathrm{Gr}(1, \mathbb{P}^\vee)$ such that, for every k-rational point $L \in U(k)$, the pair (π, f) defined in (2.8) is a good pencil.

We also record the following result.

Lemma 2.21. *Let X be a smooth scheme over a field k, and let C be a closed conical subset of T^*X. Assume that both X and C are purely of dimension n. Let $C = \bigcup_a C_a$ be the decomposition into irreducible components.*

Then, for each irreducible component C_a, there exists a diagram of k-schemes

$$X \xleftarrow{j_a} U_a \xrightarrow{f_a} Y_a,$$

where j_a is étale and Y_a is a smooth curve over k, together with an isolated C-characteristic point $u_a \in U_a$ of f_a, such that df_a meets C_a at u_a and does not meet any other component of C at u_a.

Proof. Let $X' \subset X$ be an affine open subscheme such that $C_a|_{X'}$ is nonempty. Choose a closed immersion $i\colon X' \hookrightarrow \mathbb{A}^m_k$. Replacing X by $\mathbb{A}^m_k$ and C by $i_\circ(C|_{X'})$, we may assume that $X = \mathbb{A}^m_k$. Next, replacing $\mathbb{A}^m_k$ by $\mathbb{P}^m_k$ and C by its closure in $T^*\mathbb{P}^m_k$, we reduce to the case where X is smooth projective.

Assume that X is smooth projective. The case where k is infinite follows from Lemma 2.20.

Assume now that k is finite. By a standard limit argument, there exists a finite extension k'/k such that $(X \times_k k', C \times_k k')$ admits the required diagram. Viewing the resulting smooth k'-schemes as smooth k-schemes, we obtain the desired diagram over k. This proves the assertion. □

3 Local Epsilon Factors (cf. [3], [25], [26])

In §3.1, we review Yasuda's generalization of the theory of local epsilon factors for henselian traits of equal-characteristic. In §3.3, we explain how to reduce various computations for ℓ-adic sheaves on varieties of characteristic 0 to the case of positive characteristic. In §3.4, we compute the local epsilon factors of the convolutions of vanishing cycles.

3.1 Generalities on local epsilon factors

Let k be a perfect field of characteristic p, and let T be a henselian trait isomorphic to the henselization of $\mathbb{A}^1_k$ at the origin. Denote its generic point by η, and the absolute Galois groups of k and η by G_k and G_η, respectively. Let Λ be a finite local ring with residue characteristic $\ell \neq p$.

Let (ρ, V) consists of a finite free Λ-module V and a continuous group homomorphism $\rho\colon G_\eta \to \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ is equipped with the discrete topology. To such a pair, Yasuda [25] associates a continuous character

$$\tilde{\varepsilon}_{0,\Lambda}(V, \tilde{\psi})\colon G_k \to \Lambda^\times,$$

called the local epsilon factor, generalizing the Langlands–Deligne theory [3]. This character depends on the choice of $\tilde{\psi}$, which is called a *non-trivial additive character sheaf* in [25].

In the sequel, we explain how to attach an invertible additive character sheaf $\tilde{\psi}_\omega$ to an additive character $\psi\colon \mathbb{F}_p \to \Lambda^\times$ and a differential $\omega \in \Omega^1_\eta$. Then we put

$$\varepsilon_{0,\Lambda}(T, V, \omega) = \tilde{\varepsilon}_{0,\Lambda}(V, \tilde{\psi}_\omega).$$

This notation is analogous to that introduced by Laumon in [15].

Let F be the completion of the function field of T. For the definition of non-trivial additive character sheaf on F, see [25, 4.1]. Let $\psi\colon \mathbb{F}_p \to \Lambda^\times$ be a non-trivial additive character, and let ω be a non-zero rational 1-form on η. We define a non-trivial additive character sheaf $\tilde{\psi}_\omega$ as follows.

When k is finite, the differential form ω defines an additive character

$$\psi_\omega\colon F \to \Lambda^\times, \qquad a \mapsto \psi(\mathrm{tr}_{k/\mathbb{F}_p} \circ \mathrm{Res}(a\omega)).$$

Then $\tilde{\psi}_\omega$ is the additive character sheaf corresponding to ψ_ω, which is constructed in [25, Corollary 4.3].

In general, choose a uniformizer $\pi \in F$, which determines an inclusion $\mathbb{F}_p((\pi)) \to F$. When $\omega = d\pi$, define the non-trivial additive character sheaf $\tilde{\psi}_\omega$ on F to be the pullback of $\tilde{\psi}_{d\pi}$ on $\mathbb{F}_p((\pi))$ along this inclusion. When $\omega = a d\pi$ for some non-zero element $a \in F$, define $\tilde{\psi}_\omega$ to be the pullback of $\tilde{\psi}_{d\pi}$ on F by the multiplication-by-a map $F \to F$. Using [25, Corollary 4.4(2)], we can show that this definition of $\tilde{\psi}_\omega$ is independent of the choice of π.

The assignment $(T, (\rho, V), \omega) \mapsto \varepsilon_{0,\Lambda}(T, V, \omega)$ satisfies the following properties.

Theorem 3.1. *([26], [25, §4.12]) Let the notation be as above. Let $(T, (\rho, V), \omega)$ be a triple consisting of a henselian trait T, a finite free Λ-module V equipped with a continuous homomorphism $\rho\colon G_\eta \to \mathrm{GL}(V)$, and a non-zero rational 1-form $\omega \in \Omega^1_\eta$.*

Then the local epsilon factor $\varepsilon_{0,\Lambda}(T, V, \omega)\colon G_k^{ab} \to \Lambda^\times$ satisfies the following properties.

1. *The character depends only on the isomorphism class of $(T, (\rho, V), \omega)$.*

2. *For every short exact sequence $0 \to V' \to V \to V'' \to 0$ of representations of G_η, we have*
$$\varepsilon_{0,\Lambda}(T, V, \omega) = \varepsilon_{0,\Lambda}(T, V', \omega) \cdot \varepsilon_{0,\Lambda}(T, V'', \omega).$$

3. *For every local ring homomorphism $f\colon \Lambda \to \Lambda'$, we have*
$$f \circ \varepsilon_{0,\Lambda}(T, V, \omega) = \varepsilon_{0,\Lambda'}(T, V \otimes_\Lambda \Lambda', \omega)$$
as characters $G_k \to \Lambda'^\times$. Here the additive character $\mathbb{F}_p \to \Lambda'^\times$ used in the definition of $\varepsilon_{0,\Lambda'}$ is taken to be $f \circ \psi$.

4. *We have*
$$\varepsilon_{0,\Lambda}(T, V, \omega) \cdot \varepsilon_{0,\Lambda}(T, V, \omega')^{-1} = \det(V)_{[\frac{\omega}{\omega'}]} \chi_{\mathrm{cyc}}^{(\mathrm{ord}(\omega') - \mathrm{ord}(\omega))\, \mathrm{rk}(V)}.$$
Here χ_{cyc} denotes the composition of the ℓ-adic cyclotomic character $G_k^{ab} \to \mathbb{Z}_\ell^\times$ with the canonical homomorphism $\mathbb{Z}_\ell^\times \to \Lambda^\times$. Moreover,
$$k(\eta)^\times \times \mathrm{H}^1(\eta, \Lambda^\times) \to \mathrm{H}^1(k, \Lambda^\times), \qquad (a, \chi) \mapsto \chi_{[a]}$$
denotes the pairing defined in [25, §4.2] and [22, Definition 3.12].

5. *Let W be an unramified representation of G_η on a finite free Λ-module. Then*
$$\varepsilon_{0,\Lambda}(T, V \otimes W, \omega) = \det(W)^{\otimes a(T,V,\omega)} \cdot \varepsilon_{0,\Lambda}(T, V, \omega)^{\mathrm{rk}(W)},$$
where $a(T, V, \omega) := \mathrm{Sw}(V) + \mathrm{rk}(V)\big(\mathrm{ord}(\omega) + 1\big)$.

6. *Assume that k is finite and that there exists a local ring morphism $f\colon \mathcal{O}_E \to \Lambda$, where E is a finite field extension of $\mathbb{Q}_\ell$. Let V' be a finite free $\mathcal{O}_E$-module equipped with a continuous action of G_η, and set $V := V' \otimes_{\mathcal{O}_E} \Lambda$. Then we have*
$$\varepsilon_{0,\Lambda}(T, V, \omega)(\mathrm{Frob}_k) = (-1)^{\mathrm{rk}(V) + \mathrm{Sw}(V)} f(\varepsilon_0(T, V' \otimes_{\mathcal{O}_E} E, \omega)).$$
Here $\varepsilon_0(T, V' \otimes_{\mathcal{O}_E} E, \omega)$ on the right-hand side is the classical local epsilon factor considered in [15, Théorème (3.1.5.4)].

If no confusion arises, we omit the subscript Λ from $\varepsilon_{0,\Lambda}(T, V, \omega)$. In what follows, we identify $D(\eta, \Lambda)$ with the derived category of discrete Λ-modules with continuous G_η-actions, via the canonical equivalence between the étale topos of η and the category of discrete sets with continuous G_η-actions.

By the multiplicativity property in Theorem 3.1.2, the local epsilon factos $\varepsilon_{0,\Lambda}(T, K, \omega)$ is defined for every constructible complex $K \in D_{\mathrm{ctf}}(\eta, \Lambda)$. In the following lemma, we recall this definition and prove several basic properties.

Before proceeding, recall that K is said to be *strictly perfect* if each K^i is a finite free Λ-module and $K^i = 0$ for all but finitely many i.

Lemma 3.2. *Let $K \in D_{\mathrm{ctf}}(\eta, \Lambda)$. Then the following hold.*

1. *There exists a strictly perfect complex $K' \in D_{\mathrm{ctf}}(\eta, \Lambda)$ quasi-isomorphic to K. Moreover, if K is bounded below (i.e., $K^i = 0$ for $i \ll 0$), then there exists an actual quasi-morphism $K' \to K$. Here, by an actual morphism, we means a morphism of complexes, as opposed to a morphism in the homotopy category or the derived category.*

2. *Define*
$$\varepsilon_{0,\Lambda}(T, K, \omega) := \prod_i \varepsilon_{0,\Lambda}(T, (K')^i, \omega)^{(-1)^i},$$
where K' is a strictly perfect complex as in 1. *Then $\varepsilon_{0,\Lambda}(T, K, \omega)$ is independent of the choice of K'.*

3. *For every distinguished triangle $K_1 \to K_2 \to K_3 \to$ in $D_{\mathrm{ctf}}(\eta, \Lambda)$, we have*
$$\varepsilon_{0,\Lambda}(T, K_2, \omega) = \varepsilon_{0,\Lambda}(T, K_1, \omega) \cdot \varepsilon_{0,\Lambda}(T, K_3, \omega).$$

4. *Let $f : \Lambda \to \Lambda'$ be a local homomorphism between finite local rings. Then*
$$f \circ \varepsilon_{0,\Lambda}(T, K, \omega) = \varepsilon_{0,\Lambda'}(T, K \otimes^L_\Lambda \Lambda', \omega).$$

Proof. 1. Let M be a Λ-module endowed with a continuous G_η-action, where M is equipped with the discrete topology. For any finite sequence $v_1, \ldots, v_n$ of elements in M, there exists an open subgroup $H < G_\eta$ such that the homomorphism $\Lambda[G_\eta]^{\oplus n} \to M$ of Λ-representations sending the i-th standard basis to v_i factors through the discrete quotient $\Lambda[G_\eta/H]^{\oplus n}$.

Using this fact and the finiteness of $H^i(K)$, we can construct a bounded above complex K_1 of finite free Λ-modules endowed with a continuous G_η-action together with an actual quasi-isomorphism $K_1 \to K$. Suppose that K has tor-amplitude in $[a, b]$. Then, for any integer $c \leq a$, the truncation $K' := \tau_{\geq c} K_1$ has the desired properties.

Suppose now that K is bounded below. Choose an integer $c \leq a$ such that $K^i = 0$ for $i < c$. Then, the morphism $K_1 \to K$ factors through an actual morphism $\tau_{\geq c} K_1 \to K$. This proves the last assertion.

2. Let K'' be another choice. We must show that
$$\prod_i \varepsilon_{0,\Lambda}(T, (K')^i, \omega)^{(-1)^i} = \prod_i \varepsilon_{0,\Lambda}(T, (K'')^i, \omega)^{(-1)^i}.$$

Since K' and K'' are quasi-isomorphic in the derived category, there exist actual morphisms of complexes
$$K' \leftarrow L \to K''$$
that are quasi-isomorphisms. Since K' and K'' are bounded, we may assume that L is bounded. By 1, there exists an actual quasi-isomorphism $L' \to L$ where L' is strictly perfect. Thus, we obtain actual quasi-isomorphisms of strictly perfect complexes $K' \leftarrow L' \to K''$.

Therefore, we may assume that there exists an actual quasi-isomorphism $K' \to K''$. Let C denote its mapping cone. Since C is acyclic, it decomposes into finitely many short exact sequences of finite free Λ-modules with continuous G_η-actions. By Theorem 3.1.2, it follows that $\prod_i \varepsilon_{0,\Lambda}(T, C^i, \omega)^{(-1)^i} = 1$. Since $C^i = (K'')^i \oplus (K')^{i+1}$, the desired equality follows.

3. By the same argument as in 2, we may assume that the morphism $K_1 \to K_2$ in $D_{\mathrm{ctf}}(\eta, \Lambda)$ comes from an actual morphism. Then K_3 is quasi-isomorphic to the mapping cone of this morphism. The assertion follows from the well-definedness proved in 2.

4. Let K' be a strictly perfect complex quasi-isomorphic to K. Then $K \otimes^L_\Lambda \Lambda'$ is quasi-isomorphic to $K' \otimes_\Lambda \Lambda'$. The assertion follows from Theorem 3.1.3. □

Let $\mathcal{O}_E$ be the ring of integers of a finite extension $E/\mathbb{Q}_\ell$. Fix a non-trivial character $\psi\colon \mathbb{F}_p \to \mathcal{O}_E^\times$. Using Lemma 3.2, we define, for every $\mathcal{F} \in D^b_c(\eta, \mathcal{O}_E)$, a local epsilon factor

$$\varepsilon_{0,\mathcal{O}_E}(T, \mathcal{F}, \omega)\colon G_k \to \mathcal{O}_E^\times$$

as follows. Put $\Lambda_n := \mathcal{O}_E/\ell^{n+1}$. Then the reduction $\mathcal{F}_n := \mathcal{F} \otimes^L_{\mathcal{O}_E} \Lambda_n$ belongs to $D_{\mathrm{ctf}}(\eta, \Lambda_n)$, and ψ induces a non-trivial character $\mathbb{F}_p \to \Lambda_n^\times$. By Lemma 3.2, we obtain a character $\varepsilon_{0,\Lambda_n}(T, \mathcal{F}_n, \omega)\colon G_k \to \Lambda_n^\times$. By Lemma 3.2.4, the characters $\varepsilon_{0,\Lambda_n}(T, \mathcal{F}_n, \omega)$ and $\varepsilon_{0,\Lambda_{n+1}}(T, \mathcal{F}_{n+1}, \omega)$ are compatible with the quotient map $\Lambda_{n+1} \to \Lambda_n$. Therefore, we may define $\varepsilon_{0,\mathcal{O}_E}(T, \mathcal{F}, \omega) := \varprojlim_n \varepsilon_{0,\Lambda_n}(T, \mathcal{F}_n, \omega)$. This is a continuous character of $G_k \to \mathcal{O}_E^\times$.

Finally, we extend the definition of local epsilon factor to $\overline{\mathbb{Z}_\ell}$-sheaves.

Definition-Lemma 3.3. *Let the notation be as above. Let $\overline{\mathbb{Z}_\ell}$ be the integral closure of $\mathbb{Z}_\ell$ in the algebraic closure $\overline{\mathbb{Q}_\ell}$ of $\mathbb{Q}_\ell$.*

1. *For an object $\mathcal{F} \in D^b_c(\eta, \overline{\mathbb{Z}_\ell})$, we define a character*

$$\varepsilon_{0,\overline{\mathbb{Z}_\ell}}(T, \mathcal{F}, \omega)\colon G_k^{ab} \to \overline{\mathbb{Z}_\ell}^\times$$

as follows. Choose a finite extension $E/\mathbb{Q}_\ell$ contained in $\overline{\mathbb{Q}_\ell}$ and an object $\mathcal{F}_E \in D^b_c(\eta, \mathcal{O}_E)$ such that $\mathcal{F}_E \otimes^L_{\mathcal{O}_E} \overline{\mathbb{Z}_\ell} \cong \mathcal{F}$. We define $\varepsilon_{0,\overline{\mathbb{Z}_\ell}}(T, \mathcal{F}, \omega)$ as the composition of

$$G_k \xrightarrow{\varepsilon_{0,\mathcal{O}_E}(T, \mathcal{F}_E, \omega)} \mathcal{O}_E^\times \hookrightarrow \overline{\mathbb{Z}_\ell}^\times.$$

This definition is independent of the choices of E and $\mathcal{F}_E$.

2. *For an object $\mathcal{F} \in D^b_c(T, \overline{\mathbb{Z}_\ell})$, we define $\varepsilon(T, \mathcal{F}, \omega)\colon G_k^{ab} \to \overline{\mathbb{Z}_\ell}^\times$ by*

$$\varepsilon(T, \mathcal{F}, \omega) := \varepsilon_{0,\overline{\mathbb{Z}_\ell}}(T, \mathcal{F}_\eta, \omega) \cdot \det(\mathcal{F}_s)^{-1},$$

where s denotes the closed point of T.

Proof. We verify that the definition is independent of the choice of $(E, \mathcal{F}_E)$. Let $(E', \mathcal{F}_{E'})$ be another choice. Then there exists a finite extension E'' of both E and E' such that

$$\mathcal{F}_E \otimes_{\mathcal{O}_E} \mathcal{O}_{E''} \cong \mathcal{F}_{E'} \otimes_{\mathcal{O}_{E'}} \mathcal{O}_{E''}.$$

The desired independence therefore follows from Lemma 3.2.4. □

Remark 3.4. *Recently, Guignard has given another definition and construction of local epsilon factors [7], using the Gabber–Katz canonical extension. The local epsilon factors defined in Definition 3.3.2 coincide with his, since both coincide with the local epsilon factor defined via Laumon's local Fourier transform. See [7, Theorem 11.8] and [27, Proposition 8.3].*

Using the local epsilon factors constructed above, we obtain a generalization of Laumon's product formula [15, Théorème (3.2.1.1)] to arbitrary perfect fields.

Theorem 3.5 ([7, Theorem 11.1],[25, Theorem 4.50])**.** *Let X be a smooth connected projective curve over a perfect field k of characteristic $p > 0$. Let $\mathcal{F} \in D^b_c(X, \overline{\mathbb{Z}_\ell})$. Fix a non-trivial character $\mathbb{F}_p \to \overline{\mathbb{Z}_\ell}^\times$ and a non-zero rational 1-form ω on X. Then*

$$\det(R\Gamma(X_{\bar{k}}, \mathcal{F}))^{-1} = \chi_{\mathrm{cyc}}^{-\frac{1}{2}\chi(X)\cdot \mathrm{rk}\,\mathcal{F}} \prod_{x \in |X|} \delta_{x/k}^{a(X_{(x)},\mathcal{F})} \varepsilon(X_{(x)}, \mathcal{F}, \omega) \circ \mathrm{tr}_{x/k}$$

as characters $G_k \to \overline{\mathbb{Z}_\ell}^\times$. Here $\chi(X) = \sum_i (-1)^i \dim \mathrm{H}^i(X_{\bar{k}}, \overline{\mathbb{Q}_\ell})$ denotes the Euler-Poincaré characteristic, $\mathrm{rk}\,\mathcal{F}$ denotes the generic rank of $\mathcal{F}$, $a(X_{(x)}, \mathcal{F}) = \mathrm{rk}\,\mathcal{F}_\eta + \mathrm{Sw}_x\mathcal{F} - \mathrm{rk}\,\mathcal{F}_x$ denotes the Artin conductor. Furthermore, $\mathrm{tr}_{x/k} \colon G_k^{ab} \to G_{k(x)}^{ab}$ denotes the transfer morphism, and $\delta_{x/k}$ denotes the determinant character of the induced representation $\mathrm{Ind}_{G_{k(x)}}^{G_k} \mathbb{Q}_\ell$ of the trivial representation.

Proof. In [7, Theorem 11.1], Guignard proves the theorem for objects of $D^b_c(X, \overline{\mathbb{Q}_\ell})$. The present theorem follows by applying his result to $\mathcal{F} \otimes^L_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell}$, together with the fact that the local epsilon factors of Guignard and Yasuda coincide (see Remark 3.4).

Alternatively, in [25, Theorem 4.50], Yasuda proves the formula in the finite-coefficient case when $\mathcal{F}$ is of the form $j_!\mathcal{G}$, where $j \colon U \to X$ is a dense open immersion and $\mathcal{G}$ is a locally constant sheaf on U. The present theorem also follows from this result by the multiplicativity of local epsilon factors. □

3.1.1 Local epsilon factors in characteristic 0

We define local epsilon factors in the case of characteristic 0, at the cost of losing information up to roots of unity.

We begin by reviewing the theory of Jacobi sum characters following [18], since it will be needed in the construction. Let S be an affine (not necessarily noetherian) normal scheme on which ℓ is invertible. Consider a pair (T, χ), where $T = (T_i)_i$ is a finite family of finite étale S-schemes T_i and $\chi = (\chi_i)_i$ is a family of characters

$$\chi_i \colon \mathbb{Z}/d_i\mathbb{Z}(1) \to \overline{\mathbb{Q}_\ell}^\times$$

of étale sheaves on T_i. Here $\overline{\mathbb{Q}_\ell}^\times$ is regarded as a constant étale sheaf. The integers $d_i \geq 1$ are assumed to be invertible on S, and we assume that $\mathbb{Z}/d_i\mathbb{Z}(1) \cong \mathbb{Z}/d_i\mathbb{Z}$ as étale sheaves on T_i.

Define the character

$$N_{T_i/S}(\chi_i) \colon \mathbb{Z}/d_i\mathbb{Z}(1) \to \overline{\mathbb{Q}_\ell}^\times$$

of étale sheaves on S to be the composite morphism

$$\mathbb{Z}/d_i\mathbb{Z}(1) \to f_{i*}\mathbb{Z}/d_i\mathbb{Z}(1) \xrightarrow{f_{i*}\chi_i} f_{i*}\overline{\mathbb{Q}_\ell}^\times \xrightarrow{\mathrm{tr}} \overline{\mathbb{Q}_\ell}^\times,$$

where $f_i \colon T_i \to S$ denotes the structure morphism, and the first arrow is the unit of the adjunction. For any integer $N \geq 1$ divisible by d_i and invertible on S, we regard $N_{T_i/S}(\chi_i)$ as a character of $\mathbb{Z}/N\mathbb{Z}(1)$ via the surjection

$$\mathbb{Z}/N\mathbb{Z}(1) \to \mathbb{Z}/d_i\mathbb{Z}(1), \qquad a \mapsto a^{\frac{N}{d_i}}.$$

Assume that $\prod_i N_{T_i/S}(\chi_i)$ is trivial, where the product is taken in the group of characters of $\mathbb{Z}/N\mathbb{Z}(1)$ for some common multiple N of d_i. In this case, (T, χ) is called a Jacobi datum in [18, Section 1]. When S is the spectrum of a finite field k with q elements, Saito [18, Section 2] associates to a Jacobi datum $(T, \chi) = ((T_i)_i, (\chi_i)_i)$ a Jacobi sum $j_\chi \in \overline{\mathbb{Q}_\ell}^\times$ defined by

$$j_\chi := \prod_i (\prod_j \tau_{k_{ij}}(\bar{\chi}_{ij}, \psi_0 \circ \mathrm{tr}_{k_{ij}/k})). \tag{3.1}$$

Here, for each i, we write $T_i = \coprod_j \mathrm{Spec}(k_{ij})$, where k_{ij} are finite fields with q_{ij} elements. The character $\bar{\chi}_{ij} : k_{ij}^\times \to \overline{\mathbb{Q}_\ell}^\times$ is defined by $a \mapsto \chi_i(a^{(q_{ij}-1)/d_i})$, and $\psi_0 : k \to \overline{\mathbb{Q}_\ell}^\times$ is a non-trivial additive character. Note that $q_{ij} - 1$ is divisible by d_i, since $\mathbb{Z}/d_i\mathbb{Z}(1)$ is assumed to be constant on T_i. The Gauss sums are defined by

$$\tau_k(\chi, \psi) = -\sum_{a \in k} \chi^{-1}(a)\psi(a).$$

Since $\prod_i N_{T_i/S}(\chi_i)$ is trivial, the Jacobi sum j_χ is independent of the choice of ψ_0.

Let (T, χ) be a Jacobi datum on an affine normal scheme S. In [18, Proposition 2], Saito constructed a rank-one smooth $\overline{\mathbb{Q}_\ell}$-sheaf J_χ on S, called the Jacobi sum character associated with $T, \chi)$. It is characterized by the following properties.

- For every morphism $f : S' \to S$ of an affine normal schemes, we have $f^* J_\chi \cong J_{f^*\chi}$.
- If $S = \mathrm{Spec}(\mathbb{F}_q)$, then the geometric Frobenius acts on J_χ via multiplication by the Jacobi sum j_χ defined in (3.1).

We now define local epsilon factors for tamely ramified representations using Jacobi sum characters. Let k be a perfect field of characteristic $p \geq 0$ different from ℓ. Fix an algebraic closure $\bar{k}$ of k, and let $I := \varprojlim_n \mu_n(\bar{k})$, where n runs over the positive integers invertible in k, and $\mu_n(\bar{k})$ denotes the group of n-th roots of unity in $\bar{k}$. The group I is naturally equipped with an action of $G_k = \mathrm{Gal}(\bar{k}/k)$.

Let V be a finite free $\overline{\mathbb{Z}_\ell}$-module equipped with a continuous homomorphism

$$\rho : I \to \mathrm{GL}(V),$$

where $\mathrm{GL}(V)$ is endowed with the topology induced by the ℓ-adic topology on $\overline{\mathbb{Z}_\ell}$. Set $V_{\overline{\mathbb{Q}_\ell}} := V \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell}$. For $\sigma \in G_k$, let $\sigma^* V_{\overline{\mathbb{Q}_\ell}}$ denote the representation of I obtained by composing ρ with the action of σ on I.

When ρ factors through the quotient $I \to \mu_n(\bar{k})$ for some n, the representation $\sigma^* V_{\overline{\mathbb{Q}_\ell}}$ depends only on the image of σ in $\mathrm{Gal}(k(\mu_n(\bar{k}))/k)$. In this case, for $\tau \in \mathrm{Gal}(k(\mu_n(\bar{k}))/k)$, we write $\tau^* V_{\overline{\mathbb{Q}_\ell}}$ for $\sigma^* V_{\overline{\mathbb{Q}_\ell}}$, where $\sigma \in G_k$ is a any lift of τ.

Assume that, for every $\sigma \in G_k$, we have $\sigma^* V_{\overline{\mathbb{Q}_\ell}} \cong V_{\overline{\mathbb{Q}_\ell}}$ and that $V_{\overline{\mathbb{Q}_\ell}}$ is potentially unipotent, i.e., there exists an open subgroup $I' \subset I$ that acts unipotently on $V_{\overline{\mathbb{Q}_\ell}}$. Then the semisimplification $V^{ss}_{\overline{\mathbb{Q}_\ell}}$ admits a decomposition

$$V^{ss}_{\overline{\mathbb{Q}_\ell}} \cong \bigoplus_i (\bigoplus_{\tau \in \mathrm{Gal}(k_i/k)} \tau^* \chi_i), \tag{3.2}$$

where $\chi_i : \mu_{d_i}(\bar{k}) \hookrightarrow \overline{\mathbb{Q}_\ell}^\times$ is an injective character and $k_i := k(\mu_{d_i}(\bar{k}))$. Such a decomposition is unique up to permutation of the summands. Note that the determinant character $\det(V_{\overline{\mathbb{Q}_\ell}}) = \det(V)$ equals $\prod_i N_{k_i/k}(\chi_i)$.

Definition 3.6. *Let the notation be as above. Assume that $\sigma^* V_{\overline{\mathbb{Q}_\ell}}$ is isomorphic to $V_{\overline{\mathbb{Q}_\ell}}$ for every $\sigma \in G_k$ and that $V_{\overline{\mathbb{Q}_\ell}}$ is potentially unipotent. Let μ denote the group of roots of unity in $\overline{\mathbb{Z}_\ell}^\times$.*

1. *Assume that the determinant character $\det(V)$ of I is trivial. We define $J(V)$ to be the Jacobi sum character of the Jacobi datum $((\mathrm{Spec}(k_i))_i, (\chi_i)_i)$. Thus $J(V)$ is a character $G_k \to \overline{\mathbb{Z}_\ell}^\times$.*

2. *In general, choose an integer $n \geq 1$ such that $\det(V^{\oplus n})$ is trivial. We define*

$$J(V) := \iota_n \circ J(V^{\oplus n}),$$

where $\iota_n \colon \overline{\mathbb{Z}_\ell}^\times \to \overline{\mathbb{Z}_\ell}^\times/\mu$ is the map defined by $x \mapsto \sqrt[n]{x}$. Then $J(V)$ is a group homomorphism $G_k \to \overline{\mathbb{Z}_\ell}^\times/\mu$. Moreover, it is independent of the choice of n.

$J(V) \colon G_k \to \overline{\mathbb{Z}_\ell}^\times/\mu$ is admissible in the sense of Subsection 4.1.

Let T be the henselization of $\mathbb{A}^1_k$ at the origin. Let η be the generic point of T, and fix a separable closure $\overline{k(\eta)}$ of $k(\eta)$. Let $\bar{k}$ be the algebraic closure of k in $\overline{k(\eta)}$.

Let I be the tame inertia group of $G_\eta = \mathrm{Gal}(\overline{k(\eta)}/k(\eta))$. Then I is canonically isomorphic to $\varprojlim_{p \nmid n} \mu_n(\bar{k})$. Let V be a smooth sheaf of finite free $\overline{\mathbb{Z}_\ell}$-modules on η which is tamely ramified. Then, viewed as a representation of I, the geometric stalk $V_{\bar{\eta}}$ is isomorphic to $\sigma^* V_{\bar{\eta}}$ for every $\sigma \in G_k$.

We now define local epsilon factors modulo roots of unity, particularly in characteristic 0.

Definition 3.7. *Let the notation be as above. Let V be a smooth sheaf of finite free $\overline{\mathbb{Z}_\ell}$-modules on η which is tamely ramified and potentially unipotent. Then there exists an integer $n \geq 1$ such that $\det(V)^n$ is unramified. We use the same notation $\det(V)^n$ for the corresponding character $G_k \to \overline{\mathbb{Z}_\ell}^\times$. We define the local epsilon factor*

$$\bar{\varepsilon}_0(T, V) \colon G_k \to \overline{\mathbb{Z}_\ell}^\times/\mu$$

by

$$\bar{\varepsilon}_0(T, V) := \iota_n \circ (\det(V)^n) \cdot J(V_{\bar{\eta}}).$$

Here ι_n is as in Definition 3.6.2. The definition is independent of the choice of n.

Lemma 3.8. *Let V and W be smooth sheaves of finite free $\overline{\mathbb{Z}_\ell}$-modules on η. Assume that V is unramified and that W is tamely ramified and potentially unipotent. Then*

$$\bar{\varepsilon}_0(T, V \otimes W) = (\det V)^{\mathrm{rk}(W)} \cdot \bar{\varepsilon}_0(T, W)^{\mathrm{rk}(V)}.$$

Here, in the right-hand side, $\det V$ denotes the character of G_k induced by the unramified character $\det V$ of G_η.

Proof. Since V is unramified, we have $J((V \otimes W)_{\bar{\eta}}) = J(W_{\bar{\eta}})^{\mathrm{rk}(V)}$. On the other hand, we have $\det(V \otimes W) = (\det V)^{\mathrm{rk}(W)} \cdot (\det W)^{\mathrm{rk}(V)}$. The assertion follows. $\square$

Lemma 3.9. *Assume that k is a perfect field of characteristic $p > 0$. Let V be a smooth sheaf of finite free $\overline{\mathbb{Z}_\ell}$-modules on η which is tamely ramified and potentially unipotent. Let π be a uniformizer of T.*

Then the character $\overline{\varepsilon}_0(T, V)$ defined in Definition 3.7 coincides with the composite

$$G_k^{ab} \xrightarrow{\varepsilon_0(T,V,d\pi)} \overline{\mathbb{Z}_\ell}^\times \to \overline{\mathbb{Z}_\ell}^\times/\mu,$$

where $\varepsilon_0(T, V, d\pi)$ is the character defined in Definition 3.3.1, and the second arrow is the natural quotient map.

Proof. Put $V_{\overline{\mathbb{Q}_\ell}} := V \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell}$. We may assume that $V_{\overline{\mathbb{Q}_\ell}}$ is irreducible. Let

$$\chi \colon I \to \overline{\mathbb{Z}_\ell}^\times$$

be a character appearing in the decomposition of $V_{\overline{\mathbb{Q}_\ell}}$ as a representation of I. Since V is potentially unipotent, the character χ has finite image. Let n be the order of this image. Then n is prime to p, and χ factors as

$$I \to \mu_n(\bar{k}) \hookrightarrow \overline{\mathbb{Z}_\ell}^\times,$$

where the first map is the canonical projection.

