Weak Form Mitter Conjecture on Nonmaximal Rank Estimation Algebra: State Dimension 4 and Rank 3*

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Abstract Ever since Brockett and Clark (1980), Brockett (1981) and Mitter (1980) introduced the estimation algebra method, it becomes a powerful tool to classify finite-dimensional filtering systems. In this paper, the authors investigate estimation algebra on state dimension n and linear rank $n-1$, especially the case of $n=4$. Mitter conjecture is always a key question on classification of estimation algebra. A weak form of Mitter conjecture states that observation functions in finite dimensional filters are affine functions. In this paper, the authors shall focus on the weak form of Mitter conjecture. In the first part, it will be shown that partially constant structure of Ω is a sufficient condition for weak form Mitter conjecture to be true. In the second part, the authors shall prove partially constant structure of Ω for $n=4$ which implies the weak form Mitter conjecture for this case.

Keywords Estimation algebra, finite dimensional filter, Mitter conjecture, state estimation.

1 Introduction

Filtering problem refers to estimating the state of a stochastic dynamical system by using the information of observation history. An important progress is Kalman-Bucy filter proposed in the 1960s for linear Gaussian systems with Gaussian initial condition. Linear Kalman filtering motivated a lot of researches in the study of nonlinear filtering. Conditional expectation $\mathbb{E}[\phi(X_t)|\mathcal{Y}_t]$ is optimal estimate in the sense of mean square error, where X_t denotes the state of system and $\mathcal{Y}_t$ denotes the history of observations. Obviously, conditional density $\rho(t,x)$ contains full information on nonlinear filtering. In the 1960s, Kushner[1] and Stratonovich[2]

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derived the evolution equation of conditional density independently. For the convenience of solving Kushner equation, Duncan-Mortensen-Zakai (DMZ) equation[3–5] was proposed and described the evolution of unnormalized conditional density $\sigma(t, x)$ in the late 1960s. DMZ equation is easier to deal with because it is a linear stochastic partial differential equation.

Currently there are basically four ways of solving nonlinear filtering problems. The first one is the projection based method, e.g., extended Kalman filter and geometric projection filter[6]. The second approach is the particle evolution method. In this type of method, different particles $\{X_i\}$ will be created and evolved according to the controlled stochastic differential equation. True conditional density will be approximated by empirical distribution of discrete particles. Typical algorithms include ensemble Kalman filter[7] and feedback particle filter[8]. The third approach is the optimization-based algorithm including optimal control[9] and optimal transportation[10]. The fourth aspect is based on the solution of DMZ which will be explained in detail below.

It is noted that robust DMZ equation is a linear parabolic PDE and Wei-Norman approach can be applied to solve it in principle. Estimation algebra method proposed by [11–13] originated from Wei-Norman approach in the 1970s. Once estimation algebra of the system is a finite dimensional Lie algebra, Wei-Norman approach can represent formal solution of robust DMZ equation. Estimation algebra method has been developed for more than 40 years and has the following advantages. First, it takes account of both geometrical and algebraic aspects of nonlinear filtering. Second, it allows us to determine the forms of finite dimensional filters including drift and diffusion coefficients. This prior information allows us to find potential finite dimensional filters. Efficient algorithms can be studied based on the form of finite dimensional filters. Third, DMZ equation can be solved explicitly and universal recursive filters can be constructed if estimation algebra is finite dimensional. Fourth, it demonstrates that there exists an algorithm with polynomial computational complexity for finite dimensional filter.

In the International Congress of Mathematicians of 1983, Brockett[12] proposed the program of classifying all finite dimensional estimation algebra. Since the 1990s, through persistent efforts, Yau and his collaborators finished the complete classification of maximal rank estimation algebras[14–18]. At the beginning of the 20th century, Yau and his coworkers initiated the study of nonmaximal rank finite dimensional estimation algebra. Mitter conjecture states that all functions contained in finite dimensional estimation algebra are affine. It plays an important role in the classification of finite dimensional estimation algebra in both maximal rank case and nonmaximal rank case. In fact, one critical step of maximal rank classification is to prove the validity of Mitter conjecture. With the Mitter conjecture, differential operators contained in the estimation algebra will be determined, yielding a more clear structure of the estimation algebra. As a first step towards Mitter conjecture, we need to prove the weak form of Mitter conjecture which states that observation functions are all affine functions. Recently, a lot of attentions have turned to nonmaximal rank estimation algebra with linear rank $n-1$ and state space dimension n. More precisely, Mitter conjecture was proven for state space with dimension $n \leq 3$[19, 20]. Recently linear structure of Ω matrix (see Definition 2.10) has been verified for arbitrary state dimension with linear rank $n-1$ which is a critical step towards

Mitter conjecture[21]. However, in the high dimensional situation with state dimension larger than 3, Mitter conjecture is an important unsolved problem in the field. In this paper, we shall solve weak form of Mitter conjecture for state dimension 4 with rank 3.

First we shall focus on quadratic structure of nonmaximal rank estimation algebra. With the help of linear and quadratic ranks, we are able to describe any quadratic function ϕ in E. We successfully extend the quadratic structure theory from maximal rank case to nonmaximal rank case. In the second part, we demonstrate that the weak form of Mitter conjecture is implied by partially constant structure of Ω. In the final part, partially constant structure of Ω will be proven and hence weak form of Mitter conjecture holds for $n = 4$ case.

The paper is organized as follows. In Section 2, we introduce some basic concepts of nonlinear filtering and preliminary results about the estimation algebras. In Section 3, quadratic structure of any function in E is studied. In Section 4, the partially constant structure of Ω is shown to be a sufficient condition of weak form of Mitter conjecture. In Section 5, the partially constant structure of Ω will be verified for case $n = 4$. In Section 6, we shall give a summary. Appendix contains the detailed proofs of related results.

2 Preliminaries

2.1 Basic Notations

The set of real numbers is denoted by $\mathbb{R}$. $\mathbb{R}^k$ refers to k dimensional Euclidean space. $\mathbb{R}^{m\times n}$ denotes the set of matrices with size $m \times n$. $A = (a_{ij})$ denotes a matrix A with i, j-entry a_{ij}; rank(A) denotes rank of matrix A. I_n denotes identity matrix of size $n\times n$. δ_{ij} denotes Kronecker symbol which means $\delta_{ij} = 1$ if $i = j$; otherwise $\delta_{ij} = 0$. $A = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ represents a diagonal matrix A with diagonal elements $\lambda_1, \lambda_2, \cdots, \lambda_n$. Let $C^\infty(U)$ be set of smooth functions defined on U, span$\{v_1, \cdots, v_k\}$ be the linear space spanned by vectors $\{v_1, v_2, \cdots, v_k\}$, and $P_k(x_{i_1}, \cdots, x_{i_m})$ be the set of polynomials of degree no more than k in variable $x_{i_1}, \cdots, x_{i_m}$. Let $pol_k(x_{i_1}, \cdots, x_{i_m})$ be an element in the set $P_k(x_{i_1}, \cdots, x_{i_m})$, $\widetilde{P}_k(x_{i_1}, \cdots, x_{i_m})$ be the set of polynomials of degree at most k in $x_{i_1}, \cdots, x_{i_m}$ with smooth coefficient in other variables $\{x_s, s \notin \{i_1, \cdots, i_m\}\}$. For a polynomial ϕ, $\phi^{(k)}$ denotes the homogeneous degree k part of ϕ. $J_\xi = \left(\frac{\partial^2 \xi}{\partial x_i \partial x_j}\right)$ denotes the Hessian matrix of function ξ. Finally, an unspecified constant is denoted by *const*.

2.2 Basic Concepts

In this paper, we consider the following time-invariant nonlinear filtering system:

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t))dw(t), & x(0) = x_0 \in \mathbb{R}^n, \\ dy(t) = h(x(t))dt + dv(t), & y(0) \in \mathbb{R}^m, \end{cases} \tag{1}$$

where $x(t) = (x_1, \cdots, x_n) \in \mathbb{R}^n, y(t) = (y_1, \cdots, y_m) \in \mathbb{R}^m$ represents state and observation vectors in Euclidean space. $f : \mathbb{R}^n \to \mathbb{R}^n$ denotes drift mapping. $h : \mathbb{R}^n \to \mathbb{R}^m$ denotes observation function. $g : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ represents diffusion coefficient. $f = (f_i), h = (h_i), g = (g_{ij})$ are all assumed to be smooth vector fields. $w(t) \in \mathbb{R}^p, v(t) \in \mathbb{R}^m$ are mutually independent standard Wiener process, i.e., $\mathbb{E}[dwdw^{\mathrm{T}}] = I_n dt, \mathbb{E}[dvdv^{\mathrm{T}}] = I_m dt$. Define coefficient matrix $C = (C_{ij}) := gg^{\mathrm{T}} \in \mathbb{R}^{n\times n}$.

For a continuous filtering system, the ultimate goal is to determine the conditional expectation $\mathbb{E}[\phi(x_t)|\mathcal{F}_t]$, where ϕ is a C^∞ function and $\mathcal{Y}_t := \sigma\{y_s : 0 \le s \le t\}$ is the sigma algebra generated by observation. It is well known that conditional expectation $\mathbb{E}[x_t|\mathcal{F}_t]$ is the optimal estimate with respect to the least variance criterion. Therefore, conditional density $\rho(t,x)$ given the observation history includes complete information of the filtering system.

Mathematically, unnormalized conditional density $\sigma(t,x)$ is described by the following Duncan-Mortensen-Zakai (DMZ) equation[3–5]:

$$d\sigma(t,x) = L_0\sigma dt + \sigma(t,x)h_t^{\mathrm{T}} \circ dy_t, \tag{2}$$

where

$$L_0(\circ) := \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}\frac{\partial^2(\circ)}{\partial x_i \partial x_j} - \sum_{i=1}^{n}\frac{\partial(f_i\circ)}{\partial x_i} - \frac{1}{2}h^{\mathrm{T}}h(\circ). \tag{3}$$

Note that DMZ equation is formulated in the form of Stratonovich stochastic integral. Degeneracy of the matrix $C = (C_{ij})$ will influence the behavior of L_0. In this paper, we assume that the diffusion coefficient g is an orthogonal matrix which will lead to $C = I_n$.

Next we can reformulate forward differential operator L_0 as

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2. \tag{4}$$

And we define $L_i := h_i, 1 \le i \le m$ as the zero degree differential operator of multiplication by h_i.

Let

$$D_i := \frac{\partial}{\partial x_i} - f_i, \quad 1 \le i \le n, \quad \eta := \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2. \tag{5}$$

Then we can obtain a more compact form of L_0,

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} D_i^2 - \eta\right). \tag{6}$$

Next we give some basic concepts related to Lie algebra.

Definition 2.1 If X and Y are differential operators, the Lie bracket of X and Y, $[X,Y]$, is defined by $[X,Y]\phi = X(Y\phi) - Y(X\phi)$ for any C^∞ function ϕ.

Definition 2.2 A vector space $\mathcal{F}$ with the Lie bracket operation $\mathcal{F}\times\mathcal{F}\to\mathcal{F}$ denoted by $(x,y)\longmapsto[x,y]$ is called a Lie algebra if the following axioms are satisfied:

1) The Lie bracket operation is bilinear;
2) $[x,x] = 0$ for all $x \in \mathcal{F}$;
3) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0,\ x,y,z \in \mathcal{F}$.

Definition 2.3 Let $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ be two Lie algebras. An isomorphism $f : \mathfrak{g} \to \widetilde{\mathfrak{g}}$ is a linear map and satisfies

1) f is a bijection.

2) f is a homomorphism of Lie algebras, i.e., $f[g_1, g_2] = [f(g_1), f(g_2)]$ for any $g_1, g_2 \in \mathfrak{g}$.

$\mathfrak{g}$ is isomorphic to $\widetilde{\mathfrak{g}}$, i.e., $\mathfrak{g} \cong \widetilde{\mathfrak{g}}$, if there exists an isomorphism between $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$.

Remark 2.4 If two Lie algebras are isomorphic, then they have the same Lie algebra structure.

Next we introduce the concept of estimation algebra related to filtering system.

Definition 2.5 The estimation algebra E of a filtering system (1) is defined to be the Lie algebra generated by $\{L_0, L_1, \cdots, L_m\}$, i.e., $E = \langle L_0, h_1, \cdots, h_m \rangle_{L.A.}$.

Remark 2.6 In the whole paper, we assume that E is finite dimensional. Brockett[22] proved that if one performs a smooth non-singular change of variable $z = F(x)$, this mapping will lead to an isomorphism of estimation algebra. Therefore, for the purpose of classification of estimation algebras, we can freely use orthogonal transformations and translations.

Definition 2.7 Let $L(E) \subset E$ be the vector space consisting of all the homogeneous degree 1 polynomials in E. Then the linear rank of estimation algebra E is defined by $r := \dim L(E)$. If $r = n$, we call that E has maximal rank. Otherwise, E has non-maximal rank. In what follows, we shall use (linear) rank for short.

Remark 2.8 Without loss of generality, we can assume $x_1, x_2, \cdots, x_r \in E$ and $x_{r+1}, \cdots, x_n \notin E$ by an orthogonal transformation if necessary. More details can be found in [19].

Next we define quadratic rank in order to describe the structure of quadratic function in E.

Definition 2.9 For a given function $h \in E$, we consider homogeneuous quadratic part $h^{(2)} = x^{\mathrm{T}} A x$. We define quadratic rank of h as $\lambda(h) := \mathrm{rank}(A)$. Then quadratic rank of estimation algebra E is defined as the maximal rank of functions in E, i.e., $\lambda(E) := \max_{h \in E} \lambda(h)$. $h^* := \arg\max_{h \in E} \lambda(h)$ is called maximal rank quadratic polynomial.

We would like to remark that structure of linear rank and quadratic rank play quite important roles in classification of estimation algebras.

Definition 2.10 The Wong's Ω-matrix is the matrix $\Omega = (\omega_{ij})$, where

$$\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}, \quad \forall 1 \leq i, j \leq n. \tag{7}$$

Obviously, $\omega_{ij} = -\omega_{ji}$, i.e., Ω is an antisymmetric matrix.

It is worth noticing that elements of Ω matrix satisfy the following cyclical condition, which can be obtained by direct calculations.

$$\frac{\partial \omega_{ij}}{\partial x_l} + \frac{\partial \omega_{jl}}{\partial x_i} + \frac{\partial \omega_{li}}{\partial x_j} = 0, \quad \text{for } 1 \leq i, j, l \leq n. \tag{8}$$

Remark 2.11 The structure of Ω matrix influences the form of drift term f. If $\Omega = 0$, it corresponds to Benes filter, i.e., $f = \nabla \phi$ where $\phi \in C^\infty(\mathbb{R}^n)$. In Benes filtering system, drift vector field has a potential which can be think of as electronic field. If Ω is a constant matrix, then the drift vector field corresponds to $f(x) = Lx + \nabla\phi$, where $L \in \mathbb{R}^{n\times n}, \phi \in C^\infty(\mathbb{R}^n)$. This type of filter is called Yau filtering system which contains Kalman-Bucy filter and Benes filter as special cases. Yau filtering system plays an important role in study of maximal and non-maximal rank estimation algebra.

Definition 2.12 Let U be the vector space of differential operators in the form

$$A = \sum_{(i_1,i_2,\cdots,i_n)\in I_A} a_{i_1,i_2,\cdots,i_n} D_1^{i_1} D_2^{i_2} \cdots D_n^{i_n}, \tag{9}$$

where nonzero functions $a_{i_1,i_2,\cdots,i_n} \in C^\infty(\mathbb{R}^n)$ and $I_A \subset N^n$ is a finite set. For $i = (i_1, i_2, \cdots, i_n) \in N^n$, denote $|i| := \sum_{k=1}^n i_k$. The order of A is defined by $ord(A) := \max_i |i|$. Let U_k denote differential operator in E with order no more than k. Especially, U_0 denotes smooth functions in E.

2.3 Basic Computation Rule

Some basic notations of Lie bracket are also given as below. Let $A, B \in E$ and $V \subset E$. then we denote $A = B, \mod V$ if and only if $A - B \in V$; We define adjoint map $Ad : E \times E \to E$ by $Ad_A B = [A, B]$ and $Ad_A^k B = [A, Ad_A^{k-1} B]$; Euler operator $E_S := \sum_{l\in S} x_l \frac{\partial}{\partial x_l}$, where S is an index subset of $\{1, 2, \cdots, n\}$.

Estimation algebra is an operator algebra. The following basic formulas are useful in exploring the algebraic structure.