Let η_n be the unramified extension of η whose residue field is $k(\mu_n(\bar{k}))$. Since G_{η_n} acts trivially on $\mu_n(\bar{k})$, the χ-isotypic component V_χ of $V_{\overline{\mathbb{Q}_\ell}}$ is stable under the action of G_{η_n}. Moreover, we have

$$V_{\overline{\mathbb{Q}_\ell}} \cong \operatorname{Ind}_{G_{\eta_n}}^{G_\eta} V_\chi.$$

By the induction formula of local epsilon factors with respect to unramified extensions, we may reduce to the case when $\eta_n = \eta$.

Assume that $\eta_n = \eta$. Define a character

$$\chi_1 \colon G_\eta \to \mathrm{Gal}(k(\eta)[\pi^{\frac{1}{n}}]/k(\eta)) \xrightarrow{\cong} \mu_n(\bar{k}) \xrightarrow{\chi} \overline{\mathbb{Z}_\ell}^\times,$$

where the first arrow is the canonical surjection and the second arrow is the canonical isomorphism. Then

$$V_{\overline{\mathbb{Q}_\ell}} \cong \chi_1 \otimes V_0$$

for some unramified representation V_0 of G_η. By [15, Proposition (2.5.3.1)], we have

$$F^{(0,\infty)}(V_{\overline{\mathbb{Q}_\ell}}) \cong V_{\overline{\mathbb{Q}_\ell}} \otimes G(\chi, \psi),$$

where $F^{(0,\infty)}$ denotes the local Fourier transform defined in [15, Définition (2.4.2.3)], and $G(\chi, \psi)$ is the one-dimensional representation corresponding to the Gauss sum $\tau_k(\chi^{-1}, \psi)$. See loc. cit. for the precise definitions.

Taking determinants, we obtain

$$\det F^{(0,\infty)}(V_{\overline{\mathbb{Q}_\ell}}) \cong \det(V_{\overline{\mathbb{Q}_\ell}}) \otimes G(\chi, \psi)^{\dim V_{\overline{\mathbb{Q}_\ell}}}.$$

On the other hand, if $m \geq 1$ is an integer such that $(\det V)^m$ is unramified, then $J(V^{\oplus m}) \cong G(\chi, \psi)^{\otimes m \cdot \mathrm{rk}\, V}$. The assertion now follows from Laumon's formula [15, Théorème (3.5.1.1)]. $\square$

3.2 Reminder on oriented products and the local Fourier transform ([22])

In this preliminary subsection, we briefly recall the notions of oriented products and the local Fourier transform, formulated in terms of oriented products, following [22]. The material reviewed here will be used only in Subsections 3.3 and 3.4.

For morphisms $f\colon X \to S$ and $g\colon Y \to S$ of topoi, the oriented product $X \overset{\leftarrow}{\times}_S Y$ is a topos equipped with projection morphisms $p_X\colon X \overset{\leftarrow}{\times}_S Y \to X$ and $p_Y\colon X \overset{\leftarrow}{\times}_S Y \to Y$ and a natural transformation $\sigma\colon gp_Y \to fp_X$. It is characterized by a universal property with respect to such data. We refer to [11] for its precise definition and basic properties.

Let Λ be a finite local ring whose characteristic is invertible on all schemes considered below. Let $f\colon X \to S$ be a morphism of schemes. By abuse of notation, we write X and S for their associated étale topoi . By the universal property of $X \overset{\leftarrow}{\times}_S S$, there exists a morphism of topoi $\Psi_f\colon X \to X \overset{\leftarrow}{\times}_S S$ fitting into the following 2-commutative diagram:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{\mathrm{id}_X} & \downarrow{\scriptstyle \Psi_f} & \searrow^{f} & \\ X & \xleftarrow{p_X} & X \overset{\leftarrow}{\times}_S S & \xrightarrow{p_S} & S. \end{array}$$

The derived functor $R\Psi_f\colon D^+(X,\Lambda) \to D^+(X \overset{\leftarrow}{\times}_S S, \Lambda)$ is called *the nearby cycles functor*. Since $p_X \circ \Psi_f = \mathrm{id}_X$, adjunction yields a natural morphism $p_X^* \to R\Psi_f$. *The vanishing cycles functor* $R\Phi_f\colon D^+(X,\Lambda) \to D^+(X \overset{\leftarrow}{\times}_S S, \Lambda)$ is a triangulated functor fitting into a distinguished triangle

$$p_X^* \to R\Psi_f \to R\Phi_f \to . \tag{3.3}$$

A cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & S' \\ \downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S \end{array}$$

of schemes induces a morphism of topoi

$$\overset{\leftarrow}{g}\colon X' \overset{\leftarrow}{\times}_{S'} S' \to X \overset{\leftarrow}{\times}_S S.$$

For $K \in D^+(X,\Lambda)$, there is a canonical base change morphism $\overset{\leftarrow}{g}{}^* R\Psi_f(K) \to R\Psi_{f'}(g^*K)$.

The relation between $R\Psi_f$, $R\Phi_f$, and the classical nearby and vanishing cycles formalisms of [2] is explained in [10, §1.2].

When S is a henselian trait with closed point s and generic point η, the closed subtopos $X_s \overset{\leftarrow}{\times}_S S \subset X \overset{\leftarrow}{\times}_S S$, where X_s denotes the special fiber, is canonically identified with the topos $X_s \times_s S$ appearing in [2]. Moreover, the restrictions of $R\Psi_f$ and $R\Phi_f$ to the open subtopos $X_s \overset{\leftarrow}{\times}_S \eta \subset X_s \overset{\leftarrow}{\times}_S S$ recover the classical nearby and vanishing cycles functors.

In Definition 3.10 below, we recall the generalization of the local Fourier transforms introduced in [15] to the relative setting using oriented product. We refer to [22, Section 3] for a detailed account.

Let S be a noetherian scheme over $\mathbb{F}_p$. For a non-trivial character $\psi\colon \mathbb{F}_p \to \Lambda^\times$, let $\mathcal{L}_\psi(x)$ denote the Artin–Schreier sheaf on $\mathbb{A}^1_{\mathbb{F}_p}$. We write $\mathcal{L}_\psi(x \cdot x')$ for its pullback along the multiplication morphism $\mathbb{A}^1_{\mathbb{F}_p} \times_{\mathbb{F}_p} \mathbb{A}^1_{\mathbb{F}_p} \to \mathbb{A}^1_{\mathbb{F}_p}$, where x and x' denote the standard coordinates of the first and second factors, respectively. By abuse of notation, we use the same symbol for its pullback to $\mathbb{A}^1_S \times_S \mathbb{A}^1_S$ via the canonical morphism

$$\mathbb{A}^1_S \times_S \mathbb{A}^1_S \to \mathbb{A}^1_{\mathbb{F}_p} \times_{\mathbb{F}_p} \mathbb{A}^1_{\mathbb{F}_p}.$$

We denote by 0_S (resp. ∞_S) the section $S \to \mathbb{A}^1_S$ (resp. $S \to \mathbb{P}^1_S$) corresponding to the origin (resp. the point at infinity). Consider the morphisms of topoi

$$0_S \overleftarrow{\times}_{\mathbb{A}^1_S} \mathbb{A}^1_S \xleftarrow{\overleftarrow{p}_1} (0_S \times_S \infty_S) \overleftarrow{\times}_{\mathbb{A}^1_S \times_S \mathbb{P}^1_S} (\mathbb{A}^1_S \times_S \mathbb{A}^1_S) \xrightarrow{\overleftarrow{p}_2} \infty_S \overleftarrow{\times}_{\mathbb{P}^1_S} \mathbb{A}^1_S$$

induced by the projections $\mathbb{A}^1_S \xleftarrow{p_1} \mathbb{A}^1_S \times_S \mathbb{P}^1_S \xrightarrow{p_2} \mathbb{P}^1_S$. Let

$$q\colon (0_S \times_S \infty_S) \overleftarrow{\times}_{\mathbb{A}^1_S \times_S \mathbb{P}^1_S} (\mathbb{A}^1_S \times_S \mathbb{A}^1_S) \to \mathbb{A}^1_S \times_S \mathbb{A}^1_S$$

denote the second projection.

For the notion of constructibility for sheaves on oriented products, we refer to [16, 9.1] or [10, 1.6].

Definition 3.10 ([22, Definition 3.3]). *Let $K \in D_{\mathrm{ctf}}(0_S \overleftarrow{\times}_{\mathbb{A}^1_S} \mathbb{G}_{m,S}, \Lambda)$ be a constructible complex of finite tor-dimension whose cohomology sheaves are locally constant. We define the local Fourier transform $F^{(0,\infty)}(K)$ by*

$$F^{(0,\infty)}(K) := R\overleftarrow{p}_{2*}(\overleftarrow{p}_1^{*} K_! \otimes^L_\Lambda q^*\mathcal{L}_\psi(x \cdot x'))[1] \in D^b(\infty_S \overleftarrow{\times}_{\mathbb{P}^1_S} \mathbb{A}^1_S, \Lambda).$$

Here $K_!$ denotes the extension by zero of K along $0_S \overleftarrow{\times}_{\mathbb{A}^1_S} \mathbb{G}_{m,S} \hookrightarrow 0_S \overleftarrow{\times}_{\mathbb{A}^1_S} \mathbb{A}^1_S$.

Proposition 3.11 (cf. [15, Proposition (2.4.2.2)]). *Let the notation be as above. Let Λ_0 denote the residue field of Λ.*

1. *The local Fourier transform $F^{(0,\infty)}(K)$ has finite tor-dimension. Let $\Lambda \to \Lambda'$ be a local homomorphism of finite local rings. Then the canonical morphism*

$$F^{(0,\infty)}(K) \otimes^L_\Lambda \Lambda' \to F^{(0,\infty)}(K \otimes^L_\Lambda \Lambda')$$

   is a quasi-isomorphism.

2. *The formation of $F^{(0,\infty)}(K)$ commutes with arbitrary base change $g\colon S' \to S$. More precisely, the canonical morphism*

$$g_1{}^* F^{(0,\infty)}(K) \to F^{(0,\infty)}(g_2{}^* K)$$

   is a quasi-isomorphism, where

$$g_1\colon \infty_{S'} \overleftarrow{\times}_{\mathbb{P}^1_{S'}} \mathbb{A}^1_{S'} \to \infty_S \overleftarrow{\times}_{\mathbb{P}^1_S} \mathbb{A}^1_S \qquad \text{and} \qquad g_2\colon 0_{S'} \overleftarrow{\times}_{\mathbb{A}^1_{S'}} \mathbb{G}_{m,S'} \to 0_S \overleftarrow{\times}_{\mathbb{A}^1_S} \mathbb{G}_{m,S}$$

   are the morphisms induced by g.

3. *The local Fourier transform* $F^{(0,\infty)}(K)$ *is constructible. Moreover, it is locally constant if and only if, for every* $i \in \mathbb{Z}$*, the function*

$$S \to \mathbb{Z}, \qquad s \mapsto \operatorname{dimtot} \mathcal{H}^i(K \otimes^L_\Lambda \Lambda_0)|_{\eta_{\bar{s}}}$$

is locally constant. Here $\bar{s}$ *denotes the spectrum of an algebraic closure of* $k(s)$*, and* $\eta_{\bar{s}} \cong 0 \overleftarrow{\times}_{\mathbb{A}^1_{k(\bar{s})}} \mathbb{G}_{m,k(\bar{s})}$ *denotes the generic point of the henselization* $\mathbb{A}^1_{k(\bar{s}),(0)}$ *at the origin.*

4. *When* K *has tor-amplitude in* $[0,0]$*, so does* $F^{(0,\infty)}(K)$*.*

Proof. See [22, Proposition 3.4]. □

Let k be a perfect field of characteristic $p > 0$. Let C be a smooth curve over k, and let $x \in C$ be a closed point. Recall that the oriented product $x \overleftarrow{\times}_C (C \setminus \{x\})$ is canonically equivalent to the étale topos of the generic point of the henselization $C_{(x)}$ ([22, Lemma 2.2]).

Let T and T' be henselian traits isomorphic to the henselization of $\mathbb{A}^1_k$ at the origin. Fix uniformizers π and π' of T and T', respectively. By sending $x \mapsto \pi$ (resp. $x' \mapsto 1/\pi'$), where x (resp. x') denotes the standard coordinate of $\mathbb{A}^1_k$ (resp. $\mathbb{A}^1_k \subset \mathbb{P}^1_k$), we identify T (resp. T') with the henselization of $\mathbb{A}^1_k$ (resp. $\mathbb{P}^1_k$) at 0 (resp. at ∞).

Let η and η' denote the generic points of T and T', respectively. Under the canonical identifications $\eta \cong 0 \overleftarrow{\times}_{\mathbb{A}^1_k} \mathbb{G}_{m,k}$ and $\eta' \cong \infty \overleftarrow{\times}_{\mathbb{P}^1_k} \mathbb{A}^1_k$, we may regard $F^{(0,\infty)}$ as a triangulated functor $D_{\mathrm{ctf}}(\eta, \Lambda) \to D_{\mathrm{ctf}}(\eta', \Lambda)$. It is explained in [22, §3.2] that $F^{(0,\infty)}$ satisfies properties analogous to those established by Laumon in [15]. We refer to [22, §3.2] for details.

3.3 Reduction to the case of positive characteristic

To compute the local epsilon factors of vanishing cycles in characteristic 0, we develop a method that reduces the problem to the positive characteristic case. This method may be compared with spreading-out arguments commonly used for étale sheaves with finite coefficients.

The results of this subsection are needed only in the characteristic 0 case.

Remark 3.12. *The following technique is needed because we work with* ℓ*-adic sheaves. If one could develop a theory of epsilon cycles for local epsilon factors without passing to the quotient modulo roots of unity, and if one could work with* Λ*-sheaves for a finite local ring* Λ*, then this technique would seem unnecessary.*

We begin with some general lemmas.

Fix a prime number ℓ. Let R be a discrete valuation ring in which ℓ is invertible. Denote by K and F its fraction field and residue field, respectively.

Fix a uniformizer $\pi \in R$. For each integer $m \geq 0$, let

$$R_m := R[\pi^{1/\ell^m}],$$

and denote by K_m the fraction field of R_m. We write R_∞ and K_∞ for the unions

$$R_\infty = \bigcup_{m \geq 0} R_m, \qquad K_\infty = \bigcup_{m \geq 0} K_m,$$

respectively. The rings R_m (including the case $m = \infty$) are valuation rings whose residue fields are canonically isomorphic to F.

Let $\mathcal{X}$ be a scheme over R, and let $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Consider the diagram

$$X_m \xrightarrow{j_m} \mathcal{X}_m \xleftarrow{i_m} \mathcal{X}_F, \tag{3.4}$$

where j_m is obtained from

$$X := \mathcal{X} \times_R K \xrightarrow{j} \mathcal{X} \tag{3.5}$$

by base change along $R \to R_m$, and i_m is the canonical lift of the closed immersion $\mathcal{X}_F := \mathcal{X} \times_R F \xrightarrow{i} \mathcal{X}$.

Lemma 3.13. *Let the notation be as above. Let Λ be a finite local ring whose residue characteristic is ℓ.*

1. *Let $C \in D^+(\mathcal{X}, \Lambda)$ be a bounded below complex such that the structure morphism $\mathcal{X} \to \mathrm{Spec}(R)$ is locally acyclic relative to C. Then the canonical morphism*

$$i^* C \to i_\infty^* R j_{\infty *} j_\infty^* C$$

   induced by (3.4) is a quasi-isomorphism.

2. *Assume that $\mathcal{X}$ is of finite type over R. Then the functor $Rj_{\infty *}$ has finite cohomological dimension. Moreover, for every object $C \in D_c^b(X, \Lambda)$, the complex $i_\infty^* Rj_{\infty *}(C|_{X_\infty})$ on $\mathcal{X}_F$ is constructible.*

In the situation of 2, for $C \in D_c^b(X, \Lambda)$, we define $\langle C, -\pi \rangle := i_\infty^* Rj_{\infty *}(C|_{X_\infty})$. When both $\mathcal{X}$ and $\mathcal{X}_F$ are regular, and C is a locally constant sheaf such that the inertia groups at the generic points of the divisor $\mathcal{X}_F$ act through finite ℓ-groups, this notion coincides with that in [22, Definition 2.11].

Proof. We may assume that R is strictly henselian. In particular, F is separably closed. Let $\overline{K}$ be a separable closure of K, and let $\overline{R}$ be the normalization of R in $\overline{K}$. Then the residue field $\overline{F}$ of $\overline{R}$ is an algebraic closure of F.

Similarly to (3.4), we consider the diagram

$$\overline{X} \xrightarrow{\bar{j}} \overline{\mathcal{X}} \xleftarrow{\bar{i}} \mathcal{X}_{\overline{F}},$$

where $\overline{X} := X \times_K \overline{K}$, $\overline{\mathcal{X}} := \mathcal{X} \times_R \overline{R}$, and $\mathcal{X}_{\overline{F}} := \mathcal{X} \times_R \overline{F}$.

Fix an embedding $K_\infty \hookrightarrow \overline{K}$ over K. Then the above schemes fit into a commutative diagram

$$\begin{array}{ccccc}
\overline{X} & \xrightarrow{\bar{j}} & \overline{\mathcal{X}} & \xleftarrow{\bar{i}} & \mathcal{X}_{\overline{F}} \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle f_F} \\
X_\infty & \xrightarrow{j_\infty} & \mathcal{X}_\infty & \xleftarrow{i_\infty} & \mathcal{X}_F.
\end{array}$$

The left square is cartesian. The right square becomes cartesian if we replace $\mathcal{X}_{\overline{F}}$ by $\mathcal{X} \times_R (\overline{R} \otimes_{R_\infty} F)$. Note that the étale topoi of these two schemesare canonically equivalent.

Let $I' = \mathrm{Gal}(\overline{K}/K_\infty)$ be the Galois group of $\overline{K}/K_\infty$. Note that the functor $\Gamma(I', -)$ on discrete $\Lambda[I']$-modules is exact, since every finite quotient of I' has order prime to ℓ.

1. By local acyclicity, the canonical morphism $f_F^* i^* C \to \bar{i}^* R\bar{j}_* \bar{j}^* C$ is an isomorphism. Taking I'-invariants $R\Gamma(I', -) = \Gamma(I', -)$, we obtain an isomorphism

$$\Gamma(I', f_F^* i^* C) \to \Gamma(I', \bar{i}^* R\bar{j}_* \bar{j}^* C).$$

Since the action of I' on $f_F^* i^* C$ is trivial, the source is isomorphic to $f_F^* i^* C$.

On the other hand, we have

$$\bar{i}^* R\bar{j}_* \bar{j}^* C \cong f_F^* i_\infty^* Rj_{\infty *} f_* f^* j_\infty^* C.$$

Hence the target is isomorphic to

$$f_F^* i_\infty^* Rj_{\infty *} \Gamma(I', f_* f^* j_\infty^* C) \cong f_F^* i_\infty^* Rj_{\infty *} j_\infty^* C.$$

This proves the assertion.

2. Let $\mathcal{G}$ be an étale sheaf of Λ-modules on X_∞. We claim that

$$R^n j_{\infty *} \mathcal{G} = 0 \qquad (n > 2 \dim X_\infty).$$

Let $x \to \mathcal{X}_F$ be a geometric point. Then we have an isomorphism

$$(Rj_{\infty *} \mathcal{G})_x \cong \Gamma(I', R\Gamma((X_\infty \times_{\mathcal{X}_\infty} \mathcal{X}_{\infty(x)}) \times_{K_\infty} \overline{K}, \mathcal{G})).$$

Since $\mathrm{H}^n((X_\infty \times_{\mathcal{X}_\infty} \mathcal{X}_{\infty(x)}) \times_{K_\infty} \overline{K}, \mathcal{G}) = 0$ for $n > 2 \dim X_\infty$, it follows that $(R^n j_{\infty *} \mathcal{G})_x = 0$ for $n > 2 \dim X_\infty$. This proves the first assertion.

Let $C \in D_c^b(X, \Lambda)$. We have

$$\begin{aligned} f_F^* i_\infty^* Rj_{\infty *}(C|_{X_\infty}) &\cong \Gamma(I', f_F^* i_\infty^* Rj_{\infty *} f_* f^* (C|_{X_\infty})) \\ &\cong \Gamma(I', \bar{i}^* R\bar{j}_* (C|_{\overline{X}})). \end{aligned}$$

The second assertion follows from the constructibility of the nearby cycles complex $\bar{i}^* R\bar{j}_*(C|_{\overline{X}})$. Indeed, since $\Gamma(I', -)$ is exact, the cohomology sheaf $\mathcal{H}^i(f_F^* i_\infty^* Rj_{\infty *}(C|_{X_\infty}))$ is a subsheaf of $\mathcal{H}^i(\bar{i}^* R\bar{j}_*(C|_{\overline{X}}))$, and is therefore constructible. □

Let $E/\mathbb{Q}_\ell$ be a finite field extension, and let $\mathcal{O}_E$ denote its ring of integers.

Corollary 3.14. *Let the notation be as in Lemma 3.13. Assume that $\mathcal{X}$ is of finite type over R. We use the terminology and notation of Section 6.*

Let $\mathcal{F} \in D(X^{\mathbb{N}^{\mathrm{op}}}, \mathcal{O}_{E,\bullet})$ be a normalized constructible complex on X. Then $i_\infty^* Rj_{\infty *} \mathcal{F} \in D(\mathcal{X}_F^{\mathbb{N}^{\mathrm{op}}}, \mathcal{O}_{E,\bullet})$ is a normalized $\mathcal{O}_E$-complex, and $i_\infty^* Rj_{\infty *} \mathcal{F} \otimes_{\mathcal{O}_E}^L \mathcal{O}_E/\ell\mathcal{O}_E$ is constructible, where the functor $- \otimes_{\mathcal{O}_E} \mathcal{O}_E/\ell\mathcal{O}_E$ is defined in Definition 6.7.2. In other words, $i_\infty^* Rj_{\infty *} \mathcal{F}$ defines a constructible complex of $\mathcal{O}_E$-sheaves on $\mathcal{X}_F$, in the sense of Definition 6.9.2.

We define $\langle \mathcal{F}, -\pi \rangle := i_\infty^* Rj_{\infty *} \mathcal{F}$.

Proof. Let $\mathcal{F}_n := \mathcal{F} \otimes_{\mathcal{O}_E}^L \mathcal{O}_E/\ell^{n+1}\mathcal{O}_E$. By the first assertion of Lemma 3.13.2, we have

$$Rj_{\infty *} \mathcal{F}_{n+1} \otimes_{\mathcal{O}_E/\ell^{n+2}\mathcal{O}_E}^L \mathcal{O}_E/\ell^{n+1}\mathcal{O}_E \cong Rj_{\infty *}(\mathcal{F}_{n+1} \otimes_{\mathcal{O}_E/\ell^{n+2}\mathcal{O}_E}^L \mathcal{O}_E/\ell^{n+1}\mathcal{O}_E) \cong Rj_{\infty *} \mathcal{F}_n.$$

It follows that $i_\infty^* Rj_{\infty *} \mathcal{F}$ is normalized.

By the second assertion of Lemma 3.13.2, the complex $i_\infty^* Rj_{\infty *} \mathcal{F} \otimes_{\mathcal{O}_E}^L \mathcal{O}_E/\ell\mathcal{O}_E \cong i_\infty^* Rj_{\infty *} \mathcal{F}_0$ is constructible. □

Lemma 3.15. *Let the notation be as in Lemma 3.13. Let $r \geq 0$ be an integer. Let $L, N \in D^+(\mathcal{X}_r, \Lambda)$ be bounded below complexes. Assume that $i_r^* N$ is bounded and constructible, and that the structure morphism $\mathcal{X}_r \to \mathrm{Spec}(R_r)$ is locally acyclic relative to L. Let*

$$L|_{X_r} \to M' \to N|_{X_r} \to \tag{3.6}$$

be a distinguished triangle on X_r. Then there exists an integer $n \geq r$ and a distinguished triangle $L|_{\mathcal{X}_n} \to M \to N|_{\mathcal{X}_n} \to$ on $\mathcal{X}_n$ whose pullback to X_n is isomorphic to the pullback of (3.6).

Proof. Let $\phi\colon N|_{X_r} \to L|_{X_r}[1]$ be the morphism corresponding to (3.6). Let

$$C_n := i_{n*} i_n^! L|_{\mathcal{X}_n}[2].$$

Then C_n fits into a distinguished triangle $L|_{\mathcal{X}_n}[1] \to Rj_{n*}L|_{X_n}[1] \to C_n \to$.

It suffices to show that, for some integer $n \geq r$, the composite morphism

$$N|_{\mathcal{X}_n} \to Rj_{n*}N|_{X_n} \xrightarrow{Rj_{n*}\phi} Rj_{n*}L|_{X_n}[1] \to C_n$$

vanishies. Since C_n is supported on $\mathcal{X}_F$, it suffices to show that the restriction

$$i_r^* N \cong i_n^* N|_{\mathcal{X}_n} \to i_n^* C_n$$

vanishes. Since $\mathcal{X}_r \to \mathrm{Spec}(R_r)$ is locally acyclic relative to L, the colimit $\varinjlim_{n \geq r} i_n^* C_n$ is acyclic by Lemma 3.13.1 applied to $C = L$. Since $i_r^* N$ is constructible, it follows that the above morphism vanishes for sufficiently large n. □

Let S be a regular connected scheme of finite type over $\mathbb{Z}[1/\ell]$. Let k be the perfection of the function field of S.

Let $s \in S$ be a closed point, and let S' be the blow-up of S at s. Denote by $\mathfrak{s}$ the generic point of the exceptional divisor. Let R be the henselization of $\mathcal{O}_{S',\mathfrak{s}}$, and fix a uniformizer $\pi \in R$.

We follow the notation introduced above. Thus, for each integer $m \geq 0$, we set $R_m = R[\pi^{1/\ell^m}]$. We denote by K_m the fraction field of R_m. We write $R_\infty := \varinjlim_m R_m$ and $K_\infty := \varinjlim_m K_m$. The rings R_m are valuation rings whose residue fields are canonically isomorphic to $k(\mathfrak{s})$.

Lemma 3.16. *The conjugates of the images of $G_{K_\infty} \to G_k$, for all closed points $s \in S$ and all uniformizers $\pi \in R$, topologically generate G_k.*

Proof. Let H be a finite quotient of G_k. After shrinking S, the quotient map $G_k \to H$ factors through $\pi_1(S)$. By the Chebotarev density theorem, the group H is generated by the geometric Frobenius elements at the closed points $s \in S$. Since the composite homomorphism $G_{K_\infty} \to G_k \to \pi_1(S)$ factors as

$$G_{K_\infty} \to G_{k(\mathfrak{s})} \to \pi_1(s) \to \pi_1(S),$$

and the homomorphism $G_{K_\infty} \to \pi_1(s)$ is surjective, the assertion follows. □

Consider the following commutative diagram

$$
\begin{array}{ccccccc}
Z & \hookrightarrow & U & \xrightarrow{\quad f \quad} & Y & \xrightarrow{t} & \mathbb{A}^1_k \\
& & & \searrow^{g} \quad \swarrow & & \swarrow & \\
& & & \mathrm{Spec}(k) & & &
\end{array}
\tag{3.7}
$$

of k-schemes of finite type, and let $\mathcal{F} \in D^b_c(U, \mathcal{O}_E)$. Assume that the following conditions are satisfied:

1. The morphism $t \colon Y \to \mathbb{A}^1_k$ is étale. The scheme U is smooth over k, and Z is a closed subscheme of U that is finite étale over k.

2. The morphism $f|_{U \setminus Z}$ is $SS(\mathcal{F})$-transversal in the sense of Definition 2.17.

Assume that the data above, except for $\mathcal{F}$, are defined over S. More precisely, suppose that there exists a commutative diagram

$$
\begin{array}{ccccccc}
\mathcal{Z} & \hookrightarrow & \mathcal{U} & \xrightarrow{\quad \tilde{f} \quad} & \mathcal{Y} & \xrightarrow{\tilde{t}} & \mathbb{A}^1_S \\
& & & \searrow^{\tilde{g}} \quad \swarrow & & \swarrow & \\
& & & S & & &
\end{array}
\tag{3.8}
$$

of S-schemes of finite type whose base change along $\mathrm{Spec}(k) \to S$ is identified with (3.7).

Assume moreover that there exists $\tilde{\mathcal{F}}_0 \in D_{\mathrm{ctf}}(\mathcal{U}, \mathcal{O}_E/\ell\mathcal{O}_E)$ whose pullback to U is isomorphic to $\mathcal{F} \otimes^L_{\mathcal{O}_E} \mathcal{O}_E/\ell\mathcal{O}_E$. We further assume that the above data satisfy the following conditions.

1. The morphism $\tilde{t} \colon \mathcal{Y} \to \mathbb{A}^1_S$ is étale. The scheme $\mathcal{U}$ is smooth over S, and $\mathcal{Z}$ is a closed subscheme of $\mathcal{U}$ that is finite étale over S.

2. The relative singular support $SS(\tilde{\mathcal{F}}_0, \mathcal{U}/S)$ exists and satisfies condition 2 of Theorem 2.7. In particular, $\tilde{g}$ is universally locally acyclic relative to $\tilde{\mathcal{F}}_0$ (cf. Remark 2.8).

3. The morphism $\tilde{f}|_{\mathcal{U} \setminus \mathcal{Z}}$ is $SS(\tilde{\mathcal{F}}_0, \mathcal{U}/S)$-transversal.

4. The restriction of $R\Phi_{\tilde{t} \circ \tilde{f}}(\tilde{\mathcal{F}}_0)$ to

$$\mathcal{Z} \overleftarrow{\times}_{\mathbb{A}^1_S} (\mathbb{A}^1_S \setminus \tilde{t} \circ \tilde{f}(\mathcal{Z})) \subset \mathcal{Z} \overleftarrow{\times}_{\mathbb{A}^1_S} \mathbb{A}^1_S \cong \mathcal{Z} \overleftarrow{\times}_{\mathbb{A}^1_{\mathcal{Z}}} \mathbb{A}^1_{\mathcal{Z}}$$

   is locally constant. Moreover, for each $i \in \mathbb{Z}$, the function

$$z \mapsto \mathrm{dimtot}\, R^i\Phi_{\tilde{t} \circ \tilde{f}}(\tilde{\mathcal{F}}_0 \otimes^L_{\mathcal{O}_E/\ell\mathcal{O}_E} \mathbb{F})|_{\eta_{\bar{z}}}$$

   on $\mathcal{Z}$ is locally constant (cf. Proposition 3.11.3). Here $\mathbb{F}$ denotes the residue field of $\mathcal{O}_E$. For a point $z \in \mathcal{Z}$, we write $\bar{z}$ for the spectrum of an algebraic closure of $k(z)$ and $\eta_{\bar{z}}$ for the generic point of the strict henselization of $\mathbb{A}^1_{k(\bar{z})}$ at $\bar{z} \xrightarrow{\tilde{t} \circ \tilde{f}} \mathbb{A}^1_{k(\bar{z})}$.

Let

$$\mathcal{Z}_{\mathfrak{s}} \hookrightarrow \mathcal{U}_{\mathfrak{s}} \xrightarrow{\tilde{f}_{\mathfrak{s}}} \mathcal{Y}_{\mathfrak{s}}$$

denote the base change of $\mathcal{Z} \hookrightarrow \mathcal{U} \to \mathcal{Y}$ along $\mathfrak{s} \to S$. For $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, let

$$U_m \to \mathcal{U}_m \leftarrow \mathcal{U}_{\mathfrak{s}}$$

be obtained from $\mathcal{U}$ by base change along $\operatorname{Spec}(K_m) \to \operatorname{Spec}(R_m) \leftarrow \mathfrak{s}$ over S.

Proposition 3.17. *Let the notation be as above. Let $s \in S$ be a closed point. Denote by S' the blow-up of S at s, and by $\mathfrak{s}$ the generic point of the exceptional divisor. Let R be the henselization of the local ring $\mathcal{O}_{S',\mathfrak{s}}$. Fix a uniformizer π of R, and let K_∞ be the fraction field of*

$$\bigcup_{m \geq 0} R[\pi^{1/\ell^m}].$$

If k is of positive characteristic, then there exists a commutative diagram

$$\begin{array}{ccc} G_k & \longrightarrow & \overline{\mathbb{Z}_\ell}^\times \\ \uparrow & & \uparrow \\ G_{K_\infty} & \longrightarrow & G_{k(\mathfrak{s})}. \end{array} \tag{3.9}$$

Here the top horizontal arrow is given by

$$\prod_{z \in Z} \varepsilon_0(Y_{(z)}, R\Phi_f(\mathcal{F})_z, dt) \circ \operatorname{tr}_{z/k}$$

and the right vertical arrow is given by

$$\prod_{z \in \mathcal{Z}_{\mathfrak{s}}} \varepsilon_0(\mathcal{Y}_{\mathfrak{s},(z)}, R\Phi_{\tilde{f}_{\mathfrak{s}}}(\langle \mathcal{F}, -\pi \rangle)_z, d\tilde{t}) \circ \operatorname{tr}_{z/\mathfrak{s}}\,.$$

The object $\langle \mathcal{F}, -\pi \rangle$ is defined immediately after the statement of Corollary 3.14.