Lemma 2.13 (see [17]) *Let E be an estimation algebra for the filtering problem* (1). *$\Omega = (\omega_{ij})$ is defined as in Definition* 2.10. *Assume $X, Y, Z \in E$ and $g, h \in C^\infty(R^n)$. Then*

1) $[XY, Z] = X[Y, Z] + [X, Z]Y$;
2) $[gD_i, h] = g\frac{\partial h}{\partial x_i}$;
3) $[gD_i, hD_j] = gh\omega_{ji} + g\frac{\partial h}{\partial x_i}D_i - h\frac{\partial g}{\partial x_j}D_i$;
4) $[gD_i^2, h] = 2g\frac{\partial h}{\partial x_i}D_i + g\frac{\partial^2 h}{\partial x_i^2}$;
5) $[D_i^2, hD_j] = 2\frac{\partial h}{\partial x_i}D_iD_j + 2h\omega_{ji}D_i + \frac{\partial^2 h}{\partial x_i^2}D_j + h\frac{\partial \omega_{ji}}{\partial x_i}$;
6) $[D_i^2, D_j^2] = 4\omega_{ji}D_jD_i + 2\frac{\partial \omega_{ji}}{\partial x_j}D_i + 2\frac{\partial \omega_{ji}}{\partial x_i}D_j + \frac{\partial^2 \omega_{ji}}{\partial x_i \partial x_j} + 2\omega_{ji}^2$;
7) $[D_k^2, hD_iD_j] = 2\frac{\partial h}{\partial x_k}D_kD_iD_j + 2h\omega_{jk}D_iD_k + 2h\omega_{ik}D_kD_j + \frac{\partial^2 h}{\partial x_k^2}D_iD_j + 2h\frac{\partial \omega_{jk}}{\partial x_i}D_k + h\frac{\partial \omega_{jk}}{\partial x_k}D_i + h\frac{\partial \omega_{ik}}{\partial x_k}D_j + h\frac{\partial^2 \omega_{jk}}{\partial x_i \partial x_k}$;
8) $[gD_iD_j, hD_k] = g\frac{\partial h}{\partial x_j}D_iD_k + g\frac{\partial h}{\partial x_i}D_jD_k - h\frac{\partial g}{\partial x_k}D_iD_j + gh\omega_{kj}D_i + gh\omega_{ki}D_j + g\frac{\partial^2 h}{\partial x_i \partial x_j}D_k + gh\frac{\partial \omega_{kj}}{\partial x_i}$.

The following technical results are frequently used in our paper.

Lemma 2.14 (see [21]) 1) $[L_0, x_i] = D_i$;
2) $[[L_0, \phi], \phi] = |\nabla\phi|^2 = \sum_{i=1}^n (\frac{\partial \phi}{\partial x_i})^2$;
3) $[L_0, D_j] = \sum_{i=1}^n \omega_{ji}D_i + \frac{1}{2}\frac{\partial \eta}{\partial x_j} + \frac{1}{2}\sum_{i=1}^n \frac{\partial \omega_{ji}}{\partial x_i}$;
4) $[L_0, x_j^2] = 2x_jD_j + 1$;

Lemma 2.15 (see [21]) *Suppose that E is finite dimensional estimation algebra. Let*

$$\begin{aligned}
K &:= const\cdot D_n^{l_1+2}+(B_1\overline{x}+const)D_1D_n^{l_1+1}+(B_2\overline{x}+const)D_2D_n^{l_1+1}\\
&\quad+\cdots+(B_{n-1}\overline{x}+const)D_{n-1}D_n^{l_1+1}+\textit{terms with lower order in } D_n,\ mod\ U_{l_1+1}\in E,\\
Z_1 &:= (B_1\overline{x}+const)D_n^{l_2+1}+\textit{terms with lower order in } D_n,\ mod\ U_{l_2}\in E,\\
Z_2 &:= (B_2\overline{x}+const)D_n^{l_2+1}+\textit{terms with lower order in } D_n,\ mod\ U_{l_2}\in E,\\
&\ \vdots\\
Z_{n-1} &:= (B_{n-1}\overline{x}+const)D_n^{l_2+1}+\textit{terms with lower order in } D_n,\ mod\ U_{l_2}\in E,
\end{aligned} \tag{10}$$

where const means constant number, $\overline{x}=(x_1,x_2,\cdots,x_{n-1})^{\mathrm{T}}\in\mathbb{R}^{n-1}$. $l_1,l_2\geq 0$ are nonnegative integers. $B_i\in\mathbb{R}^{1\times(n-1)}$ are constant vector for $1\leq i\leq n-1$. Define matrix $B=(B_{ij})$ as below:

$$B:=\begin{pmatrix} B_1\\ B_2\\ \vdots\\ B_{n-1}\end{pmatrix}. \tag{11}$$

If B is a symmetric matrix, then B must be equal to 0.

Remark 2.16 Notice that Lemma 2.15 holds only under the assumption of finite dimensionality of E without any linear rank condition. It can be applied in any nonmaximal rank estimation algebra. This lemma is an important tool in studying the structure of estimation algebra.

Notice that each x_i plays the same role in Lemma 2.15. Therefore, we can simply replace index n by any α, $\overline{x}$ by $(x_1,\cdots,x_{\alpha-1},x_{\alpha+1},\cdots,x_n)$, the same results still holds.

Lemma 2.17 (see [21]) *Suppose that E is finite dimensional estimation algebra and $1\leq\alpha\leq n$,*

$$\begin{aligned}
K &:= const\cdot D_\alpha^{l_1+2}+(B_1\overline{x}+const)D_1D_\alpha^{l_1+1}+\cdots+(B_{\alpha-1}\overline{x}+const)D_{\alpha-1}D_\alpha^{l_1+1}\\
&\quad+(B_{\alpha+1}\overline{x}+const)D_{\alpha+1}D_\alpha^{l_1+1}+(B_n\overline{x}+const)D_nD_\alpha^{l_1+1}\\
&\quad+\textit{terms with lower order in } D_\alpha,\ mod\ U_{l_1+1}\in E,\\
Z_1 &:= (B_1\overline{x}+const)D_\alpha^{l_2+1}+\textit{terms with lower order in } D_\alpha,\ mod\ U_{l_2}\in E,\\
&\ \vdots\\
Z_{\alpha-1} &:= (B_{\alpha-1}\overline{x}+const)D_\alpha^{l_2+1}+\textit{terms with lower order in } D_\alpha,\ mod\ U_{l_2}\in E,\\
&\ \vdots\\
Z_{\alpha+1} &:= (B_{\alpha+1}\overline{x}+const)D_\alpha^{l_2+1}+\textit{terms with lower order in } D_\alpha,\ mod\ U_{l_2}\in E,\\
&\ \vdots\\
Z_n &:= (B_n\overline{x}+const)D_\alpha^{l_2+1}+\textit{terms with lower order in } D_\alpha,\ mod\ U_{l_2}\in E,
\end{aligned} \tag{12}$$

where const means constant number, $\overline{x} = (x_1, \cdots, x_{\alpha-1}, x_{\alpha+1}, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^{n-1}$. $l_1, l_2 \geq 0$ *are nonnegative integers.* $B_i \in \mathbb{R}^{1\times(n-1)}$ *are constant row vectors. Define block matrix* B *as below:*

$$B := \begin{pmatrix} B_1 \\ \vdots \\ B_{\alpha-1} \\ \vdots \\ B_{\alpha+1} \\ \vdots \\ B_{n-1} \end{pmatrix}. \tag{13}$$

If B *is a symmetric matrix, then* B *must be equal to* 0.

Theorem 2.18 (see [21]) *Suppose that* E *is a finite dimensional estimation algebra of state dimension* n *and linear rank* $n-1$. *If the following differential operator is contained in estimation algebra,*

$$M_0 = \alpha(x_1, x_2, \cdots, x_n)D_n, \ \ mod\ U_0 \in E, \tag{14}$$

where α *is a polynomial of* $x_1, x_2, \cdots, x_n$, *then* α *is an affine function.*

Theorem 2.19 (Linear structure of Ω[21]) *Let* E *be the finite dimensional estimation algebra with state dimension* n *and linear rank* $n-1$. *Then Wong's* Ω*-matrix has linear structure; i.e., all the entries in the* Ω*-matrix are degree* 1 *polynomials. Furthermore,* $\omega_{ij} \in P_1(x_1, \cdots, x_{n-1})$ *for* $1 \leq i, j \leq n-1$.

Theorem 2.20 (Ocone[23]) *Let* E *be a finite-dimensional estimation algebra. If a function* ϕ *is in* E, *then* ϕ *is a polynomial of degree at most two.*

Detailed proof can be found in Theorem 2[17].

Actually, Mitter[13] proposed the following conjecture that is stronger than Ocone's result.

Mitter Conjecture Let E be a finite dimensional estimation algebra. If ϕ is a function in E, then ϕ is a polynomial of degree at most 1.

Observation functions h_i's play a role as generators in estimation algebra E. Their structures directly influence the solution of nonlinear filtering system. So, it is important to prove the weak form of Mitter conjecture which states that all observation functions are affine functions.

Weak form Mitter Conjecture Let E be a finite dimensional estimation algebra. Then all observations h_i's, are affine functions.

2.4 Underdetermined Partial Differential Equation

Next we summarize the known results related to underdetermined partial differential equation. Considering the following first order partial differential equation:

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = F. \tag{15}$$

Theorem 2.21 (see [19]) *Let d and $r \le n$ be two positive integers and*

$$F(x) = \sum_{|i|\le d} a_i(x_{r+1}, \cdots, x_n) x_1^{i_1} \cdots x_r^{i_r}, \tag{16}$$

where $i = (i_1, \cdots, i_r)$ and where a_i's are smooth functions in $x_{r+1}, \cdots, x_n$. The homogeneous degree d part in $x_1, \cdots, x_r$ of F is denoted by

$$F_d = \sum_{|i|=d} a_i(x_{r+1}, \cdots, x_n) x_1^{i_1} \cdots x_r^{i_r}.$$

If there exist n numbers $b_1, \cdots, b_n$ such that $F_d(b_1, \cdots, b_n) < 0$, there are no smooth functions $f_i, 1 \le i \le n$ on $\mathbb{R}^n$ satisfying the equation (15).

2.5 Euler Operator

Euler operator technique plays an important role in structure deduction of estimation algebra. This technique is frequently used with underdetermined partial differential equation. The following partial differential equation plays an important role in our paper:

$$E_S(\zeta) + m\zeta = \gamma(x), \tag{17}$$

where $E_S := \sum_{l\in S} x_l \frac{\partial}{\partial x_l}$ denotes partial Euler operator. We shall write $E_S = E_{i_1,i_2,\cdots,i_k}$, if $S = \{i_1 < i_2 < \cdots < i_k\}$. If $(i_1, i_2, \cdots, i_k) = (1, 2, \cdots, k)$, we shall write $E_S = E_{1:k}$. If $S = \{1, 2, \cdots, n\}$ is the whole index dataset, $E_S = E$ is the usual Euler operator.

The result as below says that the solution ζ of the equation (17) has polynomial structure in variables $x_1, \cdots, x_l$, if $\gamma(x) \in \widetilde{P}_{x_1,\cdots,x_l}$.

Theorem 2.22 (see [19]) *Let m be a constant integer and $\zeta \in C^\infty(\mathbb{R}^n)$ such that $E_{1:l}(\zeta) + m\zeta$ is a polynomial of degree k in $x_1, \cdots, x_l$ with smooth coefficients of $x_{l+1}, \cdots, x_n$.*

1) *If $m + k + 1 > 0$, $\zeta \in \widetilde{P}_k(x_1, \cdots, x_l)$;*
2) *If $m + k + 1 \le 0$, $\zeta \in \widetilde{P}_k(x_1, \cdots, x_l)$ or $\zeta \in \widetilde{P}_{-m}(x_1, \cdots, x_l)$.*

It is trivial to directly get the following corollary. The following corollary is helpful for us to determine the degree of solution function.

Corollary 2.23 *Let m be a constant integer and $\zeta \in C^\infty(\mathbb{R}^n)$ such that $E_{1:l}(\zeta) + m\zeta$ is a polynomial of degree k in $x_1, \cdots, x_l$ with smooth coefficients of $x_{l+1}, \cdots, x_n$. Then ζ must be a polynomial of variables $x_1. \cdots, x_l$ with smooth coefficient of $x_{l+1}, \cdots, x_n$. Moreover, degree of ζ in $x_1, \cdots, x_l$ is less than or equal to* $\max\{k, -m\}$.

3 Quadratic Structure of Nonmaximal Rank Estimation Algebra

In this section, we focus on the quadratic structure of functions in nonmaximal rank estimation algebra. Our main point is to describe the quadratic part of any function in E by means of linear rank and quadratic rank. The results of this section are good for general nonmaximal rank case and will be used for the rest of our paper.

First, we make an assumption of linear rank:

Assumption 3.1 Linear rank of E is less than or equal to $n-1$, i.e., $r \leq n-1$.

It can be easily checked that $1 \in E$ since $[[L_0, x_1], x_1] = 1 \in E$. In view of the structure of linear rank, we shall characterize the homogeneous quadratic part of any function in E. Detailed proof can be found in the Appendix.

Lemma 3.2 *Let $\phi(x) \in E$ be a function in E. Then the homogeneous quadratic part of ϕ must be in a block diagonal form, i.e.,*

$$\phi^{(2)}(x) = x^{\mathrm{T}} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} x, \tag{18}$$

where A_1 and A_2 are symmetric matrices with size $r \times r$ and $(n-r)\times(n-r)$, $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$.

Lemma 3.2 implies that maximal rank quadratic polynomial $\phi \in E$ defined in Definition 2.9 must be of the following form:

$$\phi^{(2)}(x) = x^{\mathrm{T}} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} x, \tag{19}$$

and $\mathrm{rank}(A_1) + \mathrm{rank}(A_2) = k$, where k is quadratic rank defined in Definition 2.9. Under an appropriate block diagonal orthogonal transformation

$$T = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \tag{20}$$

where $U_1 \in \mathbb{R}^{r\times r}$ and $U_2 \in \mathbb{R}^{(n-r)\times(n-r)}$.

In general, any quadratic part $\phi^{(2)}$ can be diagonalized as

$$\phi^{(2)} = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_2 \end{pmatrix}, \tag{21}$$

where D_1, D_2 are diagonal matrices with non-zero diagonal elements. Theorem 3.7 in [19] proved that there exists a quadratic function $p_0 = \sum_{i=1}^{k_1} x_i^2 + \sum_{i=n-k_2+1}^{n} x_i^2 \in E$, where $k_1 + k_2 = k, k_1 \leq r$ and $k_2 \leq n-r$, using the technique of translation of variable and Vandermonde matrix. Notice that p_0 has maximal quadratic rank k in E, i.e., $\lambda(p_0) = k$. We summarize this result as the following theorem.

Theorem 3.3 *Let E be a finite-dimensional estimation algebra with linear rank r and quadratic rank k. There exists $p_0 = \sum_{i=1}^{k_1} x_i^2 + \sum_{i=n-k_2+1}^{n} x_i^2 \in E$, where $k_1 + k_2 = k, k_1 \leq r$ and $k_2 \leq n-r$.*

Remark 3.4 Notice that such orthogonal transformation (20) and translation of variable $x_i \longmapsto x_i + const$ do not change the basis of $L(E)$. In fact, in view of Lemma 2.14 1), we have that $x_j \in E \Longleftrightarrow x_j + c \in E$ where c is a constant.

Let $S := \{1, \cdots, k_1, n-k_2+1, \cdots, n\}$. We use the following conventions:
(i) if $k_1 = 0$, $S = \{n-k_2+1, \cdots, n\}$;
(ii) if $k_2 = 0$, $S = \{1, \cdots, k_1\}$;
(iii) if $k_1 = k_2 = 0$, $S = \emptyset$.

The following lemma describes the structure of homogeneous degree 2 part of any function in E. The proof will appear in the Appendix.

Lemma 3.5 *If $p \in E$ is a quadratic function, then homogeneous quadratic part $p^{(2)}(x)$ is indepedent of x_j for $j \notin S$, i.e., $\frac{\partial p^{(2)}}{\partial x_j}(x) = 0$ for $j = k_1+1, \cdots, n-k_2$.*

For the sake of exploring more information of quadratic polynomials in E, we consider quadratic polynomials with least quadratic rank. Let $p_0 = \sum_{i=1}^{k_1} x_i^2 + \sum_{i=n-k_2+1}^{n} x_i^2 \in E$ have maximal quadratic rank and $p_1 \in E$ have least quadratic rank. In view of (21), we can write

$$p_1 = \sum_{1 \le i,j \le k_1} p_{1ij} x_i x_j + \sum_{n-k_2+1 \le i,j \le n} p_{1ij} x_i x_j + pol_1(x) \in E. \tag{22}$$

By block orthogonal transformation,

$$T = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \tag{23}$$

where

$$U_1 = \begin{pmatrix} \widetilde{U}_1 & 0 \\ 0 & I_{r-k_1} \end{pmatrix}, \quad U_2 = \begin{pmatrix} I_{n-r-k_2} & 0 \\ 0 & \widetilde{U}_2 \end{pmatrix}, \tag{24}$$

we obtain

$$p_1 = \sum_{1 \le i \le \widetilde{k}_1} d_i x_i^2 + \sum_{n-\widetilde{k}_2+1 \le i \le n} d_i x_i^2 + pol_1(x) \in E, \tag{25}$$

where $\widetilde{k}_1 \le k_1$ and $\widetilde{k}_2 \le k_2$ and $d_i \ne 0$.

Remark 3.6 It is important to notice that this orthogonal transformation (23) keeps structure of linear rank and maximal rank quadratic polynomial unchange.

Next we can assume

$$p_1 = \sum_{1 \le i \le \widetilde{k}_1} d_i x_i^2 + \sum_{n-\widetilde{k}_2+1 \le i \le n} d_i x_i^2 + \sum_{r+1 \le i \le n} e_i x_i \in E, \tag{26}$$

We get the following Euler operator from maximal rank quadratic polynomial.

$$Z := \frac{1}{2}([L_0, p_0] - k) = \left(\sum_{i=1}^{k_1} + \sum_{i=n-k_2+1}^{n} \right) x_i D_i \in E. \tag{27}$$

Since $[Z, \phi] = E_S(\phi)$ for any smooth function ϕ, Z will play the same role as Euler operator E_S. Next

$$E_S(p_1) - p_1 = \left(\sum_{1 \le i \le \widetilde{k}_1} + \sum_{n-\widetilde{k}_2+1 \le i \le n} \right) d_i x_i^2 - \sum_{r+1 \le i \le n-k_2} e_i x_i \in E. \tag{28}$$

By the Vandermonde matrix technique, it implies

$$p_1 := \Big(\sum_{1\le i\le \widetilde{k}_1} + \sum_{n-\widetilde{k}_2+1\le i\le n}\Big)x_i^2 \in E, \tag{29}$$

where we still use p_1 to denote the least rank quadratic polynomial.

Next we shall see the impact of minimal rank quadratic polynomial on other quadratic polynomials in E. Define index set $S_1 := \{1, \cdots, \widetilde{k}_1, n-\widetilde{k}_2+1, \cdots, n\}$ and set $Q := \mathrm{span}\{x_ix_j : i, j \in S_1\}$.

Theorem 3.7 *Assume that* $p_1 := (\sum_{1\le i\le\widetilde{k}_1} + \sum_{n-\widetilde{k}_2+1\le i\le n})x_i^2 \in E$ *has minimal quadratic rank. For any quadratic function* $p \in E$*, if* $p^{(2)}(x) \in Q$*, then* $p^{(2)}(x) = \lambda p_1$ *for some constant* λ.

4 Finite Dimensional Estimation Algebra: Linear Observation

4.1 Basic Techniques

Notice that results in this subsection are independent of rank condition. For fixed index (i, j), ω_{ij} must be a constant number when some specific functions exist in E. The proofs will appear in the Appendix.

Lemma 4.1 *If* $x_i^2 \in E$ *and* $x_j^2 \in E$ *where* $1 \le i < j \le n$*, then* ω_{ij} *is a constant.*

Lemma 4.2 *For* $1 \le i < j \le n$*, assume* $\omega_{ji} = c_ix_i + c_jx_j + c_0 \in P_1(x_i, x_j)$ *where* c_i, c_j, c_0 *are constant coefficients. If the following operators are contained in* E*,*

$$N_0 := x_i^2 + x_j^2 \in E, \quad N_1 := c_ix_ix_j + c_jx_j^2 \in E, \quad N_2 := c_ix_i^2 + c_jx_ix_j \in E, \tag{30}$$

then ω_{ij} *is equal to* c_0.

4.2 Partially Constant Structure of Ω

It has been proven that constant structure of Ω implies Mitter conjecture, in particular the linear structure of observations, under the maximal rank setting. In this subsection, we consider finite dimensional estimation algebra on arbitrary state dimension n and linear rank $r = n-1$. In the non-maximal rank case, we shall prove the partial constant structure of Ω. This certain part constant structure of Ω depends on maximal rank quadratic polynomial. Using the partial constant structure of Ω, we shall establish a sufficient condition for observation function $\{h_i(x)\}$ be affine functions.

It is critical to notice that the structure of η will influence structure of $\{h_i(x)\}$ largely. In the next two lemmas, we shall use tools of underdetermined partial differential equations to obtain restriction from η to $\{h_i(x)\}$. Main technique used in this section is Euler operator method and underdetermined partial differential equation combined with maximal rank quadratic polynomial. The proof can be found in the Appendix.

Lemma 4.3 *Let* Q *be a subset of index* $\{1, 2, \cdots, n\}$ *and* $\eta \in \widetilde{P}_3(x_s, s \in Q)$ *be a degree at most* 3 *polynomial in variables* $x_s, s \in Q$ *with coefficients smooth function in* $x_j, j \notin Q$*. Then* $\{h_i^{(2)}\}$ *do not contain terms* x_px_q *for* $\forall p, q \in Q$.

By combining maximal rank quadratic polynomial and Lemma 4.3, we obtain the following result. The proof will appear in the Appendix.

Lemma 4.4 *Let $p_0 = \sum_{l\in S} x_l^2 \in E$ be a degree 2 polynomial that has maximal quadratic rank. If $\eta \in \widetilde{P}_3(x_l, l \in S)$ is a degree at most 3 polynomial in $x_l, l \in S$ with coefficient smooth function in $x_s, s \notin S$, then observations h_i's are affine functions in x.*

Finally based on technical Lemmas 4.3 and 4.4, we can demonstrate that given maximal rank quadratic polynomial, if Ω has corresponding partially constant structure, then $\{h_i(x)\}$ can be proven affine.

Theorem 4.5 (Partially constant structure of Ω) *Let E be a finite dimensional estimation algebra with dimension n and linear rank $r = n - 1$. Let $p_0 = \sum_{l\in S} x_l^2 \in E$ be a degree 2 polynomial that has maximal quadratic rank. If ω_{ij}'s are constant numbers for $i \in S$ or $j \in S$, then observations h_i's are affine functions in x.*

Remark 4.6 Theorem 4.5 is a general result and points out a relaxed sufficient condition to prove linear structure of observations. This is the new breakthrough of our paper since we no longer require the whole constant structure of Ω like the proof of maximal rank case.

5 Finite Dimensional Estimation Algebra with State Dimension 4 and Rank 3

In this section, we shall focus on finite dimensional estimation algebra with state dimension 4 and linear rank 3. We shall establish a sufficient condition of partially constant structure of Ω for state dimension 4 and rank 3. The main point is to classify estimation algebra by using quadratic rank. Under different maximal rank quadratic polynomials, we shall demonstrate that Ω possesses corresponding partially constant structures. Main technique used here is construction of infinite sequence. Notice that the result of this section is built on our previously established result of linear structure of Ω. The result in this section will provide some insights for case of higher or arbitrary state dimension.

We first classify maximal rank quadratic polynomials for state dimension 4 and rank 3 by using quadratic rank. The following classification is obtained by using Theorem 3.3.

Lemma 5.1 *Let k be quadratic rank of estimation algebra with state dimension 4 and rank 3. Maximal rank quadratic polynomials can be classified to following 7 cases:*

$k = 1$:
(I) $p_0 = x_1^2 \in E$;
(II) $p_0 = x_4^2 \in E$.
$k = 2$:
(III) $p_0 = x_1^2 + x_2^2 \in E$;
(IV) $p_0 = x_1^2 + x_4^2 \in E$.
$k = 3$:
(V) $p_0 = x_1^2 + x_2^2 + x_3^2 \in E$;
(VI) $p_0 = x_1^2 + x_2^2 + x_4^2 \in E$.

$k=4$:

(VII) $p_0 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \in E$.

In the following subsections, we will consider different cases of maximal rank quadratic polynomials respectively. In view of Theorem 2.19, Ω has linear structure for our case state dimension 4 and rank 3.

In each case of Lemma 5.1, we shall start with linear representation of entries of Ω. By applying Theorem 2.19 to $n=4$ case, we get

$$\begin{cases} \omega_{12}, \omega_{13}, \omega_{23} \in P_1(x_1, x_2, x_3), \\ \omega_{14}, \omega_{24}, \omega_{34} \in P_1(x_1, x_2, x_3, x_4). \end{cases} \tag{31}$$

Based on the linear structure representation, the corresponding partially constant structure will be proven. More precisely, we shall use quadratic rank and structure to simplify part of coefficients of (ω_{ij}). Next we construct infinite sequence to prove partially constant structure of $\{\omega_{ij}\}$ for $i \in S$ or $j \in S$.

Finally, we apply Theorem 4.5 in each case to obtain the affine structure of $\{h_i(x)\}$.

5.1 Case (I): Quadratic Rank $k=1$ and $p_0 = x_1^2$

In this subsection, we discuss first case of Lemma 5.1: Quadratic rank $k=1$ and $p_0 = x_1^2 \in E$. Our goal is to prove that the three entries $\omega_{12}, \omega_{13}, \omega_{14}$ are constant numbers. With this statement, Theorem 4.5 will imply that observation terms are affine functions.

Based on affine structure of Ω of $n=4$ case (see (31)), we can assume

$$\begin{cases} \omega_{21} = \widetilde{a}_1 x_1 + \widetilde{a}_2 x_2 + \widetilde{a}_3 x_3 + \widetilde{a}_0, \\ \omega_{31} = \widetilde{b}_1 x_1 + \widetilde{b}_2 x_2 + \widetilde{b}_3 x_3 + \widetilde{b}_0, \\ \omega_{41} = \widetilde{c}_1 x_1 + \widetilde{c}_2 x_2 + \widetilde{c}_3 x_3 + \widetilde{c}_4 x_4 + \widetilde{c}_0, \end{cases} \tag{32}$$

where $\{\widetilde{a}_i\}, \{\widetilde{b}_i\}, \{\widetilde{c}_i\}$ are constant numbers.

Lemma 5.2 *If quadratic rank $k=1$ and the corresponding maximal rank quadratic polynomial is $p_0 = x_1^2 \in E$, then $\widetilde{a}_2 = \widetilde{a}_3 = \widetilde{b}_2 = \widetilde{b}_3 = 0$.*

Theorem 5.3 *If quadratic rank $k=1$ and the corresponding maximal rank quadratic polynomial is $p_0 = x_1^2 \in E$, then $\{\omega_{1i}\}$ are constant numbers for $2 \le i \le 4$.*

5.2 Case (II): Quadratic Rank $k=1$ and $p_0 = x_4^2$

In this subsection, we need to prove that $\omega_{14}, \omega_{24}, \omega_{34}$ are constant numbers.

First, we start with affine structure of Ω for $n=4$ case.

$$\begin{cases} \omega_{14} = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + a_0, \\ \omega_{24} = b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4 + b_0, \\ \omega_{34} = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 + c_0. \end{cases} \tag{33}$$

Lemma 5.4 *Assume quadratic rank $k=1$ and the corresponding maximal rank quadratic polynomial $p_0 = x_4^2 \in E$. Then $\omega_{14}, \omega_{24}, \omega_{34}$ only depend on variable x_4.*

Theorem 5.5 *If quadratic rank is one and the corresponding maximal rank quadratic polynomial is $p_0 = x_4^2 \in E$, then ω_{i4}'s are constant numbers for $1 \leq i \leq 3$.*

5.3 Case (III): Quadratic Rank $k = 2$ and $p_0 = x_1^2 + x_2^2$

In this subsection, we shall demonstrate that $\omega_{12}, \omega_{13}, \omega_{14}, \omega_{23}, \omega_{24}$ are constants.

Lemma 5.6 *Assume quadratic rank $k = 2$ and the corresponding maximal rank quadratic polynomial $p_0 \in x_1^2 + x_2^2 \in E$. Then ω_{12} is a constant number and ω_{13}, ω_{23} are degree at most one polynomial in x_1, x_2, i.e., $\omega_{13}, \omega_{23} \in P_1(x_1, x_2)$.*

Theorem 5.7 *If quadratic rank is one and the corresponding maximal rank quadratic polynomial is $p_0 = x_1^2 + x_2^2 \in E$, then $\{\omega_{1i}\}$ for $2 \leq i \leq 4$ and $\{\omega_{2j}\}$ for $3 \leq j \leq 4$ are constants.*

5.4 Case (IV): Quadratic Rank $k = 2$ and $p_0 = x_1^2 + x_4^2$

In this subsection, we shall prove that $\omega_{12}, \omega_{13}, \omega_{14}, \omega_{24}, \omega_{34}$ are constants.

Lemma 5.8 *Assume quadratic rank $k = 2$ and the corresponding maximal rank quadratic polynomial $p_0 \in x_1^2 + x_4^2 \in E$. Then $\omega_{4i} \in P_1(x_4)$ for $1 \leq i \leq 3$ and $\omega_{1j} \in P_1(x_1)$ for $2 \leq j \leq 3$.*

Next we can assume that the following affine structure:

$$\begin{cases} \omega_{21} = ax_1 + a_0, \\ \omega_{31} = bx_1 + b_0, \\ \omega_{14} = cx_4 + c_0, \\ \omega_{24} = dx_4 + d_0, \\ \omega_{34} = ex_4 + e_0. \end{cases} \tag{34}$$

Lemma 5.9 *For Case* (IV), *$\omega_{14} = 0$ is a constant.*

Lemma 5.10 *For Case* (IV), *ω_{1i} for $2 \leq i \leq 3$ and ω_{j4} for $2 \leq j \leq 3$ are constants.*

Theorem 5.11 *For Case* (IV), *ω_{1i} for $2 \leq i \leq 4$ and ω_{j4} for $2 \leq j \leq 3$ are constants.*

5.5 Case (V): Quadratic Rank $k = 3$ and $p_0 = x_1^2 + x_2^2 + x_3^2$

In this subsection, our aim is to prove that Ω is a constant matrix. In exploring structure of $\{\omega_{ij}\}$ for $i, j \in S$, we shall utilize tool of least quadratic rank structure which will provide more informtion of quadratic functions in E.

Lemma 5.12 *For Case* (V), *$\omega_{21}, \omega_{31}, \omega_{32}$ are constant numbers.*

Lemma 5.13 *For Case* (V), *$\omega_{14}, \omega_{24}, \omega_{34}$ are constant numbers.*

Theorem 5.14 *For Case* (V), *Ω is a constant matrix.*

5.6 Case (VI): Quadratic Rank $k = 3$ and $p_0 = x_1^2 + x_2^2 + x_4^2$

In this subsection, our goal is to prove Ω is a constant matrix.

Lemma 5.15 *For Case* (VI), *$\omega_{4i} \in P_1(x_4)$ for $1 \leq i \leq 3$ and $\omega_{21}, \omega_{31}, \omega_{23} \in P_1(x_1, x_2)$.*

Based on the above lemma, we can assume affine expression of Ω as

$$\begin{cases} \omega_{21} = a_1x_1 + a_2x_2 + a_0, \\ \omega_{31} = b_1x_1 + b_2x_2 + b_0, \\ \omega_{32} = c_1x_1 + c_2x_2 + c_0, \\ \omega_{34} = dx_4 + d_0, \\ \omega_{41} = ex_4 + e_0, \\ \omega_{42} = lx_4 + l_0. \end{cases} \tag{35}$$

Lemma 5.16 *For Case* (VI), *ω_{ij}'s are constants for $i, j \in \{1, 2, 4\}$. Moreover, $\omega_{41} = \omega_{42} = 0$.*

Theorem 5.17 *For Case* (VI), *Ω is a constant matrix. Furthermore, $\omega_{34} = 0$.*

5.7 Case (VII): Quadratic Rank $k = 4$ and $p_0 = x_1^2 + x_2^2 + x_3^2 + x_4^2$

In this subsection, our goal is to prove Ω is a constant matrix. First we do some basic computations

$$K_0 := \frac{1}{2}[L_0, p_0] - 2 = \sum_{i=1}^{4} x_i D_i \in E \tag{36}$$

and

$$\begin{cases} A_1 :=[D_1, K_0] - D_1 = \sum_{i\neq 1} x_i\omega_{i1} \in E, \\ A_2 :=[D_2, K_0] - D_2 = \sum_{i\neq 2} x_i\omega_{i2} \in E, \\ A_3 :=[D_3, K_0] - D_3 = \sum_{i\neq 3} x_i\omega_{i3} \in E. \end{cases} \tag{37}$$

By applying Lemma 3.2, we get the following result.

Lemma 5.18 *For Case* (VII), *$\omega_{4i} \in P_1(x_4)$ for $1 \le i \le 3$.*

In the following, we can assume affine expression of Ω,

$$\begin{cases} \omega_{21} = a_1x_1 + a_2x_2 + a_3x_3 + a_0, \\ \omega_{31} = b_1x_1 + b_2x_2 + b_3x_3 + b_0, \\ \omega_{32} = c_1x_1 + c_2x_2 + c_3x_3 + c_0, \\ \omega_{41} = l_1x_4 + l_0, \\ \omega_{42} = m_1x_4 + m_0, \\ \omega_{43} = n_1x_4 + n_0. \end{cases} \tag{38}$$

Then

$$\begin{cases} A_1^{(2)} = a_1x_1x_2 + a_2x_2^2 + b_1x_1x_3 + b_3x_3^2 + (a_3 + b_2)x_2x_3 + l_1x_4^2, \\ A_2^{(2)} = -a_1x_1^2 - a_2x_1x_2 + (c_1 - a_3)x_1x_3 + c_2x_2x_3 + c_3x_3^2 + m_1x_4^2, \\ A_3^{(2)} = -b_1x_1^2 - (b_2 + c_1)x_1x_2 - b_3x_1x_3 - c_2x_2^2 - c_3x_2x_3 + n_1x_4^2. \end{cases} \tag{39}$$

Next we demonstrate constant structure of Ω.