When k is of characteristic 0, there is a similar commutative diagram to (3.9), with ε_0 and $\overline{\mathbb{Z}_\ell}^\times$ replaced by $\overline{\varepsilon}_0$ (Definition 3.7) and $\overline{\mathbb{Z}_\ell}^\times/\mu$, respectively.

Proof. In the course of the proof, we use the notion of oriented products and of local Fourier transforms in the relative setting, which we briefly recall in Subsection 3.2.

Replacing S and the closed immersion $\mathcal{Z} \hookrightarrow \mathcal{U}$ by $\mathcal{Z}$ and the graph embedding $\mathcal{Z} \hookrightarrow \mathcal{Z} \times_S \mathcal{U}$, respectively, we may assume that $\mathcal{Z} \to S$ is an isomorphism. We may also replace $(\mathcal{Y}, \tilde{t})$ by $(\mathbb{A}^1_S, \mathrm{id})$, as the local epsilon factors in the statement do not change. After a change of coordinates on $\mathbb{A}^1_S$, we may assume that $S \cong \mathcal{Z} \xrightarrow{\tilde{f}|_{\mathcal{Z}}} \mathbb{A}^1_S$ is the origin.

By induction on $m \geq 0$, we find a sequence of integers $0 = n_0 \leq n_1 \leq n_2 \leq \cdots$ and objects $\tilde{\mathcal{F}}_m \in D_{\mathrm{ctf}}(\mathcal{U}_{n_m}, \mathcal{O}_E/\ell^{m+1}\mathcal{O}_E)$ such that $\tilde{\mathcal{F}}_{m+1} \otimes^L_{\mathcal{O}_E/\ell^{m+2}\mathcal{O}_E} \mathcal{O}_E/\ell^{m+1}\mathcal{O}_E \cong \tilde{\mathcal{F}}_m|_{\mathcal{U}_{n_{m+1}}}$.

Assume that n_m and $\tilde{\mathcal{F}}_m$ have been constructed. Applying Lemma 3.15 with $L = \tilde{\mathcal{F}}_0$ and $N = \tilde{\mathcal{F}}_m$, we obtain an integer $n \geq n_m$ and an object $\tilde{\mathcal{F}}_{m+1} \in D^b_c(\mathcal{U}_n, \mathcal{O}_E/\ell^{m+2}\mathcal{O}_E)$ fitting into a distinguished triangle

$$\tilde{\mathcal{F}}_0 \to \tilde{\mathcal{F}}_{m+1} \to \tilde{\mathcal{F}}_m \to$$

whose restriction to U_n is isomorphic to the pullback of

$$\mathcal{F}_0 \to \mathcal{F}_{m+1} \to \mathcal{F}_m \to .$$

We show that $\tilde{\mathcal{F}}_{m+1}$ and $n_{m+1} = n$ satisfy the required properties. By induction, the morphism $\mathcal{U}_n \to \operatorname{Spec}(R_n)$ is universally locally acyclic relative to both $\tilde{\mathcal{F}}_m$ and $\tilde{\mathcal{F}}_0$. It follows that it is also universally locally acyclic relative to $\tilde{\mathcal{F}}_{m+1}$. We claim that the canonical morphism

$$\tilde{\phi}\colon \tilde{\mathcal{F}}_{m+1} \otimes^L_{\mathcal{O}_E/\ell^{m+2}\mathcal{O}_E} \mathcal{O}_E/\ell^{m+1}\mathcal{O}_E \to \tilde{\mathcal{F}}_m$$

is a quasi-isomorphism. This will also imply that $\tilde{\mathcal{F}}_{m+1} \in D_{\mathrm{ctf}}(\mathcal{U}_n, \mathcal{O}_E/\ell^{m+2}\mathcal{O}_E)$.

We prove that $\tilde{\phi}$ is a quasi-isomorphism. The restriction $\tilde{\phi}|_{U_n}$ is identified with the canonical morphism $\phi\colon \mathcal{F}_{m+1} \otimes^L_{\mathbb{Z}_\ell/\ell^{m+2}\mathcal{O}_E} \mathcal{O}_E/\ell^{m+1}\mathcal{O}_E \to \mathcal{F}_m$, which is a quasi-isomorphism since $\mathcal{F}$ is normalized.

On the other hand, by Lemma 3.13.1 and the fact that $\mathcal{U}_n \to \operatorname{Spec}(R_n)$ is locally acyclic relative to $\tilde{\mathcal{F}}_i$ for $i = m, m+1$, the canonical morphism

$$\tilde{\mathcal{F}}_i|_{\mathcal{U}_{\mathfrak{s}}} \to i^*_\infty Rj_{\infty *}\mathcal{F}_i =: \langle \mathcal{F}_i, -\pi\rangle$$

is a quasi-isomorphism. Therefore, the restriction $i_n^*\tilde{\phi}$ to the special fiber is identified with

$$i^*_\infty Rj_{\infty *}\mathcal{F}_{m+1} \otimes^L_{\mathcal{O}_E/\ell^{m+2}\mathcal{O}_E} \mathcal{O}_E/\ell^{m+1}\mathcal{O}_E \to i^*_\infty Rj_{\infty *}\mathcal{F}_m.$$

Since the cohomological dimension of $Rj_{\infty *}$ is finite, this morphism is further identified with $i^*_\infty Rj_{\infty *}\phi$. As ϕ is a quasi-isomorphism, so is $i_n^*\tilde{\phi}$. Since $\tilde{\phi}$ is a quasi-isomorphism on both the generic and special fibers, it follows that $\tilde{\phi}$ itself is a quasi-isomorphism. This proves the claim.

Choose one integer m and put $n = n_m$. Let $\tilde{f}_n\colon \mathcal{U}_n \to \mathbb{A}^1_{R_n}$ be the base change of $\tilde{f}$. By the commutativity of the formation of $R\Phi_f$ ([16, Proposition 6.1]), the restrictions of $R\Phi_{\tilde{f}_n}\tilde{\mathcal{F}}_m$, viewed as a complex on $0_{R_n} \overleftarrow{\times}_{\mathbb{A}^1_{R_n}} \mathbb{G}_{m,R_n}$, to

$$0_{\mathfrak{s}} \overleftarrow{\times}_{\mathbb{A}^1_{\mathfrak{s}}} \mathbb{G}_{m,\mathfrak{s}} \quad \text{and} \quad 0_{K_n} \overleftarrow{\times}_{\mathbb{A}^1_{K_n}} \mathbb{G}_{m,K_n}$$

are identified with $R\Phi_{\tilde{f}_{\mathfrak{s}}}\langle \mathcal{F}_m, -\pi\rangle$ and $R\Phi_f\mathcal{F}_m$, respectively. By assumption 4, $R\Phi_{\tilde{f}_n}\tilde{\mathcal{F}}_m$ is locally constant on $0_{R_n} \overleftarrow{\times}_{\mathbb{A}^1_{R_n}} \mathbb{G}_{m,R_n}$, and its total dimension is locally constant. In particular, if the generic fiber $R\Phi_f\mathcal{F}_m$ is tamely ramified, then so is $R\Phi_{\tilde{f}_{\mathfrak{s}}}\langle \mathcal{F}_m, -\pi\rangle$.

We prove the assertion in the case where k is of positive characteristic. By Proposition 3.11, the complex $F^{(0,\infty)}(R\Phi_{\tilde{f}_n}\tilde{\mathcal{F}}_m)$ is locally constant. Moreover, its restrictions to $\infty_{\mathfrak{s}} \overleftarrow{\times}_{\mathbb{P}^1_{\mathfrak{s}}} \mathbb{A}^1_{\mathfrak{s}}$ and $\infty_{K_n} \overleftarrow{\times}_{\mathbb{P}^1_{K_n}} \mathbb{A}^1_{K_n}$ are isomorphic to $F^{(0,\infty)}(R\Phi_{\tilde{f}_{\mathfrak{s}}}\langle \mathcal{F}_m, -\pi\rangle)$ and $F^{(0,\infty)}(R\Phi_f\mathcal{F}_m)$, respectively. By [22, Corollary 3.9], the determinant $\det F^{(0,\infty)}(R\Phi_{\tilde{f}_n}\tilde{\mathcal{F}}_m)$ is tamely ramified. Applying the construction of [22, Definition 2.16], we obtain a character

$$\langle \det F^{(0,\infty)}(R\Phi_{\tilde{f}_n}\tilde{\mathcal{F}}_m), 1/x\rangle$$

of $\pi_1(\operatorname{Spec}(R_n))$ (with the notation of loc. cit.), where x denotes the standard coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$. The assertion then follows from [22, Lemma 2.19] together with Laumon's cohomological interpretation [22, Theorem 3.10].

The assertion in characteristic 0 is proved as follows. Let x denote the standard coordinate of $\mathbb{A}^1$. It suffices to prove the commutativity of the diagram

$$\begin{array}{ccc} G_k & \longrightarrow & \overline{\mathbb{Z}_\ell}^{\times}/\mu \\ \uparrow & & \uparrow \\ G_{K_\infty} & \longrightarrow & G_{k(\mathfrak{s})}, \end{array}$$

where the top horizontal arrow is given by $J(R\Phi_f\mathcal{F})$ and the right vertical arrow is given by $J\left(R\Phi_{\tilde{f}_\mathfrak{s}}\langle\mathcal{F},-\pi\rangle\right)$.

Fix a separable closure $\overline{K}_\infty$ of K_∞. Define

$$I := \varprojlim_N \mu_N(\overline{K}_\infty) \qquad \text{and} \qquad I^p := \varprojlim_{p\nmid N} \mu_N(\overline{K}_\infty),$$

where p denotes the residue characteristic of R.

Let $\eta_{k(\mathfrak{s})}$ (resp. η_{K_∞}) denote the function field of the henselization $\mathbb{A}^1_{k(\mathfrak{s}),(0)}$ (resp. $\mathbb{A}^1_{K_\infty,(0)}$). Fix geometric points $\overline{\eta}_\mathfrak{s}$ and $\overline{\eta}_\infty$ lying over $\eta_{k(\mathfrak{s})}$ and η_{K_∞}, respectively. We also fix a specialization $\overline{\eta}_\infty \to \overline{\eta}_\mathfrak{s}$ of geometric points of $\mathbb{A}^1_{R_n,(0_\mathfrak{s})}$.

The groups I^p and I are naturally identified with the tame inertia groups $I^t_{\eta_{k(\mathfrak{s})}}$ and $I^t_{\eta_{K_\infty}}$, respectively.

Let π_1^t denote the fundamental group classifying finite étale covers of

$$\mathbb{A}^1_{R_n,(0_\mathfrak{s})} \setminus 0_{R_n,(0_\mathfrak{s})}$$

that are tamely ramified along $0_{R_n,(0_\mathfrak{s})}$. There is a natural embedding $I^p \hookrightarrow \pi_1^t$ and a commutative diagram

$$\begin{array}{ccccc} I & \longrightarrow & I^p & & \\ \downarrow\cong & & \downarrow & \searrow^{\cong} & \\ I^t_{\eta_{K_\infty}} & \longrightarrow & \pi_1^t & \longleftarrow & I^t_{\eta_{k(\mathfrak{s})}}, \end{array}$$

where the top horizontal arrow is the natural projection $I \to I^p$ and the bottom horizontal arrows are the canonical homomorphisms.

Since the generic characteristic of R_n is zero, the restriction of $R\Phi_{\tilde{f}_n}\tilde{\mathcal{F}}_m$ to

$$\mathbb{A}^1_{R_n,(0_\mathfrak{s})} \setminus 0_{R_n,(0_\mathfrak{s})} \cong 0_\mathfrak{s} \overleftarrow{\times}_{\mathbb{A}^1_{R_n}} \mathbb{G}_{m,R_n} \subset 0_{R_n} \overleftarrow{\times}_{\mathbb{A}^1_{R_n}} \mathbb{G}_{m,R_n}$$

is locally constant with tamely ramified cohomology sheaves. Since $R\Phi_{\tilde{f}_n}\tilde{\mathcal{F}}_m$ is locally constant, the specialization morphism

$$(R\Phi_{\tilde{f}_\mathfrak{s}}\langle\mathcal{F}_m,-\pi\rangle)_{\overline{\eta}_\mathfrak{s}} \cong (R\Phi_{\tilde{f}_n}\tilde{\mathcal{F}}_m)_{\overline{\eta}_\mathfrak{s}} \to (R\Phi_{\tilde{f}_n}\tilde{\mathcal{F}}_m)_{\overline{\eta}_\infty} \cong (R\Phi_f\mathcal{F}_m)_{\overline{\eta}_\infty}$$

is an isomorphism. We regard it as an isomorphism of complexes of I-representations. This isomorphism is compatible with the transition morphisms

$$R\Phi_f\mathcal{F}_{m+1} \to R\Phi_f\mathcal{F}_m \quad \text{and} \quad R\Phi_{\tilde{f}_\mathfrak{s}}\langle\mathcal{F}_{m+1},-\pi\rangle \to R\Phi_{\tilde{f}_\mathfrak{s}}\langle\mathcal{F}_m,-\pi\rangle.$$

Hence $C_\infty := R\Phi_f\mathcal{F} \otimes_{\mathcal{O}_E} E$ and $C_\mathfrak{s} := R\Phi_{\tilde{f}_\mathfrak{s}}\langle\mathcal{F},-\pi\rangle \otimes_{\mathcal{O}_E} E$ are quasi-isomorphic as complexes of finite-dimensional E-representations of I.

Let $I \to \mu_N(\overline{K}_\infty)$ be a finite quotient through which I acts on the semisimplifications of the cohomology groups of $C_\infty \cong C_{\mathfrak{s}}$. We may assume that $p \nmid N$. For each i, the semisimplification of $H^i(C_\infty)^{\oplus N}$ defines a Jacobi datum over K_∞ as in Definition 3.6. Since N is invertible in R_∞, this Jacobi datum extends to a Jacobi datum over R_∞.

The resulting Jacobi datum over R_∞ restricts both to the Jacobi datum over $k(\mathfrak{s})$ associated with $H^i(C_{\mathfrak{s}})^{\oplus N}$ and to the Jacobi datum over K_∞ associated with $H^i(C_\infty)^{\oplus N}$. Therefore, the assertion follows. □

Remark 3.18. *By repeatedly applying the method used above, one can reduce various problems concerning ℓ-adic sheaves on schemes of finite type over (the perfections of) finitely generated fields to the corresponding cases over finite fields. For example, Theorem 1 in [18] can be proved unconditionally, that is, without assuming that the sheaf $\mathcal{F}$ in loc. cit. is defined over a scheme of finite type over $\mathbb{Z}$, provided that the function field of the base scheme S is a purely inseparable extension of a finitely generated field. Nevertheless, it would be desirable to obtain such a result directly by developing a theory of Jacobi sum characters for representations with torsion coefficients.*

3.4 Local epsilon factors of convolutions

In this subsection, we compute the local epsilon factors of convolutions of vanishing cycles. To this end, we recall the Thom–Sebastiani theorem for étale sheaves, proved in [10]. For the basic theory of oriented products used in this subsection, we refer the reader to [11] and [10].

Let k be a perfect field of characteristic $p \geq 0$, and fix a prime number $\ell \neq p$. Let Λ be a finite local ring with residue characteristic ℓ.

Following the notation of [10], let A_h and A_h^2 denote the henselizations of $\mathbb{A}_k^1$ at 0 and of $\mathbb{A}_k^2$ at $(0,0)$, respectively. Let $a\colon A_h^2 \to A_h$ be the morphism induced by the addition map $\mathbb{A}_k^2 \to \mathbb{A}_k^1$.

Let $f_1\colon X_1 \to A_h$ and $f_2\colon X_2 \to A_h$ be morphisms of schemes of finite type. Let

$$X := (X_1 \times X_2) \times_{A_h \times A_h} A_h^2,$$

and let $f\colon X \to A_h^2$ be the projection. We regard X as an A_h-scheme via the composition $X \xrightarrow{f} A_h^2 \xrightarrow{a} A_h$.

Definition-Lemma 3.19 ([10, Definition 4.1 and Proposition 4.3])**.** *For each $i = 1, 2$, let $K_i \in D_{\mathrm{ctf}}(X_i \overleftarrow{\times}_{A_h} A_h, \Lambda)$. We define the* local convolution

$$K_1 *_\Lambda^L K_2 \in D(X \overleftarrow{\times}_{A_h} A_h, \Lambda)$$

of K_1 and K_2 by

$$K_1 *_\Lambda^L K_2 := R\overleftarrow{a}_*(\overleftarrow{\mathrm{pr}}_{1*}K_1 \otimes_\Lambda^L \overleftarrow{\mathrm{pr}}_{2*}K_2)[1],$$

where

$$\overleftarrow{\mathrm{pr}}_i\colon X \overleftarrow{\times}_{A_h^2} A_h^2 \to X_i \overleftarrow{\times}_{A_h} A_h \quad \text{and} \quad \overleftarrow{a}\colon X \overleftarrow{\times}_{A_h^2} A_h^2 \to X \overleftarrow{\times}_{A_h} A_h$$

are the morphisms induced by the i-th projections and by a, respectively.

If no confusion arises, we omit Λ from the notation and write $*^L$ for $*_\Lambda^L$. By [10, Proposition 4.3], the complex $K_1 *^L K_2$ belongs to $D_{\mathrm{ctf}}(X \overleftarrow{\times}_{A_h} A_h, \Lambda)$.

We note that this definition is slightly different from that of [10], in that the resulting complex is shifted by 1.

We next show that the formation of $K_1 *^L K_2$ commutes with base changes in both X_1 and X_2.

Lemma 3.20. *Let $g_i \colon Y_i \to X_i$ be morphisms of A_h-schemes of finite type for $i = 1, 2$. Put*

$$Y := (Y_1 \times Y_2) \times_{A_h \times A_h} A_h^2,$$

and let $g \colon Y \to X$ be the morphism induced by g_i. Let $\overleftarrow{g}_i \colon Y_i \overleftarrow{\times}_{A_h} A_h \to X_i \overleftarrow{\times}_{A_h} A_h$ and $\overleftarrow{g} \colon Y \overleftarrow{\times}_{A_h} A_h \to X \overleftarrow{\times}_{A_h} A_h$ be the induced morphisms. Then, for $K_i \in D_{\mathrm{ctf}}(X_i \overleftarrow{\times}_{A_h} A_h, \Lambda)$, the canonical morphism

$$\overleftarrow{g}^*(K_1 *^L K_2) \to (\overleftarrow{g}_1^* K_1) *^L (\overleftarrow{g}_2^* K_2)$$

in $D_{\mathrm{ctf}}(Y \overleftarrow{\times}_{A_h} A_h, \Lambda)$ is an isomorphism.

Proof. The assertion follows once one verifies that the base change morphism

$$\overleftarrow{g}^* R\overleftarrow{a}_* \to R\overleftarrow{a}_* \overleftarrow{g}^*$$

associated with the commutative diagram

$$\begin{array}{ccc} Y \overleftarrow{\times}_{A_h^2} A_h^2 & \xrightarrow{\overleftarrow{a}} & Y \overleftarrow{\times}_{A_h} A_h \\ \downarrow{\scriptstyle \overleftarrow{g}} & & \downarrow{\scriptstyle \overleftarrow{g}} \\ X \overleftarrow{\times}_{A_h^2} A_h^2 & \xrightarrow{\overleftarrow{a}} & X \overleftarrow{\times}_{A_h} A_h \end{array}$$

is an isomorphism. This follows from [22, Corollary 2.5]. □

We now recall the Thom–Sebastiani theorem for étale sheaves proved in [10].

Theorem 3.21 ([10, Theorem 4.5])**.** *With the above notation, let K_1, K_2 be objects of $D_{\mathrm{ctf}}(X_1, \Lambda), D_{\mathrm{ctf}}(X_2, \Lambda)$ respectively. Let $K := (K_1 \boxtimes^L K_2)|_X$. Then there is a functorial isomorphism*

$$(R\Phi_{f_1}(K_1)) *^L (R\Phi_{f_2}(K_2))\big|_{X_0 \overleftarrow{\times}_{A_h} A_h} \cong R\Phi_{af}(K)[1]\big|_{X_0 \overleftarrow{\times}_{A_h} A_h}$$

in $D_{\mathrm{ctf}}(X_0 \overleftarrow{\times}_{A_h} A_h, \Lambda)$, where X_0 denotes the closed fiber of $X \to A_h$.

If one takes each X_i to be the closed point of A_h, then $X_i \overleftarrow{\times}_{A_h} A_h$ are canonically equivalent to the étale topos of A_h. Hence $*^L$ may be regarded as a functor

$$D_{\mathrm{ctf}}(A_h, \Lambda) \times D_{\mathrm{ctf}}(A_h, \Lambda) \to D_{\mathrm{ctf}}(A_h, \Lambda).$$

We now consider the ℓ-adic analogue of this convolution functor on the derived category $D_c^b(A_h, E)$, where E is a finite extension of $\mathbb{Q}_\ell$. It is defined as follows. For the notation and conventions used in the construction of $D_c^b(-, E)$, we refer to Section 6.

Let $K_1, K_2 \in D_{\mathrm{ctf}}(A_h, \Lambda)$. Since the cohomological dimension of $R\overleftarrow{a}_*$ is finite (this can be proved in the same way as [16, Proposition 3.1], using [10, Proposition 1.13]), the canonical morphism

$$(K_1 *^L_\Lambda K_2) \otimes^L_\Lambda \Lambda' \to (K_1 \otimes^L_\Lambda \Lambda') *^L_{\Lambda'} (K_2 \otimes^L_\Lambda \Lambda')$$

is an isomorphism for every local homomorphism $\Lambda \to \Lambda'$ of finite local rings. Therefore, the functor

$$D^b(A_h^{\mathbb{N}^{\mathrm{op}}}, \mathcal{O}_{E\bullet}) \times D^b(A_h^{\mathbb{N}^{\mathrm{op}}}, \mathcal{O}_{E\bullet}) \to D^b(A_h^{\mathbb{N}^{\mathrm{op}}}, \mathcal{O}_{E\bullet}), \quad (K_1, K_2) \mapsto R\overleftarrow{a}_*(\overleftarrow{\mathrm{pr}}_{1*} K_1 \otimes^L_\Lambda \overleftarrow{\mathrm{pr}}_{2*} K_2)[1]$$

preserves the full subcategory $D_{c,\mathrm{norm}}$ (Definition 6.9.1). Hence it induces a functor

$$*^L \colon D^b_c(A_h, \mathcal{O}_E) \times D^b_c(A_h, \mathcal{O}_E) \to D^b_c(A_h, \mathcal{O}_E).$$

Passing to the 2-colimit $\varinjlim_E$, where E runs through the finite extensions of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}_\ell}$, and then inverting ℓ, we obtain convolution functors on $D^b_c(A_h, \overline{\mathbb{Z}_\ell})$ and $D^b_c(A_h, \overline{\mathbb{Q}_\ell})$.

Let η be the generic point of A_h. For $\mathcal{F} \in D^b_c(\eta, \overline{\mathbb{Q}_\ell})$, let $\mathcal{F}_!$ denote the extension by zero of $\mathcal{F}$ to A_h. As explained in [10, §3.B], the functor $(\mathcal{F}_1, \mathcal{F}_2) \mapsto (\mathcal{F}_{1!} *^L \mathcal{F}_{2!})|_\eta$ is naturally isomorphic to the convolution functor considered in [15, (2.7)]; recall that our definition of $*^L$ differs from that of [10] by a shift of 1.

To state Lemma 3.22, we slightly modify the notation. Let k be a perfect field as above. Let

$$f_i \colon X_i \to \mathbb{A}^1_k \qquad (i = 1, 2)$$

be morphisms of finite type over k. Let $X := X_1 \times_k X_2$, and let $af \colon X \to \mathbb{A}^1_k$ denote the composite

$$X = X_1 \times_k X_2 \xrightarrow{f_1 \times f_2} \mathbb{A}^2_k \xrightarrow{a} \mathbb{A}^1_k,$$

where a is the addition map.

For $i = 1, 2$, let $\mathcal{F}_i \in D^b_c(X_i, \overline{\mathbb{Z}_\ell})$, and let $x_i \in X_i$ be an at most isolated $SS(\mathcal{F}_i)$-characteristic k-rational point of f_i such that $f_i(x_i) = 0$.

Lemma 3.22. *Let the notation be as above. Let $x := (x_1, x_2) \in X$ be the k-rational point over x_1 and x_2. Let $\mathcal{F} := \mathcal{F}_1 \boxtimes^L \mathcal{F}_2$. Let t denote the standard coordinate of $\mathbb{A}^1_k$, and let A_h be the henselization of $\mathbb{A}^1_k$ at the origin.*

1. *Assume that $p > 0$. Then*

$$\varepsilon_0(A_h, R\Phi_{af}(\mathcal{F})_x, dt)^{-1} = \varepsilon_0(A_h, R\Phi_{f_1}(\mathcal{F}_1)_{x_1}, dt)^{\mathrm{dimtot}\, R\Phi_{f_2}(\mathcal{F}_2)_{x_2}} \cdot \varepsilon_0(A_h, R\Phi_{f_2}(\mathcal{F}_2)_{x_2}, dt)^{\mathrm{dimtot}\, R\Phi_{f_1}(\mathcal{F}_1)_{x_1}}.$$

2. *Assume that k is finitely generated over $\mathbb{Q}$. Then*

$$\overline{\varepsilon}_0(A_h, R\Phi_{af}(\mathcal{F})_x)^{-1} = \overline{\varepsilon}_0(A_h, R\Phi_{f_1}(\mathcal{F}_1)_{x_1})^{\mathrm{dimtot}\, R\Phi_{f_2}(\mathcal{F}_2)_{x_2}} \cdot \overline{\varepsilon}_0(A_h, R\Phi_{f_2}(\mathcal{F}_2)_{x_2})^{\mathrm{dimtot}\, R\Phi_{f_1}(\mathcal{F}_1)_{x_1}}.$$

Here $\overline{\varepsilon_0}$ is as in Definition 3.7.

Proof. 1. By the $\overline{\mathbb{Z}_\ell}$-analogue of Theorem 3.21, we have an isomorphism

$$(R\Phi_{f_1}(\mathcal{F}_1)) *^L (R\Phi_{f_2}(\mathcal{F}_2))|_{X_0 \overleftarrow{\times}_{A_h} A_h} \cong R\Phi_{af}(\mathcal{F})[1]|_{X_0 \overleftarrow{\times}_{A_h} A_h} \tag{3.10}$$

in $D_c^b(X_0 \overleftarrow{\times}_{A_h} A_h, \overline{\mathbb{Z}_\ell})$. After shrinking X_i around x_i, we may assume that $X_i \setminus \{x_i\} \to \mathbb{A}_k^1$ is universally locally acyclic relative to $\mathcal{F}_i$ for $i = 1, 2$. It follows that $R\Phi_{f_i}(\mathcal{F}_i)$ is supported on

$$x_i \overleftarrow{\times}_{\mathbb{A}_k^1} \mathbb{A}_k^1 \cong A_h.$$

Since $R\Phi_{f_i}(\mathcal{F}_i)$ is isomorphic to $j_! R\Phi_{f_i}(\mathcal{F}_i)_{x_i}$, where j denotes the open immersion $\eta \to A_h$, Lemma 3.20 together with (3.10) yields an isomorphism

$$(R\Phi_{f_1}(\mathcal{F}_1)_{x_1}) * (R\Phi_{f_2}(\mathcal{F}_2)_{x_2}) \cong R\Phi_{af}(\mathcal{F})_x[1],$$

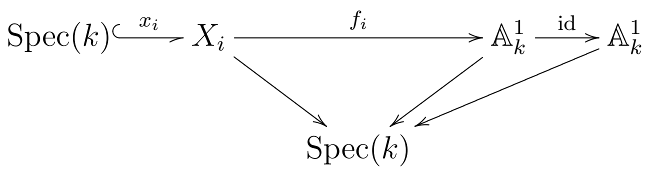
where $*$ denotes the convolution functor considered in [15, (2.7)]. By [10, §3.B], this convolution functor is naturally isomorphic to our convolution functor.

By [15, Proposition (2.7.2.2)], we have

$$F^{(0,\infty)}(R\Phi_{f_1}(\mathcal{F}_1)_{x_1}) \otimes F^{(0,\infty)}(R\Phi_{f_2}(\mathcal{F}_2)_{x_2}) \cong F^{(0,\infty)}(R\Phi_{af}(\mathcal{F})_x)[1].$$

The assertion then follows from this isomorphism together with [15, Théorème (3.5.1.1)].

2. Apply Proposition 3.17 to the commutative diagrams



and the sheaves $\mathcal{F}_i$ for $i = 1, 2$, as well as to the analogous diagram for X and $\mathcal{F}$. The assertion then follows from 1 and Lemma 3.16. □

4 Epsilon Cycles of ℓ-adic Sheaves

In this section, we construct epsilon cycles that compute local epsilon factors modulo roots of unity. Let μ denote the group of roots of unity in $\overline{\mathbb{Z}_\ell}^\times$.

4.1 Group of characters modulo torsion

For a field E, let μ_E denote the group of roots of unity in E.

Definition 4.1. *Let G be a compact Hausdorff abelian group.*

1. *For a finite extension E of $\mathbb{Q}_\ell$, define*

$$\Theta_{G,E} := \mathrm{Hom}_{\mathrm{conti}}(G, \mathcal{O}_E^\times/\mu_E)$$

to be the group of continuous homomorphisms. Here $\mathcal{O}_E^\times/\mu_E$ is endowed with the quotient topology induced by the ℓ-adic topology on $\mathcal{O}_E$.

2. *Define*

$$\Theta_G := \varinjlim_{E} \Theta_{G,E},$$

where E runs over the finite subextensions of $\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell$.

3. *When G is the abelianization of the absolute Galois group of a field k, we also write $\Theta_{k,E}$ and Θ_k for $\Theta_{G,E}$ and Θ_G, respectively.*

We shall usually identify Θ_G with a subgroup of $\mathrm{Hom}(G, \overline{\mathbb{Z}_\ell}^\times/\mu)$. A group homomorphism $G \to \overline{\mathbb{Z}_\ell}^\times/\mu$ is said to be *admissible* if it belongs to Θ_G.

By Lemma 4.2 below, every compact subgroup of $\overline{\mathbb{Z}_\ell}^\times$ is contained in $\mathcal{O}_E^\times$ for some finite subextension E of $\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell$, where $\overline{\mathbb{Z}_\ell}^\times$ is endowed with the topology induced by the valuation of $\overline{\mathbb{Q}_\ell}$. Therefore, every continuous homomorphism $G \to \overline{\mathbb{Z}_\ell}^\times$ induces an admissible homomorphism $G \to \overline{\mathbb{Z}_\ell}^\times/\mu$.

Lemma 4.2. *Let $K \subset \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ be a compact subgroup. Then there exists a finite subextension E of $\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell$ such that $K \subset \mathrm{GL}_n(E)$.*

Proof. We include a proof for completeness. Fix a bijection from the set of integers ≥ 0 to the set of finite subextensions of $\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell$, which is denoted by $m \mapsto E_m$. For each integer $m \geq 0$, put $K_m := K \cap \mathrm{GL}_n(E_m)$. Then K_m is a closed subgroup of K, and

$$\bigcup_m K_m = K.$$

Since K is a compact Hausdorff space, the Baire category theorem applies. Hence there exists $m \geq 0$ such that K_m contains a non-empty open subset of K. Since K_m is a subgroup, it follows that K_m is an open subgroup. Therefore $[K : K_m]$ is finite. This proves the assertion. □

Lemma 4.3. *Let G be a compact Hausdorff abelian group.*

1. *The group Θ_G is uniquely divisible.*

2. *Let $\mathrm{Hom}_{\mathrm{conti}}(G, \overline{\mathbb{Z}_\ell}^\times)$ denote the group of continuous group homomorphisms from G to $\overline{\mathbb{Z}_\ell}^\times$. Then the kernel and the cokernel of the natural map $\mathrm{Hom}_{\mathrm{conti}}(G, \overline{\mathbb{Z}_\ell}^\times) \to \Theta_G$ are torsion.*

Proof. 1. Since the group $\mathcal{O}_E^\times/\mu_E$ is torsion-free, so is $\Theta_{G,E}$. Hence Θ_G is torsion-free.

It remains to show that Θ_G is divisible. Let $\chi \in \Theta_{G,E}$. For every integer $n \geq 1$, we shall find a finite extension E'/E and a continuous homomorphism $\xi\colon G \to \mathcal{O}_{E'}^\times/\mu_{E'}$ such that $\xi^n = \chi$.

Let E' be a finite extension of E containing the n-th roots of all elements in $\mathcal{O}_E^\times$; such an extension exists since $\mathcal{O}_E^\times/(\mathcal{O}_E^\times)^n$ is finite. Then the composite of

$$G \xrightarrow{\chi} \mathcal{O}_E^\times/\mu_E \hookrightarrow \mathcal{O}_{E'}^\times/\mu_{E'}$$

factors through the injection

$$\mathcal{O}_{E'}^\times/\mu_{E'} \to \mathcal{O}_{E'}^\times/\mu_{E'}, \qquad a \mapsto a^n.$$

Since this map is a homeomorphism onto its image, there exists a desired ξ.