Theorem 5.19 *For Case* (VII), *Ω is a constant matrix. Furthermore,* $\omega_{4i} = 0$ *for* $1 \leq i \leq 3$.

In summary, we conclude all results obtained from Cases (I) to (VII) in Table 1. In particular, we have proven that the weak form Mitter conjecture holds for $n = 4$ case.

Table 1 Summary of partially constant structure of Ω. $\lambda(E)$ denotes quadratic rank of E. k_1, k_2 are defined in the maximal rank quadratic polynomial in Theorem 3.3. const denotes a constant number. Other $*$ terms appeared in the table denote the irrelevant terms with partially constant structure

$\lambda(E)$	k_1	k_2	ω_{21}	ω_{31}	ω_{32}	ω_{41}	ω_{42}	ω_{43}
1	1	0	const	const	$*$	const	$*$	$*$
	0	1	$*$	$*$	$*$	0	0	0
2	2	0	const	const	const	const	const	$*$
	1	1	const	const	$*$	0	0	0
3	3	0	const	const	const	const	const	const
	2	1	const	const	const	0	0	0
4	3	1	const	const	const	0	0	0

Theorem 5.20 *Let E be a finite dimensional estimation algebra with state dimension* 4 *and linear rank* 3. *Then all observations h_i's are affine functions.*

Finally, we provide a concrete example of finite dimensional filter in which weak Mitter conjecture holds.

$$\begin{cases} dx_1 = \left(x_1 + \dfrac{e^{x_1}}{e^{x^1} + e^{x^2}}\right)dt + dw_1, \\ dx_2 = \left(x_1 + \dfrac{e^{x_2}}{e^{x^1} + e^{x^2}}\right)dt + dw_2, \\ dx_3 = \left(x_3 + \dfrac{e^{x_3}}{e^{x^3} + e^{x^4}}\right)dt + dw_3, \\ dx_4 = \left(x_3 + \dfrac{e^{x_4}}{e^{x^3} + e^{x^4}}\right)dt + dw_4, \\ dy_i = x_i dt + dv_i, \quad 1 \leq i \leq 3. \end{cases} \tag{40}$$

It is easy to obtain

$$\Omega = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{41}$$

is a constant matrix. Further it can be verified that $\eta = 3x_1^2 + x_2^2 + 3x_3^2 + 2x_1 + 2x_3 + 4$ is a quadratic polynomial. By direct computation, estimation algebra associated with such example is of dimension 9 and contains the following basis:

$$E = \text{span}\{1, x_1, x_2, x_3, D_1, D_2, D_3, D_4, L_0\}. \tag{42}$$

6 Conclusion

In this paper, we mainly focus on weak Mitter conjecture for nonmaximal rank estimation algebra. First we establish the concept of partially constant structure of Ω, which lead us to prove weak Mitter conjecture. Our second main contribution is to verify partially constant condition for $n = 4$ case. In the whole proof, we develop a lot of algebraic techniques which works not only for maximal rank case, but also for nonmaximal rank case. For example, we extend the Euler operator technique, and quadratic structure theory to nonmaximal rank case. In our future work, we shall prove the weak Mitter conjecture for higher dimension n.

Conflict of Interest

The authors declare no conflict of interest.

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Appendix

In this section, we will provide detailed proofs about results in this paper.

A.1 Proof of Lemma 3.2

Due to linear rank property and Ocone's lemma, we can assume any function $\phi \in E$ has the form:

$$\phi(x) = x^{\mathrm{T}} A x + [0, p^{\mathrm{T}}] x \in E, \tag{A.1}$$

where $A = (A_{ij}) \in \mathbb{R}^{n \times n}$ is a symmetric matrix, $p \in \mathbb{R}^{n-r}$ is a vector. Next we consider first order differential operator $[L_0, x_i] = D_i$ for $1 \le i \le r$. Then we consider bracket between D_i and ϕ:

$$[D_i, \phi(x)] = \frac{\partial \phi(x)}{\partial x_i} = \sum_{j=1}^{n} A_{ij} x_j \in E. \tag{A.2}$$

Then again by linear rank property, we get $A_{ij} = 0$ for $1 \le i \le r, r+1 \le j \le n$. ∎

A.2 Proof of Lemma 3.5

Assume $\frac{\partial p^{(2)}}{\partial x_j} \neq 0$ for some $k_1 + 1 \le j \le n - k_2$. We shall derive a contradiction.

First

$$p_0 = x^{\mathrm{T}} \begin{pmatrix} I_{k_1 \times k_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{k_2 \times k_2} \end{pmatrix} x \tag{A.3}$$

and

$$p^{(2)} = x^{\mathrm{T}} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^{\mathrm{T}} & A_{22} & A_{23} \\ A_{13}^{\mathrm{T}} & A_{23}^{\mathrm{T}} & A_{33} \end{pmatrix} x, \tag{A.4}$$

where $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$, $A_{11} \in R^{k_1 \times k_1}, A_{22} \in R^{(n-k) \times (n-k)}, A_{33} \in R^{k_2 \times k_2}$ are symmetric matrices and $A_{12} \in R^{k_1 \times (n-k)}, A_{13} \in R^{k_1 \times k_2}, A_{23} \in R^{(n-k) \times k_2}$.

Next we consider $tp_0 + p$. Since p_0 and p are in E, we have $tp_0 + p \in E$.

$$(tp_0 + p)^{(2)} = tp_0 + p^{(2)} = x^{\mathrm{T}} \begin{pmatrix} tI + A_{11} & A_{12} & A_{13} \\ A_{12}^{\mathrm{T}} & A_{22} & A_{23} \\ A_{13}^{\mathrm{T}} & A_{23}^{\mathrm{T}} & tI + A_{33} \end{pmatrix} x. \tag{A.5}$$

Then we have

$$\begin{aligned} \operatorname{rank}(tp_0 + p) &= \operatorname{rank} \begin{pmatrix} tI + A_{11} & A_{12} & A_{13} \\ A_{12}^{\mathrm{T}} & A_{22} & A_{23} \\ A_{13}^{\mathrm{T}} & A_{23}^{\mathrm{T}} & tI + A_{33} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} tI + A_{11} & A_{13} & A_{12} \\ A_{13}^{\mathrm{T}} & tI + A_{33} & A_{23}^{\mathrm{T}} \\ A_{12}^{\mathrm{T}} & A_{23} & A_{22} \end{pmatrix} \\ &= \operatorname{rank} \left(\begin{array}{c|c} tI + \begin{pmatrix} A_{11} & A_{13} \\ A_{13}^{\mathrm{T}} & A_{33} \end{pmatrix} & \begin{matrix} A_{12} \\ A_{23}^{\mathrm{T}} \end{matrix} \\ \hline \begin{matrix} A_{12}^{\mathrm{T}} & A_{23} \end{matrix} & A_{22} \end{array} \right). \end{aligned} \tag{A.6}$$

Since $\frac{\partial p^{(2)}}{\partial x_j} \neq 0$ for some $k_1+1 \leq j \leq n-k_2$, then A_{12}, A_{23}, A_{22} are not all zero. By an elementary matrix transformation, we always can put a nonzero element in A_{12}, A_{23}, A_{22} into position (i,j) of A, where $i = k+1, 1 \leq j \leq k+1$ or $j = k+1, 1 \leq i \leq k+1$. Then we can derive

$$\operatorname{rank}(tp_0 + p) \geq \operatorname{rank}\left(\begin{array}{c|c} tI_k + \begin{pmatrix} A_{11} & A_{13} \\ A_{13}^{\mathrm{T}} & A_{33} \end{pmatrix} & c \\ \hline c^{\mathrm{T}} & b \end{array}\right), \tag{A.7}$$

where $c \in R^{k\times 1}, b \in R$ and b, c are not both zero. Next we denote

$$\widetilde{A} := tI_k + \begin{pmatrix} A_{11} & A_{13} \\ A_{13}^{\mathrm{T}} & A_{33} \end{pmatrix}. \tag{A.8}$$

Since $\left(\begin{smallmatrix} A_{11} & A_{13} \\ A_{13}^{\mathrm{T}} & A_{33} \end{smallmatrix}\right)$ is a real symmetric matrix, it has orthogonal diagonalized decomposition

$$\begin{pmatrix} A_{11} & A_{13} \\ A_{13}^{\mathrm{T}} & A_{33} \end{pmatrix} = U \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k \end{pmatrix} U^{\mathrm{T}} := U\Lambda U^{\mathrm{T}}, \tag{A.9}$$

where $U \in R^{k\times k}$ is an orthogonal matrix and $\lambda_1, \cdots, \lambda_k$ are eigenvalues. Then

$$\begin{pmatrix} U^{\mathrm{T}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \widetilde{A} & c \\ c^{\mathrm{T}} & b \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} tI + \Lambda & U^{\mathrm{T}}c \\ c^{\mathrm{T}}U & b \end{pmatrix}. \tag{A.10}$$

Next we denote $U^{\mathrm{T}}c = (\widetilde{c}_1, \cdots, \widetilde{c}_k)^{\mathrm{T}}$. Since b, c are not both zero, then $b, \widetilde{c}_1, \cdots, \widetilde{c}_k$ are not all zero. It is easy to see that for large enough t,

$$\operatorname{rank}\begin{pmatrix} tI + \Lambda & U^{\mathrm{T}}c \\ c^{\mathrm{T}}U & b \end{pmatrix} = k + \operatorname{rank}\left(b - \sum_{i=1}^{k} \frac{\widetilde{c}_i^2}{t+\lambda_i}\right) = k+1, \tag{A.11}$$

where the first equation comes from rank formula of block matrix.

Then

$$\operatorname{rank}(tp_0 + p) \geq \operatorname{rank}\begin{pmatrix} \widetilde{A} & c \\ c^{\mathrm{T}} & b \end{pmatrix} = \operatorname{rank}\begin{pmatrix} tI + \Lambda & U^{\mathrm{T}}c \\ c^{\mathrm{T}}U & b \end{pmatrix} = k+1. \tag{A.12}$$

This is contradictory to that p_0 has greatest quadratic rank k in E. Then $\frac{\partial p^{(2)}}{\partial x_j} = 0$ for $j = k_1+1, \cdots, n-k_2$ holds. ∎

A.3 Proof of Lemma 4.1

It can be obtained that

$$\frac{1}{4}[[L_0, x_i^2], [L_0, x_j^2]] = x_i x_j \omega_{ji} \in E. \tag{A.13}$$

Ocone's lemma implies ω_{ji} is a constant. ∎

A.4 Proof of Theorem 3.7

First

$$p^{(2)}(x) = \sum_{p,q\in S_1} a_{pq}x_p x_q = \widetilde{x}^{\mathrm{T}} A\widetilde{x}, \tag{A.14}$$

where $A = (a_{pq}) \in \mathbb{R}^{(\widetilde{k}_1+\widetilde{k}_2)\times(\widetilde{k}_1+\widetilde{k}_2)}$ is a symmetric matrix and

$$\widetilde{x} = (x_1, \cdots, x_{\widetilde{k}_1}, x_{n-\widetilde{k}_2+1}, \cdots, x_n)^{\mathrm{T}}.$$

If $p^{(2)} \neq \lambda p_1$ for any λ, then

$$\mathrm{rank}(p - \lambda p_1) = \mathrm{rank}(A - \lambda I) > 0, \tag{A.15}$$

for any λ. If we pick $\lambda = \lambda_1$ is an eigenvalue of A, then matrix $A-\lambda_1 I$ is not of full rank which leads to $\mathrm{rank}(A - \lambda_1 I) < \widetilde{k}_1 + \widetilde{k}_2$. It follows that $p - \lambda_1 p_1$ has a lower positive rank than p_1. A contradiction! ▮

A.5 Proof of Lemma 4.2

First we calculate

$$\mathrm{Det}\begin{bmatrix} 1 & 1 & 0 \\ 0 & c_j & c_i \\ c_i & 0 & c_j \end{bmatrix} = c_i^2 + c_j^2. \tag{A.16}$$

We claim $c_i = c_j = 0$. Otherwise, previous determinant is nonzero which implies $x_i^2 \in E, x_j^2 \in E$. Lemma 4.1 leads to ω_{ij} is a constant, which is a contradiction. ▮

A.6 Proof of Lemma 4.3

Based on definition of η, we have

$$\sum_{i=1}^n f_i^2 + \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} = \eta - \sum_{i=1}^m h_i^2. \tag{A.17}$$

Next we define function $F(x) := \eta - \sum_{i=1}^m h_i^2$.

Without loss of generality, we assume $Q := \{1, 2, \cdots, k\}$ where $k \leq n$, and we can assume that η has the form

$$\eta = \sum_{0\leq|\alpha|\leq 3} \eta_\alpha(x_s) x_1^{\alpha_1} \cdots x_k^{\alpha_k}, \tag{A.18}$$

where $x_s := \{x_{k+1}, \cdots, x_n\}$.

Notice $\eta \in \widetilde{P}_3(x_s, s \in Q)$ and $h_i \in P_2(x)$ which implies $F(x) \in \widetilde{P}_4(x_s, s \in Q)$. Then it can be written as

$$F = \sum_{0\leq|\alpha|\leq 4} a_\alpha(x_s) x_1^{\alpha_1} \cdots x_k^{\alpha_k}. \tag{A.19}$$

We denote F_4 the homogeneous degree 4 part of F in variable $x_1, \cdots, x_k$.

If there exists h_i which contains ceratin terms of $x_p x_q$ where $p, q \in Q$. Without loss of generality, we assume that h_1 has this property and we can expand h_1 in terms of variables $x_s, s \in Q$. $h_1^{(2)}$ can be assumed as $\sum_{1\leq i\leq j\leq k} h_{ij} x_i x_j \not\equiv 0$.

Then

$$F_4 = -\sum_{i=1}^{m}(h_i^2)^{(4)} \leq -(h_1^2)^{(4)} = -(h_1^{(2)})^2 = -\Big(\sum_{1\leq i\leq j\leq k} h_{ij}x_ix_j\Big)^2. \tag{A.20}$$

That means there exists $(b_1, \cdots, b_n)$ such that $F_4(b_1, \cdots, b_n) < 0$. By Theorem 2.21, there does not exist smooth functions $f_1, \cdots, f_n$ satisfying the equation (A.17), which yields a contradiction. ∎

A.7 Proof of Lemma 4.4

By Lemma 4.3, $\eta \in \widetilde{P}_2(x_l, l \in S)$ will yield that $\{h_i^{(2)}\}$ do not contain terms x_px_q for any $p, q \in S$, i.e.,

$$\frac{\partial^2 h_i^{(2)}}{\partial x_i \partial x_j} = 0, \quad \text{for } \forall p, q \in S. \tag{A.21}$$

On the other hand, by considering the structure of maximal quadratic rank polynomial p_0, Lemma 3.5 implies that $h_1^{(2)}$ only depends on $x_j, j \in S$. That means $h_1^{(2)} \equiv 0$ and observations are all affine functions in x. ∎

A.8 Proof of Theorem 4.5

Recall that the index set $S := \{1, \cdots, k_1, n-k_2+1, \cdots, n\}$ of maximal rank quadratic polynomial p_0. And $|S| = k = k_1 + k_2$. Based on whether set S contains index n, there are two cases: Case [1]. $n \in S$; Case [2]. $n \notin S$. For the simplicity, we denote linear subspace

$$E_0 := \text{span}\{L_0, D_1, \cdots, D_{n-1}, x_1, \cdots, x_{n-1}, 1, p_0\} \subset E$$

consisting of some known operators in E.

Case [1] $n \in S$.

Case [1.1] $k = 1$. Corresponding $p_0 = x_n^2 \in E$ and by assumption $\{\omega_{in}\}$ are constants for $1 \leq i \leq n-1$.

Next we calculate bracket between L_0 and maximal rank quadratic function p_0.

$$Z_1 = \frac{1}{2}[L_0, p_0] = x_nD_n \in E \tag{A.22}$$

and by using L_0 again, we get

$$\begin{aligned} Z_2 = [L_0, Z_1] =& D_n^2 + \sum_{i=1}^{n-1} x_n\omega_{ni}D_i + \sum_{i=1}^{n-1} x_n\frac{\partial\omega_{ni}}{\partial x_i} + \frac{1}{2}E_n(\eta) \\ =& D_n^2 + \sum_{i=1}^{n-1} x_n\omega_{ni}D_i + \frac{1}{2}E_n(\eta) \in E \end{aligned} \tag{A.23}$$

and

$$\begin{aligned} [Z_1, Z_2] =& -2D_n^2 + \sum_{i=1}^{n-1} x_n\omega_{ni}D_i - x_n^2 \cdot \sum_{i=1}^{n-1}\omega_{in}^2 + \frac{1}{2}E_n^2(\eta) \\ =& -2D_n^2 + \sum_{i=1}^{n-1} x_n\omega_{ni}D_i + \frac{1}{2}E_n^2(\eta), \quad \text{mod } E_0 \in E. \end{aligned} \tag{A.24}$$

Next we denote

$$Z_3 = -2D_n^2 + \sum_{i=1}^{n-1} x_n\omega_{ni}D_i + \frac{1}{2}E_n^2(\eta) \in E. \tag{A.25}$$

Basic computations show that

$$Z_4 = 2Z_2 + Z_3 = 3\sum_{i=1}^{n-1} x_n\omega_{ni}D_i + p \in E, \tag{A.26}$$

where $p := \frac{1}{2}E_n^2(\eta) + E_n(\eta)$. Continue the computation,

$$[Z_1, Z_4] = 3\sum_{i=1}^{n-1}(x_n\omega_{ni}D_i - x_n^2\omega_{ni}^2) + E_n(p) \in E. \tag{A.27}$$

By adding certain constant multiple of p_0 to $[Z_1, Z_4]$, we can define

$$Z_5 := 3\sum_{i=1}^{n-1} x_n\omega_{ni}D_i + E_n(p) \in E. \tag{A.28}$$

Notice that Z_4 and Z_5 possess the same first order differential operator. Subtracting Z_4 and Z_5 will yield the following function contained in E.

$$Z_5 - Z_4 = E_n(p) - p \in E. \tag{A.29}$$

Next we will use technique of Euler operators and deduce structure of η step by step. By Corollary 2.23, we deduce $p \in \widetilde{P}_2(x_n)$ is a polynomial of degree at most 2 in x_n with smooth coefficient of other variables. Again by using Corollary 2.23, $E_n(\eta) + 2\eta \in \widetilde{P}_2(x_n)$. The same technique is applied and we get $\eta \in \widetilde{P}_2(x_n)$ is a polynomial of degree at most 2 in x_n with smooth coefficient of other variables. By Lemma 4.4, we conclude that the linear structure of observations $\{h_i\}$.

Case [1.2] $k > 1$. Corresponding $p_0 = \sum_{i=1}^{k-1} x_i^2 + x_n^2$ and by assumption $\{\omega_{in}\}$ are constants for $1 \le i \le n-1$ and $\{\omega_{ij}\}$ are constants for $1 \le i \le k-1$ and $1 \le j \le n$.

Similarly, we calculate

$$Z_1 = \frac{1}{2}[L_0, p_0] = \sum_{i=1}^{k-1} x_iD_i + x_nD_n \in E \tag{A.30}$$

and

$$[D_i, Z_1] = \sum_{j=1}^{k-1} x_j\omega_{ji} + D_i + x_n\omega_{ni} \in E = x_n\omega_{ni}, \quad \text{mod } E_0 \in E, \text{ for } 1 \le i \le n-1. \tag{A.31}$$

Since $x_n \notin E$ by linear rank condition, we obtain $\omega_{ni} = 0$ for $1 \le i \le n-1$. Next we calculate the first order operators for $1 \le j \le k-1$,

$$[L_0, D_j] = \sum_{i=1}^{n} \omega_{ji}D_i + \frac{1}{2}\frac{\partial\eta}{\partial x_j} \in E, \quad 1 \le j \le k-1. \tag{A.32}$$

Notice $D_i \in E$ for $1 \le i \le k-1$ and $\omega_{ni} = 0$ for $1 \le i \le n-1$. It implies

$$Y_j := \frac{1}{2}\frac{\partial \eta}{\partial x_j} \in E, \quad \text{for } 1 \le j \le k-1. \tag{A.33}$$

Therefore by the equation (A.33), we can infer $\eta \in \widetilde{P}_3(x_1, \cdots, x_{k-1})$ is a polynomial of degree at most 3 in $x_1, \cdots, x_{k-1}$ with coefficients of $x_k, \cdots, x_n$. Then Lemma 4.3 implies that $\{h_i\}$ does not contain $x_p x_q, 1 \le p, q \le k-1$. If observation functions $\{h_i\}$ are not all affine, without loss of generality, we assume $\deg(h_1) = 2$. In view of the structure of maximal rank quadratic polynomial, $h_1^{(2)}$ can be written down

$$h_1^{(2)} = h_{nn}x_n^2 + h_{1n}x_1x_n + \cdots + h_{(k-1)n}x_{k-1}x_n. \tag{A.34}$$

By considering the linear rank condition, Lemma 3.2 implies that $h_1^{(2)}$ cannot contain $x_i x_n$ term for $1 \le i \le n-1$. Namely, $h_{1n} = \cdots = h_{(k-1)n} = 0$ then $h_1^{(2)} = h_{nn}x_n^2$ with $h_{nn} \neq 0$. Then we can assume

$$h_1 = x_n^2 - cx_n \in E, \tag{A.35}$$

where c is certain constant. Then by subtracting h_1 by p_0, it follows that $p_0 - h_1 = \sum_{i=1}^{k-1} x_i^2 + cx_n \in E$. Due to

$$[[L_0, p_0 - h_1], p_0 - h_1] = |\nabla(p_0 - h_1)|^2 = \sum_{i=1}^{n} \left(\frac{\partial(p_0 - h_1)}{\partial x_i}\right)^2 = \sum_{i=1}^{k-1} 4x_i^2 + c^2 \in E, \tag{A.36}$$

it follows $\sum_{i=1}^{k-1} x_i^2 \in E$. Subtracting by p_0, it implies that $x_n^2 \in E$. Repeat the process of Case [1.1] and it results in $\eta \in \widetilde{P}_2(x_n)$. Then by Lemma 4.3, $h_i^{(2)}$ does not contain x_n^2, which leads to a contradition. Then observations are all affine.

Case [2] $n \notin S$. Corresponding $p_0 = \sum_{i=1}^{k} x_i^2, 1 \le k \le n-1$ and by assumption $\{\omega_{ji}\}$ are constant numbers for $1 \le j \le k$ and $1 \le i \le n$.

First we calculate the typical first order operator:

$$[L_0, D_j] = \sum_{i=1}^{n} \omega_{ji} D_i + \frac{1}{2}\frac{\partial \eta}{\partial x_j} + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial \omega_{ji}}{\partial x_i}, \quad 1 \le j \le k. \tag{A.37}$$

Assumption that ω_{ji}'s are constant numbers for $1 \le j \le k$, it leads to

$$Y_j := \omega_{jn} D_n + \frac{1}{2}\frac{\partial \eta}{\partial x_j} \in E, \quad 1 \le j \le k. \tag{A.38}$$

Next we calculate

$$Z_1 := \frac{1}{2}([L_0, p_0] - k) = \sum_{i=1}^{k} x_i D_i \in E \tag{A.39}$$

and

$$[Z_1, Y_j] = \sum_{i=1}^{k} \omega_{jn}\omega_{ni} x_i + \frac{1}{2} E_{1:k}\left(\frac{\partial \eta}{\partial x_j}\right) = \frac{1}{2} E_{1:k}\left(\frac{\partial \eta}{\partial x_j}\right), \quad \text{mod } E_0 \in E, \ 1 \le j \le k. \tag{A.40}$$

By Corollary 2.23, we derive that $\frac{\partial\eta}{\partial x_j} \in \widetilde{P}_2(x_1,\cdots,x_k)$ is a polynomial of degree at most 2 in $x_1,\cdots,x_k$ with smooth coefficients of $x_{k+1},\cdots,x_n$. Then $\eta \in \widetilde{P}_3(x_1,\cdots,x_k)$ is a degree at most 3 polynomial in $x_1,\cdots,x_k$ with smooth coefficients of $x_{k+1},\cdots,x_n$. Therefore, Lemma 4.4 implies that the result of linear observation functions. ∎

A.9 Proof of Lemma 5.2

First we calculate $[L_0, \frac{1}{2}p_0] - \frac{1}{2} = x_1D_1 \in E$. Then

$$[D_2, x_1D_1] = -\widetilde{a}_1x_1^2 - \widetilde{a}_2x_1x_2 - \widetilde{a}_3x_1x_3 - \widetilde{a}_0x_1 \in E \tag{A.41}$$

and

$$[D_3, x_1D_1] = -\widetilde{b}_1x_1^2 - \widetilde{b}_2x_1x_2 - \widetilde{b}_3x_1x_3 - \widetilde{b}_0x_1 \in E. \tag{A.42}$$

Lemma 3.5 implies for any function $\phi \in E$, then $\phi^{(2)}$ is independent of x_2, x_3, x_4. It leads to $\widetilde{a}_2 = \widetilde{a}_3 = \widetilde{b}_2 = \widetilde{b}_3 = 0$. ∎

A.10 Proof of Theorem 5.3

In the following, we make some basic calculations firstly.

$$K_0 = \frac{1}{2}([L_0, p_0] - 1) = x_1D_1 \in E, \tag{A.43}$$

$$K_1 = [L_0, K_0] = D_1^2 - \sum_{i=2}^{4}\alpha_iD_i - \frac{1}{2}\sum_{i=2}^{4}\frac{\partial\alpha_i}{\partial x_i} + \frac{1}{2}E_1(\eta) \in E, \tag{A.44}$$

$$K_2 = [K_1, K_0] = 2D_1^2 + \sum_{i=2}^{4}E_1(\alpha_i)D_i + \gamma(x) \in E, \tag{A.45}$$

where $\alpha_i := x_1\omega_{i1}$ and $E_1 := x_1\frac{\partial}{\partial x_1}$ and

$$\gamma(x) := -\sum_{i=2}^{4}\alpha_ix_1\omega_{1i} + \frac{1}{2}\sum_{i=2}^{4}E_1\left(\frac{\partial\alpha_i}{\partial x_i}\right) - \frac{1}{2}E_1^2(\eta). \tag{A.46}$$

It will derive that

$$K_2 - 2K_1 = \sum_{i=2}^{4}[E_1(\alpha_i) + 2\alpha_i]D_i, \quad \text{mod } U_0 \in E. \tag{A.47}$$

In the following, the basic definition yields

$$\begin{aligned}
\alpha_2 &= \widetilde{a}_1x_1^2 + \widetilde{a}_2x_1x_2 + \widetilde{a}_3x_1x_3 + \widetilde{a}_0x_1, \\
\alpha_3 &= \widetilde{b}_1x_1^2 + \widetilde{b}_2x_1x_2 + \widetilde{b}_3x_1x_3 + \widetilde{b}_0x_1, \\
\alpha_4 &= \widetilde{c}_1x_1^2 + \widetilde{c}_2x_1x_2 + \widetilde{c}_3x_1x_3 + \widetilde{c}_4x_1x_4 + \widetilde{c}_0x_1.
\end{aligned} \tag{A.48}$$

Then the coefficients of (A.47) can be calculated as following:

$$\begin{aligned}
E_1(\alpha_2) + 2\alpha_2 &= 4\widetilde{a}_1x_1^2 + 3\widetilde{a}_2x_1x_2 + 3\widetilde{a}_3x_1x_3 + 3\widetilde{a}_0x_1, \\
E_1(\alpha_3) + 2\alpha_3 &= 4\widetilde{b}_1x_1^2 + 3\widetilde{b}_2x_1x_2 + 3\widetilde{b}_3x_1x_3 + 3\widetilde{b}_0x_1, \\
E_1(\alpha_4) + 2\alpha_4 &= 4\widetilde{c}_1x_1^2 + 3\widetilde{c}_2x_1x_2 + 3\widetilde{c}_3x_1x_3 + 3\widetilde{c}_4x_1x_4 + 3\widetilde{c}_0x_1.
\end{aligned} \tag{A.49}$$

Next the equation (A.47) will become

$$\begin{aligned}K_2 - 2K_1 =&(4\widetilde{a}_1 x_1^2 + 3\widetilde{a}_2 x_1 x_2 + 3\widetilde{a}_3 x_1 x_3 + 3\widetilde{a}_0 x_1)D_2\\&+ (4\widetilde{b}_1 x_1^2 + 3\widetilde{b}_2 x_1 x_2 + 3\widetilde{b}_3 x_1 x_3 + 3\widetilde{b}_0 x_1)D_3\\&+ (4\widetilde{c}_1 x_1^2 + 3\widetilde{c}_2 x_1 x_2 + 3\widetilde{c}_3 x_1 x_3 + 3\widetilde{c}_4 x_1 x_4 + 3\widetilde{c}_0 x_1)D_4, \quad \text{mod } U_0 \in E.\end{aligned} \tag{A.50}$$

For variable substitution, we can define

$$\begin{aligned}Y_1 :=&(a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_1 x_3 + a_0 x_1)D_2\\&+ (b_1 x_1^2 + b_2 x_1 x_2 + b_3 x_1 x_3 + b_0 x_1)D_3\\&+ (c_1 x_1^2 + c_2 x_1 x_2 + c_3 x_1 x_3 + c_4 x_1 x_4 + c_0 x_1)D_4, \quad \text{mod } U_0 \in E,\end{aligned} \tag{A.51}$$

where $a_1 = 4\widetilde{a}_1, a_2 = 3\widetilde{a}_2, a_3 = 3\widetilde{a}_3, a_0 = 3\widetilde{a}_0, b_1 = 4\widetilde{b}_1, b_2 = 3\widetilde{b}_2, b_3 = 3\widetilde{b}_3, b_0 = 3\widetilde{b}_0, c_1 = 4\widetilde{c}_1, c_2 = 3\widetilde{c}_2, c_3 = 3\widetilde{c}_3, c_4 = 3\widetilde{c}_4, c_0 = 3\widetilde{c}_0$.

Next we only need to prove $a_i = 0, b_i = 0, c_j = 0$ for $1 \le i \le 3, 1 \le j \le 4$ and it will lead to the conclusion. We will prove it by several steps.

Step [1] If there at least one is nonzero in entries c_1, c_2, c_3.

We have the following three cases.

(i) If $c_1 \neq 0$, then $Ad^2_{D_1} Y_1 = 2a_1 D_2 + 2b_1 D_3 + 2c_1 D_4,\ \text{mod } U_0 \in E \Longrightarrow D_4,\ \text{mod } U_0 \in E$.

(ii) If $c_2 \neq 0$, then $[D_1, [D_2, Y_1]] = a_2 D_2 + b_2 D_3 + c_2 D_4,\ \text{mod } U_0 \in E \Longrightarrow D_4,\ \text{mod } U_0 \in E$.

(iii) If $c_3 \neq 0$, then $[D_1, [D_3, Y_1]] = a_3 D_2 + b_3 D_3 + c_3 D_4,\ \text{mod } U_0 \in E \Longrightarrow D_4,\ \text{mod } U_0 \in E$.

So we always obtain that D_4, mod $U_0 \in E$ for any situations.

Step [1.1] We claim $c_4 = 0$.

Otherwise, if $c_4 \neq 0$, we will obtain

$$[D_4, \text{mod } U_0, Y_1] = c_4 x_1 D_4, \text{mod } U_0 \in E \Longrightarrow T := x_1 D_4, \text{mod } U_0 \in E. \tag{A.52}$$

Next we can construct an infinite sequence by using operator T and L_0.

$$\begin{cases} T = x_1 D_4, \quad \text{mod } U_0 \in E,\\ [L_0, T] = D_1 D_4, \quad \text{mod } U_1 \in E,\\ [[L_0, T], T] = D_4^2, \quad \text{mod } U_1 \in E \end{cases} \tag{A.53}$$

and

$$\begin{cases} [D_4^2, \text{mod } U_1, Y_1] = 2c_4 x_1 D_4^2, \quad \text{mod } U_1 \in E \Longrightarrow x_1 D_4^2, \quad \text{mod } U_1 \in E,\\ [L_0, x_1 D_4^2, \text{mod } U_1] = D_1 D_4^2, \quad \text{mod } U_2 \in E,\\ [D_1 D_4^2, \text{mod } U_2, x_1 D_4^2, \text{mod } U_1] = D_4^4, \quad \text{mod } U_1 \in E. \end{cases} \tag{A.54}$$

By repeating this process, we have $D_4^{2^k}$, mod $U_{2^k-1} \in E$. A contradiction! Hence, $c_4 = 0$.

Step [1.2] We claim $c_1 = c_2 = c_3 = 0$.

We rewrite following two operators:

$$\begin{aligned} M_1 &:= Y_1 = (c_1x_1^2 + c_2x_1x_2 + c_3x_1x_3 + c_0x_1)D_4 \\ &\quad + \text{terms with lower order in } D_4, \quad \mathrm{mod}\ U_0 \in E, \\ M_2 &:= [L_0, Y_1] = (2c_1x_1 + c_2x_2 + c_3x_3 + c_0)D_1D_4 + c_2x_1D_2D_4 + c_3x_1D_3D_4 \\ &\quad + \text{terms with lower order in } D_4, \quad \mathrm{mod}\ U_1 \in E. \end{aligned} \tag{A.55}$$

Next we calculate

$$\begin{cases} [D_1, M_1] = (2c_1x_1 + c_2x_2 + c_3x_3 + c_0)D_4 \\ \qquad + \text{terms with lower order in } D_4, \quad \mathrm{mod}\ U_0 \in E, \\ [D_2, M_1] = c_2x_1D_4 + \text{terms with lower order in } D_4, \quad \mathrm{mod}\ U_0 \in E, \\ [D_3, M_1] = c_3x_1D_4 + \text{terms with lower order in } D_4, \quad \mathrm{mod}\ U_0 \in E. \end{cases} \tag{A.56}$$

By Lemma 2.15 and combining operators $M_2, [D_1, M_1], [D_2, M_1], [D_3, M_1]$, we deduce $c_1 = c_2 = c_3 = 0$. This is a contradiction to assumption of Step [1]. Therefore, we obtain $c_1 = c_2 = c_3 = 0$.