2. The kernel is torsion, since every compact subgroup of $\mu \subset \overline{\mathbb{Z}_\ell}^\times$ is finite by Lemma 4.2.

Let E be a finite extension of $\mathbb{Q}_\ell$ and let $\chi\colon G \to \mathcal{O}_E^\times/\mu_E$ be a continuous homomorphism. We will show that there exist a continuous homomorphism $\xi\colon G \to \mathcal{O}_E^\times$ and an integer $n \geq 1$ such that the composite of ξ with the quotient map $\mathcal{O}_E^\times \to \mathcal{O}_E^\times/\mu_E$ is equal to χ^n.

Choose an open subgroup $U \subset \mathcal{O}_E^\times$ such that $U \cap \mu_E = \{1\}$. Then the composite map $U \to \mathcal{O}_E^\times \to \mathcal{O}_E^\times/\mu_E$ is an isomorphism onto an open subgroup of $\mathcal{O}_E^\times/\mu_E$, which we again denote by U. Let H be the inverse image of $U \subset \mathcal{O}_E^\times/\mu_E$ under χ. Then H is an open subgroup of G. Let $n := [G : H]$, and define ξ as the composite

$$G \xrightarrow{n} H \xrightarrow{\chi} U \to \mathcal{O}_E^\times.$$

Then ξ and n satisfy the required condition. □

4.2 Constructions of epsilon cycles

In this subsection, we construct epsilon cycles for $\overline{\mathbb{Z}_\ell}$-sheaves (Theorem 4.10) by applying Proposition 2.13. To apply this proposition, we need to study the variation of local epsilon factors in families of isolated characteristic points. In the case of positive characteristic, this was done in [22].

Let k be a field, and let ℓ be a prime number invertible in k. In Theorem 4.10, we define epsilon cycles for ℓ-adic sheaves on smooth varieties over k under the following assumption:

k is the perfection of a finitely generated field over its prime field.

Before proceeding, we recall two results that are key ingredients in our construction. The first result is the following theorem of Katz–Lang. Since it will be used crucially at several points, we need to impose the above assumption on k.

Theorem 4.4 ([13, Theorem 1]). *Let k be the perfection of a finitely generated field of characteristic $p \geq 0$. Let X be a geometrically connected smooth scheme over k. Then the natural homomorphism*

$$\pi_1^{ab}(X) \to \pi_1^{ab}(k) = G_k^{ab}$$

is surjective. The kernel is the product of a finite group and a pro-p group if $p > 0$, and is finite if $p = 0$.

The second result is the "continuity" of local epsilon factors established in [22].

Theorem 4.5 ([22, Theorem 4.9.2]). *Let S be a connected scheme of finite type over a perfect field k of characteristic $p > 0$. Consider a commutative diagram*

$$\begin{array}{ccccc} Z & \hookrightarrow & U & \xrightarrow{f} & Y \\ & & & {\scriptstyle g}\searrow \quad \swarrow & \\ & & & S & \end{array}$$

of S-schemes of finite type. Let $\mathcal{F} \in D_c^b(U, \overline{\mathbb{Z}_\ell})$. Assume that

- *Y is a smooth S-curve.*
- *Z is a closed subscheme of U finite over S.*
- *g and $f|_{U\setminus Z}$ are universally locally acyclic relative to $\mathcal{F}$.*

Let $t\colon Y \to \mathbb{A}^1_S$ be an étale S-morphism. Then there exists a continuous character

$$\rho_{\mathcal{F},t}\colon \pi_1^{ab}(S) \to \overline{\mathbb{Z}_\ell}^\times$$

with the following property: for every perfect field k' with a morphism $\operatorname{Spec}(k') \to S$, the composite $G_{k'}^{ab} \to \pi_1^{ab}(S) \xrightarrow{\rho_{\mathcal{F},t}} \overline{\mathbb{Z}_\ell}^\times$ is equal to

$$\prod_{z\in Z_{k'}} \delta_{k(z)/k'}^{\operatorname{dimtot}(R\Phi_{f_{k'}}(\mathcal{F}|_{U_{k'}})_z)} \cdot \varepsilon_0(Y_{k'(z)}, R\Phi_{f_{k'}}(\mathcal{F}|_{U_{k'}})_z, dt) \circ \operatorname{tr}_{k(z)/k'}\,.$$

Here $(-)_{k'}$ denotes base change to k'.

Combining these results, we first show that local epsilon factors modulo roots of unity are independent of the choice of a uniformizer.

Lemma 4.6. *Let k be the perfection of a field finitely generated over $\mathbb{F}_p$. Let X be a smooth scheme of finite type over k, and let $\mathcal{F}$ be an object of $D_c^b(X, \overline{\mathbb{Z}}_\ell)$. Let*

(4.1)
$$\begin{array}{ccc} U & \xrightarrow{f} & Y \\ {\scriptstyle j}\big\downarrow & & \\ X & & \end{array}$$

be a diagram as in (2.2). Let $u \in U$ be an at most isolated $SS(j^*\mathcal{F})$-characteristic point of f. For two local parameters t and t' of Y around $f(u)$, the ratio

$$\varepsilon_0(Y_{(u)}, R\Phi_f(\mathcal{F})_u, dt) \cdot \varepsilon_0(Y_{(u)}, R\Phi_f(\mathcal{F})_u, dt')^{-1} = (\det R\Phi_f(\mathcal{F})_u)_{[\frac{dt}{dt'}]}$$

of the characters of $G_{k(u)}^{ab}$ (see Theorem 3.1.4) is of finite order.

Proof. We may assume that $u \to \operatorname{Spec}(k)$ is an isomorphism and that $U \setminus \{u\} \to Y$ is universally locally acyclic relative to $\mathcal{F}$. We show that $(\det R\Phi_f(\mathcal{F})_u)_{[\frac{dt}{dt'}]}$ is of finite order. Let n denote the Swan conductor of $\det R\Phi_f(\mathcal{F})_u$. If $dt - dt'$ vanishes at $f(u)$, then the above character is annihilated by p^n (see [25, Lemma 4.8]).

Let $a \in k^\times$ be such that $d(at) - dt'$ vanishes at $f(u)$. Consider the diagram

(4.2)
$$\begin{array}{ccccc} \mathbb{G}_{m,k} & \hookrightarrow & U \times_k \mathbb{G}_{m,k} & \xrightarrow{f\times\mathrm{id}} & Y \times_k \mathbb{G}_{m,k} \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow & \swarrow & \\ & & \mathbb{G}_{m,k} & & \end{array}$$

obtained by taking the product of

$$\begin{array}{ccccc} u & \hookrightarrow & U & \xrightarrow{f} & Y \\ & {\scriptstyle \cong}\searrow & \downarrow & \swarrow & \\ & & \operatorname{Spec}(k) & & \end{array}$$

with $\mathbb{G}_{m,k}$. Let x denote the standard coordinate of $\mathbb{G}_{m,k}$, and set $t'' := xt$. Applying Theorem 4.5 to the diagram (4.2) and the sheaf $\mathrm{pr}^*\mathcal{F}$, where pr denotes the projection $U \times_k \mathbb{G}_{m,k} \to U$, we obtain a continuous character

$$\rho_{\mathrm{pr}^*\mathcal{F},t''} \colon \pi_1^{ab}(\mathbb{G}_{m,k}) \to \overline{\mathbb{Z}_\ell}^\times$$

satisfying the property stated in the theorem. By Theorem 4.4, the composite

$$\pi_1^{ab}(\mathbb{G}_{m,k}) \xrightarrow{\rho_{\mathrm{pr}^*\mathcal{F},t''}} \overline{\mathbb{Z}_\ell}^\times \to \overline{\mathbb{Z}_\ell}^\times/\mu$$

factors through teh natural surjection $\pi_1^{ab}(\mathbb{G}_{m,k}) \to G_k^{ab}$. Specializing to $x = 1$ and $x = a$, we obtain the assertion. □

Definition 4.7. *Let the notation and assumptions be as in Lemma 4.6. We define*

$$\bar{\varepsilon}_0(Y_{(u)}, R\Phi_f(\mathcal{F})_u) \colon G_{k(u)}^{ab} \to \overline{\mathbb{Z}_\ell}^\times/\mu$$

to be the composite of $\varepsilon_0(Y_{(u)}, R\Phi_f(\mathcal{F})_u, dt)$ with the quotient map $\overline{\mathbb{Z}_\ell}^\times \to \overline{\mathbb{Z}_\ell}^\times/\mu$. By Lemma 4.6, the character $\bar{\varepsilon}_0(Y_{(u)}, R\Phi_f(\mathcal{F})_u)$ is independent of the choice of the local parameter t. It belongs to $\Theta_{k(u)}$.

Lemma 4.8. *Let k be the perfection of a finitely generated field of positive characteristic. Let X be a smooth scheme of finite type over k, and let $\mathcal{F}$ be an object of $D_c^b(X, \overline{\mathbb{Z}}_\ell)$. Let C denote the singular support of $\mathcal{F}$. For a diagram as in (2.2) and an at most isolated C-characteristic point $u \in U$ of f, define*

$$\varphi(f, u) := \bar{\varepsilon}_0(Y_{(u)}, R\Phi_f(\mathcal{F})_u)^{-1} \circ \mathrm{tr}_{k(u)/k}\,.$$

This assignment defines a Θ_k-valued function on isolated C-characteristic points in the sense of Definition 2.12.1. Moreover, this function is flat in the sense of Definition 2.12.2.

Proof. First, we verify that φ is a Θ_k-valued function on isolated C-characteristic points.

If u is not an isolated C-characteristic point, then $R\Phi_f(\mathcal{F})_u = 0$, and hence

$$\varphi(f, u) = 1.$$

Therefore condition (a) of Definition 2.12.1 is satisfied.

We next verify condition (b). Consider a diagram of k-schemes as in (2.3) and an isolated C-characteristic point $u' \in U'$ of f'. Since the restriction of $\bar{\varepsilon}_0(Y_{(u)}, R\Phi_f(\mathcal{F})_u)$ to $G_{k(u')}^{ab}$ coincides with $\bar{\varepsilon}_0(Y'_{(u')}, R\Phi_{f'}(\mathcal{F})_{u'})$, the assertion follows from the fact that the composite

$$G_{k(u)}^{ab} \xrightarrow{\mathrm{tr}_{k(u')/k(u)}} G_{k(u')}^{ab} \to G_{k(u)}^{ab}$$

is multiplication by $\deg(u'/u)$.

We now prove the flatness assertion. Consider a diagram of k-schemes as in (2.4). We must show that the function

$$\varphi_f \colon |Z| \to \Theta_k, \qquad z \mapsto \varphi(f_s, z),$$

where $s \in S$ denotes the image of z, is flat over S. Let g denote the structure morphism $Z \to S$ and take a closed point $z \in Z$.

After replacing S by an étale neighborhood of $g(z)$ and shrinking Z around z, we may assume that Z is finite over S. After further replacing S by an open neighborhood of $g(z)$ and Y by an open neighborhood of the image of z, we may assume that there exists a section $t \in \Gamma(Y, \mathcal{O}_Y)$ inducing an étale morphism $Y \to \mathbb{A}^1_S$. Finally, we may assume that S is connected. Then Theorem 4.5 yields a continuous group homomorphism

$$\rho_{\mathrm{pr}_1^*\mathcal{F},t} \colon \pi_1^{ab}(S) \to \overline{\mathbb{Z}}_\ell^\times,$$

where pr_1 denotes the map $U \to X$ in (2.4), satisfying the property described in loc. cit.

Let k' be the normalization of k in S. By Theorem 4.4, the composite of $\rho_{\mathrm{pr}_1^*\mathcal{F},t}$ with the quotient map $\overline{\mathbb{Z}}_\ell^\times \to \overline{\mathbb{Z}}_\ell^\times/\mu$ factors through $G_{k'}^{ab}$. We denote the resulting character by

$$\xi \colon G_{k'}^{ab} \to \overline{\mathbb{Z}}_\ell^\times/\mu.$$

For a closed point $s \in S$, it follows from Theorem 4.5 that

$$\prod_{z\in Z_s} \varphi_f(z) = \prod_{z\in Z_s} \overline{\varepsilon}_0(Y_{s,(z)}, R\Phi_{f_s}(\mathcal{F}_s)_z)^{-1} \circ \mathrm{tr}_{k(z)/k} = \xi|_{G_{k(s)}^{ab}} \circ \mathrm{tr}_{k(s)/k}\,.$$

Since

$$\xi|_{G_{k(s)}^{ab}} \circ \mathrm{tr}_{k(s)/k} = (\xi \circ \mathrm{tr}_{k'/k})^{\deg(k(s)/k')},$$

the assertion follows. □

The following lemma allows us to reduce the existence of epsilon cycles in characteristic 0 to the positive characteristic case.

Lemma 4.9. *Let S be a regular noetherian scheme. Let X be a smooth scheme purely of relative dimension n over S. Let $Z \subset X$ be an integral closed subscheme, flat over S and purely of relative dimension $n-c$ over S. Let W be a smooth scheme purely of relative dimension m over S, and let $h\colon W \to X$ be a closed immersion of S-schemes.*

Assume that each irreducible component C_a of $Z \times_X W = \bigcup_a C_a$, equipped with the reduced subscheme structure, is flat purely of relative dimension $m-c$ over S. Then, after replacing S by a dense open subscheme, there exists integers $(t_a)_a$, indexed by the set of irreducible components of $Z \times_X W$, such that, for every morphism $s \to S$ from the spectrum of a field, we have

$$h_s^![Z_s] = \sum_a t_a[C_{a,s}]$$

as cycles supported on $(Z \times_X W)_s$, where $(-)_s$ denotes the base change $(-) \times_S s$.

Proof. After shrinking S, we may assume that, for any distinct indices a, b, the morphism $C_a \cap C_b \to S$ has relative dimension $< m-c$.

Let $K = \mathcal{O}_Z \otimes^L_{\mathcal{O}_X} \mathcal{O}_W$. Since $h\colon W \to X$ is a regular immersion, the complex K is bounded. Moreover, its support is contained in $Z \times_X W$.

Let $U \subset W$ be an open neighborhood of the generic points of $Z \times_X W$ such that the subschemes $U \cap C_a$ are pairwise disjoint and, for every $i \in \mathbb{Z}$, the sheaf $H^i(K|_{U\cap C_a})$ is an iterated extension of finite free $\mathcal{O}_{U\cap C_a}$-modules.

Let η_a denote the generic point of C_a. Let t_a be the length of K_{η_a}, namely

$$t_a = \sum_i (-1)^i \operatorname{length}_{\mathcal{O}_{X,\eta_a}}\big(H^i(K_{\eta_a})\big).$$

Let $s \to S$ be a morphism from the spectrum of a field. Then

$$h_s^![Z_s]\big|_{U_s} = [K \otimes^L_{\mathcal{O}_S} k(s)]\big|_{U_s} = \sum_a t_a[U_s \cap C_{a,s}].$$

Therefore, the family $(t_a)_a$ has the desired property. □

Theorem 4.10. *Let k be the perfection of a finitely generated field of characteristic $p \geq 0$. Let X be a smooth scheme of finite type over k, and let $\mathcal{F}$ be an object of $D^b_c(X, \overline{\mathbb{Z}}_\ell)$. Let $SS(\mathcal{F}) = \cup_a C_a$ be the decomposition of the singular support into irreducible components.*

Let $Z_n(T^*X)$ denote the group of n-cycles on T^*X, and let Θ_k be as in Definition 4.1. Then there exists a unique cycle

$$\mathcal{E}(\mathcal{F})_k = \sum_a \xi_a \otimes [C_a] \in \Theta_k \otimes_{\mathbb{Z}} Z_n(T^*X)$$

satisfying the following property. For every diagram as in (2.2) and every at most isolated $SS(\mathcal{F})$-characteristic point $u \in U$ of f, we have

$$\bar{\varepsilon}_0(Y_{(u)}, R\Phi_f(\mathcal{F})_u)^{-1} \circ \mathrm{tr}_{k(u)/k} = (\mathcal{E}(\mathcal{F})_k, df)_u^{\deg(u/k)}.$$

Proof. The case $p > 0$ follows from Lemma 4.8 and Proposition 2.13.

Assume that $p = 0$. For each irreducible component C_a of $SS(\mathcal{F})$, choose a diagram of k-schemes

$$X \xleftarrow{j_a} U_a \xrightarrow{f_a} Y_a,$$

and an isolated $SS(\mathcal{F})$-characteristic point $u_a \in U_a$ of f_a as in Lemma 2.21. Let $\xi_a \in \Theta_k$ be an element satisfying

$$\xi_a^{\deg(u_a/k)(C_a, df_a)_{u_a}} = \bar{\varepsilon}_0(Y_{a,(u_a)}, R\Phi_{f_a}(\mathcal{F})_{u_a})^{-1} \circ \mathrm{tr}_{k(u_a)/k}\,.$$

Such an element exists, since the character on the right-hand side belongs to Θ_k and Θ_k is divisible.

We claim that the cycle $\sum_a \xi_a \otimes [C_a]$ satisfies the required condition. Let

$$\begin{array}{ccc} U & \xrightarrow{f} & Y \\ {\scriptstyle j}\downarrow & & \\ X & & \end{array}$$

be a diagram of k-schemes where j is étale and Y is a smooth k-curve. Let $u \in U$ be an at most isolated $SS(\mathcal{F})$-characteristic point of f. We need to prove that

$$\bar{\varepsilon}_0(Y_{(u)}, R\Phi_f(\mathcal{F}))^{-\frac{1}{\deg(u/k)}} \circ \mathrm{tr}_{u/k} = \prod_a \bar{\varepsilon}_0(Y_{a,(u_a)}, R\Phi_{f_a}(\mathcal{F}))^{-\frac{(C_a, df)_u}{\deg(u_a/k)(C_a, df_a)_{u_a}}} \circ \mathrm{tr}_{u_a/k}\,. \tag{4.3}$$

Replacing k with a finite extension, we may assume that both u_a and u are k-rational.

After shrinking Y_a and Y, we choose étale k-morphisms $Y_a \to \mathbb{A}^1_k$ and $Y \to \mathbb{A}^1_k$. Applying Proposition 3.17 to the diagram

$$\begin{array}{ccccccc} u_a & \hookrightarrow & U_a & \xrightarrow{f_a} & Y_a & \longrightarrow & \mathbb{A}^1_k \\ & \searrow^{\cong} & \searrow & & \swarrow & \swarrow & \\ & & & \mathrm{Spec}(k) & & & \end{array}$$

and to the analogous diagram associated with $u \hookrightarrow U \to Y$, we obtain commutative diagrams of topological groups as in the proposition. By Lemma 3.16 and Lemma 4.9, the equality (4.3) follows from the case of positive characteristic. □

Definition 4.11. *Let k be the perfection of a finitely generated field. We call the cycle $\mathcal{E}(\mathcal{F})_k$ constructed in Theorem 4.10 the* epsilon cycle *of $\mathcal{F}$. If no confusion arises, we omit the subscript k and write $\mathcal{E}(\mathcal{F})$ instead.*

Remark 4.12. *In what follows, we write the group law of $\Theta_k \otimes Z_n(T^*X)$ additively, whereas the group law of Θ_k is written multiplicatively. We also use the following notation. For $\chi \in \Theta_k$ and $Z = \sum_b n_b[D_b] \in \mathbb{Q} \otimes_{\mathbb{Z}} Z_n(T^*X)$, we write*

$$\chi^Z := \sum_b \chi^{n_b} \otimes [D_b] \in \Theta_k \otimes Z_n(T^*X).$$

Definition 4.13. *Let X be a smooth scheme of finite type over k. For a constructible complex $\mathcal{F} \in D_c^b(X, \overline{\mathbb{Z}_\ell})$ and a rational number r, we define the r-*twisted epsilon cycle *$\mathcal{E}(\mathcal{F})(r)$ by*

$$\mathcal{E}(\mathcal{F})(r) := \chi_{\mathrm{cyc}}^{rCC(\mathcal{F})} + \mathcal{E}(\mathcal{F}) \in \Theta_k \otimes Z_n(T^*X).$$

Equivalently, if $\mathcal{E}(\mathcal{F}) = \sum_a \xi_a \otimes [C_a]$ and $CC(\mathcal{F}) = \sum_a m_a[C_a]$, then

$$\mathcal{E}(\mathcal{F})(r) = \sum_a \chi_{\mathrm{cyc}}^{rm_a} \xi_a \otimes [C_a].$$

4.3 Properties of epsilon cycles

In this subsection, we establish several basic properties of epsilon cycles.

Lemma 4.14. *Let k be the perfection of a finitely generated field. Let X be a smooth scheme of finite type over k, and let $\mathcal{F} \in D_c^b(X, \overline{\mathbb{Z}_\ell})$.*

1. *Let $\mathcal{G}$ be a smooth $\overline{\mathbb{Z}_\ell}$-sheaf of finite free $\overline{\mathbb{Z}_\ell}$-modules on X, and assume that X is connected. Let k' be the normalization of k in the function field of X. By Theorem 4.4, the composite $\pi_1^{ab}(X) \xrightarrow{\det(\mathcal{G})} \overline{\mathbb{Z}_\ell}^\times \to \overline{\mathbb{Z}_\ell}^\times/\mu$ factors through $G_{k'}^{ab}$. We denote by the same symbol $\det(\mathcal{G})$ the resulting element of $\Theta_{k'}$.*

   Then
   $$\mathcal{E}(\mathcal{G} \otimes^L \mathcal{F}) = (\det(\mathcal{G}) \circ \mathrm{tr}_{k'/k})^{\frac{1}{\deg(k'/k)} \cdot CC(\mathcal{F})} + \mathrm{rk}(\mathcal{G}) \cdot \mathcal{E}(\mathcal{F}).$$
   Here, for $CC(\mathcal{F}) = \sum_a m_a[C_a]$, we set

   $$(\det(\mathcal{G}) \circ \mathrm{tr}_{k'/k})^{\frac{1}{\deg(k'/k)} \cdot CC(\mathcal{F})} := \sum_a (\det(\mathcal{G}) \circ \mathrm{tr}_{k'/k})^{\frac{m_a}{\deg(k'/k)}} \otimes [C_a].$$

   See Remark 4.12 for our conventions.

   In particular, we have $\mathcal{E}(\mathcal{F}(n)) = \mathcal{E}(\mathcal{F})(n)$.

2. *Let k_1 be a subfield of k such that $\deg(k/k_1)$ is finite (in particular, k_1 is perfect). We regard X also as a smooth scheme over k_1. Then*

   $$\mathcal{E}(\mathcal{F})_k \circ \mathrm{tr}_{k/k_1} = \deg(k/k_1) \cdot \mathcal{E}(\mathcal{F})_{k_1},$$

where, for $\mathcal{E}(\mathcal{F})_k = \sum_a \xi_a \otimes [C_a]$, we set

$$\mathcal{E}(\mathcal{F})_k \circ \mathrm{tr}_{k/k_1} := \sum_a (\xi_a \circ \mathrm{tr}_{k/k_1}) \otimes [C_a].$$

3. *Let k'/k be an extension of fields such that k' is the perfection of a finitely generated field. For each irreducible component C_a of $SS(\mathcal{F})$, let $(C'_{a,b})_b$ be the set of irreducible components of $C_a \times_k k'$, and set $[C_a \times_k k'] := \sum_b [C'_{a,b}]$. Let $\mathcal{E}(\mathcal{F})_k = \sum_a \xi_a \otimes [C_a]$. Then*
$$\mathcal{E}(\mathcal{F}|_{X_{k'}}) = \sum_a \xi_a|_{G^{ab}_{k'}} \otimes [C_a \times_k k']$$
provided that one of the following conditions holds:
   - *(a) k is algebraically closed in k'.*
   - *(b) k is a finite field.*

Proof. 1. Choose a diagram as in (2.2) and an at most isolated $SS(\mathcal{F})$-characteristic point $u \in U$ of f. We have

$$\begin{aligned}
(\mathcal{E}(\mathcal{G} \otimes^L \mathcal{F}), df)_u^{\deg(u/k)} &= \overline{\varepsilon}_0(Y_{(u)}, R\Phi_f(\mathcal{G} \otimes \mathcal{F})_u)^{-1} \circ \mathrm{tr}_{u/k} \\
&= \overline{\varepsilon}_0(Y_{(u)}, \mathcal{G}_{\bar{u}} \otimes R\Phi_f(\mathcal{F})_u)^{-1} \circ \mathrm{tr}_{u/k} \\
&= (\det \mathcal{G} \circ \mathrm{tr}_{u/k})^{-\operatorname{dimtot} R\Phi_f(\mathcal{F})_u} \cdot \overline{\varepsilon}_0(Y_{(u)}, R\Phi_f(\mathcal{F})_u)^{-\operatorname{rk}(\mathcal{G})} \circ \mathrm{tr}_{u/k} \\
&= (\det \mathcal{G} \circ \mathrm{tr}_{k'/k})^{\deg(u/k') \cdot (CC(\mathcal{F}), df)_u} (\mathcal{E}(\mathcal{F}), df)_u^{\deg(u/k) \cdot \operatorname{rk}(\mathcal{G})} \\
&= \left((\det \mathcal{G} \circ \mathrm{tr}_{k'/k})^{\frac{1}{\deg(k'/k)} \cdot CC(\mathcal{F})} + \mathcal{E}(\mathcal{F})^{\operatorname{rk}(\mathcal{G})}, df\right)_u^{\deg(u/k)}.
\end{aligned}$$

2. We show that
$$\deg(k/k_1)^{-1} \cdot (\mathcal{E}(\mathcal{F})_k \circ \mathrm{tr}_{k/k_1})$$
satisfies the property of Theorem 4.10. Consider a diagram of k_1-schemes $X \overset{j}{\leftarrow} U \overset{f}{\rightarrow} Y$ where j is étale and Y is a smooth k_1-curve. Since local epsilon factors are unchanged after replacing Y by $Y \times_{k_1} k$, we may assume that the diagram is defined over k. The assertion then follows from the characterization of epsilon cycles in Theorem 4.10.

3. First, we consider case (a). Then $C_a \times_k k'$ is irreducible. Fix an irreducible component C_a.

By Lemma 2.21, we may choose a diagram of k-schemes
$$X \overset{j}{\leftarrow} U \overset{f}{\rightarrow} Y,$$
where j is étale and Y is a smooth k-curve, together with an isolated $SS(\mathcal{F})$-characteristic point $u \in U$ of f such that df only meets C_a at u. Set $k'(u) := k' \otimes_k k(u)$, which is a field extension of k'. Then the assertion follows from Theorem 4.10 and the commutativity of the diagram
$$\begin{array}{ccc}
G^{ab}_{k'} & \longrightarrow & G^{ab}_{k} \\
\Big\downarrow{\scriptstyle \mathrm{tr}_{k'(u)/k'}} & & \Big\downarrow{\scriptstyle \mathrm{tr}_{k(u)/k}} \\
G^{ab}_{k'(u)} & \longrightarrow & G^{ab}_{k(u)}.
\end{array}$$

Next, we consider case (b). Let k'' be the algebraic closure of k in k'. Since k' is the perfection of a finitely generated field, k'' is a finite field. As the case k'/k'' has already been treated in (a), we may replace k' by k''. Hence we may assume that k' is a finite field.

Let $\mathcal{E}(\mathcal{F}|_{X_{k'}})_{k'} = \sum_{a,b} \xi_{a,b} \otimes [C'_{a,b}]$. By assertion 2, we have

$$\deg(k'/k) \cdot \mathcal{E}(\mathcal{F}|_{X_{k'}})_k = \sum_{a,b} (\xi_{a,b} \circ \mathrm{tr}_{k'/k}) \otimes [C'_{a,b}].$$

Since epsilon cycles are étale local, we have $\xi_a^{\deg(k'/k)} = \xi_{a,b} \circ \mathrm{tr}_{k'/k}$ for every b. Since k' is finite, the composite

$$G_{k'}^{ab} \to G_k^{ab} \xrightarrow{\mathrm{tr}} G_{k'}^{ab}$$

is multiplication by $\deg(k'/k)$. Therefore

$$\xi_{a,b}^{\deg(k'/k)} = (\xi_{a,b} \circ \mathrm{tr}_{k'/k})|_{G_{k'}^{ab}} = \xi_a^{\deg(k'/k)}|_{G_{k'}^{ab}}.$$

The assertion follows. □

4.3.1 Pushforward along closed immersions

In this subsection, we study the behavior of epsilon cycles under pushforward along closed immersions.

Let $f\colon X \to Y$ be a morphism of smooth k-schemes, and let $C \subset T^*X$ be a closed conical subset. Assume that X is purely of dimension n and that Y is purely of dimension m. Assume further that f is proper on the base $C \cap T^*_X X$ of C. Under these assumptions, Saito [19] defines a group homomorphism

$$f_!\colon \mathrm{CH}_n(C) \to \mathrm{CH}_m(f_\circ C) \tag{4.4}$$

as follows. Consider the diagram

$$\begin{array}{ccccc} C & \leftarrow & C' & \rightarrow & f_\circ C \\ \downarrow & & \downarrow & & \downarrow \\ T^*X & \xleftarrow{df} & T^*Y \times_Y X & \xrightarrow{\mathrm{pr}} & T^*Y, \end{array}$$

where the left square is cartesian. Intersection theory defines a pullback morphism

$$(df)^!\colon \mathrm{CH}_n(C) \to \mathrm{CH}_m(C')$$

and a pushforward morphism

$$\mathrm{pr}_*\colon \mathrm{CH}_m(C') \to \mathrm{CH}_m(f_\circ C).$$

The map $f_!$ is then defined by $f_! = \mathrm{pr}_*(df)^!$. For an abelian group A, we also denote by

$$f_!\colon A \otimes \mathrm{CH}_n(C) \to A \otimes \mathrm{CH}_m(f_\circ C)$$

the homomorphism obtained from $f_!$ by tensoring with A.

Assume that every irreducible component of C has dimension n and that every irreducible component of $f_\circ C$ has dimension m. Then $f_!$ can be defined at the level of cycles, since $\mathrm{CH}_n(C) = Z_n(C)$ and $\mathrm{CH}_m(f_\circ C) = Z_m(f_\circ C)$.

Lemma 4.15. *Let $i\colon X \to X'$ be a closed immersion of smooth k-schemes. Assume that X is purely of dimension n and that X' is purely of dimension m. Let $\mathcal{F} \in D_c^b(X, \overline{\mathbb{Z}_\ell})$. Then we have an equality*

$$i_!\mathcal{E}(\mathcal{F}) = \mathcal{E}(i_*\mathcal{F})$$

in $\mathrm{CH}_m(i_\circ SS(\mathcal{F})) = Z_m(i_\circ SS(\mathcal{F}))$.

Proof. Since i is a closed immersion, the morphism $di\colon T^*X' \times_{X'} X \to T^*X$ is smooth of relative dimension $m - n$. It follows that every irreducible component of $i_\circ SS(\mathcal{F})$ has dimension m.

Consider a commutative diagram of k-schemes

$$\begin{array}{ccccc} X & \xleftarrow{j} & U & & \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} & \searrow^{f} & \\ X' & \xleftarrow[j']{} & U' & \xrightarrow[f']{} & Y, \end{array}$$

where the left horizontal arrows are étale, the square is cartesian, and Y is a smooth curve.

Let $u' \in U'$ be an isolated $SS(i_*\mathcal{F})$-characteristic point of f'. Since

$$SS(i_*\mathcal{F}) = i_\circ SS(\mathcal{F}),$$

we have $u' \in U$. It suffices to show that

$$\overline{\varepsilon}_0(Y_{(u')}, R\Phi_{f'}(i_*\mathcal{F})_{u'}) = \overline{\varepsilon}_0(Y_{(u')}, R\Phi_f(\mathcal{F})_{u'}).$$

This follows from the fact that the canonical morphism $R\Phi_{f'}(i_*\mathcal{F})_{u'} \to i'_*R\Phi_f(\mathcal{F})_{u'}$ is an isomorphism. □

4.3.2 Thom–Sebastiani theorem and pullback by smooth morphisms

Proposition 4.16. *Let X_1 and X_2 be smooth schemes of finite type over k, where k is the perfection of a finitely generated field. Let $\mathcal{F}_i \in D_c^b(X_i, \overline{\mathbb{Z}_\ell})$ for each $i = 1, 2$. Then*

$$\mathcal{E}(\mathcal{F}_1 \boxtimes \mathcal{F}_2) = \big(\mathcal{E}(\mathcal{F}_1) \boxtimes CC(\mathcal{F}_2)\big) + \big(CC(\mathcal{F}_1) \boxtimes \mathcal{E}(\mathcal{F}_2)\big).$$

Here $\mathcal{E}(\mathcal{F}_1) \boxtimes CC(\mathcal{F}_2)$ is defined by

$$\mathcal{E}(\mathcal{F}_1) \boxtimes CC(\mathcal{F}_2) := \sum_{a,b} \xi_a^{n_b} \otimes [C_a \times D_b],$$

where

$$\mathcal{E}(\mathcal{F}_1) = \sum_a \xi_a \otimes [C_a] \qquad \textit{and} \qquad CC(\mathcal{F}_2) = \sum_b n_b \cdot [D_b].$$

The definition of $CC(\mathcal{F}_1) \boxtimes \mathcal{E}(\mathcal{F}_2)$ is similar.

Proof. By Lemma 4.14.2 applied to $k_1 = k$, it suffices to prove the statement after base change to a finite extension of k. Hence we may assume that every irreducible component of $SS(\mathcal{F}_i)$ for $i = 1, 2$ is geometrically irreducible.

The cycle $\mathcal{E}(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ is supported on $SS(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$. By Proposition 2.9.2, we have $SS(\mathcal{F}_1 \boxtimes \mathcal{F}_2) = SS(\mathcal{F}_1) \times SS(\mathcal{F}_2)$. Therefore, it suffices to compare the coefficients of $[C_1 \times C_2]$ for irreducible components $C_1 \subset SS(\mathcal{F}_1)$ and $C_2 \subset SS(\mathcal{F}_2)$.

After a field extension of k, we may assume that, for each $i = 1, 2$, there exists a diagram

$$\begin{array}{ccc} U_i & \xrightarrow{f_i} & \mathbb{A}^1_k \\ \downarrow & & \\ X_i & & \end{array}$$

and a k-rational isolated $SS(\mathcal{F}_i)$-characteristic point $u_i \in U_i$ such that df_i meets only C_i at u_i.

Let $f\colon U_1 \times U_2 \to \mathbb{A}^2_k$ be the product of f_1 and f_2, and let $a\colon \mathbb{A}^2_k \to \mathbb{A}^1_k$ be the addition map. Set $u := (u_1, u_2) \in U_1 \times U_2$. Since u is an isolated $SS(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$-characteristic point of af, we have

$$(\mathcal{E}(\mathcal{F}_1 \boxtimes \mathcal{F}_2), d(af))_{T^*(U_1 \times U_2),u} = \overline{\varepsilon}_0(\mathbb{A}^1_{k,(0)}, R\Phi_{af}(\mathcal{F}_1 \boxtimes \mathcal{F}_2)_u)^{-1}.$$

Let ξ_i be the coefficient of C_i in $\mathcal{E}(\mathcal{F}_i)$, and ξ be the coefficient of $C_1 \times C_2$ in $\mathcal{E}(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$. Since $d(af)$ meets only $C_1 \times C_2$ at u, the left-hand side is equal to $\xi^{(C_1, df_1)_{T^*U_1,u_1} \cdot (C_2, df_2)_{T^*U_2,u_2}}$. On the other hand, by Lemma 3.22, the right-hand side is equal to

$$\overline{\varepsilon}_0(\mathbb{A}^1_{k,(0)}, R\Phi_{f_1}(\mathcal{F}_1)_{u_1})^{\operatorname{dimtot} R\Phi_{f_2}(\mathcal{F}_2)_{u_2}} \cdot \overline{\varepsilon}_0(\mathbb{A}^1_{k,(0)}, R\Phi_{f_2}(\mathcal{F}_2)_{u_2})^{\operatorname{dimtot} R\Phi_{f_1}(\mathcal{F}_1)_{u_1}},$$

which is equal to

$$\xi_1^{(C_1, df_1)_{u_1} \cdot (CC(\mathcal{F}_2), df_2)_{u_2}} \cdot \xi_2^{(C_2, df_2)_{u_2} \cdot (CC(\mathcal{F}_1), df_1)_{u_1}}.$$

The assertion follows. □

We recall the definition of the pullback of cycles by a properly transversal morphism, following [20, Definition 7.1.2].

Let X and W be smooth schemes over a field k, and let C be a closed conical subset of T^*X. Assume that X is purely of dimension n and that W is purely of dimension m.

Let $h\colon W \to X$ be a C-transversal k-morphism, and consider the diagram

$$\begin{array}{ccccc} h^\circ C & \longleftarrow & W \times_X C & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ T^*W & \xleftarrow{dh} & W \times_X T^*X & \xrightarrow{\mathrm{pr}} & T^*X. \end{array}$$

Intersection theory defines a pullback morphism

$$\mathrm{pr}^!\colon \mathrm{CH}_n(C) \to \mathrm{CH}_m(W \times_X C).$$

Note that $dh\colon W \times_X C \to h^\circ C$ is finite by Lemma 2.2.

Definition 4.17. *Let the notation and assumptions be as above. We define*

$$h^!\colon \mathrm{CH}_n(C) \to \mathrm{CH}_m(h^\circ C)$$

to be the composite

$$\mathrm{CH}_n(C) \xrightarrow{\mathrm{pr}^!} \mathrm{CH}_m(W \times_X C) \xrightarrow{dh_*} \mathrm{CH}_m(h^\circ C)$$

multiplied by $(-1)^{n-m}$. *For an abelian group* A*, we also denote by*

$$h^!\colon A \otimes \mathrm{CH}_n(C) \to A \otimes \mathrm{CH}_m(h^\circ C)$$

the homomorphism obtained from $h^!$ *by tensoring with* A*.*

Assume that every irreducible component of C *has dimension* n *and that* h *is properly* C*-transversal. Then* $h^!$ *can be defined at the level of cycles, since* $\mathrm{CH}_n(C) = Z_n(C)$ *and* $\mathrm{CH}_m(h^\circ C) = Z_m(h^\circ C)$*.*

Proposition 4.18. *Let* k *be the perfection of a finitely generated field. Let* $h\colon W \to X$ *be a smooth morphism of smooth schemes of finite type over* k*. Assume that* X *is purely of dimension* n *and that* W *is purely of dimension* m*. Let* $\mathcal{F} \in D^b_c(X, \overline{\mathbb{Z}_\ell})$*. Then we have an equality*

$$\mathcal{E}(h^*\mathcal{F}) = h^!\big(\mathcal{E}(\mathcal{F})(\frac{n-m}{2})\big)$$

in $\mathrm{CH}_m(h^\circ SS(\mathcal{F})) = Z_m(h^\circ SS(\mathcal{F}))$*, where the twist* $\mathcal{E}(\mathcal{F})(r)$ *for a rational number* r *is defined in Definition 4.13.*

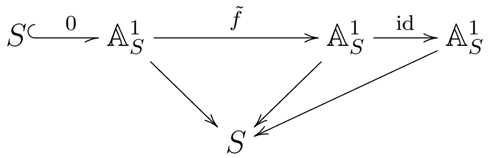
Proof. Since a smooth morphism is properly C-transversal for any closed conical subset C, we have $\mathrm{CH}_m(h^\circ SS(\mathcal{F})) = Z_m(h^\circ SS(\mathcal{F}))$.

Since the assertion is étale local on W, we may assume that $W = X \times \mathbb{A}^{m-n}_k$ and that h is the projection. By induction on $m-n$, we are reduced to the case $W = X \times \mathbb{A}^1_k$. By Proposition 4.16, it suffices to show that, for the constant $\overline{\mathbb{Z}_\ell}$-sheaf $\mathcal{G} := \overline{\mathbb{Z}_\ell}$ on $\mathbb{A}^1_k$,

$$\mathcal{E}(\mathcal{G}) = \chi_{\mathrm{cyc}}^{\frac{1}{2}} \otimes [T^*_{\mathbb{A}^1_k}\mathbb{A}^1_k].$$

First, we prove the assertion when $p \neq 2$. Consider the Kummer covering $f\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$ defined by $t \mapsto t^2$. The vanishing cycles complex $R\Phi_f(\mathcal{G})_0$ is concentrated in degree 0 and has rank 1. The corresponding character is a quadratic tamely ramified character. Hence, by Definition 3.7 and Lemma 3.9, we have $\overline{\varepsilon}_0(\mathbb{A}^1_{k,(0)}, R\Phi_f(\mathcal{G})_0) = \chi_{\mathrm{cyc}}^{\frac{-1}{2}}$ in Θ_k. On the other hand, the intersection number $(T^*_{\mathbb{A}^1_{\mathbb{F}_q}}\mathbb{A}^1_{\mathbb{F}_q}, df)_0$ is equal to 1. Therefore the assertion follows.

When $p = 2$, we argue as follows. Let $S := \mathrm{Spec}(\mathbb{Z}[\frac{1}{3\ell}])$ and consider the commutative diagram



where $\tilde{f}$ is defined by $t \mapsto t^3$. This diagram, together with the constant $\overline{\mathbb{Z}_\ell}$-sheaf on $\mathbb{A}^1_S$, satisfies the conditions from 1 to 5 given after the diagram (3.8). Then the assertion follows from Lemma 3.16, Proposition 3.17, and the case $p \neq 2$. □

Corollary 4.19. *Let* X *be a smooth connected scheme of finite type over* k*, and set* $n := \dim X$*. For a smooth* $\overline{\mathbb{Z}_\ell}$*-sheaf* $\mathcal{F}$ *on* X*, we have*

$$\mathcal{E}(\mathcal{F}) = (\det(\mathcal{F}) \circ \mathrm{tr}_{k'/k})^{\frac{(-1)^n}{\deg(k'/k)}} \cdot \chi_{\mathrm{cyc}}^{\frac{(-1)^{n+1} n \cdot \mathrm{rk}\,\mathcal{F}}{2}} \otimes [T^*_X X].$$

Here k' *denotes the normalization of* k *in* X*.*

Proof. This follows from Lemma 4.14.1 and Proposition 4.18. Recall that, by the Katz–Lang theorem (Theorem 4.4), the character $\det(\mathcal{F})\colon \pi_1^{ab}(X) \to \overline{\mathbb{Z}_\ell}^\times/\mu$ factors through $G_{k'}^{ab} \to \overline{\mathbb{Z}_\ell}^\times/\mu$. $\square$

Example 4.20. *Let X be a smooth connected curve over k, and let $\mathcal{F} \in D_c^b(X, \overline{\mathbb{Z}_\ell})$. Let $U \subset X$ be a dense open subset on which $\mathcal{F}$ is smooth. Then*

$$\mathcal{E}(\mathcal{F}) = (\det(\mathcal{F}|_U)\circ\mathrm{tr}_{k'/k})^{\frac{-1}{\deg(k'/k)}} \cdot \chi_{\mathrm{cyc}}^{\frac{\mathrm{rk}(\mathcal{F}|_U)}{2}} \otimes [T_X^*X] + \sum_{x \in X\setminus U} (\bar{\varepsilon}(X_{(x)}, \mathcal{F})^{-1}\circ\mathrm{tr}_{x/k})^{\frac{1}{\deg(x/k)}} \otimes [T_x^*X].$$

Here $\bar{\varepsilon}(X_{(x)}, \mathcal{F}) = \bar{\varepsilon}_0(X_{(x)}, \mathcal{F}_{\eta_x}) \cdot \det(\mathcal{F}_x)^{-1}$.

Lemma 4.21 (cf. [15, Théorème (3.2.1.1)], [7, Theorem 11.1])**.** *Let X be a smooth projective curve over k, and let $\mathcal{F} \in D_c^b(X, \overline{\mathbb{Z}_\ell})$. Then we have*

$$\det R\Gamma(X_{\bar{k}}, \mathcal{F}) = (\mathcal{E}(\mathcal{F}), T_X^*X)_{T^*X}$$

in Θ_k.

Proof. When k is of positive characteristic, the assertion follows from Theorem 3.5 and Example 4.20.

Assume now that k is of characteristic 0. Let Z be a closed subscheme of X such that $\mathcal{F}$ is smooth outside Z. Then the assertion follows from the case of positive characteristic by applying Proposition 3.17 to the diagram

$$\begin{array}{ccccc} Z & \hookrightarrow & X & \xrightarrow{\mathrm{id}} & X \\ & & \searrow & & \swarrow \\ & & & \mathrm{Spec}(k) & \end{array}$$

together with Lemma 3.16. $\square$

4.3.3 Independence of ℓ

In this subsection, we establish the ℓ-independence of epsilon cycles. Let $\mathbb{F}_q$ be the field with q elements. We identify $\Theta_{\mathbb{F}_q}$ with a subgroup of $\overline{\mathbb{Q}_\ell}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ via

$$\Theta_{\mathbb{F}_q} \xrightarrow{\cong} \overline{\mathbb{Z}_\ell}^\times/\mu \subset \overline{\mathbb{Q}_\ell}^\times/\mu = \overline{\mathbb{Q}_\ell}^\times \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad \xi \mapsto \xi(\mathrm{Frob}_q).$$

Let ℓ and ℓ' be (not necessarily distinct) prime numbers invertible in $\mathbb{F}_q$. Let F be a field of characteristic 0, and fix field embeddings $\iota\colon F \to \overline{\mathbb{Q}_\ell}$ and $\iota'\colon F \to \overline{\mathbb{Q}_{\ell'}}$. These embeddings induce injective ring homomorphisms

$$\iota\colon F[T] \hookrightarrow \overline{\mathbb{Q}_\ell}[T], \qquad \iota'\colon F[T] \hookrightarrow \overline{\mathbb{Q}_{\ell'}}[T]$$

and injective group homomorphisms

$$\iota\colon (F^\times \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes Z_n(T^*X) \hookrightarrow (\overline{\mathbb{Q}_\ell}^\times \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes Z_n(T^*X),$$
$$\iota'\colon (F^\times \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes Z_n(T^*X) \hookrightarrow (\overline{\mathbb{Q}_{\ell'}}^\times \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes Z_n(T^*X).$$

Proposition 4.22. *Let X be a smooth scheme of finite type over $\mathbb{F}_q$, purely of dimension n. Let $\mathcal{F}$ and $\mathcal{F}'$ be objects of $D_c^b(X, \overline{\mathbb{Z}_\ell})$ and $D_c^b(X, \overline{\mathbb{Z}_{\ell'}})$, respectively. Assume that, for every closed point x of X, there exists a polynomial $f_x(T) \in F[T]$ such that*

$$\iota(f_x(T)) = \det(T - \mathrm{Frob}_x \mid \mathcal{F}_{\bar{x}}), \qquad \iota'(f_x(T)) = \det(T - \mathrm{Frob}_x \mid \mathcal{F}'_{\bar{x}}).$$

Then there exists an element

$$\mathcal{E}_F \in (F^\times \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes Z_n(T^*X)$$

such that

$$\iota(\mathcal{E}_F) = \mathcal{E}(\mathcal{F}), \qquad \iota'(\mathcal{E}_F) = \mathcal{E}(\mathcal{F}').$$

Proof. We first prove the case where X is a smooth projective curve. Let U be a dense open subset of X on which $\mathcal{F}$ and $\mathcal{F}'$ are smooth. Let $x \in X$ be a closed point, and let η_x denote the generic point of $X_{(x)}$. By assumption, if $x \in U$, the semisimplifications of $\mathcal{F}_{\eta_x}$ and $\mathcal{F}'_{\eta_x}$ are compatible in the sense of [3, Définition 8.8]. Therefore, by [3, Théorème 9.8], the semisimplifications of $\mathcal{F}_{\eta_x}$ and $\mathcal{F}'_{\eta_x}$ are compatible for any closed point x. Hence, for any nonzero element $\omega \in \Omega^1_{X_{(x)}}$, there exists an element $s \in F^\times$ such that

$$\iota(s) = \varepsilon(X_{(x)}, \mathcal{F}, \omega), \qquad \iota'(s) = \varepsilon(X_{(x)}, \mathcal{F}', \omega).$$

The assertion then follows from Example 4.20.

We now prove the general case. Set

$$C := SS(\mathcal{F}) \cup SS(\mathcal{F}').$$

We show that, for each irreducible component C_a of C, there exists an element $\xi \in F^\times \otimes \mathbb{Q}$ such that $\iota(\xi)$ is equal to the coefficient of $[C_a]$ in $\mathcal{E}(\mathcal{F})$ and $\iota'(\xi)$ is equal to that of $[C_a]$ in $\mathcal{E}(\mathcal{F}')$. Since this is étale local, we may assume that X is affine. Taking an immersion of X into a pojective space and replacing $\mathcal{F}, \mathcal{F}'$ with their extensions by zero, we may assume that X is smooth projective.

By Lemma 2.20, after replacing $\mathbb{F}_q$ by a finite extension, we have a good pencil

$$X \xleftarrow{\pi} X_L \xrightarrow{f} \mathbb{P}^1_{\mathbb{F}_q}.$$

By property 6 of Definition 2.19, there exists an isolated $\pi^\circ C$-characteristic point $x \in X_L$ of f such that df meets only C_a at x. By property 5, the point x does not lie in the exceptional locus of π. It follows that $C_a \cap T^*_X X$ is not contained in the exceptional locus. Thus it suffices to prove the assertion for $\pi^*\mathcal{F}$ and $\pi^*\mathcal{F}'$.

By properties 4 and 6, together with Theorem 4.10, it suffices to show that there exists $s \in F^\times \otimes \mathbb{Q}$ such that

$$\iota(s) = \varepsilon_0(\mathbb{P}^1_{\mathbb{F}_q,(x)}, R\Phi_f(\mathcal{F})_x, dt), \qquad \iota'(s) = \varepsilon_0(\mathbb{P}^1_{\mathbb{F}_q,(x)}, R\Phi_f(\mathcal{F}')_x, dt).$$

By the proper base change theorem, the quantities on the right-hand side are equal to the local epsilon factors of $Rf_*\pi^*\mathcal{F}$ and $Rf_*\pi^*\mathcal{F}'$, respectively. Since the curve case has already been established, it remains to verify that $Rf_*\pi^*\mathcal{F}$ and $Rf_*\pi^*\mathcal{F}'$ satisfy the assumptions of the proposition. This follows from [5, Theorem 2].

This completes the proof. □

4.3.4 Radon transform and pullback by properly transversal morphisms

We prove the compatibility of epsilon cycles with pullback by properly transversal morphisms. We follow the method used for characteristic cycles in [20], due to Beilinson. We use the theory of universal hyperplane sections and follow the notation of (2.7).

Lemma 4.23 ([20, Lemma 3.11])**.** *We follow the notation of (2.7). Let $\mathbb{P} = \mathbb{P}^n$ be a projective space and $\mathbb{P}^\vee$ be its dual projective space. Let $C^\vee \subset T^*\mathbb{P}^\vee$ be a closed conical subset whose irreducible components have dimension n. Let*

$$C = \boldsymbol{p}_\circ \boldsymbol{p}^{\vee\circ} C^\vee \subset T^*\mathbb{P}.$$

Then every irreducible component of C is of dimension n.

Proof. See [20, Lemma 3.11]. □

Proposition 4.24. *Let k be the perfection of a finitely generated field. Let $\mathbb{P} = \mathbb{P}^n_k$ be a projective space, and let $\mathbb{P}^\vee$ be its dual projective space. Let $\mathcal{G} \in D^b_c(\mathbb{P}^\vee, \overline{\mathbb{Z}_\ell})$ and set $\mathcal{F} := R\boldsymbol{p}_*\boldsymbol{p}^{\vee*}\mathcal{G}$. Let $C^\vee \subset T^*\mathbb{P}^\vee$ be a closed conical subset whose irreducible components have dimension n, and set $C := \boldsymbol{p}_\circ\boldsymbol{p}^{\vee\circ}C^\vee \subset T^*\mathbb{P}$. Assume that $\mathcal{G}$ is micro-supported on $C^\vee$.*

Let X be a smooth subscheme of $\mathbb{P}$ purely of dimension m. Assume that the immersion $h\colon X \to \mathbb{P}$ is properly C-transversal.

1. *The complex $\mathcal{F}$ is micro-supported on C.*
2. *We have*
$$p_\circ p^{\vee\circ} C^\vee = h^\circ C.$$
Consequently, both $p_!\mathcal{E}(p^{\vee*}\mathcal{G})$ and $p_!p^{\vee!}\big(\mathcal{E}(\mathcal{G})(\frac{1-m}{2})\big)$ may be regarded as elements of
$$\Theta_k \otimes Z_m(h^\circ C) \subset \Theta_k \otimes Z_m(T^*X).$$
3. *For every element $\mathcal{E} \in \Theta_k \otimes Z_m(T^*X)$, define*
$$\mathcal{E}^0 := \mathcal{E} - \xi_0 \otimes [T^*_X X],$$
where ξ_0 denotes the coefficient of the zero section $T^*_X X$ in $\mathcal{E}$. Then
$$\mathcal{E}(Rp_*p^{\vee*}\mathcal{G})^0 = \big(p_!\mathcal{E}(p^{\vee*}\mathcal{G})\big)^0 = \big(p_!p^{\vee!}\big(\mathcal{E}(\mathcal{G})(\frac{1-m}{2})\big)\big)^0. \tag{4.5}$$
In particular, we have
$$\mathcal{E}(R\boldsymbol{p}_*\boldsymbol{p}^{\vee*}\mathcal{G})^0 = \big(\boldsymbol{p}_!\mathcal{E}(\boldsymbol{p}^{\vee*}\mathcal{G})\big)^0 = \big(\boldsymbol{p}_!\boldsymbol{p}^{\vee!}(\mathcal{E}(\mathcal{G})(\frac{1-n}{2}))\big)^0.$$
4. *We have*
$$\mathcal{E}(h^*\mathcal{F}) = h^!\big(\mathcal{E}(\mathcal{F})(\frac{n-m}{2})\big).$$

Proof. 1. The assertion follows from [1, §2.2, Lemma].

2. By [20, Corollary 3.13.2], the morphism $p^\vee$ is $C^\vee$-transversal and we have

$$p_\circ p^{\vee\circ} C^\vee = h^\circ C.$$

Therefore, by the construction in (4.4) and Definition 4.17, the classes $p_!\mathcal{E}(p^{\vee *}\mathcal{G})$ and $p_! p^{\vee !}\big(\mathcal{E}(\mathcal{G})(\frac{1-m}{2})\big)$ are defined in $\Theta_k \otimes \mathrm{CH}_m(h^\circ C)$. Since h is properly C-transversal, we have $\mathrm{CH}_m(h^\circ C) = Z_m(h^\circ C)$. The assertion follows.

3. First, we prove the second equality of (4.5). Since $p^\vee$ is $C^\vee$-transversal, it follows that $p^{\vee *}\mathcal{G}$ is micro-supported on $p^{\vee\circ}C^\vee$. Since $p^\vee$ is smooth outside $\Delta_X := \mathbb{P}(T^*_X\mathbb{P}) \subset X \times_{\mathbb{P}} Q$, by Proposition 4.18, we have

$$\mathcal{E}(p^{\vee *}\mathcal{G}) = p^{\vee !}(\mathcal{E}(\mathcal{G})(\frac{1-m}{2}))$$

on $(X \times_{\mathbb{P}} Q) \setminus \Delta_X$. Let D be the union of the irreducible components of $p^{\vee\circ}C^\vee$ contained in $T^*(X \times_{\mathbb{P}} Q)|_{\Delta_X}$. Then the element

$$\mathcal{E}(p^{\vee *}\mathcal{G}) - p^{\vee !}\big(\mathcal{E}(\mathcal{G})(\frac{1-m}{2})\big) \in \Theta_k \otimes \mathrm{CH}_{n+m-1}(p^{\vee\circ}C^\vee)$$

lies in the image of the canonical map $\Theta_k \otimes \mathrm{CH}_{n+m-1}(D) \to \Theta_k \otimes \mathrm{CH}_{n+m-1}(p^{\vee\circ}C^\vee)$. Hence, to prove the second equality of (4.5), it suffices to show that $p_\circ D$ is contained in the zero section. Since $h\colon X \to \mathbb{P}$ is C-transversal, by [20, Corollary 3.13.1], the pair $(p^\vee, p)$ is $C^\vee$-transversal around $\Delta_X \subset X \times_{\mathbb{P}} Q$. The claim follows from this fact.

We now prove the first equality in (4.5). Since both $\mathcal{E}(Rp_* p^{\vee *}\mathcal{G})$ and $p_!\mathcal{E}(p^{\vee *}\mathcal{G})$ are supported on $h^\circ C = p_\circ p^{\vee\circ}C^\vee$, it suffices to prove the equality

$$(\mathcal{E}(Rp_* p^{\vee *}\mathcal{G}), df)_u^{\deg(u/k)} = (p_!\mathcal{E}(p^{\vee *}\mathcal{G}), df)_u^{\deg(u/k)} \tag{4.6}$$

for every diagram

$$X \xleftarrow{j} U \xrightarrow{f} \mathbb{A}^1_k,$$

where j is étale and f is smooth, and for every at most isolated $h^\circ C$-characteristic point $u \in U$ of f. By Theorem 4.10, the left-hand side of (4.6) is equal to

$$\bar{\varepsilon}_0(\mathbb{A}^1_{k(u)}, R\Phi_f(h^*\mathcal{F})_u)^{-1} \circ \mathrm{tr}_{u/k}\,.$$

We next compute the right-hand side of (4.6). After replacing U by an open neighborhood of u, we may assume that $U \setminus \{u\} \to \mathbb{A}^1_k$ is $h^\circ C$-transversal. Then, by [20, Corollary 3.15], the morphism $fp\colon U \times_{\mathbb{P}} Q \to \mathbb{A}^1_k$ has only finitely many $p^{\vee\circ}C^\vee$-characteristic points.

Therefore, the right-hand side of (4.6) is equal to

$$\prod_v (\mathcal{E}(p^{\vee *}\mathcal{G}), d(fp))_v^{\deg(v/k)},$$

where v runs through the $p^{\vee\circ}C^\vee$-characteristic points of fp lying over u. By Theorem 4.10, this is equal to

$$\prod_v \bar{\varepsilon}_0(\mathbb{A}^1_{k(v)}, R\Phi_{fp}(p^{\vee *}\mathcal{G})_v)^{-1} \circ \mathrm{tr}_{v/k}\,.$$

Hence, the equality (4.6) follows from the isomorphism

$$R\Phi_f(Rp_*p^{\vee *}\mathcal{G})_u \xrightarrow{\cong} \bigoplus_v \operatorname{Ind}_{G_v}^{G_u} R\Phi_{fp}(p^{\vee *}\mathcal{G})_v$$

obtained from the proper base change theorem.

4. By the proper base change theorem, we have an isomorphism $h^*\mathcal{F} \xrightarrow{\cong} Rp_*p^{\vee *}\mathcal{G}$. Therefore, by 2, we have

$$\begin{aligned}\mathcal{E}(h^*\mathcal{F})^0 &= \big(p_!p^{\vee !}\big(\mathcal{E}(\mathcal{G})(\frac{1-m}{2})\big)\big)^0 \\ &= \big(h^!\boldsymbol{p}_!\boldsymbol{p}^{\vee !}\big(\mathcal{E}(\mathcal{G})(\frac{1-m}{2})\big)\big)^0 = \big(h^!\big(\mathcal{E}(\mathcal{F})(\frac{n-m}{2})\big)\big)^0.\end{aligned}$$

Since h is properly C-transversal, the subscheme $X \subset \mathbb{P}$ meets the smooth locus of $\mathcal{F}$. Hence, by Corollary 4.19, the coefficient of the zero section in $\mathcal{E}(h^*\mathcal{F})$ coincides with that in $h^!(\mathcal{E}(\mathcal{F})(\frac{n-m}{2}))$. Thus the assertion follows. □

Before stating Corollary 4.25, we recall the Radon transform of ℓ-adic sheaves and the Legendre transform of closed conical subsets.

Let $\mathbb{P} = \mathbb{P}^n_k$. We define the Radon transform $R\colon D^b_c(\mathbb{P}, \overline{\mathbb{Z}_\ell}) \to D^b_c(\mathbb{P}^\vee, \overline{\mathbb{Z}_\ell})$ by

$$R\mathcal{F} := R\boldsymbol{p}^\vee_*\boldsymbol{p}^*\mathcal{F}[n-1].$$

Let C be a closed conical subset of $T^*\mathbb{P}$, and let

$$C = \bigcup_a C_a$$

be its decomposition into irreducible components. Assume that each C_a has dimension n. For

$$A := \sum_a \beta_a \otimes [C_a] \in \Theta_k \otimes Z_n(T^*\mathbb{P}),$$

we define its Legendre transform by

$$LA := (-1)^{n-1} \cdot \boldsymbol{p}^\vee_! \boldsymbol{p}^! A.$$

By Lemma 4.23, the above definition yields an element of $\Theta_k \otimes Z_n(T^*\mathbb{P}^\vee)$. We include the sign $(-1)^{n-1}$ so as to cancel the sign appearing in the definition of $\boldsymbol{p}^!A$.

Corollary 4.25. *Let $\mathcal{F}$ be an object of $D^b_c(\mathbb{P}, \overline{\mathbb{Z}_\ell})$. We use the notation $\mathcal{E}^0$ introduced in Proposition 4.24.2. Then*

$$\mathcal{E}(R\mathcal{F})^0 = \big(L\big(\mathcal{E}(\mathcal{F})(\frac{1-n}{2})\big)\big)^0.$$

We will prove the equality $\mathcal{E}(R\mathcal{F}) = L(\mathcal{E}(\mathcal{F})(\frac{1-n}{2}))$ in Corollary 5.8.

Proof. This is a restatement of Proposition 4.24.2 with $\mathbb{P}$ and $\mathbb{P}^\vee$ exchanged. □

Theorem 4.26. *Let k be the perfection of a finitely generated field. Let X be a smooth scheme of finite type over k, and let $\mathcal{F} \in D^b_c(X, \overline{\mathbb{Z}_\ell})$. Let $h\colon W \to X$ be a properly $SS(\mathcal{F})$-transversal k-morphism from a smooth k-scheme W of finite type. Assume that X is purely of dimension n and that W is purely of dimension m. Then*

$$\mathcal{E}(h^*\mathcal{F}) = h^!\big(\mathcal{E}(\mathcal{F})(\frac{n-m}{2})\big).$$

Proof. Decomposing h as $W \to W \times X \to X$, we may assume that h is either a smooth morphism or an immersion. The smooth case follows from Proposition 4.18.

Suppose that h is an immersion. We first consider the case where X is a projective space $\mathbb{P}$. Let $R^\vee$ denote the dual Radon transform $R\boldsymbol{p}_*\boldsymbol{p}^{\vee *}[n-1]$. Then, by [14, Lemma 1.4], there is a canonical isomorphism

$$R^\vee R\mathcal{F} \cong \mathcal{F}(1-n) \oplus \mathcal{H},$$

where $\mathcal{H}$ is an object of $D^b_c(\mathbb{P}, \overline{\mathbb{Z}_\ell})$ whose cohomology sheaves are smooth. We prove the assertion by showing that both $R^\vee R\mathcal{F}$ and $\mathcal{H}$ satisfy the theorem.

Set $C := SS(\mathcal{F}) \cup T^*_{\mathbb{P}}\mathbb{P}$. Then h is properly C-transversal. Since the cohomology sheaves of $\mathcal{H}$ are smooth, $\mathcal{H}$ is micro-supported on C. Moreover, by Corollary 4.19, we have

$$\mathcal{E}(h^*\mathcal{H}) = h^!\big(\mathcal{E}(\mathcal{H})(\frac{n-m}{2})\big).$$

By the dual version of Proposition 4.24.1, the complex $R\mathcal{F}$ is micro-supported on $C^\vee := \boldsymbol{p}^\vee_\circ \boldsymbol{p}^\circ C$. Hence, $R^\vee R\mathcal{F}$ is micro-supported on $\boldsymbol{p}_\circ \boldsymbol{p}^{\vee\circ} C^\vee$. We claim that

$$C = \boldsymbol{p}_\circ \boldsymbol{p}^{\vee\circ} C^\vee. \tag{4.7}$$

Indeed, by [1, §1.6.2], the canonical identification

$$\mathbb{P}(T^*\mathbb{P}) \cong Q \cong \mathbb{P}(T^*\mathbb{P}^\vee)$$

identifies the projectivizations $\mathbb{P}(C) \subset \mathbb{P}(T^*\mathbb{P})$ and $\mathbb{P}(C^\vee) \subset \mathbb{P}(T^*\mathbb{P}^\vee)$. It follows that the equality (4.7) holds outside the zero section $T^*_{\mathbb{P}}\mathbb{P}$. Since, by construction, both C and $C^\vee$ contain the zero section, the equality (4.7) follows.

The theorem for $R^\vee R\mathcal{F}$ now follows from Proposition 4.24.4 applied to $\mathcal{G} = R\mathcal{F}$. This completes the proof of the theorem for $\mathcal{F}$.

We now prove the general case. Since the assertion is local on W, we may assume that X is affine. Fix an immersion $i \colon X \to \mathbb{P}$. After shrinking W further if necessary, we may assume that there exists a smooth subscheme $V \subset \mathbb{P}$ such that $X \cap V = W$ and V intersects X transversally. Then the immersion $\tilde{h} \colon V \to \mathbb{P}$ is properly $SS(i_!\mathcal{F})$-transversal in a neighborhood of W. Hence the assertion follows from the equality

$$\mathcal{E}(\tilde{h}^* i_!\mathcal{F}) = \tilde{h}^!\big(\mathcal{E}(i_!\mathcal{F})(\frac{n-m}{2})\big),$$

which holds in a neighborhood of W. □

4.4 Epsilon cycles for tamely ramified sheaves

Let k be the perfection of a finitely generated field of characteristic $p \neq \ell$. In this subsection, we compute the epsilon cycles of tamely ramified $\overline{\mathbb{Z}_\ell}$-sheaves.

Let X be a smooth scheme of finite type over k, and let $D \subset X$ be a simple normal crossings divisor. Denote by $U = X \setminus D$ its complement. Let $(D_a)_{a \in A}$ be the irreducible components of D. For each subset $B \subset A$, we set

$$D_B := \bigcap_{a \in B} D_a.$$

For simplicity, we assume that X is connected and f dimension n. Then, for every subset $B \subset A$, the scheme D_B is a smooth closed subscheme of X purely of dimension $n - |B|$.