Step [2] We claim $a_2 = a_3 = b_2 = b_3 = 0$.

This is followed from Lemma 5.2 directly. Finally we only need to prove $a_1 = b_1 = c_4 = 0$. That will finish the whole proof.

Step [3] We claim $a_1 = b_1 = c_4 = 0$.

We rewrite

$$\begin{aligned} A_0 &= [L_0, Y_1] = (2a_1x_1 + a_0)D_1D_2 + (2b_1x_1 + b_0)D_1D_3 + (c_4x_4 + c_0)D_1D_4 \\ &\quad + c_4x_1D_4^2, \quad \mathrm{mod}\ U_1 \in E, \\ \widetilde{A}_1 &= \frac{1}{2}Ad_{L_0}^2Y_1 = a_1D_1^2D_2 + b_1D_1^2D_3 + c_4D_1D_4^2, \quad \mathrm{mod}\ U_2 \in E. \end{aligned} \tag{A.57}$$

Step [3.1] We claim $a_1 = 0$.

$$\begin{cases} \widetilde{A}_1 = a_1D_1^2D_2 + \text{terms with lower order in } D_2, \quad \mathrm{mod}\ U_2 \in E, \\ A_0 = (2a_1x_1 + a_0)D_1D_2 + \text{terms with lower order in } D_2, \quad \mathrm{mod}\ U_1 \in E, \\ A_2 = [\widetilde{A}_1, A_0] = 2^2a_1^2D_1^2D_2^2 + \text{terms with lower order in } D_2, \quad \mathrm{mod}\ U_3 \in E, \\ \vdots \\ A_k = 2^{2(k-1)}a_1^kD_1^2D_2^k + \text{terms with lower order in } D_2, \quad \mathrm{mod}\ U_{k+1} \in E. \end{cases} \tag{A.58}$$

By $\dim E < \infty$, we obtain $a_1 = 0$.

Step [3.2] We claim $b_1 = 0$.

Again we rewrite $A_0, \widetilde{A}_1$ as below:

$$\begin{cases} \widetilde{A}_1 = b_1 D_1^2 D_3 + \text{terms with lower order in } D_3, \quad \bmod U_2 \in E, \\ A_0 = (2b_1x_1 + b_0) D_1 D_3 + \text{terms with lower order in } D_3, \quad \bmod U_1 \in E, \\ A_2 = [\widetilde{A}_1, A_0] = 2^2 b_1^2 D_1^2 D_3^2 + \text{terms with lower order in } D_3, \quad \bmod U_3 \in E, \\ \vdots \\ A_k = 2^{2(k-1)} b_1^k D_1^2 D_3^k + \text{terms with lower order in } D_3, \quad \bmod U_{k+1} \in E. \end{cases} \tag{A.59}$$

By $\dim E < \infty$, we obtain $b_1 = 0$.

Step [3.3] We claim $c_4 = 0$.

Otherwise, if $c_4 \neq 0$, we obtain

$$\begin{cases} V_1 = \frac{1}{c_4}\widetilde{A}_1 = D_1 D_4^2, \quad \bmod U_2 \in E, \\ V_0 = \frac{1}{c_4} A_0 = x_1 D_4^2 + (x_4 + c.t.) D_1 D_4 \\ \qquad +\text{order 2 terms with constant coefficients} \in E, \\ [V_1, V_0] = D_4^4 + 2D_1^2 D_4^2, \quad \bmod U_3 \in E, \\ [[V_1, V_0], V_0] = 8 D_1 D_4^4, \quad \bmod U_4 \in E \Longrightarrow V_2 = D_1 D_4^4, \quad \bmod U_4 \in E. \end{cases} \tag{A.60}$$

By repeating the same process, we obtain

$$\begin{cases} [V_2, V_0] = D_4^6 + 4 D_1 D^2 D_4^4, \quad \bmod U_5 \in E, \\ [[V_2, V_0], V_0] = 14 D_1 D_4^4, \quad \bmod U_6 \in E \Longrightarrow V_2 = D_1 D_4^6, \quad \bmod U_6 \in E, \\ \vdots \\ V_k = D_1 D_4^{2k}, \quad \bmod U_{2k} \in E. \end{cases} \tag{A.61}$$

A contradiction! So we have $c_4 = 0$.

Finally, we prove the original claim. ∎

A.11 Proof of Lemma 5.4

Firstly, we calculate the first order operator:

$$K_0 = \frac{1}{2}[L_0, p_0] - \frac{1}{2} = x_4 D_4 \in E. \tag{A.62}$$

Then we have

$$\begin{aligned} [D_1, K_0] &= -a_1x_1x_4 - a_2x_2x_4 - a_3x_3x_4 - a_4x_4^2 - a_0x_4 \in E, \\ [D_2, K_0] &= -b_1x_1x_4 - b_2x_2x_4 - b_3x_3x_4 - b_4x_4^2 - b_0x_4 \in E, \\ [D_3, K_0] &= -c_1x_1x_4 - c_2x_2x_4 - c_3x_3x_4 - c_4x_4^2 - c_0x_4 \in E. \end{aligned} \tag{A.63}$$

Quadratic structure of E (3.5) implies that any function $\phi \in E$ is a quadratic polynomial, then homogeneous quadratic part $\phi^{(2)}$ is independent of x_1, x_2, x_3. It derives that $a_i = 0, b_i = 0, c_i = 0$ for $1 \leq i \leq 3$. That means $\omega_{i4} \in P_1(x_4)$ for $1 \leq i \leq 3$. ∎

A.12 Proof of Theorem 5.5

Starting with the proof of Lemma 5.4, we obtain

$$
\begin{aligned}
&[D_1, K_0] = -a_4x_4^2 - a_0x_4 \in E, \quad [D_2, K_0] = -b_4x_4^2 - b_0x_4 \in E,\\
&[D_3, K_0] = -c_4x_4^2 - c_0x_4 \in E.
\end{aligned} \tag{A.64}
$$

Considering $p_0 = x_4^2 \in E$ and linear rank equals 3, we obtain $a_0 = b_0 = c_0 = 0$ which means

$$
\omega_{14} = a_4x_4, \quad \omega_{24} = b_4x_4, \quad \omega_{34} = c_4x_4. \tag{A.65}
$$

Next we calculate

$$
\begin{aligned}
K_1 = [L_0, K_0] =& D_4^2 - \sum_{i=1}^{3} \alpha_i D_i - \frac{1}{2}\sum_{i=1}^{3}\frac{\partial \alpha_i}{\partial x_i} + \frac{1}{2}E_4(\eta)\\
=& D_4^2 - a_4x_4^2D_1 - b_4x_4^2D_2 - c_4x_4^2D_3, \quad \text{mod } U_0 \in E
\end{aligned} \tag{A.66}
$$

and

$$
\begin{aligned}
K_2 = [K_1, K_0] =& 2D_4^2 + \sum_{i=1}^{3} E_4(\alpha_i)D_i, \quad \text{mod } U_0\\
=& 2D_4^2 + 2a_4x_4^2D_1 + 2b_4x_4^2D_2 + 2c_4x_4^2D_3, \quad \text{mod } U_0 \in E.
\end{aligned} \tag{A.67}
$$

Linear combination of two operators K_1 and K_2 implies

$$
\frac{1}{2}\left(\frac{1}{2}K_2 + K_1\right) = D_4^2, \quad \text{mod } U_0 \in E \tag{A.68}
$$

and

$$
Z := \frac{1}{2}\left(\frac{1}{2}K_2 - K_1\right) = a_4x_4^2D_1 + b_4x_4^2D_2 + c_4x_4^2D_3, \quad \text{mod } U_0 \in E. \tag{A.69}
$$

Next we can construct an infinite sequence as below:

$$
\begin{cases}
Z_1 =& \frac{1}{2}[L_0, Z]\\
&= a_4x_4D_1D_4 + b_4x_4D_2D_4 + c_4x_4D_3D_4, \quad \text{mod } U_1 \in E\\
&= a_4x_4D_1D_4 + \text{terms with lower order in } D_1, \quad \text{mod } U_1 \in E,\\
Z_2 =& \frac{1}{2}Ad_{L_0}^2 Z\\
&= a_4D_1D_4^2 + b_4D_2D_4^2 + c_4D_3D_4^2, \quad \text{mod } U_2 \in E\\
&= a_4D_1D_4^2 + \text{terms with lower order in } D_1, \quad \text{mod } U_2 \in E,\\
Z_3 =& [Z_2, Z_1]\\
&= 2a_4^2D_1^2D_4^2 + \text{terms with lower order in } D_1, \quad \text{mod } U_3 \in E,\\
&\ \vdots\\
Z_k =& [Z_{k-1}, Z_1]\\
&= 2^{k-2}a_4^{k-2}D_1^{k-1}D_4^2 + \text{terms with lower order in } D_1, \quad \text{mod } U_k \in E.
\end{cases} \tag{A.70}
$$

By finite dimensionality of E, we obtain $a_4 = 0$. Then we have

$$\begin{aligned} Z_1 =& b_4 x_4 D_2 D_4 + c_4 x_4 D_3 D_4, \quad \text{mod } U_1 \in E, \\ Z_2 =& b_4 D_2 D_4^2 + c_4 D_3 D_4^2, \quad \text{mod } U_2 \in E. \end{aligned} \tag{A.71}$$

In the following, repeating the same process, we can obtain $b_4 = c_4 = 0$. It naturally implies that $\omega_{14} = \omega_{24} = \omega_{34} = 0$. ∎

A.13 Proof of Lemma 5.6

First we calculate $K_0 = [L_0, \frac{1}{2}p_0] - 1 = x_1 D_1 + x_2 D_2 \in E$ and obtain

$$\begin{aligned} A_1 :=& [D_1, K_0] = x_2 \omega_{21}, \quad \text{mod } E_0 \in E, \\ A_2 :=& [D_2, K_0] = x_1 \omega_{12}, \quad \text{mod } E_0 \in E, \\ A_3 :=& [D_3, K_0] = x_1 \omega_{13} + x_2 \omega_{23} \in E, \end{aligned} \tag{A.72}$$

where $E_0 := \text{span}\{L_0, x_i, D_i, 1\}, 1 \leq i \leq 3$ is a linear subspace of E. By assuming $\omega_{12} = k_1 x_1 + k_2 x_2 + k_3 x_3 + k_0$, from A_1, A_2, we can derive

$$\begin{aligned} & k_1 x_1 x_2 + k_2 x_2^2 + k_3 x_3 x_2 + k_0 x_2 \in E, \\ & k_1 x_1^2 + k_2 x_1 x_2 + k_3 x_3 x_1 + k_0 x_1 \in E. \end{aligned} \tag{A.73}$$

Structure of maximal rank quadratic polynomial implies $k_3 = 0$. Then we define

$$\widetilde{A}_1 = k_1 x_1 x_2 + k_2 x_2^2 \in E, \quad \widetilde{A}_2 = k_1 x_1^2 + k_2 x_1 x_2 \in E. \tag{A.74}$$

Considering $p_0, \widetilde{A}_1, \widetilde{A}_2$, Theorem 4.2 implies ω_{12} is a constant. Considering structure of maximal rank quadratic polynomial for function $A_3 \in E$ implies $\omega_{13}, \omega_{23} \in P_1(x_1, x_2)$. ∎

A.14 Proof of Theorem 5.7

In the following, we do more computations of Lie brackets in E.

$$K_0 := [L_0, \frac{1}{2}p_0] - 1 = x_1 D_1 + x_2 D_2 \in E \tag{A.75}$$

and

$$\begin{aligned} K_1 := [L_0, K_0] =& D_1^2 + D_2^2 - \beta_1 D_1 - \alpha_2 D_2 - (\alpha_3 + \beta_3) D_3 - (\alpha_4 + \beta_4) D_4 \\ & - \frac{1}{2} \sum_{i \neq 1} \frac{\partial \alpha_i}{\partial x_i} - \frac{1}{2} \sum_{j \neq 2} \frac{\partial \beta_j}{\partial x_j} + \frac{1}{2} E_{1,2}(\eta) \in E, \end{aligned} \tag{A.76}$$

where $\alpha_i := x_1 \omega_{i1}$ and $\beta_j := x_2 \omega_{j2}$ and $E_{1,2} := x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$.

$$\begin{aligned} K_2 = [K_1, K_0] =& 2D_1^2 + 2D_2^2 + (E_2(\beta_1) - 3\beta_1) D_1 + (E_1(\alpha_2) - 3\alpha_2) D_2 \\ & + E_{1,2}(\alpha_3 + \beta_3) D_3 + E_{1,2}(\alpha_4 + \beta_4) D_4, \quad \text{mod } U_0 \in E. \end{aligned} \tag{A.77}$$

Then we calculate

$$\begin{aligned}K_3 := K_2 - 2K_1 =&(E_2(\beta_1) - \beta_1)D_1 + (E_1(\alpha_2) - \alpha_2)D_2 + [E_{1,2}(\alpha_3 + \beta_3) + 2(\alpha_3 + \beta_3)]D_3\\ &+ [E_{1,2}(\alpha_4 + \beta_4) + 2(\alpha_4 + \beta_4)]D_4, \quad \text{mod } U_0 \in E\\ =&[E_{1,2}(\alpha_3 + \beta_3) + 2(\alpha_3 + \beta_3)]D_3 + [E_{1,2}(\alpha_4 + \beta_4) + 2(\alpha_4 + \beta_4)]D_4,\\ &\text{mod } U_0 \in E, \end{aligned} \tag{A.78}$$

where the third equality used the properties $\beta_1 = x_2\omega_{12} \in P_1(x_2)$ and $\alpha_2 = x_1\omega_{21} \in P_1(x_1)$.

Next we assume

$$\begin{cases}\omega_{31} = \widetilde{a}_1x_1 + \widetilde{a}_2x_2 + \widetilde{a}_0,\\ \omega_{32} = \widetilde{b}_1x_1 + \widetilde{b}_2x_2 + \widetilde{b}_0,\\ \omega_{41} = \widetilde{c}_1x_1 + \widetilde{c}_2x_2 + \widetilde{c}_3x_3 + \widetilde{c}_4x_4 + \widetilde{c}_0,\\ \omega_{42} = \widetilde{d}_1x_1 + \widetilde{d}_2x_2 + \widetilde{d}_3x_3 + \widetilde{d}_4x_4 + \widetilde{d}_0.\end{cases} \tag{A.79}$$

By definition, we can obtain

$$E_{1,2}(\alpha_3 + \beta_3) + 2(\alpha_3 + \beta_3) = 4\widetilde{a}_1x_1^2 + 4(\widetilde{a}_2 + \widetilde{b}_1)x_1x_2 + 4\widetilde{b}_2x_2^2 + 3\widetilde{a}_0x_1 + 3\widetilde{b}_0x_2 \tag{A.80}$$

and

$$\begin{aligned}E_{1,2}(\alpha_4 + \beta_4) + 2(\alpha_4 + \beta_4) =&4\widetilde{c}_1x_1^2 + 4(\widetilde{c}_2 + \widetilde{d}_1)x_1x_2 + 4\widetilde{d}_2x_2^2\\ &+ 3\widetilde{c}_3x_1x_3 + 3\widetilde{c}_4x_1x_4 + 3\widetilde{d}_3x_2x_3 + 3\widetilde{d}_4x_2x_4\\ &+ 3\widetilde{c}_0x_1 + 3\widetilde{d}_0x_2.\end{aligned} \tag{A.81}$$

By cyclical condition,

$$\frac{\partial\omega_{31}}{\partial x_2} + \frac{\partial\omega_{12}}{\partial x_3} + \frac{\partial\omega_{23}}{\partial x_1} = 0 \Longrightarrow \widetilde{a}_2 = \widetilde{b}_1 \tag{A.82}$$

and

$$\frac{\partial\omega_{41}}{\partial x_2} + \frac{\partial\omega_{12}}{\partial x_4} + \frac{\partial\omega_{24}}{\partial x_1} = 0 \Longrightarrow \widetilde{c}_2 = \widetilde{d}_1. \tag{A.83}$$

Next we will prove that coefficients of operator K_3 are all degree at most 1 polynomial. If the claim is right, then we obtain $\widetilde{a}_i = 0, \widetilde{b}_i = 0, \widetilde{c}_i = 0, \widetilde{d}_i = 0$. It leads to $\omega_{31}, \omega_{32}, \omega_{41}, \omega_{42}$ are constants. That will derive the final result.

We rewrite operator

$$\begin{aligned}K_3 := K_2 - 2K_1 =&(a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2 + pol_1(x_1, x_2))D_3\\ &+ (b_{11}x_1^2 + b_{12}x_1x_2 + b_{22}x_2^2 + b_{13}x_1x_3 + b_{14}x_1x_4\\ &+ b_{23}x_2x_3 + b_{24}x_2x_4 + pol_1(x_1, x_2))D_4, \quad \text{mod } U_0 \in E,\end{aligned} \tag{A.84}$$

where $a_{11} = 4\widetilde{a}_1, a_{12} = 8\widetilde{a}_2, a_{22} = 4\widetilde{b}_2, b_{11} = 4\widetilde{c}_1, b_{12} = 8\widetilde{c}_2, b_{22} = 4\widetilde{d}_2, b_{13} = 3\widetilde{c}_3, b_{14} = 3\widetilde{c}_4, b_{23} = 3\widetilde{d}_3, b_{24} = 3\widetilde{d}_4$.