Let $\mathcal{F}$ be a nonzero smooth $\overline{\mathbb{Z}_\ell}$-sheaf of free $\overline{\mathbb{Z}_\ell}$-modules on U, tamely ramified along D. Let $j\colon U \to X$ denote the inclusion. Then

$$SS(j_!\mathcal{F}) = \bigcup_B T^*_{D_B}X, \tag{4.8}$$

$$CC(j_!\mathcal{F}) = \sum_B (-1)^n \operatorname{rk}\mathcal{F}\,[T^*_{D_B}X], \tag{4.9}$$

where the union and the sum are taken over all subsets $B \subset A$ (see [20, Proposition 4.11 and Theorem 7.14]).

For each $a \in A$, let ξ_a denote the generic point of D_a, and let k_a be the normalization of k in $k(\xi_a)$. Since $\mathcal{F}$ is tamely ramified, its restriction to the henselian trait $X_{(\xi_a)}$ determines a representation V_a of the tame inertia group I_a. The group I_a is canonically isomorphic to

$$\varprojlim_{n \neq p} \mu_n(\bar{k}),$$

where $\mu_n(\bar{k})$ denotes the group of n-th roots of unity in an algebraic closure $\bar{k}$ of k_a, and n runs through the positive integers prime to p. Note that, for every $\sigma \in \operatorname{Gal}(\bar{k}/k_a)$, we have $\sigma^* V_a \cong V_a$. Thus we obtain a character

$$J(V_a)\colon \operatorname{Gal}(\bar{k}/k_a) \to \overline{\mathbb{Z}_\ell}^\times/\mu$$

constructed in Definition 3.6.2. We define

$$J_a := (J(V_a) \circ \operatorname{tr}_{k_a/k})^{\frac{1}{\deg(k_a/k)}}.$$

Thus J_a is an element of Θ_k.

Proposition 4.27. *Let the notation and assumptions be as above. Assume that X is connected and of dimension n, and that $\mathcal{F}$ is tamely ramified along D. Let k' be the normalization of k in the function field of X. For every subset $B \subset A$, define*

$$\chi_B := (\det(\mathcal{F}) \circ \operatorname{tr}_{k'/k})^{\frac{(-1)^n}{\deg(k'/k)}} \cdot \chi_{\mathrm{cyc}}^{\frac{|B|-n}{2}(-1)^n \operatorname{rk}(\mathcal{F})} \cdot \prod_{a \in B} J_a^{(-1)^n}.$$

Then

$$\mathcal{E}(j_!\mathcal{F}) = \sum_B \chi_B \otimes [T^*_{D_B}X].$$

Proof. By (4.8), it suffices to determine the coefficient of $[T^*_{D_B}X]$ for each subset $B \subset A$. Fix a subset $B \subset A$. Replacing X by

$$X \setminus \bigcup_{a \in A \setminus B} D_a$$

and D by its restriction to this open subscheme, we may assume that $A = B$.

Assume that $A = B$. If $D = D_B$ is empty, there is nothing to prove. Suppose that D is nonempty. By Lemma 2.21, there exists a diagram of k-schemes

$$X \xleftarrow{j'} V \xrightarrow{f} Y,$$

where j' is étale and Y is a smooth curve over k, together with an isolated $SS(j_!\mathcal{F})$-characteristic point $x \in V$ of f, such that df meets $T^*_{D_A}X$ at x and does not meet any other component of $SS(j_!\mathcal{F})$ at x. Replacing X by V, we may assume that $V = X$ and $j' = \mathrm{id}_X$.

By Theorem 4.10, to determine the coefficient of $[T^*_{D_A}X]$, it suffices to show that

$$\bar{\varepsilon}_0(Y_{(x)}, R\Phi_f(j_!\mathcal{F})_x)^{-1} \circ \mathrm{tr}_{k(x)/k} = \chi_B^{(T^*_{D_A}X, df)_x^{\deg(x/k)}}. \tag{4.10}$$

Observe that both sides depend only on the restriction of $j_!\mathcal{F}$ to the henselization $X_{(x)}$. Thus, if $\mathcal{G}$ is a smooth $\overline{\mathbb{Z}_\ell}$-sheaf on U such that

$$j_!\mathcal{G}\big|_{X_{(x)}} \cong j_!\mathcal{F}\big|_{X_{(x)}},$$

then (4.10) for $\mathcal{F}$ is equivalent to (4.10) for $\mathcal{G}$. In the sequel, we will find such $\mathcal{G}$ for which (4.10) is easy to verify.

Set $m := |A|$. For each $1 \leq i \leq n$, let $E_i \subset \mathbb{A}^n_{k(x)}$ denote the hyperplane defined by the i-th coordinate function, and set

$$E := \bigcup_{1 \leq i \leq m} E_i.$$

After replacing X by an étale neighborhood of x, we may find an étale morphism

$$\pi \colon X \to \mathbb{A}^n_{k(x)}$$

sending x to the origin such that, after a suitable numbering of the elements of A, the pullbacks of $(E_i)_{1 \leq i \leq m}$ are equal to $(D_a)_{a \in A}$.

The henselization $D_{(x)}$ is a simple normal crossings divisor of $X_{(x)}$. Let

$$\pi_1^{\mathrm{tame}}(X_{(x)} \setminus D_{(x)})$$

denote the fundamental group classifying finite étale coverings of $X_{(x)} \setminus D_{(x)}$ that are tamely ramified along $D_{(x)}$. We also write

$$\pi_1^{\mathrm{tame}}(\mathbb{A}^n_{k(x)} \setminus E)$$

for the fundamental group classifying finite étale coverings of $\mathbb{A}^n_{k(x)} \setminus E$ that are tamely ramified along both E and $\mathbb{P}^n_{k(x)} \setminus \mathbb{A}^n_{k(x)}$. The morphism π induces an isomorphism

$$\pi_1^{\mathrm{tame}}(X_{(x)} \setminus D_{(x)}) \to \pi_1^{\mathrm{tame}}(\mathbb{A}^n_{k(x)} \setminus E).$$

Hence there exists a smooth $\overline{\mathbb{Z}_\ell}$-sheaf $\mathcal{G}$ on $\mathbb{A}^n_{k(x)} \setminus E$, tamely ramified along both E and $\mathbb{P}^n_{k(x)} \setminus \mathbb{A}^n_{k(x)}$, such that

$$j_!\pi^*\mathcal{G}\big|_{X(x)} \cong j_!\mathcal{F}\big|_{X(x)}.$$

Thus we may assume that $X = \mathbb{A}^n_K$ for some finite extension K/k, that $D = E$, and that $\mathcal{F}$ is tamely ramified along $\mathbb{P}^n_K \setminus \mathbb{A}^n_K$ as well.

Replacing $\mathcal{F}$ by its subquotients, we may further assume that $\mathcal{F} \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell}$ is irreducible. For each $1 \leq i \leq m$, let I_i be the tame inertia group of the henselization of $\mathbb{A}^n_K$ at the generic point of E_i, and let V_i denote the representation of I_i associated with $\mathcal{F}$.

Since the canonical homomorphism $I_i \to \pi_1^{\mathrm{tame}}(\mathbb{A}^n_K \setminus E)$ is injective and its image is a normal subgroup, the irreducibility of $\mathcal{F} \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell}$ implies that the I_i-representation $V_i \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell}$ is semisimple. Moreover, there exists a character $\chi_i \colon I_i \to \overline{\mathbb{Q}_\ell}^\times$ such that $V_i \otimes \overline{\mathbb{Q}_\ell}$ is a direct sum of copies of χ_i and its conjugates.

Since K contains only finitely many roots of unity, χ_i factors through a finite quotient $I_i \to \mu_{d_i}(\bar{k})$ for some integer $d_i \geq 1$ prime to p. After replacing K by a finite extension, we may assume that $\mu_{d_i}(\bar{k}) \subset K$ for all i. Further replacing $\mathcal{F}$ by its subquotients, we may assume that, for each i,

$$V_i \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell} \cong \chi_i^{\oplus n_i}$$

for some integer $n_i \geq 1$.

After these replacements, there is a canonical homomorphism

$$\pi_1^{\mathrm{tame}}(\mathbb{A}^n_K \setminus E) \to \mu_{d_i}(\bar{k}) \times \mathrm{Gal}(\bar{k}/K)$$

such that the composite

$$I_i \to \pi_1^{\mathrm{tame}}(\mathbb{A}^n_K \setminus E) \to \mu_{d_i}(\bar{k}) \times \mathrm{Gal}(\bar{k}/K) \to \mu_{d_i}(\bar{k}),$$

where the last map is the first projection, is the natural surjection. Therefore, for each i, there exists a rank-one smooth $\overline{\mathbb{Z}_\ell}$-sheaf $\mathcal{G}_i$ of finite order on $\mathbb{A}^1_K \setminus 0$, tamely ramified at 0 and ∞, such that its pullback to $\mathbb{A}^n_K \setminus E$ by the i-th projection $\mathrm{pr}_i \colon \mathbb{A}^n_K \to \mathbb{A}^1_K$ has the same local monodromy at the generic point of E_i as χ_i. It follows that $\mathcal{F} \otimes \bigotimes_i \mathrm{pr}_i^* \mathcal{G}_i^{-1}$ is unramified along E, and hence extends to a smooth $\overline{\mathbb{Z}_\ell}$-sheaf $\mathcal{H}$ on $\mathbb{A}^n_K$.

By Lemma 4.14.1, Propositions 4.16, and 4.18, the coefficient of $[T^*_{\bigcap_{1 \leq i \leq m} E_i} \mathbb{A}^n_K]$ in $\mathcal{E}(j_! \mathcal{F})$ is

$$(\det(\mathcal{H}) \circ \mathrm{tr}_{K/k})^{\frac{1}{\deg(K/k)} \cdot (-1)^n} \cdot \chi_{\mathrm{cyc}}^{\frac{m-n}{2}(-1)^n \operatorname{rk}(\mathcal{F})} \cdot \prod_{1 \leq i \leq m} (\bar{\varepsilon}_0(\mathbb{A}^1_{K,(0)}, \mathcal{G}_i) \circ \mathrm{tr}_{K/k})^{\frac{(-1)^n}{\deg(K/k)} \operatorname{rk}(\mathcal{F})}.$$

Since each $\mathcal{G}_i$ has finite order, we have $\det(\mathcal{H}) = \det(\mathcal{F})$ as elements of Θ_K. Moreover, $(\bar{\varepsilon}_0(\mathbb{A}^1_{K,(0)}, \mathcal{G}_i))^{\operatorname{rk}(\mathcal{F})} = J(V_i)$. The assertion follows. □

5 Radon Transform and Product Formula

Let k be a field. In Subsections 5.2 and 5.3, we assume that k is the perfection of a finitely generated field and fix a prime number ℓ invertible in k.

5.1 Reminder on the Chow groups of projective space bundles

In this preliminary subsection, we briefly recall the necessary facts on the Chow groups of projective space bundles, following [20, §6.1].

Let X be a scheme of finite type over k. We write $\mathrm{CH}_\bullet(X) = \bigoplus_i \mathrm{CH}_i(X)$ for the Chow group of X. Let $\mathbb{Z}[h]$ be the polynomial ring in one variable h over $\mathbb{Z}$, and put

$$\mathrm{CH}_\bullet(X)[h] := \mathrm{CH}_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Z}[h].$$

We regard $\mathrm{CH}_\bullet(X)[h]$ as a left module over the ring

$$\mathrm{End}\big(\mathrm{CH}_\bullet(X)\big)[h] := \mathrm{End}\big(\mathrm{CH}_\bullet(X)\big) \otimes_{\mathbb{Z}} \mathbb{Z}[h].$$

For a vector bundle E of rank $n+1$ over X, define

$$c_h(E) := \sum_{q=0}^{n+1} c_q(E) h^{n+1-q} \in \mathrm{End}\big(\mathrm{CH}_\bullet(X)\big)[h].$$

Lemma 5.1. *Let X be a scheme of finite type over k, and let E be a vector bundle of rank $n+1$ over X.*

1. *Let $\pi\colon \mathbb{P}(E) \to X$ denote the projection. Then the homomorphism*

   $$\alpha_E\colon \mathrm{CH}_\bullet(X)[h] \to \mathrm{CH}_\bullet(\mathbb{P}(E))$$

   given by $ah^q \mapsto c_1(\mathcal{O}(1))^q \cap \pi^* a$ is surjective, and its kernel is $c_h(E) \cdot \mathrm{CH}_\bullet(X)[h]$.

2. *([20, Lemma 6.2]) Let $i\colon F \to E$ be an injection of vector bundles over X.*

   (a) The diagram

   $$\begin{array}{ccc} \mathrm{CH}_\bullet(\mathbb{P}(E)) & \xrightarrow{\quad i^* \quad} & \mathrm{CH}_\bullet(\mathbb{P}(F)) \\ {\scriptstyle \alpha_E}\nwarrow & & \nearrow{\scriptstyle \alpha_F} \\ & \mathrm{CH}_\bullet(X)[h] & \end{array}$$

   is commutative.

   (b) Let K denote the cokernel of i. Then the diagram

   $$\begin{array}{ccc} \mathrm{CH}_\bullet(\mathbb{P}(F)) & \xrightarrow{i_*} & \mathrm{CH}_\bullet(\mathbb{P}(E)) \\ \uparrow{\scriptstyle \alpha_F} & & \uparrow{\scriptstyle \alpha_E} \\ \mathrm{CH}_\bullet(X)[h] & \xrightarrow[c_h(K)\cdot]{} & \mathrm{CH}_\bullet(X)[h] \end{array}$$

   is commutative.

Proof. 1. Since $c_0(E) = 1$, we have a decomposition

$$\mathrm{CH}_\bullet(X)[h] = \Big(\bigoplus_{i=0}^{n} \mathrm{CH}_\bullet(X) \cdot h^i\Big) \oplus \big(c_h(E) \cdot \mathrm{CH}_\bullet(X)[h]\big).$$

By [6, Theorem 3.3(b)], the restriction of α_E to $\bigoplus_{i=0}^{n} \mathrm{CH}_\bullet(X) \cdot h^i$ is an isomorphism.

It remains to show that, for every $a \in \mathrm{CH}_\bullet(X)$ and every $j \geq 0$, the map α_E annihilates $c_h(E) \cdot ah^j$. Since the restriction of α_E to $\bigoplus_{i=0}^n \mathrm{CH}_\bullet(X) \cdot h^i$ is an isomorphism, there are unique elements $b_0, \ldots, b_n \in \mathrm{CH}_\bullet(X)$ such that

$$\alpha_E\big(c_h(E) \cdot ah^j\big) = \sum_{i=0}^n c_1(\mathcal{O}(1))^i \cap \pi^* b_i.$$

We prove that $b_i = 0$ by descending induction on i. Assume that $b_{i'} = 0$ for all $i' > i$ (where we set $b_{i'} = 0$ if $i' > n$). By [6, Proposition 3.1(a)], we have

$$b_i = \pi_*\big(c_1(\mathcal{O}(1))^{n-i} \cap \alpha_E\big(c_h(E) \cdot ah^j\big)\big).$$

The right-hand side can be computed as

$$\begin{aligned}\pi_*\big(c_1(\mathcal{O}(1))^{n-i} \cap \alpha_E\big(c_h(E) \cdot ah^j\big)\big) &= \pi_*\big(\sum_{q=0}^{n+1} c_1(\mathcal{O}(1))^{n+(n+1+j-q-i)} \cap \pi^*(c_q(E) \cap a)\big) \\ &= \sum_{q=0}^{n+1} s_{n+1+j-q-i}(E) \cap (c_q(E) \cap a).\end{aligned}$$

Since $s(E)c(E) = 1$ and $(n+1+j-q-i) + q = n+1+j-i > 0$, the last expression is zero. This completes the proof.

2. See [20, Lemma 6.2]. □

5.2 Epsilon class and product formula

In this subsection, we introduce the notion of epsilon class, which is an analogue of the characteristic class introduced in [20, Section 6]. Many results concerning characteristic classes extend to epsilon classes with essentially the same proofs. In particular, following Beilinson's method in [20, Section 7], we describe the epsilon classes of the Radon transforms in Proposition 5.7. Using this result, we formulate and prove our product formula in Theorem 5.9.

Let X be a scheme of finite type over k. We say that X is *embeddable* if there exists a closed immersion $i\colon X \to M$ into a smooth k-scheme M.

Let X be an embeddable scheme of finite type over k, and let $i\colon X \to M$ be a closed immersion into a smooth k-scheme. Assume that M is purely of dimension n. By Lemma 5.1.1, we identify $\mathrm{CH}_\bullet(X) = \oplus_{i=0}^n \mathrm{CH}_i(X)$ with $\mathrm{CH}_n\left(\mathbb{P}\big((X \times_M T^*M) \oplus \mathbb{A}^1_X\big)\right)$ via the isomorphism

$$\text{(5.1)} \qquad \mathrm{CH}_\bullet(X) \xrightarrow{\cong} \mathrm{CH}_n\left(\mathbb{P}\big((X \times_M T^*M) \oplus \mathbb{A}^1_X\big)\right), \qquad (a_i)_i \mapsto \sum_i c_1(\mathcal{O}(1))^i \cap \pi^* a_i,$$

where $\pi\colon \mathbb{P}\big((X \times_M T^*M) \oplus \mathbb{A}^1_X\big) \to X$ denotes the projection. Tensoring with Θ_k, we obtain an induced isomorphism

$$\Theta_k \otimes \mathrm{CH}_\bullet(X) \xrightarrow{\cong} \Theta_k \otimes \mathrm{CH}_n\left(\mathbb{P}\big((X \times_M T^*M) \oplus \mathbb{A}^1_X\big)\right).$$

Let $\mathcal{F}$ be an object of $D^b_c(X, \overline{\mathbb{Z}_\ell})$, and write

$$\mathcal{E}(i_*\mathcal{F}) = \sum_a \xi_a \otimes [C_a],$$

where the C_a are the irreducible components of $SS(i_*\mathcal{F})$. Since $i_*\mathcal{F}$ is supported on X, each C_a is contained in $X \times_M T^*M$.

Definition 5.2. *Let the notation be as above. For $\mathcal{F} \in D^b_c(X, \overline{\mathbb{Z}_\ell})$, we define the* epsilon class $\varepsilon_X(\mathcal{F})$ *by*

$$\varepsilon_X(\mathcal{F}) := \sum_a \xi_a \otimes [\mathbb{P}(C_a \times_k \mathbb{A}^1_k)] \in \Theta_k \otimes \mathrm{CH}_n\left(\mathbb{P}\big((X \times_M T^*M) \oplus \mathbb{A}^1_X\big)\right) = \Theta_k \otimes \mathrm{CH}_\bullet(X).$$

First, we show that the above definition is independent of the choice of the auxiliary closed immersion.

Lemma 5.3. *The element $\varepsilon_X(\mathcal{F}) \in \Theta_k \otimes \mathrm{CH}_\bullet(X)$ constructed in Definition 5.2 is independent of the choice of the closed immersion $i\colon X \to M$.*

Proof. The proof is similar to that of [20, Lemma 6.6]. Let $j\colon X \to N$ be another closed immersion into a smooth k-scheme N purely of dimension m. We show that the epsilon classes $\varepsilon_X(\mathcal{F})$ constructed via i and j coincide.

By considering the product $M \times_k N$ and the projections, we may assume that there exists a smooth k-morphism $f\colon M \to N$ such that $f \circ i = j$. Let

$$df\colon X \times_N T^*N \hookrightarrow X \times_M T^*M$$

be the induced injection of vector bundles. By Lemma 5.1.2(a), the diagram

$$\begin{array}{ccc} \Theta_k \otimes \mathrm{CH}_\bullet(X) & \xrightarrow{\cong} & \Theta_k \otimes \mathrm{CH}_n\left(\mathbb{P}\big((X \times_M T^*M) \oplus \mathbb{A}^1_X\big)\right) \\ & {\searrow}^{\cong} & \big\downarrow \mathrm{id}_{\Theta_k} \otimes (df)^* \\ & & \Theta_k \otimes \mathrm{CH}_m\left(\mathbb{P}\big((X \times_N T^*N) \oplus \mathbb{A}^1_X\big)\right) \end{array}$$

is commutative. Therefore, it suffices to show that each irreducible component of $SS(i_*\mathcal{F})$ meets the image of df properly and that the pullback of $\mathcal{E}(i_*\mathcal{F})$ is equal to $\mathcal{E}(j_*\mathcal{F})$. Since the assertion is étale local on N, we may assume that there exists a section $s\colon N \to M$ of f such that $s \circ j = i$. The assertion then follows from the equalities

$$s_\circ SS(j_*\mathcal{F}) = SS(i_*\mathcal{F}) \qquad \text{and} \qquad s_!\mathcal{E}(j_*\mathcal{F}) = \mathcal{E}(i_*\mathcal{F}).$$

The second equality is proved in Lemma 4.15. □

Let $K(X, \overline{\mathbb{Z}_\ell})$ denote the Grothendieck group of the triangulated category $D^b_c(X, \overline{\mathbb{Z}_\ell})$. The epsilon class defines a group homomorphism

$$\varepsilon_X\colon K(X, \overline{\mathbb{Z}_\ell}) \to \Theta_k \otimes \mathrm{CH}_\bullet(X).$$

For $\mathcal{F} \in D^b_c(X, \overline{\mathbb{Z}_\ell})$, write

$$\varepsilon_X(\mathcal{F}) = \sum_{i \geq 0} \varepsilon_{X,i}(\mathcal{F}),$$

where $\varepsilon_{X,i}(\mathcal{F})$ denotes the component of $\varepsilon_X(\mathcal{F})$ in $\Theta_k \otimes \mathrm{CH}_i(X)$. The following lemma computes $\varepsilon_{X,i}(\mathcal{F})$ in the extremal cases.

Lemma 5.4 (cf. [20, Lemma 6.9]). *Let X be a smooth k-scheme purely of dimension n, and let $\mathcal{F} \in D_c^b(X, \overline{\mathbb{Z}_\ell})$.*

1. *Let $0_X \colon X = T_X^*X \hookrightarrow T^*X$ denote the zero section. Define*

$$(-, T_X^*X)_{T^*X} := \mathrm{id}_{\Theta_k} \otimes 0_X^* \colon \Theta_k \otimes Z_n(T^*X) \to \Theta_k \otimes \mathrm{CH}_0(X),$$

where $0_X^* \colon Z_n(T^*X) \to \mathrm{CH}_0(X)$ is the pullback homomorphism. Then

$$\varepsilon_{X,0}(\mathcal{F}) = (\mathcal{E}(\mathcal{F}), T_X^*X)_{T^*X}.$$

2. *Assume that X is connected. Let $U \subset X$ be a dense open subset on which $\mathcal{F}$ is smooth. Let $\mathrm{rk}^\circ(\mathcal{F})$ denote the rank of $\mathcal{F}|_U$.*

Let k' be the normalization of k in X. By Theorem 4.4, the composite

$$\pi_1^{ab}(U) \xrightarrow{\det(\mathcal{F}|_U)} \overline{\mathbb{Z}_\ell}^\times \to \overline{\mathbb{Z}_\ell}^\times/\mu$$

factors through $G_{k'}^{ab}$. We denote the induced homomorphism $G_{k'}^{ab} \to \overline{\mathbb{Z}_\ell}^\times/\mu$ by $\det^\circ(\mathcal{F})$. Then we have

$$\varepsilon_{X,n}(\mathcal{F}) = (\det{}^\circ(\mathcal{F}) \circ \mathrm{tr}_{k'/k})^{\frac{(-1)^n}{\deg(k'/k)}} \cdot \chi_{\mathrm{cyc}}^{\frac{(-1)^{n+1}n}{2}\mathrm{rk}^\circ(\mathcal{F})} \cdot [X]$$

in $\Theta_k \otimes \mathrm{CH}_n(X) = \Theta_k \cdot [X]$.

Proof. 1. Let $j \colon T^*X \to \mathbb{P}(T^*X \oplus \mathbb{A}_X^1)$ be the open immersion given by $v \mapsto (v, 1)$. Since $\mathcal{O}(1)$ is trivial on $j(T^*X)$, the projection $\mathrm{CH}_n(\mathbb{P}(T^*X \oplus \mathbb{A}_X^1)) \cong \mathrm{CH}_\bullet(X) \to \mathrm{CH}_0(X)$ is equal to the composite $0_X^* \circ j^*$. The assertion follows.

2. Let $\pi \colon \mathbb{P}(T^*X \oplus \mathbb{A}_X^1) \to X$ denote the projection. By [6, Proposition 3.1(a)], we have

$$\pi_*\big(c_1(\mathcal{O}(1))^i \cap \pi^*a\big) = \begin{cases} 0 & \text{if } i < n, \\ a & \text{if } i = n. \end{cases}$$

Hence the projection $\mathrm{CH}_n(\mathbb{P}(T^*X \oplus \mathbb{A}_X^1)) \cong \mathrm{CH}_\bullet(X) \to \mathrm{CH}_n(X)$ is equal to π_*.

Now fix a closed point $x \in U$ and consider the cartesian square

$$\begin{array}{ccc} \mathbb{P}_x^n & \xrightarrow{\pi} & x \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ \mathbb{P}(T^*X \oplus \mathbb{A}_X^1) & \xrightarrow{\pi} & X. \end{array}$$

The assertion then follows from $i^*\pi_* = \pi_* i^*$ together with Corollary 4.19. □

In the following lemma, we compute the epsilon class of the pullback under a properly transversal immersion.

Lemma 5.5. *Let X be a smooth scheme of finite type over k, and let $\mathcal{F} \in D_c^b(X, \overline{\mathbb{Z}_\ell})$. Let $i \colon W \to X$ be a properly $SS(\mathcal{F})$-transversal closed immersion of smooth k-schemes. Assume that X is purely of dimension n and that W is purely of dimension $m = n - c$.*

Then we have

$$\chi_{\mathrm{cyc}}^{-c/2} \otimes cc_W(i^*\mathcal{F}) + \varepsilon_W(i^*\mathcal{F}) = (-1)^c \cdot c(T^*_W X)^{-1} \cap i^*\varepsilon_X(\mathcal{F})$$

in $\Theta_k \otimes \mathrm{CH}_\bullet(W)$. *Here,* cc_W *on the left-hand side denotes the characteristic class defined in [20, Definition 6.7], and* $i^*\varepsilon_X(\mathcal{F})$ *on the right-hand side denotes the image of* $\varepsilon_X(\mathcal{F})$ *under the homomorphism*

$$\mathrm{id} \otimes i^* \colon \Theta_k \otimes \mathrm{CH}_\bullet(X) \to \Theta_k \otimes \mathrm{CH}_\bullet(W).$$

Proof. Set

$$E := (T^*X \times_X W) \oplus \mathbb{A}^1_W, \qquad F := T^*W \oplus \mathbb{A}^1_W, \qquad K := T^*_W X.$$

These fit into a short exact sequence of vector bundles on W

$$0 \to K \to E \xrightarrow{\theta} F \to 0.$$

Let $\pi' \colon \mathbb{P}(E)' \to \mathbb{P}(E)$ be the blow-up of $\mathbb{P}(E)$ along the closed subscheme $\mathbb{P}(K) \hookrightarrow \mathbb{P}(E)$. Then the canonical morphism

$$\mathbb{P}(E) \setminus \mathbb{P}(K) \to \mathbb{P}(F),$$

induced by θ, extends to a morphism $\theta' \colon \mathbb{P}(E)' \to \mathbb{P}(F)$. Moreover, the morphism θ' identifies $\mathbb{P}(E)'$ with the projective bundle associated with the vector bundle $\theta^{-1}(L)$ on $\mathbb{P}(F)$, where $L \subset F \times_W \mathbb{P}(F)$ denotes the universal line subbundle and $\theta \colon E \times_W \mathbb{P}(F) \to F \times_W \mathbb{P}(F)$ is the pullback of the surjection $E \to F$.

We have a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}_\bullet(X)[h] & \xrightarrow{i^* \otimes \mathrm{id}_{\mathbb{Z}[h]}} & \mathrm{CH}_\bullet(W)[h] \\ \downarrow{\scriptstyle \alpha_{T^*X \oplus \mathbb{A}^1_X}} & & \downarrow{\scriptstyle \alpha_E} \\ \mathrm{CH}_\bullet\big(\mathbb{P}(T^*X \oplus \mathbb{A}^1_X)\big) & \xrightarrow{i^*} & \mathrm{CH}_\bullet\big(\mathbb{P}(E)\big). \end{array} \tag{5.2}$$

Moreover, by the proof of [20, Lemma 6.4.1], the diagram

$$\begin{array}{ccc} \mathrm{CH}_\bullet(W)[h] & \xrightarrow{\alpha_E} & \mathrm{CH}_\bullet\big(\mathbb{P}(E)\big) \\ \uparrow{\scriptstyle c_h(K)\cdot} & & \downarrow{\scriptstyle \theta'_*\pi'^*} \\ \mathrm{CH}_\bullet(W)[h] & \xrightarrow{\alpha_F} & \mathrm{CH}_\bullet\big(\mathbb{P}(F)\big) \end{array} \tag{5.3}$$

is commutative.

For an element $\beta = \sum_{j \geq 0} \beta_j \in \Theta_k \otimes \mathrm{CH}_\bullet(X) = \bigoplus_{j \geq 0} \Theta_k \otimes \mathrm{CH}_j(X)$, define

$$\Phi(\beta) := \sum_{j \geq 0} \beta_j \cdot h^j \in \big(\Theta_k \otimes \mathrm{CH}_\bullet(X)\big)[h].$$

We use the same notation for elements of $\Theta_k \otimes \mathrm{CH}_\bullet(W)$.

By the commutativity of the diagrams (5.2) and (5.3), we obtain

$$\begin{aligned}\alpha_F\left(\Phi\big(c(T^*_W X)^{-1}\cap i^*\varepsilon_X(\mathcal{F})\big)\right) &= \theta'_*\pi'^*\alpha_E\left(\Phi\big(i^*\varepsilon_X(\mathcal{F})\big)\cdot h^c\right)\\ &= \theta'_*\pi'^*\alpha_E\left((i^*\otimes \mathrm{id}_{\mathbb{Z}[h]})\circ\Phi\big(\varepsilon_X(\mathcal{F})\big)\right)\\ &= \theta'_*\pi'^* i^*\alpha_{T^*X\oplus\mathbb{A}^1_X}\left(\Phi\big(\varepsilon_X(\mathcal{F})\big)\right),\end{aligned}$$

where the second equality follows from the identity $(i^*\otimes \mathrm{id}_{\mathbb{Z}[h]})\circ\Phi(\beta)=\Phi(i^*\beta)\cdot h^c$.

Write $\mathcal{E}(\mathcal{F})=\sum_a \xi_a\otimes[C_a]$. Then, by the definition of the epsilon class,

$$\alpha_{T^*X\oplus\mathbb{A}^1_X}\left(\Phi\big(\varepsilon_X(\mathcal{F})\big)\right)=\sum_a \xi_a\otimes[\mathbb{P}(C_a\times_k\mathbb{A}^1_k)].$$

Since $i\colon W\to X$ is properly $SS(\mathcal{F})$-transversal, for each a, the pullback

$$i^*\mathbb{P}(C_a\times_k\mathbb{A}^1_k)=\mathbb{P}(C_a\times_k\mathbb{A}^1_k)\times_X W$$

has dimension m. Moreover, this pullback is disjoint from $\mathbb{P}(K)$. The assertion therefore follows from Theorem 4.26 and [20, Proposition 7.8]. $\square$

For the theory of the universal family of pencils, we follow the notation introduced in Subsection 2.3. Let $\mathbb{P}=\mathbb{P}^n$ be the projective n-space, and let $\mathbb{P}^\vee$ be its dual projective space. We identify $\mathbb{P}(T^*\mathbb{P})=Q=\mathbb{P}(T^*\mathbb{P}^\vee)$, and let $\boldsymbol{p}\colon Q\to\mathbb{P}$ and $\boldsymbol{p}^\vee\colon Q\to\mathbb{P}^\vee$ denote the projections. The Radon transforms

$$R=R\boldsymbol{p}^\vee_!\boldsymbol{p}^*[n-1]\qquad\text{and}\qquad R^\vee=R\boldsymbol{p}_!\boldsymbol{p}^{\vee *}[n-1](n-1)$$

induce group homomorphisms

$$R\colon K(\mathbb{P},\overline{\mathbb{Z}_\ell})\to K(\mathbb{P}^\vee,\overline{\mathbb{Z}_\ell}),\quad R^\vee\colon K(\mathbb{P}^\vee,\overline{\mathbb{Z}_\ell})\to K(\mathbb{P},\overline{\mathbb{Z}_\ell}). \tag{5.4}$$

Set $\tilde{\Theta}_k:=\mathrm{Hom}_{\mathrm{conti}}(G_k^{ab},\overline{\mathbb{Z}_\ell}^\times)$. For an algebraic variety X over k and $\mathcal{F}\in D^b_c(X,\overline{\mathbb{Z}_\ell})$, define

$$\chi(\mathcal{F}):=\chi(X_{\bar{k}},\mathcal{F}\otimes_{\overline{\mathbb{Z}_\ell}}\overline{\mathbb{Q}_\ell})\qquad\text{and}\qquad \varepsilon^{-1}(\mathcal{F}):=\det\left(R\Gamma(X_{\bar{k}},\mathcal{F}\otimes_{\overline{\mathbb{Z}_\ell}}\overline{\mathbb{Q}_\ell})\right).$$

These invariant induce a group homomorphism $(\chi,\varepsilon^{-1})\colon K(X,\overline{\mathbb{Z}_\ell})\to\mathbb{Z}\times\tilde{\Theta}_k$.

Lemma 5.6. *Let $n\geq 1$ be an integer, and let $\mathbb{P}=\mathbb{P}^n$.*

1. *The diagram*

$$\begin{array}{ccc} K(\mathbb{P},\overline{\mathbb{Z}_\ell}) & \xrightarrow{(\chi,\varepsilon^{-1})} & \mathbb{Z}\times\tilde{\Theta}_k\\ {\scriptstyle R}\downarrow & & \downarrow\\ K(\mathbb{P}^\vee,\overline{\mathbb{Z}_\ell}) & \xrightarrow[(\chi,\varepsilon^{-1})]{} & \mathbb{Z}\times\tilde{\Theta}_k \end{array} \tag{5.5}$$

commutes, where the right vertical arrow is given by

$$(a,b)\mapsto\big((-1)^{n-1}na,\ \chi_{\mathrm{cyc}}^{(-1)^n\cdot\frac{n(n-1)}{2}a}b^{(-1)^{n-1}n}\big).$$

2. *The composite*

$$K(\mathbb{P}, \overline{\mathbb{Z}_\ell}) \xrightarrow{R^\vee R} K(\mathbb{P}, \overline{\mathbb{Z}_\ell}) \xrightarrow{(\chi, \varepsilon^{-1})} \mathbb{Z} \times \tilde{\Theta}_k$$

sends $\mathcal{F}$ *to*

$$(n^2\chi(\mathcal{F}), \varepsilon^{-1}(\mathcal{F})^{n^2}).$$

Proof. 1. For $\mathcal{F} \in D_c^b(\mathbb{P}, \overline{\mathbb{Z}_\ell})$, we have

$$R\Gamma(\mathbb{P}^\vee_{\bar{k}}, R\mathcal{F}) = R\Gamma(Q_{\bar{k}}, \boldsymbol{p}^*\mathcal{F})[n-1] = R\Gamma(\mathbb{P}_{\bar{k}}, \mathcal{F} \otimes^L R\boldsymbol{p}_*\overline{\mathbb{Z}_\ell})[n-1],$$

where the second equality follows from the projection formula. Hence the assertion follows from

$$R^q\boldsymbol{p}_*\overline{\mathbb{Z}_\ell} = \begin{cases} \overline{\mathbb{Z}_\ell}(-q/2) & \text{if } q \text{ is even and } 0 \leq q \leq 2(n-1), \\ 0 & \text{otherwise.} \end{cases}$$

2. By assertion 1, for $\mathcal{G} \in D_c^b(\mathbb{P}^\vee, \overline{\mathbb{Z}_\ell})$, we have

$$(\chi, \varepsilon^{-1})R^\vee\mathcal{G} = \left((-1)^{n-1}n\chi(\mathcal{G}),\, \chi_{\mathrm{cyc}}^{(-1)^n\frac{n(n-1)}{2}\chi(\mathcal{G})} \cdot \varepsilon^{-1}(\mathcal{G}(n-1))^{(-1)^{n-1}n}\right).$$

Applying this to $\mathcal{G} = R\mathcal{F}$ and using assertion 1 once more, we obtain the assertion. □

Define the Legendre transforms

$$L\colon \mathrm{CH}_\bullet(\mathbb{P}) \to \mathrm{CH}_\bullet(\mathbb{P}^\vee), \qquad L^\vee\colon \mathrm{CH}_\bullet(\mathbb{P}^\vee) \to \mathrm{CH}_\bullet(\mathbb{P})$$

by

$$L(a) := \boldsymbol{p}^\vee_*\left(c\left(T^*(Q/\mathbb{P})\right) \cap \boldsymbol{p}^*a\right), \qquad L^\vee(a) := \boldsymbol{p}_*\left(c\left(T^*(Q/\mathbb{P}^\vee)\right) \cap \boldsymbol{p}^{\vee *}a\right).$$

Here, $c(T^*(Q/\mathbb{P}))$ and $c(T^*(Q/\mathbb{P}^\vee))$ denote the total Chern classes of the relative cotangent bundles of Q over $\mathbb{P}$ and $\mathbb{P}^\vee$, respectively.

We also use the same notation for the induced homomorphisms

$$\Theta_k \otimes \mathrm{CH}_\bullet(\mathbb{P}) \to \Theta_k \otimes \mathrm{CH}_\bullet(\mathbb{P}^\vee) \qquad \text{and} \qquad \Theta_k \otimes \mathrm{CH}_\bullet(\mathbb{P}^\vee) \to \Theta_k \otimes \mathrm{CH}_\bullet(\mathbb{P}).$$

The following proposition refines [20, Proposition 7.11].

Proposition 5.7. *Let* $n \geq 1$ *be an integer and let* $\mathbb{P} = \mathbb{P}^n$. *Let* cc_X *denote the characteristic class defined in [20, Definition 6.7].*

1. *The diagram*

(5.6)
$$\begin{array}{ccc} K(\mathbb{P}, \overline{\mathbb{Z}_\ell}) & \xrightarrow{(cc_{\mathbb{P}}, \varepsilon_{\mathbb{P}})} & (\mathbb{Z} \oplus \Theta_k) \otimes \mathrm{CH}_\bullet(\mathbb{P}) \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle \tilde{L}} \\ K(\mathbb{P}^\vee, \overline{\mathbb{Z}_\ell}) & \xrightarrow{(cc_{\mathbb{P}^\vee}, \varepsilon_{\mathbb{P}^\vee})} & (\mathbb{Z} \oplus \Theta_k) \otimes \mathrm{CH}_\bullet(\mathbb{P}^\vee) \end{array}$$

is commutative, where [2]

$$\tilde{L}(a, b) := (L(a), L(\chi_{\mathrm{cyc}}^{\frac{1-n}{2}a} + b)).$$

The analogous diagram with R *replaced by* $R^\vee$ *and* $\tilde{L}$ *by*

$$\tilde{L}^\vee(a, b) := (L^\vee(a), L^\vee(\chi_{\mathrm{cyc}}^{\frac{n-1}{2}a} + b))$$

is also commutative.

[2] Here, $\chi_{\mathrm{cyc}}^{\frac{1-n}{2}a}$ denotes $\chi_{\mathrm{cyc}}^{\frac{1-n}{2}} \otimes a \in \Theta_k \otimes \mathrm{CH}_\bullet(\mathbb{P})$. This notation is consistent with Remark 4.12.

2. The diagram

$$
(5.7)\qquad
\begin{array}{ccc}
K(\mathbb{P}, \overline{\mathbb{Z}_\ell}) & \xrightarrow{(cc_{\mathbb{P}}, \varepsilon_{\mathbb{P}})} & (\mathbb{Z} \oplus \Theta_k) \otimes \mathrm{CH}_\bullet(\mathbb{P}) \\
 & \searrow^{(\chi, \varepsilon^{-1})} & \downarrow \deg \\
 & & \mathbb{Z} \oplus \Theta_k
\end{array}
$$

is commutative.

Proof. We prove the assertions by induction on n. When $n = 1$, the projections $\boldsymbol{p}\colon Q \to \mathbb{P}$ and $\boldsymbol{p}^\vee\colon Q \to \mathbb{P}^\vee$ are isomorphisms, so assertion 1 is immediate. Since

$$\deg cc_{\mathbb{P}}\mathcal{F} = (CC\mathcal{F}, T^*_{\mathbb{P}}\mathbb{P})_{T^*\mathbb{P}}$$

by [20, Lemma 6.9.1] and

$$\deg \varepsilon_{\mathbb{P}}\mathcal{F} = (\mathcal{E}(\mathcal{F}), T^*_{\mathbb{P}}\mathbb{P})_{T^*\mathbb{P}}$$

by Lemma 5.4.1, assertion 2 for $n = 1$ is precisely the combination of the Grothendieck–Ogg–Shafarevich formula and the product formula (Lemma 4.21).

From now on, assume that $n \geq 2$. First, we show that assertion 2 for $n-1$ implies assertion 1 for n. Since the second part of assertion 1 follows from the first, it suffices to show that the diagram (5.6) commutes. Set

$$\theta := \tilde{L} \circ (cc_{\mathbb{P}}, \varepsilon_{\mathbb{P}}) - (cc_{\mathbb{P}^\vee}, \varepsilon_{\mathbb{P}^\vee}) \circ R.$$

We prove that $\theta = 0$ using the direct sum decomposition

$$(5.8)\qquad \mathrm{CH}_\bullet(\mathbb{P}^\vee) \cong \mathrm{CH}_n(\mathbb{P}(T^*\mathbb{P}^\vee \oplus \mathbb{A}^1_{\mathbb{P}^\vee})) \xrightarrow{\cong} \mathrm{CH}_{n-1}(\mathbb{P}(T^*\mathbb{P}^\vee)) \oplus \mathrm{CH}_n(\mathbb{P}^\vee),$$

where the projection to $\mathrm{CH}_n(\mathbb{P}^\vee)$ is the projection to the degree-n part, and the projection to $\mathrm{CH}_{n-1}(\mathbb{P}(T^*\mathbb{P}^\vee))$ is induced by the pullback along $\mathbb{P}(T^*\mathbb{P}^\vee) \hookrightarrow \mathbb{P}(T^*\mathbb{P}^\vee \oplus \mathbb{A}^1_{\mathbb{P}^\vee})$.

We first show that the composition of θ with the projection $\mathrm{CH}_\bullet(\mathbb{P}^\vee) \to \mathrm{CH}_{n-1}(\mathbb{P}(T^*\mathbb{P}^\vee))$ vanishes. Consider the following diagram:

$$
\begin{array}{ccccc}
\mathrm{CH}_\bullet(\mathbb{P}) & \longrightarrow & \mathrm{CH}_{n-1}(\mathbb{P}(T^*\mathbb{P})) & & \\
\downarrow{\scriptstyle \boldsymbol{p}^*} & & \downarrow{\scriptstyle \boldsymbol{p}^*} & & \\
\mathrm{CH}_\bullet(Q) & \longrightarrow & \mathrm{CH}_{2n-2}(\mathbb{P}(Q \times_{\mathbb{P}} T^*\mathbb{P})) & & \\
\downarrow{\scriptstyle c(T^*(Q/\mathbb{P}))\cap} & & \downarrow{\scriptstyle d\boldsymbol{p}_*} & & \\
\mathrm{CH}_\bullet(Q) & \longrightarrow & \mathrm{CH}_{2n-2}(\mathbb{P}(T^*Q)) & \xrightarrow{d\boldsymbol{p}^{\vee *}} & \mathrm{CH}_{n-1}(\mathbb{P}(Q \times_{\mathbb{P}^\vee} T^*\mathbb{P}^\vee)) \\
\downarrow{\scriptstyle \boldsymbol{p}^\vee_*} & & & & \downarrow{\scriptstyle \boldsymbol{p}^\vee_*} \\
\mathrm{CH}_\bullet(\mathbb{P}^\vee) & & \longrightarrow & & \mathrm{CH}_{n-1}(\mathbb{P}(T^*\mathbb{P}^\vee)).
\end{array}
$$

Here, each unlabeled horizontal arrow sends $a \in \mathrm{CH}_i$ to $c_1(\mathcal{O}(1))^j \cap \pi^* a$, where π denotes the structure morphism of the corresponding projective bundle, and j is the unique integer such that the resulting cycle class has the displayed degree.

We claim that this diagram is commutative. The commutativity of the upper square is immediate. The commutativity of the middle square follows from Lemma 5.1.2(b). The commutativity of the lower square follows from the projection formula.

The commutativity of the above diagram, together with Corollary 4.25 and [20, Corollary 7.5], shows that the composition

$$K(\mathbb{P},\overline{\mathbb{Z}_\ell}) \xrightarrow{\theta} (\mathbb{Z}\oplus\Theta_k)\otimes \mathrm{CH}_\bullet(\mathbb{P}^\vee) \to (\mathbb{Z}\oplus\Theta_k)\otimes \mathrm{CH}_{n-1}(\mathbb{P}(T^*\mathbb{P}^\vee))$$

vanishes.

Next, we show that the composition of θ with the projection

$$(\mathbb{Z}\oplus\Theta_k)\otimes \mathrm{CH}_\bullet(\mathbb{P}^\vee) \to \mathbb{Z}\oplus\Theta_k$$

to the degree-n part vanishes. Note that, for every k-rational point $i\colon \operatorname{Spec}(k)\to\mathbb{P}^\vee$, the projection $\mathrm{CH}_\bullet(\mathbb{P}^\vee)\to\mathrm{CH}_n(\mathbb{P}^\vee)=\mathbb{Z}$ is given by the pullback

$$i^*\colon \mathrm{CH}_\bullet(\mathbb{P}^\vee)\to\mathrm{CH}_\bullet(\operatorname{Spec}(k))\cong\mathbb{Z}.$$

Let $\mathcal{F}\in D^b_c(\mathbb{P},\overline{\mathbb{Z}_\ell})$ and set $C=SS(\mathcal{F})$. After replacing k by a finite extension, we may choose a k-rational point $i\colon \operatorname{Spec}(k)\to\mathbb{P}^\vee$ outside the image of $\mathbb{P}(C)\to\mathbb{P}^\vee$. Moreover, we may assume that the corresponding hyperplane H is not contained in the image of $\mathbb{P}(C)\to\mathbb{P}$. Then $h\colon H\hookrightarrow\mathbb{P}$ is properly C-transversal.

We have

$$i^*\tilde{L}(cc_{\mathbb{P}},\varepsilon_{\mathbb{P}})\mathcal{F} = i^*\Big(L(cc_{\mathbb{P}}(\mathcal{F})), L\big(\chi_{\mathrm{cyc}}^{\frac{1-n}{2}}\otimes cc_{\mathbb{P}}(\mathcal{F})+\varepsilon_{\mathbb{P}}(\mathcal{F})\big)\Big). \tag{5.9}$$

Pulling back the short exact sequence

$$0\to T^*_Q(\mathbb{P}\times\mathbb{P}^\vee)\to T^*\mathbb{P}^\vee\times_{\mathbb{P}^\vee}Q\to T^*(Q/\mathbb{P})\to 0$$

along $H\hookrightarrow Q$, we obtain

$$c\big(T^*(Q/\mathbb{P})\times_Q H\big)=c\big(T^*_Q(\mathbb{P}\times\mathbb{P}^\vee)\times_Q H\big)^{-1}=c\big(\mathcal{O}_H(-1)\big)^{-1}.$$

Hence, the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{CH}_\bullet(\mathbb{P}) & \xrightarrow{h^*} & \mathrm{CH}_\bullet(H)\\
\big\downarrow{\scriptstyle L} & & \big\downarrow{\scriptstyle \deg(c(\mathcal{O}_H(-1))^{-1}\cap -)}\\
\mathrm{CH}_\bullet(\mathbb{P}^\vee) & \xrightarrow{i^*} & \mathbb{Z}.
\end{array}$$

Thus, by (5.9),

(5.10)

$$i^*\tilde{L}(cc_{\mathbb{P}},\varepsilon_{\mathbb{P}})\mathcal{F} = \Big(\deg\big(c(\mathcal{O}_H(-1))^{-1}\cap h^*cc_{\mathbb{P}}(\mathcal{F})\big), \deg\big(c(\mathcal{O}_H(-1))^{-1}\cap h^*(\chi_{\mathrm{cyc}}^{\frac{1-n}{2}}\otimes cc_{\mathbb{P}}(\mathcal{F})+\varepsilon_{\mathbb{P}}(\mathcal{F}))\big)\Big).$$

By Lemma 5.5 and [20, Proposition 7.8], we have

$$c(\mathcal{O}_H(-1))^{-1} \cap h^* cc_{\mathbb{P}}\mathcal{F} = -cc_H(h^*\mathcal{F}),$$
$$c(\mathcal{O}_H(-1))^{-1} \cap h^* \varepsilon_{\mathbb{P}}(\mathcal{F}) = \chi_{\mathrm{cyc}}^{\frac{1}{2}} \otimes cc_H(h^*\mathcal{F}) - \varepsilon_H(h^*\mathcal{F}).$$

By the induction hypothesis, we have

$$\deg cc_H(h^*\mathcal{F}) = \chi(h^*\mathcal{F}), \qquad \deg \varepsilon_H(h^*\mathcal{F}) = \varepsilon^{-1}(h^*\mathcal{F}).$$

Therefore, by (5.10), we obtain

$$\begin{aligned} i^*\tilde{L}(cc_{\mathbb{P}}, \varepsilon_{\mathbb{P}})\mathcal{F} &= \big(-\deg cc_H(h^*\mathcal{F}), \deg(\chi_{\mathrm{cyc}}^{\frac{n-1}{2}} \otimes cc_H(h^*\mathcal{F}) + \chi_{\mathrm{cyc}}^{\frac{1}{2}} \otimes cc_H(h^*\mathcal{F}) - \varepsilon_H(h^*\mathcal{F}))\big) \\ &= (-\chi(h^*\mathcal{F}), \chi_{\mathrm{cyc}}^{\frac{n}{2}\chi(h^*\mathcal{F})} \cdot \det R\Gamma(H_{\bar{k}}, h^*\mathcal{F})^{-1}). \end{aligned}$$

On the other hand, by Lemma 5.4.2 and [20, Lemma 6.9.2], we have

$$i^*(cc_{\mathbb{P}^\vee}, \varepsilon_{\mathbb{P}^\vee})R\mathcal{F} = ((-1)^n \operatorname{rk}^\circ R\mathcal{F}, \det(R\mathcal{F})^{\circ(-1)^n} \cdot \chi_{\mathrm{cyc}}^{(-1)^{n+1}\frac{n}{2}\cdot \operatorname{rk}^\circ R\mathcal{F}}).$$

This completes the proof of assertion 1 for n.

We show that assertion 1 for $n \geq 2$ implies assertion 2 for n. By the commutativity of diagram (5.6) and its analogue for $(R^\vee, \tilde{L}^\vee)$, the endomorphism $R^\vee R$ of $K(\mathbb{P}, \overline{\mathbb{Z}_\ell})$ preserves the kernel

$$K(\mathbb{P}, \overline{\mathbb{Z}_\ell})^\circ := \ker\Big((cc_{\mathbb{P}}, \varepsilon_{\mathbb{P}})\colon K(\mathbb{P}, \overline{\mathbb{Z}_\ell}) \to (\mathbb{Z} \oplus \Theta_k) \otimes \mathrm{CH}_\bullet(\mathbb{P})\Big).$$

Let $\mathcal{F} \in K(\mathbb{P}, \overline{\mathbb{Z}_\ell})^\circ$. Then there exists an element $\mathcal{G} \in K(\operatorname{Spec}(k), \overline{\mathbb{Z}_\ell})$ such that $a^*\mathcal{G} = R^\vee R\mathcal{F} - \mathcal{F}$, where $a\colon \mathbb{P} \to \operatorname{Spec}(k)$ is the structure morphism.

Since $cc_{\mathbb{P}}(a^*\mathcal{G}) = 0$ and $\varepsilon_{\mathbb{P}}(a^*\mathcal{G}) = 1$, it follows that $\operatorname{rk}(\mathcal{G}) = 0$ and $\det(\mathcal{G}) = 1$. Hence

$$(\chi, \varepsilon^{-1})R^\vee R\mathcal{F} = (\chi, \varepsilon^{-1})\mathcal{F}.$$

By Lemma 5.6.2, this is equivalent to $(n^2\chi(\mathcal{F}), \varepsilon^{-1}(\mathcal{F})^{n^2}) = (\chi(\mathcal{F}), \varepsilon^{-1}(\mathcal{F}))$. This implies that $\mathcal{F}$ lies in the kernel of (χ, ε^{-1}).

We next show that the cokernel of $(cc_{\mathbb{P}}, \varepsilon_{\mathbb{P}})$ is torsion. Indeed, let $\mathbb{P}^a \subset \mathbb{P}$ be a linear subspace, and let $\xi\colon G_k \to \overline{\mathbb{Z}_\ell}^\times$ be a continuous character. Denote by $\xi_{\mathbb{P}^a}$ the smooth rank-one $\overline{\mathbb{Z}_\ell}$-sheaf on $\mathbb{P}^a$ corresponding to the composite homomorphism

$$\pi_1(\mathbb{P}^a) \to G_k \to \overline{\mathbb{Z}_\ell}^\times,$$

and regard it as an object of $D_c^b(\mathbb{P}, \overline{\mathbb{Z}_\ell})$ by extension by zero. Then

$$\mathcal{E}(\xi_{\mathbb{P}^a}[a]) = \xi \cdot \chi_{\mathrm{cyc}}^{-\frac{a}{2}} \otimes [T^*_{\mathbb{P}^a}\mathbb{P}].$$

Since the classes of $\mathbb{P}(T^*_{\mathbb{P}^a}\mathbb{P} \times_k \mathbb{A}^1_k)$ span $\mathrm{CH}_n(\mathbb{P}(T^*\mathbb{P} \otimes \mathbb{A}^1_{\mathbb{P}})) \cong \mathrm{CH}_\bullet(\mathbb{P})$, the assertion follows from Lemma 4.3.2.

Since $\mathbb{Q} \oplus \Theta_k$ is uniquely divisible, there exists a unique group homomorphism

$$\deg'\colon (\mathbb{Z} \oplus \Theta_k) \otimes \mathrm{CH}_\bullet(\mathbb{P}) \to \mathbb{Q} \oplus \Theta_k$$

making diagram (5.7) commutative after replacing deg by $\deg'$.

We need to show that $\deg = \deg'$. Since $(\mathbb{Q} \oplus \Theta_k) \otimes \mathrm{CH}_\bullet(\mathbb{P})$ is spanned, as a $\mathbb{Q}$-vector space, by the images of $\xi_{\mathbb{P}^a}[a]$, where $\xi\colon G_k \to \overline{\mathbb{Z}_\ell}^\times$ ranges over continuous characters and $\mathbb{P}^a \subset \mathbb{P}$ ranges over linear subspaces, it suffices to verify the equality for $\xi_{\mathbb{P}^a}[a]$. This follows from Lemma 5.4 and [20, Lemma 6.9.1]. □

Corollary 5.8. *Let $n \geq 1$ be an integer, and let $\mathbb{P} = \mathbb{P}^n_k$. Let $\mathcal{F} \in D^b_c(\mathbb{P}, \overline{\mathbb{Z}_\ell})$. Then*

$$\mathcal{E}(R\mathcal{F}) = L(\mathcal{E}(\mathcal{F})(\frac{1-n}{2})).$$

Proof. By Corollary 4.25, it remains to verify the equality of the coefficient of the zero section. By Proposition 5.7.1, we have

$$cc_{\mathbb{P}^\vee}(R\mathcal{F}) = L(cc_{\mathbb{P}}(\mathcal{F})), \qquad \varepsilon_{\mathbb{P}^\vee}(R\mathcal{F}) = \chi_{\mathrm{cyc}}^{\frac{1-n}{2}} \otimes L(cc_{\mathbb{P}}(\mathcal{F})) + L(\varepsilon_{\mathbb{P}}(\mathcal{F})).$$

Comparing the degree-n parts and applying Lemma 5.4.2, we obtain the desired equality. □

Finally, we prove our product formula.

Theorem 5.9. *Let k be the perfection of a finitely generated field, and let X be a smooth projective variety over k. Then, for every $\mathcal{F} \in D^b_c(X, \overline{\mathbb{Z}_\ell})$, we have*

$$\det\big(R\Gamma(X_{\bar{k}}, \mathcal{F} \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell})\big) = (\mathcal{E}(\mathcal{F}), T^*_X X)_{T^*X} \tag{5.11}$$

as elements of Θ_k.

Proof. By Lemma 4.15, we may assume that X is a projective space. Then the assertion follows from Lemma 5.4.1 and Proposition 5.7.2. □

As a consequence of the above theorem, we obtain a product formula for the absolute values of global epsilon factors.

Let K be a field endowed with an absolute value

$$|\cdot|\colon K \to \mathbb{R}_{\geq 0}.$$

Fix a field embedding $\iota\colon \overline{\mathbb{Q}_\ell} \hookrightarrow K$. The composition

$$\overline{\mathbb{Q}_\ell}^\times \xrightarrow{\iota} K^\times \xrightarrow{|\cdot|} \mathbb{R}_{>0}$$

induces a group homomorphism $\overline{\mathbb{Q}_\ell}^\times/\mu \to \mathbb{R}_{>0}$. By abuse of notation, we denote this homomorphism by $x \mapsto |\iota(x)|$.

Corollary 5.10. *Let X be a smooth projective variety over a finite field $\mathbb{F}_q$. Let $\mathcal{F} \in D^b_c(X, \overline{\mathbb{Z}_\ell})$ and write*

$$\mathcal{E}(\mathcal{F}) = \sum_a \beta_a \otimes [C_a] \in \overline{\mathbb{Z}_\ell}^\times/\mu \otimes Z_n(T^*X).$$

Here, we identify $\Theta_{\mathbb{F}_q}$ with $\overline{\mathbb{Z}_\ell}^\times/\mu$ via $\xi \mapsto \xi(\mathrm{Frob}_q)$. Then

$$\Big|\iota\big(\det\big(\mathrm{Frob}_q, R\Gamma(X_{\overline{\mathbb{F}_q}}, \mathcal{F} \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell})\big)\big)\Big| = \prod_a |\iota(\beta_a)|^{\deg(C_a, T^*_X X)_{T^*X}}. \tag{5.12}$$

Example 5.11. *Let $\mathbb{F}_q$ be a finite field with q elements. Let X be a smooth projective variety over $\mathbb{F}_q$, and let $\mathcal{F} \in D^b_c(X, \overline{\mathbb{Z}_\ell})$.*

1. *Fix a field isomorphism* $\iota\colon \overline{\mathbb{Q}_\ell} \to \mathbb{C}$. *Assume that* $\mathcal{F}_{\overline{\mathbb{Q}_\ell}} := \mathcal{F} \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell}$ *is* ι*-pure of weight* 0 *in the sense of [14, II.12.7]. Then, for every* $i \in \mathbb{Z}$ *and every Frobenius eigenvalue* α *on* $\mathrm{H}^i(X_{\overline{\mathbb{F}_q}}, \mathcal{F}_{\overline{\mathbb{Q}_\ell}})$, *we have*
$$|\iota(\alpha)| = q^{\frac{i}{2}}.$$
Hence, by (5.12), we obtain
$$\frac{1}{2}\sum_i (-1)^i i \cdot \dim \mathrm{H}^i(X_{\overline{\mathbb{F}_q}}, \mathcal{F}_{\overline{\mathbb{Q}_\ell}}) = \sum_a \deg(C_a, T^*_X X)_{T^*X} \cdot \log_q\left(|\iota(\beta_a)|\right).$$

2. *Fix a field isomorphism* $\overline{\mathbb{Q}_\ell} \to \overline{\mathbb{Q}_p}$ *(where* p *is the characteristic of* $\mathbb{F}_q$*). Then the product formula (5.12) gives an expression of the* p*-adic absolute value of the global epsilon factor* $\varepsilon(X, \mathcal{F}) = \det(-\mathrm{Frob}_q, R\Gamma(X_{\overline{\mathbb{F}_q}}, \mathcal{F}_{\overline{\mathbb{Q}_\ell}}))^{-1}$ *in terms of the* p*-adic absolute values of the local epsilon factors. Moreover, the* p*-adic valuation of the local epsilon factors of tamely ramified representations can be computed by Stickelberger's theorem ([24, Proposition 6.13]), as pointed out to the author by N. Katz.*

5.3 An axiomatic description of epsilon cycles

We give an axiomatic description of epsilon cycles. A similar axiomatic description of characteristic cycles was given in [19, Proposition 8].

Theorem 5.12. *Let* k *be the perfection of a finitely generated field of characteristic* $p \neq \ell$. *There exists a unique assignment*
$$(X, \mathcal{F}) \longmapsto \mathcal{E}(\mathcal{F})$$
satisfying the following axioms, where X *is a smooth scheme of finite type over* k, $\mathcal{F} \in D^b_c(X, \overline{\mathbb{Z}_\ell})$, *and*
$$\mathcal{E}(\mathcal{F}) = \sum_a \xi_a \otimes [C_a]$$
is a cycle with coefficients in Θ_k, *supported on the singular support* $SS(\mathcal{F})$.

1. *(Normalization) Let* $X = \mathrm{Spec}(k')$, *where* k' *is a finite extension of* k. *Then*
$$\mathcal{E}(\mathcal{F}) = (\det(\mathcal{F}) \circ \mathrm{tr}_{k'/k})^{\frac{1}{\deg(k'/k)}} \otimes [T^*_X X].$$

2. *(Tate twist) We have*
$$\mathcal{E}(\mathcal{F}(1)) = \chi_{\mathrm{cyc}} \otimes CC(\mathcal{F}) + \mathcal{E}(\mathcal{F}).$$

3. *(Additivity) For every distinguished triangle*
$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to,$$
we have $\mathcal{E}(\mathcal{F}) = \mathcal{E}(\mathcal{F}') + \mathcal{E}(\mathcal{F}'')$.

4. *(Closed immersion) For a closed immersion* $i\colon X \to P$ *of smooth* k*-schemes of finite type and* $\mathcal{F} \in D^b_c(X, \overline{\mathbb{Z}_\ell})$, *we have* $\mathcal{E}(i_*\mathcal{F}) = i_!\mathcal{E}(\mathcal{F})$.

5. *(Pullback) For a half-integer $r \in \frac{1}{2}\mathbb{Z}$, define $\mathcal{E}(\mathcal{F})(r) := \chi_{\mathrm{cyc}}^r \otimes CC(\mathcal{F}) + \mathcal{E}(\mathcal{F})$. Then, for a properly $SS(\mathcal{F})$-transversal morphism $h\colon W \to X$ from a smooth k-scheme W of finite type, we have*

$$\mathcal{E}(h^*\mathcal{F}) = h^!\big(\mathcal{E}(\mathcal{F})(\frac{\dim X - \dim W}{2})\big).$$

Here $\dim X$ and $\dim W$ are regarded as locally constant functions on X and W, respectively.

6. *(Radon transform) For a constructible complex $\mathcal{F} \in D_c^b(\mathbb{P}, \overline{\mathbb{Z}_\ell})$ on a projective space $\mathbb{P} = \mathbb{P}^n$, we have*

$$\mathcal{E}(R\mathcal{F}) = L\big(\mathcal{E}(\mathcal{F})(\frac{1-n}{2})\big).$$

7. *(Same monodromy) Let X (resp. X') be a smooth curve over k, and let x (resp. x') be a closed point of X (resp. X'). Let $\mathcal{F}$ (resp. $\mathcal{F}'$) be an object of $D_c^b(X, \overline{\mathbb{Z}_\ell})$ (resp. $D_c^b(X', \overline{\mathbb{Z}_\ell})$). Assume that there exists an isomorphism*

$$f\colon X_{(x)} \xrightarrow{\cong} X'_{(x')}$$

of k-schemes such that $\mathcal{F}|_{X_{(x)}} \cong f^*\mathcal{F}'|_{X'_{(x')}}$. Then the coefficient of $[T_x^*X]$ in $\mathcal{E}(\mathcal{F})$ coincides with the coefficient of $[T_{x'}^*X']$ in $\mathcal{E}(\mathcal{F}')$.

Moreover, the epsilon cycles constructed in Theorem 4.10 satisfy these axioms.

We first prove several lemmas.

Lemma 5.13. *Let $\mathcal{E}(-)$ be an assignment as in Theorem 5.12 satisfying the axioms therein. Let X be a smooth curve of finite type over k, and let $x \in X$ be a closed point. Set $U = X \setminus \{x\}$, and let $j\colon U \hookrightarrow X$ denote the inclusion. Let $\mathcal{F}$ be a smooth $\overline{\mathbb{Z}_\ell}$-sheaf on U. Assume that $\mathcal{F} \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell}$ has unipotent monodromy at x. Then the coefficient of $[T_x^*X]$ in $\mathcal{E}(j_!\mathcal{F})$ is equal to*

$$(\det(\mathcal{F})_x^{-1} \circ \mathrm{tr}_{k(x)/k})^{\frac{1}{\deg(x/k)}}.$$

Note that $\det(\mathcal{F})$ extends to a smooth rank-one sheaf on X, since it is unramified at x.

Proof. Set $k' := k(x)$. We regard $\mathbb{A}^1_{k'}$ as a smooth k-scheme. Fix an isomorphism $f\colon X_{(x)} \cong \mathbb{A}^1_{k',(0)}$ of k-schemes. Since the tame inertia group of $\mathbb{A}^1_{k',(0)}$ is canonically isomorphic to the tame fundamental group of $\mathbb{G}_{m,k'}$, there exists a smooth $\overline{\mathbb{Z}_\ell}$-sheaf $\mathcal{G}$ on $\mathbb{G}_{m,k'}$, tamely ramified at 0 and ∞, such that $\mathcal{F}|_{\eta_x} \cong f^*(\mathcal{G}|_{\eta_0})$, where η_x and η_0 denote the generic points of $X_{(x)}$ and $\mathbb{A}^1_{k',(0)}$, respectively.