Step [1] If $b_{11}, b_{12}, b_{22}, b_{13}, b_{23}$ are not all zero.

Without loss of generality, we assume $b_{11} \neq 0$. Then we have

$$Ad^2_{D_1} K_3 = 2a_{11} D_3 + 2b_{11} D_4, \quad \text{mod } U_0 \in E. \tag{A.85}$$

We will obtain $D_4, \text{ mod } U_0 \in E$. For other case, it can be verified that $D_4, \text{ mod } U_0 \in E$ by using similar method.

Step [1.1] We claim $b_{14} = b_{24} = 0$.

Next we calculate

$$\begin{cases} T := [D_4, \text{ mod } U_0, K_3] = (b_{14}x_1 + b_{24}x_2)D_4, \quad \text{mod } U_0 \in E, \\ [L_0, T] = b_{14}D_1D_4 + b_{24}D_2D_4, \quad \text{mod } U_1 \in E, \\ [[L_0, T], T] = (b_{14}^2 + b_{24}^2)D_4^2, \quad \text{mod } U_1 \in E. \end{cases} \tag{A.86}$$

If there exists one nonzero in b_{14}, b_{24} at least, we can obtain $D_4^2, \text{ mod } U_1 \in E$.

$$\begin{cases} T_1 := [D_4^2, \text{ mod } U_1, K_3] = 2(b_{14}x_1 + b_{24}x_2)D_4^2, \quad \text{mod } U_1 \in E, \\ [L_0, T_1] = 2b_{14}D_1D_4^2 + 2b_{24}D_2D_4^2, \quad \text{mod } U_2 \in E, \\ [[L_0, T_1], T_1] = 4(b_{14}^2 + b_{24}^2)D_4^4, \quad \text{mod } U_3 \in E. \end{cases} \tag{A.87}$$

Similarly, we get $D_4^4, \text{ mod } U_3 \in E$. Repeat this process, we can obtain

$$D_4^{2^k}, \quad \text{mod } U_{2^k-1} \in E, \quad \forall k \in \mathbb{Z}_{>0}. \tag{A.88}$$

This is contradictory to finite dimensionality of E.

Step [1.2] We claim $b_{11}, b_{12}, b_{22}, b_{13}, b_{23}$ are all zero.

We rewrite operator K_3:

$$\begin{aligned} M_1 = K_3 = &(b_{11}x_1^2 + b_{12}x_1x_2 + b_{22}x_2^2 + b_{13}x_1x_3 + b_{23}x_2x_3 + pol_1(x_1, x_2))D_4 \\ &+ \text{terms with lower order in } D_4, \quad \text{mod } U_0 \in E \end{aligned} \tag{A.89}$$

and

$$\begin{aligned} M_2 = &[L_0, M_1] \\ = &(2b_{11}x_1 + b_{12}x_2 + b_{13}x_3 + const)D_1D_4 \\ &+ (b_{12}x_1 + 2b_{22}x_2 + b_{23}x_3 + const)D_2D_4 \\ &+ (b_{13}x_1 + b_{23}x_2)D_3D_4 + \text{terms with lower order in } D_4, \quad \text{mod } U_0 \in E. \end{aligned} \tag{A.90}$$

Next we calculate

$$\begin{aligned} [D_1, M_1] = &(2b_{11}x_1 + b_{12}x_2 + b_{13}x_3 + const)D_4 \\ &+ \text{terms with lower order in } D_4, \quad \text{mod } U_0 \in E, \\ [D_2, M_1] = &(b_{12}x_1 + 2b_{22}x_2 + b_{23}x_3 + const)D_4 \\ &+ \text{terms with lower order in } D_4, \quad \text{mod } U_0 \in E, \\ [D_3, M_1] = &(b_{13}x_1 + b_{23}x_2)D_4 \\ &+ \text{terms with lower order in } D_4, \quad \text{mod } U_0 \in E. \end{aligned} \tag{A.91}$$

Combining four operators $M_2, [D_1, M_1], [D_2, M_1], [D_3, M_1]$ and using Lemma 2.15 we obtain $b_{11} = b_{12} = b_{22} = b_{13} = b_{23} = 0$.

Step [2] We claim $a_{11} = a_{12} = a_{22} = 0$.

We rewrite K_3 as

$$\begin{aligned} K_3 =&(a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2 + pol_1(x_1, x_2))D_3 \\ &+ (b_{14}x_1x_4 + b_{24}x_2x_4 + pol_1(x_1, x_2))D_4, \quad \bmod U_0 \in E \\ =&(a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2 + pol_1(x_1, x_2))D_3 \\ &+ \text{terms with lower order in } D_3, \quad \bmod U_0 \in E \end{aligned} \tag{A.92}$$

and

$$\begin{aligned} A_0 =&[L_0, K_3] \\ =&(2a_{11}x_1 + a_{12}x_2 + const)D_1D_3 + (a_{12}x_1 + 2a_{22}x_2 + const)D_2D_3 \\ &+ \text{terms with lower order in } D_3, \quad \bmod U_1 \in E. \end{aligned} \tag{A.93}$$

By using the linear rank condition, we calculate

$$\begin{aligned} [D_1, K_3] =&(2a_{11}x_1 + a_{12}x_2 + const)D_3 \\ &+ \text{terms with lower order in } D_3, \quad \bmod U_0 \in E, \\ [D_2, K_3] =&(a_{12}x_1 + 2a_{22}x_2 + const)D_3 \\ &+ \text{terms with lower order in } D_3, \quad \bmod U_0 \in E. \end{aligned} \tag{A.94}$$

Combining operators $A_0, [D_1, K_3], [D_2, K_3]$ and Lemma 2.17 implies $a_{11} = a_{12} = a_{22} = 0$.

Step [3] We claim $b_{14} = b_{24} = 0$.

Again we rewrite operator

$$K_3 = (b_{14}x_1x_4 + b_{24}x_2x_4 + pol_1(x_1, x_2))D_4 + pol_1(x_1, x_2)D_3, \quad \bmod U_0 \in E. \tag{A.95}$$

In the following, we will construct an infinite sequence:

$$\begin{aligned} V_0 =&[L_0, K_3] \\ =&(b_{14}x_4 + const)D_1D_4 + (b_{24}x_4 + const)D_2D_4 + (b_{14}x_1 + b_{24}x_2)D_4^2 \\ &+ \text{order 2 terms with constant coefficients}, \quad \bmod U_1 \in E. \end{aligned} \tag{A.96}$$

If b_{14}, b_{24} are not all zero, without loss of generality, we assume $b_{14} \neq 0$. Then we have

$$\frac{1}{b_{14}}[D_1, V_0] = D_4^2, \quad \bmod U_1 \in E, \tag{A.97}$$

$$\begin{cases} V_0 := \dfrac{1}{2}[D_4^2, \bmod U_1, K_3] = (b_{14}x_1 + b_{24}x_2)D_4^2, \quad \bmod U_1 \in E, \\ [L_0, V_0] = b_{14}D_1D_4^2 + b_{24}D_2D_4^2, \quad \bmod U_2 \in E, \\ [[L_0, V_0], V_0] = (b_{14}^2 + b_{24}^2)D_4^4, \quad \bmod U_3 \in E \Longrightarrow D_4^4, \quad \bmod U_3 \in E \end{cases} \tag{A.98}$$

and

$$\begin{cases} V_1 := \frac{1}{4}[D_4^4, \mod U_3, K_3] = (b_{14}x_1 + b_{24}x_2)D_4^4, \quad \mod U_3 \in E, \\ [L_0, V_1] = b_{14}D_1D_4^4 + b_{24}D_2D_4^4, \quad \mod U_4 \in E, \\ [[L_0, V_1], V_1] = (b_{14}^2 + b_{24}^2)D_4^8, \quad \mod U_7 \in E \Longrightarrow D_4^8, \quad \mod U_7 \in E. \end{cases} \tag{A.99}$$

Repeat the same process, we can obtain $D_4^{2^k}$, mod $U_{2^k-1} \in E, \forall k \in \mathbb{Z}_{>0}$. A contradiction! Hence b_{14} and b_{24} are both zero. ∎

A.15 Proof of Lemma 5.8

First we do some basic computations as

$$K_0 = \frac{1}{2}[L_0, p_0] - 1 = x_1D_1 + x_4D_4 \in E \tag{A.100}$$

and

$$[D_1, K_0] = x_4\omega_{41}, \quad \mod E_0 \in E, \tag{A.101}$$

$$[D_2, K_0] = x_1\omega_{12} + x_4\omega_{42} \in E, \tag{A.102}$$

$$[D_3, K_0] = x_1\omega_{13} + x_4\omega_{43} \in E. \tag{A.103}$$

Equation (A.101) and structure of quadratic rank of E (3.5), we obtain $\omega_{41} \in P_1(x_1, x_4)$. By linear rank condition, Lemma 3.2 shows for any function $\phi \in E$, $\phi^{(2)}$ does not contain term $x_ix_4, 1 \le i \le 3$. Hence, we obtain $\omega_{41} \in P_1(x_4)$. Similarly, we can obtain $\omega_{42}, \omega_{43} \in P_1(x_4)$. Similarly, we conclude $\omega_{12}, \omega_{13} \in P_1(x_1)$. ∎

A.16 Proof of Lemma 5.9

From the equation (A.101), we derive $x_4\omega_{14} = cx_4^2 + c_0x_4 \in E$. Then

$$[K_0, x_4\omega_{14}] = 2cx_4^2 + c_0x_4 \in E. \tag{A.104}$$

It is direct to obtain $cx_4^2 \in E, c_0x_4 \in E$. The latter derives $c_0 = 0$. If $c \neq 0$, then $x_4^2 \in E \Longrightarrow x_1^2 \in E$. Then Lemma 4.1 implies ω_{41} is a constant, a contradiction! Then $c = 0$. ∎

A.17 Proof of Lemma 5.10

First we can do some basic computations:

$$\begin{aligned} K_1 = [L_0, K_0] =& D_1^2 + D_4^2 - \alpha_4 D_4 - \beta_1 D_1 - (\alpha_2 + \beta_2)D_2 - (\alpha_3 + \beta_3)D_3 \\ & - \frac{1}{2}\sum_{i \neq 1} \frac{\partial \alpha_i}{\partial x_i} - \frac{1}{2}\sum_{j \neq 4} \frac{\partial \beta_j}{\partial x_j} + \frac{1}{2}E_{1,4}(\eta) \in E, \end{aligned} \tag{A.105}$$

where $\alpha_i := x_1\omega_{i1}$ and $\beta_j := x_4\omega_{j4}$ and $E_{1,4}(\cdot)$ is an Euler operator.

$$\begin{aligned}K_2 = [K_1, K_0] =& 2D_1^2 + 2D_4^2 + (E_{1,4}(\beta_1) - 3\beta_1)D_1 + (E_{1,4}(\alpha_4) - 3\alpha_4)D_4 \\ &+ E_{1,4}(\alpha_3 + \beta_3)D_3 + E_{1,4}(\alpha_2 + \beta_2)D_2, \quad \text{mod } U_0 \in E,\end{aligned} \tag{A.106}$$

$$\begin{aligned}K_3 = K_2 - 2K_1 =& (E_{1,4}(\beta_1) - \beta_1)D_1 + (E_{1,4}(\alpha_4) - \alpha_4)D_4 \\ &+ [E_{1,4}(\alpha_2 + \beta_2) + 2(\alpha_2 + \beta_2)]D_2 \\ &+ [E_{1,4}(\alpha_3 + \beta_3) + 2(\alpha_3 + \beta_3)]D_3, \quad \text{mod } U_0 \in E \\ =& [E_{1,4}(\alpha_2 + \beta_2) + 2(\alpha_2 + \beta_2)]D_2 \\ &+ [E_{1,4}(\alpha_3 + \beta_3) + 2(\alpha_3 + \beta_3)]D_3, \quad \text{mod } U_0 \in E,\end{aligned} \tag{A.107}$$

where $\omega_{14} = 0$ is used in the third equality.

$$\begin{aligned}K_3 =& (4ax_1^2 + 4dx_4^2 + pol_1(x_1, x_4))D_2 \\ &+ (4bx_1^2 + 4ex_4^2 + pol_1(x_1, x_4))D_3, \quad \text{mod } U_0 \in E.\end{aligned} \tag{A.108}$$

In the following, we will prove coefficients of K_3 are all degree at most 1 polynomials which derives the conclusion of theorem.

$$\begin{aligned}[L_0, K_3] =& (8bx_1 + const)D_1D_3 + (8ex_4 + const)D_3D_4 \\ &+ (8ax_1 + const)D_1D_2 + (8dx_4 + const)D_2D_4, \quad \text{mod } U_1 \in E.\end{aligned} \tag{A.109}$$

Next we rewrite $[L_0, K_3]$ in terms of order of D_3.

$$\begin{cases} A_0 := \frac{1}{8}[L_0, K_3] \\ \quad = (bx_1 + const)D_1D_3 + (ex_4 + const)D_3D_4 \\ \quad\quad + \text{terms with lower order in } D_3, \quad \text{mod } U_1 \in E, \\ A_1 = \frac{1}{8}Ad_{L_0}^2 K_3 \\ \quad = bD_1^2D_3 + eD_3D_4^2 + \text{terms with lower order in } D_3, \quad \text{mod } U_2 \in E, \\ A_2 = \frac{1}{2}[A_1, A_0] \\ \quad = b^2D_1^2D_3^2 + e^2D_3^2D_4^2 + \text{terms with lower order in } D_3, \quad \text{mod } U_3 \in E, \\ \quad \vdots \\ A_k = \frac{1}{2}[A_{k-1}, A_0] \\ \quad = b^kD_1^2D_3^k + e^kD_3^kD_4^2 + \text{terms with lower order in } D_3, \quad \text{mod } U_{k+1} \in E. \end{cases} \tag{A.110}$$

If one of b, e is not zero, the order A_k will be $k+2$. By finite dimension of E, we derive that $b = e = 0$.

Then

$$\begin{aligned}[L_0, K_3] =& const \cdot D_1D_3 + const \cdot D_3D_4 + (8ax_1 + const)D_1D_2 \\ &+ (8dx_4 + const)D_2D_4, \quad \text{mod } U_1 \in E\end{aligned} \tag{A.111}$$

and

$$Ad^2_{L_0}K_3 = 8aD_1^2D_2 + 8dD_2D_4^2, \quad \mod U_2 \in E. \tag{A.112}$$

Next we consider the expanding operators $[L_0, K_3]$, $Ad^2_{L_0}K_3$ in terms of D_2 and we define

$$\begin{cases} B_0 := \frac{1}{8}[L_0, K_3] = (ax_1 + const)D_1D_2 + (dx_4 + const)D_2D_4 \\ \qquad + \text{terms with lower order in } D_2, \quad \mod U_1 \in E, \\ B_1 := \frac{1}{8}Ad^2_{L_0}K_3 = aD_1^2D_2 + dD_2D_4^2, \quad \mod U_2 \in E, \\ B_2 = \frac{1}{2}[B_1, B_0] = a^2D_1^2D_2^2 + d^2D_2^2D_4^2 + \text{terms with lower order in } D_2, \quad \mod U_3 \in E, \\ \quad \vdots \\ B_k = \frac{1}{2}[B_{k-1}, B_0] = a^kD_1^2D_2^k + d^kD_2^kD_4^2 + \text{terms with lower order in } D_2, \quad \mod U_{k+1} \in E. \end{cases}$$

By finite dimensionality of E, it can be derived that $a = d = 0$. ▮

A.18 Proof of Lemma 5.12

First we assume linear expression of $\omega_{21}, \omega_{31}, \omega_{32}$ as below:

$$\begin{cases} \omega_{21} = a_1x_1 + a_2x_2 + a_3x_3 + a_0, \\ \omega_{31} = b_1x_1 + b_2x_2 + b_3x_3 + b_0, \\ \omega_{32} = c_1x_1 + c_2x_2 + c_3x_3 + c_0. \end{cases} \tag{A.113}$$

In Case (V), there exists $p_0 = x_1^2 + x_2^2 + x_3^2 \in E \cap Q$ with the greastest quadratic rank, where Q is a vector space consisting of quadratic polynomials in terms of variables $x_i, 1 \leq i \leq 4$. Due to Lemma 3.5, for $\forall p \in E \cap Q$, $p^{(2)}$ is independent of x_4. Therefore, $\forall p \in E \cap Q$, $p^{(2)} \in Q_1 := \text{span}\{x_ix_j : 1 \leq i, j \leq 3\}$.