By axiom (7), we may replace $(X, \mathcal{F})$ by $(\mathbb{A}^1_{k'}, \mathcal{G})$. By axiom (3), we may further replace $\mathcal{G}$ by its semisimplification. Since the monodromy of $\mathcal{G}$ at 0 is unipotent, its semisimplification is unramified at 0. The assertion then follows from axioms (1) and (4). □

Lemma 5.14. *Let $\mathcal{E}(-)$ be an assignment as in Theorem 5.12 satisfying the axioms therein. Let X be a smooth projective scheme over k, and let $\mathcal{F} \in D_c^b(X, \overline{\mathbb{Z}_\ell})$. Then we have*

$$\det(R\Gamma(X_{\bar{k}}, \mathcal{F})) = (\mathcal{E}(\mathcal{F}), T_X^*X)_{T^*X}$$

in Θ_k.

Proof. Since X is projective, we may assume that $X = \mathbb{P} = \mathbb{P}^n$ ($n \geq 2$) by axiom (4). Consider the universal hyperplane section $\mathbb{P} \xleftarrow{\boldsymbol{p}} Q \xrightarrow{\boldsymbol{p}^\vee} \mathbb{P}^\vee$. Let $R^\vee = R\boldsymbol{p}_*\boldsymbol{p}^{\vee *}[n-1](n-1)$ be the inverse Radon transform. By [14, IV. Lemma 1.4], we have an isomorphism

$$R^\vee R\mathcal{F} \cong \mathcal{F} \oplus \bigoplus_{i=1}^{n-1} R\Gamma(\mathbb{P}_{\bar{k}}, \mathcal{F})[2i](i),$$

where $R\Gamma(\mathbb{P}_{\bar{k}}, \mathcal{F})$ is regarded as a complex of geometrically constant sheaves. By axioms (1), (3), and (5), we have

$$\mathcal{E}(R^\vee R\mathcal{F}) - \mathcal{E}(\mathcal{F}) = \det(R\Gamma(\mathbb{P}_{\bar{k}}, \mathcal{F}))^{(-1)^n(n-1)} \otimes [T^*_{\mathbb{P}}\mathbb{P}]. \tag{5.13}$$

On the other hand, by axioms (6) and (2), the left-hand side of (5.13) is equal to

$$\begin{aligned} L^\vee\big(\mathcal{E}(R\mathcal{F}(n-1))(\tfrac{1-n}{2})\big) - \mathcal{E}(\mathcal{F}) &= L^\vee L\mathcal{E}(\mathcal{F}) - \mathcal{E}(\mathcal{F}) \\ &= (\mathcal{E}(\mathcal{F}), T^*_{\mathbb{P}}\mathbb{P})_{T^*\mathbb{P}}^{(-1)^n(n-1)} \otimes [T^*_{\mathbb{P}}\mathbb{P}]. \end{aligned}$$

Since $n \geq 2$, the assertion follows. □

Proof of Theorem 5.12. First, we show that the epsilon cycles constructed in Theorem 4.10 satisfy the axioms. Axioms (3) and (7) follow from the construction. The remaining axioms are established in Lemma 4.14.1, Lemma 4.15, Corollary 4.19, Theorem 4.26, and Corollary 5.8.

We now prove the uniqueness. Let $\mathcal{E}(-)$ be an assignment satisfying the axioms of the theorem. Let X and $\mathcal{F}$ be as in the theorem. It suffices to determine the coefficients of $\mathcal{E}(\mathcal{F})$ uniquely from the axioms. By axiom (5), we may replace X by an affine open subscheme. By axiom (4), we may further assume that X is an affine space. Then, by axiom (5), we may further replace X by a projective space and $\mathcal{F}$ by its extension by zero. Thus we may assume that X is projective.

Let $i\colon X \hookrightarrow \mathbb{P} = \mathbb{P}^n$ be a closed immersion. By Lemma 2.20, after composing i with a Veronese embedding $\mathbb{P} \hookrightarrow \mathbb{P}'$ of degree ≥ 3 if necessary, we may find a finite extension k'/k and a good pencil

$$X_{k'} \xleftarrow{\pi} X_{k',L} \xrightarrow{f} L$$

induced by a line $L \subset \mathbb{P}^\vee_{k'}$ defined over k'. In the sequel, we regard smooth k'-schemes as smooth k-schemes.

By axiom (5), we may assume that $X = X_{k'}$. Let C_a be an irreducible component of $SS(\mathcal{F})$. Since (π, f) is a good pencil, there exists a closed point $x_a \in X_{k',L}$ such that x_a is an isolated $SS(\pi^*\mathcal{F})$-characteristic point of f at which df meets C_a and does not meet any other irreducible component. Moreover, x_a does not lie in the exceptional locus of π, and $f^{-1}(f(x_a))$ does not contain any isolated $SS(\pi^*\mathcal{F})$-characteristic point of f other than x_a.

Let $\mathbb{P}_{k',L}$ denote the blow-up of $\mathbb{P}_{k'}$ along the axis A_L defined by L. Since π is $SS(\mathcal{F})$-transversal and X meets A_L transversely, the morphism $\mathbb{P}_{k',L} \to \mathbb{P}_{k'}$ is $SS(i_*\mathcal{F})$-transversal. Let $i'\colon L \hookrightarrow \mathbb{P}^\vee_{k'}$ denote the inclusion. Applying [20, Lemma 3.9.3] to the cartesian diagram

$$\begin{array}{ccc} Q_{k'} & \longleftarrow & \mathbb{P}_{k',L} \\ \downarrow & & \downarrow \\ \mathbb{P}^{\vee}_{k'} & \xleftarrow{i'} & L \end{array}$$

shows that i' is properly $LSS(i_*\mathcal{F})$-transversal. Since $SS(R(i_*\mathcal{F})) \subset LSS(i_*\mathcal{F}) \cup T^*_{\mathbb{P}^\vee}\mathbb{P}^\vee$, it follows that i' is properly $SS(R(i_*\mathcal{F}))$-transversal. Therefore, by axioms (5) and (6),

$$\begin{aligned}\mathcal{E}(i'^*R(i_*\mathcal{F})) &= i'^!(\mathcal{E}(R(i_*\mathcal{F}))(\frac{n-1}{2})) \\ &= i'^! L\mathcal{E}(i_*\mathcal{F}).\end{aligned}$$

Let ξ_a denote the coefficient of $[C_a]$ in $\mathcal{E}(\mathcal{F})$, and let $v_a = f(x_a)$ be the image of x_a in L. Then the coefficient of $[T^*_{v_a}L]$ in $i'^! L\mathcal{E}(i_*\mathcal{F})$ is equal to $\xi_a^{(-1)^{n-1}\cdot(C_a,df)_{x_a}}$. Therefore, the determination of ξ_a is reduced to the case where X is a smooth projective curve.

Suppose that X is a smooth projective curve. By axioms (1) and (4), we may assume that $\mathcal{F} = j_!\mathcal{G}$, where $j\colon U \hookrightarrow X$ is an open immersion and $\mathcal{G}$ is a smooth $\overline{\mathbb{Z}_\ell}$-sheaf on U. The coefficient of $[T^*_X X]$ can be determined from axioms (1) and (5).

Let $x \in X \setminus U$. By weak approximation, we can find a finite morphism $f\colon X' \to X$ from a smooth projective curve such that f is étale around $f^{-1}(x)$ and $f^*\mathcal{G}$ has unipotent monodromy at every point of $X' \setminus f^{-1}(U \cup \{x\})$. Then we can determine the coefficient of $[T^*_x X]$ in $\mathcal{E}(\mathcal{F})$ by axiom (5), Lemma 5.13, and Lemma 5.14. □

Remark 5.15. *The above proof shows that axiom (7) can be replaced by Lemma 5.13.*

6 Appendix: Complements on ℓ-adic formalism

In this appendix, we review the ℓ-adic formalism on a noetherian topos, following [4]. We include this appendix because we need the explicit definition of ℓ-adic sheaves in order to complete the proofs in Subsections 3.3 and 3.4.

To simplify the exposition, we restrict ourselves to the construction of bounded complexes. For a topos T, we define the category $T^{\mathbb{N}^{\mathrm{op}}}$ as follows. An object is a projective system $(M_n, \varphi_n)_{n\in\mathbb{N}}$ indexed by $\mathbb{N}$ where each M_n is an object of T and $\varphi_n\colon M_{n+1} \to M_n$ is a morphism in T. The morphisms φ_n are referred to as transition maps. If no confusion arises, we omit the transition maps φ_n and simply write $(M_n)_n$ instead.

A morphism $(M_n)_n \to (M'_n)_n$ is a family of morphisms $M_n \to M'_n$ compatible with the transition maps.

The category $T^{\mathbb{N}^{\mathrm{op}}}$ is known to be a topos. Let

$$\pi\colon T^{\mathbb{N}^{\mathrm{op}}} \to T \tag{6.1}$$

be the morphism of topoi determined by $\pi_*(M_n)_n = \varprojlim_n M_n$. The functor

$$\pi^{-1}M = (M, \mathrm{id}_M)_{n\in\mathbb{N}}$$

is left adjoint to π_*.

Definition 6.1. *1. We say that a commutative group object $(M_n)_n$ in $T^{\mathbb{N}^{\mathrm{op}}}$ is* essentially zero *if, for every $n \in \mathbb{N}$, there exists $m \geq n$ such that the transition map $M_m \to M_n$ is zero. In [12, (1.10)], such an object is called ML-zero.*

2. *We say that a complex $K \in D(T^{\mathbb{N}^{\mathrm{op}}}, \mathbb{Z})$ of sheaves of abelian groups is* essentially zero *if each cohomology sheaf of K is essentially zero.*

3. *We say that a morphism in $D(T^{\mathbb{N}^{\mathrm{op}}}, \mathbb{Z})$ is an* essential isomorphism *if its mapping cone is essentially zero.*

4. *We say that a complex $K \in D(T^{\mathbb{N}^{\mathrm{op}}}, \mathbb{Z})$ is* essentially constant *if there exist complexes $L \in D(T^{\mathbb{N}^{\mathrm{op}}}, \mathbb{Z})$ and $M \in D(T, \mathbb{Z})$ together with a diagram*

$$K \leftarrow L \to \pi^{-1}M$$

in $D(T^{\mathbb{N}^{\mathrm{op}}}, \mathbb{Z})$ such that both morphisms are essential isomorphisms.

We collect some basic properties that will be needed later.

Lemma 6.2. *The following statements hold.*

1. *Let $M \in D^b(T, \mathbb{Z})$ be a bounded complex. Then the canonical morphism $M \to R\pi_*\pi^{-1}M$ is an isomorphism.*

2. *([12, Lemma (1.11)]) Let $K \in D^b(T^{\mathbb{N}^{\mathrm{op}}}, \mathbb{Z})$ be an essentially zero complex. Then $R\pi_*K = 0$.*

3. *(cf. [4, Lemma 1.3.iv)]) Let $K \in D^b(T^{\mathbb{N}^{\mathrm{op}}}, \mathbb{Z})$ be a bounded complex. If K is essentially constant, then $R\pi_*K$ is bounded, and the canonical morphism $\pi^{-1}R\pi_*K \to K$ is an essential isomorphism.*

Proof. For a sheaf $N = (N_n)_n$ of abelian groups on $T^{\mathbb{N}^{\mathrm{op}}}$ and an object $U \in T$, we have a short exact sequence [12, Proposition (1.6)]

$$0 \to R^1 \varprojlim_n \mathrm{H}^{i-1}(U, N_n) \to \mathrm{H}^i(\pi^{-1}(U), N) \to \varprojlim_n \mathrm{H}^i(U, N_n) \to 0. \tag{6.2}$$

1. We may assume that M is a sheaf. Applying (6.2) to $N = \pi^{-1}M$, we obtain

$$\mathrm{H}^i(\pi^{-1}(U), \pi^{-1}M) \cong \mathrm{H}^i(U, M),$$

hence the assertion.

2. We may assume that $K = (K_n)$ is a sheaf. Since K is essentially zero, the projective system $(\mathrm{H}^i(U, K_n))_n$ is also essentially zero for every $U \in T$. The assertion therefore follows from the exact sequence (6.2).

3. Choose a diagram $K \leftarrow L \to \pi^{-1}M$ as in Definition 6.1.4. Since K is bounded, by replacing L, M with their truncations $\tau^{[-n,n]}L$ and $\tau^{[-n,n]}M$ for sufficiently large $n \in \mathbb{N}$, we may assume that L and M are also bounded. Then, by 2, the morphisms

$$\pi^{-1}R\pi_*K \leftarrow \pi^{-1}R\pi_*L \to \pi^{-1}R\pi_*\pi^{-1}M$$

are isomorphisms. By 1, the complex on the right-hand side is isomorphic to $\pi^{-1}M$. It follows that $R\pi_*K$ is bounded. This proves the first assertion.

To prove the second assertion, consider the following commutative diagram

$$\begin{array}{ccccccc}
\pi^{-1}R\pi_*K & \xleftarrow{\cong} & \pi^{-1}R\pi_*L & \xrightarrow{\cong} & \pi^{-1}R\pi_*\pi^{-1}M & \xleftarrow{\cong} & \pi^{-1}M \\
\downarrow & & \downarrow & & \downarrow & \swarrow_{\mathrm{id}} & \\
K & \longleftarrow & L & \longrightarrow & \pi^{-1}M. & &
\end{array}$$

Since the top horizontal arrows are isomorphisms, the assertion follows from a diagram chase. □

Let $(R, \mathfrak{m})$ be a complete discrete valuation ring, and put $R_n := R/\mathfrak{m}^{n+1}$. Let

$$R_\bullet := (R_n)_{n\in\mathbb{N}}$$

be the ring object of $T^{\mathbb{N}^{\mathrm{op}}}$, where the transition maps are given by the natural projections $R_{n+1} \to R_n$. The morphism (6.1) extends canonically to a morphism of ringed topoi

$$\pi\colon (T^{\mathbb{N}^{\mathrm{op}}}, R_\bullet) \to (T, R). \tag{6.3}$$

For a sheaf M of R-modules on T, define $\pi^*M := R_\bullet \otimes_{\pi^{-1}R} \pi^{-1}M$. Since $R_\bullet$ has finite tor-dimension as a $\pi^{-1}R$-module, the left-derived functor $L\pi^*$ is defined on the unbounded derived category $D(T, R)$.

For each $n \in \mathbb{N}$, let $i_n\colon T \to T^{\mathbb{N}^{\mathrm{op}}}$ be the morphism of topoi defined by

$$i_{n*}M = (\cdots \xrightarrow{\mathrm{id}} M \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} \overset{n\text{-th}}{M} \to * \to \cdots \to *) \qquad \text{and} \qquad i_n^{-1}(M_m)_m = M_n,$$

where $*$ denotes the final object of T. This induces a morphism of ringed topoi

$$i_n\colon (T, R_n) \to (T^{\mathbb{N}^{\mathrm{op}}}, R_\bullet).$$

Note that the canonical morphism $i_n^{-1}R_\bullet \to R_n$ is an isomorphism.

Lemma 6.3. *The following statements hold.*

1. *Let M be a sheaf of R_0-modules on T, viewed as a sheaf of R-modules. Then the composite morphism*

$$\pi^{-1}M \cong \pi^{-1}R \otimes^L_{\pi^{-1}R} \pi^{-1}M \to R_\bullet \otimes^L_{\pi^{-1}R} \pi^{-1}M = L\pi^*M$$

   and the canonical morphism

$$L\pi^*M \to H^0(L\pi^*M) \cong \pi^{-1}M$$

   are essential isomorphisms.

2. *Let $K, L \in D^-(T^{\mathbb{N}^{\mathrm{op}}}, R_\bullet)$ be bounded above complexes. If either K or L is essentially zero, then $L \otimes^L_{R_\bullet} K$ is also essentially zero.*

3. *Let $C \in D^-(T^{\mathbb{N}^{\mathrm{op}}}, R_\bullet)$ be a bounded above complex. If $R_0 \otimes^L_{R_\bullet} C$ is essentially zero, then so is C. Moreover, if $R_0 \otimes^L_{R_\bullet} C$ is acyclic, then C is also acyclic.*

Proof. 1. Let $L_1 = (R)_n = \pi^{-1}R$ and $L_2 = (\mathfrak{m}^{n+1})_n$ be sheaves of R-modules on $T^{\mathbb{N}^{\mathrm{op}}}$ whose transition maps are given by the inclusions. We have a short exact sequence

$$0 \to L_2 \to L_1 \to R_\bullet \to 0,$$

which provides an R-flat resolution of $R_\bullet$. Hence, the mapping cone of $\pi^{-1}M \to L\pi^*M$ is quasi-isomorphic to $L_2 \otimes_R \pi^{-1}M[1]$. Since the transition maps of $L_2 \otimes_R \pi^{-1}M$ are zero, it follows that the first morphism is an essential isomorphism. The assertion for the second morphism follows from the fact that the composition of $\pi^{-1}M \to L\pi^*M \to \pi^{-1}M$ coincides with the identity.

2. This follows from the spectral sequence

$$E_2^{p,q} = \bigoplus_{i+j=q} \mathrm{Tor}_{-p}^{R_\bullet}(H^i(L), H^j(K)) \Rightarrow H^{p+q}(L \otimes^L_{R_\bullet} K).$$

3. For each $n \geq 0$, let $R'_n := R_\bullet/\mathfrak{m}^{n+1}R_\bullet$. Let K_n denote the kernel of the natural surjection $R'_{n+1} \to R'_n$. Note that K_n is a sheaf of R_0-modules on $T^{\mathbb{N}^{\mathrm{op}}}$ that is essentially isomorphic to R_0.

Assume first that $R_0 \otimes^L_{R_\bullet} C$ is essentially zero. We prove by induction on n that $R'_n \otimes^L_{R_\bullet} C$ is essentially zero for every $n \geq 0$. Since $R'_0 = R_0$, the case $n = 0$ is trivial.

Suppose that $R'_n \otimes^L_{R_\bullet} C$ is essentially zero for some n. Since K_n is essentially isomorphic to R_0, it follows from 2 that $K_n \otimes^L_{R_\bullet} C$ is essentially zero. Hence, by the distinguished triangle

$$K_n \otimes^L_{R_\bullet} C \to R'_{n+1} \otimes^L_{R_\bullet} C \to R'_n \otimes^L_{R_\bullet} C \to,$$

we conclude that $R'_{n+1} \otimes^L_{R_\bullet} C$ is essentially zero, as claimed.

Consequently, for each $i \in \mathbb{Z}$ and $n \geq 0$, there exists $m \geq n$ such that the transition map

$$i_m^* H^i(R'_n \otimes^L_{R_\bullet} C) \to i_n^* H^i(R'_n \otimes^L_{R_\bullet} C) = i_n^* H^i(C)$$

is zero. Hence the composite

$$i_m^* H^i(C) \to i_m^* H^i(R'_n \otimes^L_{R_\bullet} C) \to i_n^* H^i(C)$$

is also zero. Therefore, C is essentially zero.

Suppose now that $R_0 \otimes^L_{R_\bullet} C$ is acyclic. Then

$$K_n \otimes^L_{R_\bullet} C \cong K_n \otimes^L_{R_0} (R_0 \otimes^L_{R_\bullet} C)$$

is also acyclic. It follows by induction on n that $R'_n \otimes^L_{R_\bullet} C$ is acyclic for every n. Therefore,

$$i_n^* H^i(C) \cong i_n^* H^i(R'_n \otimes^L_{R_\bullet} C) = 0$$

for all $i \in \mathbb{Z}$ and $n \geq 0$. □

We recall the notion of (normalized) R-complexes, following [4].

Definition 6.4. *1. We say that a complex $K \in D^b(T^{\mathbb{N}^{\mathrm{op}}}, R_\bullet)$ is an* R-complex *if $L\pi^* R_0 \otimes^L_{R_\bullet} K$ is essentially constant.*

2. We say that a complex $K \in D^b(T^{\mathbb{N}^{\mathrm{op}}}, R_\bullet)$ is a normalized R-complex *if, for each $n \in \mathbb{N}$, the canonical morphism $i_{n+1}^* K \otimes^L_{R_{n+1}} R_n \to i_n^* K$ is an isomorphism.*

The following lemma collects several basic properties that will be used in the construction of the derived category of $\mathfrak{m}$-adic sheaves.

Lemma 6.5. *Let $K \in D^b(T^{\mathbb{N}^{\mathrm{op}}}, R_\bullet)$. Then the following statements hold.*

1. *The canonical morphism $L\pi^* R_0 \otimes^L_{R_\bullet} K \to \pi^{-1} R_0 \otimes^L_{R_\bullet} K$ is an essential isomorphism.*

2. *If K is a normalized R-complex, then K is an R-complex.*

3. *If K is an R-complex, then $L\pi^* R\pi_* K$ is bounded, and the canonical morphism $L\pi^* R\pi_* K \to K$ is an essential isomorphism.*

4. *The following conditions are equivalent.*

   (a) The complex K is a normalized R-complex.

   (b) The canonical morphism $L\pi^* R\pi_* K \to K$ is an isomorphism.

   (c) There is a complex $M \in D(T, R)$ such that $L\pi^* M \cong K$.

   If these equivalent conditions hold, then there exists a bounded complex $M \in D^b(T, R)$ such that $L\pi^* M \cong K$.

Proof. 1. This follows from Lemma 6.3.1 and 2.

2. Suppose that K is a normalized R-complex. By 1, the morphism $L\pi^* R_0 \otimes^L_{R_\bullet} K \to \pi^{-1} R_0 \otimes^L_{R_\bullet} K$ is an essential isomorphism. Since K is normalized, the latter complex is isomorphic to $\pi^{-1} i_0^{-1} K$. Therefore, $L\pi^* R_0 \otimes^L_{R_\bullet} K$ is essentially constant, and hence K is an R-complex.

3. We first show that $L\pi^* R\pi_* K$ is bounded. It suffices to prove that $R_0 \otimes^L_R R\pi_* K$ is bounded. Since R_0 has finite projective dimension as an R-module, we have $R_0 \otimes^L_R R\pi_* K \cong R\pi_*(L\pi^* R_0 \otimes^L_{R_\bullet} K)$. Since $L\pi^* R_0 \otimes^L_{R_\bullet} K$ is bounded and essentially constant, the assertion follows from Lemma 6.2.3.

We now show that $L\pi^* R\pi_* K \to K$ is an essential isomorphism. Since both $L\pi^* R\pi_* K$ and K are bounded above, it suffices, by Lemma 6.3.3, to show that $R_0 \otimes^L_{R_\bullet} L\pi^* R\pi_* K \to R_0 \otimes^L_{R_\bullet} K$ is an essential isomorphism.

Since $L\pi^* R_0 \to \pi^{-1} R_0$ is an essential isomorphism, it remains to show that

$$L\pi^* R_0 \otimes^L_{R_\bullet} L\pi^* R\pi_* K \to L\pi^* R_0 \otimes^L_{R_\bullet} K$$

is an essential isomorphism. The former complex is isomorphic to $L\pi^* R\pi_*(L\pi^* R_0 \otimes^L_{R_\bullet} K)$, and we have a commutative diagram

$$\begin{array}{ccc} L\pi^* R\pi_*(L\pi^* R_0 \otimes^L_{R_\bullet} K) & \longrightarrow & L\pi^* R_0 \otimes^L_{R_\bullet} K \\ \uparrow & \nearrow & \\ \pi^{-1} R\pi_*(L\pi^* R_0 \otimes^L_{R_\bullet} K) & & \end{array}$$

in $D(T^{\mathbb{N}^{\mathrm{op}}}, \mathbb{Z})$, where the vertical morphism is induced by $\pi^{-1} R \to R_\bullet$ and the diagonal morphism is the counit of adjunction.

Since $L\pi^* R_0 \otimes^L_{R_\bullet} K$ is essentially constant and bounded, the diagonal morphism is an essential isomorphism. Moreover, $R\pi_*(L\pi^* R_0 \otimes^L_{R_\bullet} K)$ is bounded by Lemma 6.2.3. Since

$R\pi_*(L\pi^* R_0 \otimes^L_{R_\bullet} K) \cong R_0 \otimes^L_R R\pi_* K$, its cohomology sheaves are R_0-modules. Hence, by Lemma 6.3.1, the vertical morphism is an essential isomorphism. Therefore, the horizontal morphism is also an essential isomorphism. This proves the assertion.

4. We first prove $(a) \Rightarrow (b)$. Let K be a normalized R-complex. Then it is an R-complex by 2, and hence $L\pi^* R\pi_* K$ is bounded by 3. Applying Lemma 6.3.3 to the mapping cone of $L\pi^* R\pi_* K \to K$, it suffices to show that $R_0 \otimes^L_{R_\bullet} L\pi^* R\pi_* K \to R_0 \otimes^L_{R_\bullet} K$ is an isomorphism.

Since K is a normalized R-complex, we have $R_0 \otimes^L_{R_\bullet} K \cong \pi^{-1} i_0^{-1} K$. On the other hand,

$$R_0 \otimes^L_{R_\bullet} L\pi^* R\pi_* K \cong R_0 \otimes^L_R \pi^{-1} R\pi_* K \cong \pi^{-1} R\pi_*(L\pi^* R_0 \otimes^L_{R_\bullet} K) \cong \pi^{-1} R\pi_*(\pi^{-1} R_0 \otimes^L_{R_\bullet} K).$$

Here, the last isomorphism follows from the facts that $L\pi^* R_0 \otimes^L_{R_\bullet} K \to \pi^{-1} R_0 \otimes^L_{R_\bullet} K$ is an essential isomorphism between bounded complexes, and that $R\pi_*$ sends bounded essentially zero complexes to zero.

Under these identifications, the morphism $R_0 \otimes^L_{R_\bullet} L\pi^* R\pi_* K \to R_0 \otimes^L_{R_\bullet} K$ corresponds to the counit morphism

$$\pi^{-1} R\pi_*(R_0 \otimes^L_{R_\bullet} K) \to R_0 \otimes^L_{R_\bullet} K.$$

Since $R_0 \otimes^L_{R_\bullet} K \cong \pi^{-1} i_0^{-1} K$, this is an isomorphism by Lemma 6.2.1. This proves (b).

$(b) \Rightarrow (c)$ is obvious. We prove $(c) \Rightarrow (a)$. Let M be an object of $D(T, R)$ such that $L\pi^* M$ is bounded. We claim that there exists a diagram in $D(T, R)$

$$M \xrightarrow{\alpha} M' \xleftarrow{\beta} M'' \tag{6.4}$$

such that M'' is bounded and the cohomology sheaves of the mapping cones of α and β are uniquely divisible by a uniformizer ϖ of R.

For $N \in D(T, R)$ whose cohomology sheaves are uniquely divisible, the pullback $L\pi^* N$ is acyclic. Therefore, the claim impies that $L\pi^* M$ is quasi-isomorphic to $L\pi^* M''$. This proves (a) and the last assertion.

It remains to construct the diagram (6.4). The short exact sequence

$$0 \to R \xrightarrow{\varpi} R \to R_0 \to 0$$

induces a distinguished triangle

$$M \xrightarrow{\varpi} M \to R_0 \otimes^L_R M \to .$$

By assumption, $R_0 \otimes^L_R M \cong i_0^* L\pi^* M$ is bounded. Let n be a positive integer such that $H^i(R_0 \otimes^L_R M)$ is zero whenever $|i| \geq n$. Then, from the distinguished triangle above, it follows that the morphism

$$H^i(M) \xrightarrow{\varpi} H^i(M)$$

is an isomorphism whenever $|i| \geq n+1$. Choosing suitable truncations of M, we therefore obtain a diagram (6.4) with the required properties. □

Let $\mathcal{A}$ and $D_{\mathrm{norm}}(T^{\mathbb{N}^{\mathrm{op}}}, R_\bullet)$ denote the full subcategories of $D^b(T^{\mathbb{N}^{\mathrm{op}}}, R_\bullet)$ consisting of R-complexes and normalized R-complexes, respectively. Let $\mathcal{B}$ denote the full subcategory of $D^b(T^{\mathbb{N}^{\mathrm{op}}}, R_\bullet)$ consisting of complexes that are essentially zero when regarded as objects in $D(T^{\mathbb{N}^{\mathrm{op}}}, \mathbb{Z})$. By Lemma 6.3.2, we have $\mathcal{B} \subset \mathcal{A}$.

Since $\mathcal{B}$ is a thick triangulated subcategory of $D^b(T^{\mathbb{N}^{\rm op}}, R_\bullet)$, the Verdier quotient $D^b(T^{\mathbb{N}^{\rm op}}, R_\bullet)/\mathcal{B}$ is a triangulated category. Define $D^b(T - R)$ to be the quotient category $\mathcal{A}/\mathcal{B}$. Since the subcategory of essentially constant complexes is stable under extensions and shifts (the former follows from Lemma 6.2.3), it follows that $D^b(T - R)$ is a triangulated subcategory of $D^b(T^{\mathbb{N}^{\rm op}}, R_\bullet)/\mathcal{B}$.

Lemma 6.6. *The following statements hold.*

1. *For $K \in \mathcal{A}$, set $\widehat{K} := L\pi^* R\pi_* K$. Then $\widehat{K}$ is a normalized R-complex. Moreover, if $K \in \mathcal{B}$, then $\widehat{K}$ is acyclic.*
2. *For $K \in \mathcal{A}$, the complex $R_n \otimes^L_R R\pi_* K$ is bounded. Moreover, if $K \in \mathcal{B}$, then it is acyclic.*

Proof. 1. The first assertion follows from Lemma 6.5.3 and 4. The secondassertion follows from Lemma 6.2.2.

2. This follows from 1, since $R_n \otimes^L_R R\pi_* K \cong i_n^{-1}\widehat{K}$. □

Definition 6.7. *1. By Lemma 6.6.1, the functor*

$$\mathcal{A} \to D(T^{\mathbb{N}^{\rm op}}, R_\bullet), \qquad K \mapsto \widehat{K} := L\pi^* R\pi_* K$$

induces a functor $\Phi\colon D^b(T - R) \to D_{\rm norm}(T^{\mathbb{N}^{\rm op}}, R_\bullet)$.

2. By Lemma 6.6.2, the assignment $K \mapsto R_n \otimes^L_R R\pi_* K$ induces a functor

$$D^b(T - R) \to D^b(T, R_n).$$

For $K \in D^b(T - R)$, we denote its image under this functor by $R_n \otimes^L_R K$.

By Lemma 6.5.2, we have $D_{\rm norm}(T^{\mathbb{N}^{\rm op}}, R_\bullet) \subset \mathcal{A}$. Hence, we obtain a functor

$$\Psi\colon D_{\rm norm}(T^{\mathbb{N}^{\rm op}}, R_\bullet) \hookrightarrow \mathcal{A} \to \mathcal{A}/\mathcal{B} = D^b(T - R).$$

Lemma 6.8. *The functor Ψ is an equivalence of categories with quasi-inverse Φ.*

Proof. It suffices to show that the two compositions are naturally isomorphic to the corresponding identity functors. Let K be a normalized R-complex. Then $\Phi\Psi(K) = L\pi^* R\pi_* K$, which is isomorphic to K by Lemma 6.5.4.

Let K be an R-complex. By Lemma 6.5.3, the canonical morphism $L\pi^* R\pi_* K \to K$ is an essential isomorphism. Therefore, it induces an isomorphism in $D^b(T - R)$. This shows that $\Psi\Phi$ is naturally isomorphic to the identity functor. □

We now impose a finiteness condition on (normalized) R-complexes. From now on, we assume that T is a noetherian topos. Let $D^b_c(T, R_0)$ denote the full subcategory of $D^b(T, R_0)$ consisting of bounded complexes with constructible cohomology sheaves.

Definition 6.9. *1. Define $D_{c,\rm norm}(T^{\mathbb{N}^{\rm op}}, R_\bullet)$ to be the full subcategory of $D_{\rm norm}(T^{\mathbb{N}^{\rm op}}, R_\bullet)$ consisting of objects K such that $i_0^* K \in D^b_c(T, R_0)$.*

2. Define $D^b_c(T, R)$ to be the full subcategory of $D^b(T - R)$ consisting of objects K such that $R_0 \otimes^L_R K \in D^b_c(T, R_0)$ (the definition of $R_0 \otimes^L_R K$ is given in Definition 6.7.2). An object of $D^b_c(T, R)$ is called a constructible complex of R-sheaves.

Lemma 6.10. *The equivalence in Lemma 6.8 restricts to an equivalence*

$$D_{c,\mathrm{norm}}(T^{\mathbb{N}^{\mathrm{op}}}, R_\bullet) \cong D_c^b(T, R).$$

Proof. The assertion follows from Lemma 6.8 together with the isomorphism $i_0^* K \cong R_0 \otimes_R^L R\pi_* K$ for $K \in D_{\mathrm{norm}}(T^{\mathbb{N}^{\mathrm{op}}}, R_\bullet)$. □

Let ℓ be a prime number, and let $\overline{\mathbb{Q}_\ell}$ be an algebraic closure of $\mathbb{Q}_\ell$. We define the category

$$D_c^b(T, \overline{\mathbb{Z}_\ell})$$

to be the 2-colimit $\varinjlim_E D_c^b(T, \mathcal{O}_E)$, where E ranges over the finite extensions of $\mathbb{Q}_\ell$ contained in $\overline{\mathbb{Q}}_\ell$.

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