Next we assume that $p_1 \in E \cap Q$ has least quadratic rank in E. That means $1 \leq r(p_1) \leq r(p_0) = 3$. Specifically, we can assume

$$p_1 = \sum_{i,j \in \{1,2,3\}} A_{i,j}x_ix_j + d_0x_4 \in E, \tag{A.114}$$

where we used $x_i \in E$ for $1 \leq i \leq 3$. By an orthogonal transformation fixing x_4, quadratic part of p_1 can be diagonalized

$$p_1 = \sum_{i=1}^{k_1} d_ix_i^2 + d_0x_4 \in E, \tag{A.115}$$

where $1 \leq k_1 \leq 3$ and $d_i \neq 0$ for $1 \leq i \leq k_1$. By applying technique of Vandermonde matrix, strating from p_1 we can deduce $\sum_{i=1}^{k_1} x_i^2 \in E$. For this reason, with a little abuse of notation, we can assume that

$$p_1 = \sum_{i=1}^{k_1} x_i^2 \in E. \tag{A.116}$$

It is important to notice this orthogonal transformation keeps structure of linear rank and maximal rank quadratic polynomial unchange.

In the following, we only require to discuss three cases.

Case [1] Least quadratic rank $k_1 = 1$.

By $p_1 = x_1^2 \in E$, hence $p_2 = p_0 - p_1 = x_2^2 + x_3^2 \in E$. It follows that

$$\frac{1}{2}[L_0, p_1] - \frac{1}{2} = x_1 D_1 \in E \tag{A.117}$$

and

$$\frac{1}{2}[L_0, p_2] - 1 = x_2 D_2 + x_3 D_3 \in E. \tag{A.118}$$

Then

$$\begin{aligned}[x_1D_1, x_2D_2 + x_3D_3] =& x_1x_3\omega_{31} + x_1x_2\omega_{21} \\ =& b_1x_1^2x_3 + (b_2 + a_3)x_1x_2x_3 + b_3x_1x_3^2 \\ &+ a_1x_1^2x_2 + a_2x_1x_2^2 + b_0x_1x_3 + a_0x_1x_2 \in E.\end{aligned} \tag{A.119}$$

From $[x_1D_1, x_2D_2 + x_3D_3]$, Ocone's theorem implies $b_1 = b_3 = a_1 = a_2 = b_2 + a_3 = 0$. It shows that $\omega_{31} \in P_1(x_2), \omega_{21} \in P_1(x_3)$.

Notice that $D_i \in E, 1 \le i \le 3$. Hence,

$$[D_2, x_1D_1] = x_1\omega_{12} = -a_3x_1x_3 - a_0x_1 \in E \Longrightarrow a_3x_1x_3 \in E \tag{A.120}$$

and

$$[D_3, x_1D_1] = x_1\omega_{13} = -b_2x_1x_2 - b_0x_1 \in E \Longrightarrow b_2x_1x_2 \in E. \tag{A.121}$$

Step [1.1] We claim $a_3 = b_2 = 0$, i.e., ω_{21}, ω_{31} are constants.

We can seperately deal with a_3, b_2. First we assume $a_3 \ne 0$. Then $x_1x_3 \in E$. Then

$$[[L_0, x_1x_3], x_1x_3] = x_1^2 + x_3^2 \in E. \tag{A.122}$$

Noticing $p_0 = x_1^2 + x_2^2 + x_3^2 \in E$, then $x_2^2 \in E$ holds. Lemma 4.1 implies ω_{12} is constant, a contradition! Then $a_3 = 0$. Same statement holds for b_2. Then $a_3 = b_2 = 0$. Then ω_{21}, ω_{31} are constants.

Step [1.2] ω_{32} is a constant.

By cyclic condition, we get $c_1 = 0$. Then

$$Z_1 := [D_2, x_2D_2 + x_3D_3] = D_2 + x_3\omega_{32} = c_2x_2x_3 + c_3x_3^2 \in E. \tag{A.123}$$

Similarly,

$$Z_2 := [D_3, x_2D_2 + x_3D_3] = D_3 + x_2\omega_{23} = c_2x_2^2 + c_3x_2x_3 \in E. \tag{A.124}$$

Notice $p_2 = x_2^2 + x_3^2 \in E$ and Lemma 4.2 implies ω_{32} is a constant.

Therefore, in Case [1], $\omega_{21}, \omega_{31}, \omega_{32}$ are constants.

Case [2] Least quadratic rank $k_1 = 2$.

In this case, $p_1 = x_1^2 + x_2^2 \in E \Longrightarrow p_2 = p_0 - p_1 = x_3^2 \in E$. Considering x_1, x_2, x_3 play an equal role in estimation algebra, we can obtain the same result by repeating procedure of Case [1].

Case [3] Least quadratic rank $k_1 = 3$.

We claim that for any $p \in E \cap Q$, $p^{(2)} = \lambda p_1$ for some λ, where $p_1 = x_1^2 + x_2^2 + x_3^2$ has greatest and least qudratic rank. We denote set $S = \{1, 2, 3\}$ then $p \in E \cap Q$ can be written as $p^{(2)} = \sum_{p,q \in S} a_{pq} x_p x_q = x^{\mathrm{T}} A x$, where A is a symmetric matrix. If $p^{(2)} \neq \lambda p_1$ for any λ, then $r(A - \lambda I) > 0$ for any λ. If we pick λ_0 is an eigenvalue of A, then $0 < r(A - \lambda_0 I) < 3$. Then $p - \lambda_0 p_0$ has a lower positive rank than p_0. A contradiction!

In the following, we calculate

$$K_0 := \frac{1}{2}[L_0, p_0] - \frac{3}{2} = \sum_{i=1}^{3} x_i D_i \in E. \tag{A.125}$$

By linear rank condition, we use $D_i, 1 \leq i \leq 3$ to get bracket with K_0 and obtain some functions in E,

$$\begin{aligned} A_1 :=& [D_1, K_0] = x_2\omega_{21} + x_3\omega_{31}, \quad \text{mod } E_0 \in E, \\ A_2 :=& [D_2, K_0] = x_1\omega_{12} + x_3\omega_{32}, \quad \text{mod } E_0 \in E, \\ A_3 :=& [D_3, K_0] = x_1\omega_{13} + x_2\omega_{23}, \quad \text{mod } E_0 \in E, \end{aligned} \tag{A.126}$$

where E_0 is a vector space generated by operators $L_0, x_i, D_i, 1$ $(1 \leq i \leq 3)$. By using the concrete expression form of $\omega_{21}, \omega_{31}, \omega_{32}$, we can get

$$\begin{aligned} A_1 \in& E \Longrightarrow \widetilde{A}_1 = [0, a_2, b_3, a_1, a_3 + b_2, b_1] \cdot X_{23} \in E, \\ A_2 \in& E \Longrightarrow \widetilde{A}_2 = [-a_1, 0, c_3, -a_2, c_2, c_1 - a_3] \cdot X_{23} \in E, \\ A_3 \in& E \Longrightarrow \widetilde{A}_3 = [b_1, c_2, 0, b_2 + c_1, c_3, b_3] \cdot X_{23} \in E, \end{aligned} \tag{A.127}$$

where $X_{23} = [x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3, x_1x_3]^{\mathrm{T}}$. Due to the property of least quadratic rank, we get

$$\begin{cases} a_1 = a_2 = b_1 = b_3 = c_2 = c_3 = 0, \\ a_3 + b_2 = c_1 - a_3 = b_2 + c_1 = 0. \end{cases} \tag{A.128}$$

It shows that $\omega_{21} \in P_1(x_3), \omega_{31} \in P_1(x_2), \omega_{32} \in P_1(x_1)$. Considering cyclic condition satisfied by Ω, we get $a_3 - b_2 + c_1 = 0$. By solving the linear equations satisfied by coefficients a_3, b_2, c_1, we get $a_3 = b_2 = c_1 = 0$. It means that $\omega_{21}, \omega_{31}, \omega_{32}$ are constants. ∎

A.19 Proof of Lemma 5.13

Through some basic computations, we get

$$K_0 := \frac{1}{2}[L_0, p_0] - \frac{3}{2} = \sum_{i=1}^{3} x_i D_i \in E \tag{A.129}$$

and

$$[L_0, K_0] = \sum_{i=1}^{3} D_i^2 - \sum_{i\neq 1} \alpha_i D_i - \sum_{i\neq 2} \beta_i D_i - \sum_{i\neq 3} \gamma_i D_i \\ - \frac{1}{2} \sum_{i\neq 1} \frac{\partial \alpha_i}{\partial x_i} - \frac{1}{2} \sum_{i\neq 2} \frac{\partial \beta_i}{\partial x_i} - \frac{1}{2} \sum_{i\neq 3} \frac{\partial \gamma_i}{\partial x_i} + \frac{1}{2} E_{1,2,3}(\eta) \in E, \tag{A.130}$$

where $\alpha_i := x_1\omega_{i1}, \beta_i := x_2\omega_{i2}$ and $\gamma_i := x_3\omega_{i3}$.

$$L_2 := Ad_{L_0} K_0 - 2L_0 = - D_4^2 - (\beta_1 + \gamma_1) D_1 - (\alpha_2 + \gamma_2) D_2 \\ - (\alpha_3 + \beta_3) D_3 - (\alpha_4 + \beta_4 + \gamma_4) D_4, \quad \mathrm{mod}\ U_0 \in E \tag{A.131}$$

and

$$L_3 := [K_0, L_2] = \left[2 \sum_{i=1}^{3} x_i \omega_{i4} - E_{1,2,3}(\alpha_4 + \beta_4 + \gamma_4) \right] D_4, \quad \mathrm{mod}\ U_0 \in E, \tag{A.132}$$

where we used the result of Lemma 5.12 that $\omega_{21}, \omega_{31}, \omega_{32}$ are constants.

In the following, we assume the linear form of $\omega_{14}, \omega_{24}, \omega_{34}$ concretely,

$$\begin{cases} \omega_{14} = \sum_{i=1}^{4} p_i x_i + p_0, \\ \omega_{24} = \sum_{i=1}^{4} q_i x_i + p_0, \\ \omega_{34} = \sum_{i=1}^{4} l_i x_i + p_0. \end{cases} \tag{A.133}$$

By the linear form of $\omega_{14}, \omega_{24}, \omega_{34}$, L_3 can be further calculated as below:

$$L_3 = [4p_1 x_1^2 + 4(p_2 + q_1) x_1 x_2 + 4(p_3 + l_1) x_1 x_3 \\ + 4q_2 x_2^2 + 4(q_3 + l_2) x_2 x_3 + 4l_3^2 x_3^2 + 3p_4 x_4 x_1 \\ + 3q_4 x_4 x_2 + 3l_4 x_4 x_3 + pol_1(x_1, x_2, x_3)] D_4, \quad \mathrm{mod}\ U_0 \in E. \tag{A.134}$$

Theorem 2.18 shows that coefficients of D_4 in L_3 are all degree at most 1 polynomials. Hence

$$p_1 = p_2 + q_1 = p_3 + l_1 = q_2 = q_3 + l_2 = l_3 = p_4 = q_4 = l_4 = 0. \tag{A.135}$$

It only remains to prove $p_2 = q_1 = p_3 = l_1 = q_3 = l_2 = 0$. Similarly, by cyclical condition satisfied by Ω,

$$\begin{cases} \frac{\partial \omega_{14}}{\partial x_2} + \frac{\partial \omega_{42}}{\partial x_1} + \frac{\partial \omega_{21}}{\partial x_4} = 0 \Longrightarrow p_2 - q_1 = 0, \\ \frac{\partial \omega_{14}}{\partial x_3} + \frac{\partial \omega_{43}}{\partial x_1} + \frac{\partial \omega_{31}}{\partial x_4} = 0 \Longrightarrow p_3 - l_1 = 0, \\ \frac{\partial \omega_{24}}{\partial x_3} + \frac{\partial \omega_{43}}{\partial x_2} + \frac{\partial \omega_{32}}{\partial x_4} = 0 \Longrightarrow q_3 - l_2 = 0, \end{cases} \tag{A.136}$$

where we used the result of Lemma 5.12 that $\omega_{21}, \omega_{31}, \omega_{32}$ are constants.

Combining the equations (A.135) and (A.136), it is directly derived that $\omega_{14}, \omega_{24}, \omega_{34}$ are constants. ∎

A.20 Proof of Lemma 5.15

First we do some basic computations.

$$K_0 := \frac{1}{2}[L_0, p_0] - \frac{3}{2} = x_1 D_1 + x_2 D_2 + x_4 D_4 \in E. \tag{A.137}$$

By using $D_i, 1 \le i \le 3$ to get bracket with K_0, we obtain

$$A_1 := [D_1, K_0] - D_1 = x_2\omega_{21} + x_4\omega_{41} \in E, \tag{A.138}$$

$$A_2 := [D_2, K_0] - D_2 = x_1\omega_{12} + x_4\omega_{42} \in E, \tag{A.139}$$

$$A_3 := [D_3, K_0] = x_1\omega_{13} + x_2\omega_{23} + x_4\omega_{43} \in E. \tag{A.140}$$

By the restriction of maximal rank quadratic polynomial, for any function $p \in E \cap Q$, then $p^{(2)}$ is independent of x_3. By using $A_1 \in E$, it derives $\omega_{21} \in P_1(x_1, x_2)$ and $\omega_{41} \in P_1(x_1, x_2, x_4)$. Similarly, by using $A_2 \in E$, it derives $\omega_{42} \in P_1(x_1, x_2, x_4)$. By using $A_3 \in E$, we have $\omega_{13} \in P_1(x_1.x_2), \omega_{23} \in P_1(x_1.x_2), \omega_{43} \in P_1(x_1, x_2, x_4)$.

Next due to linear rank condition, Lemma 3.2 implies for any function $p \in E \cap Q$, $p^{(2)}$ does not contain term $x_i x_4, 1 \le i \le 3$. Then by using $A_1 \in E$, we derive $\omega_{41} \in P_1(x_4)$. Similarly, $\omega_{42} \in P_1(x_4)$ and $\omega_{43} \in P_1(x_4)$. ∎

A.21 Proof of Lemma 5.16

In this proof, we still start with the tool of minimal qudratic structure proposed in Section 3. We assume $p_1 \in E$ has minimal quadratic rank, i.e., $1 \le r(p_1) \le r(p_0) = 3$. Following the discussion of Section 3, minimal rank quadratic polynomial can be following two forms:

$$E \ni p_1 = \begin{cases} \sum_{i=1}^{k_1} x_i^2, & 1 \le k_1 \le 2, \\ \sum_{i=1}^{k_1} x_i^2 + x_4^2, & 0 \le k_1 \le 2. \end{cases} \tag{A.141}$$

In the following, we discuss three cases and determine more structure of Ω.

Case [1] $r(p_1) = 3$.

It means that $p_1 = x_1^2 + x_2^2 + x_4^2 \in E$ has maximal and minimal quadratic rank. Then Theorem 3.7 implies for any quadratic polynomial $p \in E$, $p^{(2)}(x) = \lambda p_1$ for some λ. Recall

$$A_1 = x_2\omega_{21} + x_4\omega_{41} \in E \tag{A.142}$$

and

$$A_2 = x_1\omega_{12} + x_4\omega_{42} \in E. \tag{A.143}$$

It derives that ω_{ij}'s are constants for $i, j \in \{1, 2, 4\}$.

Case [2] $r(p_1) = 2$.

Case [2.1] $p_1 = x_1^2 + x_2^2 \in E$.

Then $p_2 := p_0 - p_1 = x_4^2 \in E$. It follows that

$$\frac{1}{2}[L_0, x_1^2 + x_2^2] - 1 = x_1 D_1 + x_2 D_2 \in E \tag{A.144}$$

and

$$\frac{1}{2}[L_0, x_4^2] - \frac{1}{2} = x_4 D_4 \in E. \tag{A.145}$$

Then

$$\begin{aligned}[x_1 D_1 + x_2 D_2, x_4 D_4] =& x_1 x_4 \omega_{41} + x_2 x_4 \omega_{42} \in E \\ =& d x_1 x_4^2 + e x_2 x_4^2 + pol_2(x_1, x_2, x_4) \in E.\end{aligned} \tag{A.146}$$

Then by Ocone's lemma, it leads to $d = e = 0$, i.e., ω_{41}, ω_{42} are constant. Next we calculate

$$[D_1, x_1 D_1 + x_2 D_2] = D_1 + x_2 \omega_{21} \in E \Longrightarrow a_1 x_1 x_2 + a_2 x_2^2 \in E, \tag{A.147}$$

$$[D_2, x_1 D_1 + x_2 D_2] = D_2 + x_1 \omega_{12} \in E \Longrightarrow a_1 x_1^2 + a_2 x_1 x_2 \in E. \tag{A.148}$$

Considering $p_1 = x_1^2 + x_2^2 \in E$ and Lemma 4.2 implies ω_{21} is a constant.

Case [2.2] $p_1 = x_1^2 + x_4^2 \in E$.

Then $p_2 := p_0 - p_1 = x_2^2$. Then we calculate

$$\frac{1}{2}[L_0, p_1] - 1 = x_1 D_1 + x_4 D_4 \in E, \tag{A.149}$$

$$\frac{1}{2}[L_0, p_2] - \frac{1}{2} = x_2 D_2 \in E. \tag{A.150}$$

Then

$$\begin{aligned}[x_2 D_2, x_1 D_1 + x_4 D_4] =& x_1 x_2 \omega_{12} + x_2 x_4 \omega_{42} \\ =& - a_1 x_1^2 x_2 - a_2 x_1 x_2^2 + e x_2 x_4^2 + pol_2(x_1, x_2, x_4) \in E.\end{aligned} \tag{A.151}$$

It follows that $a_1 = a_2 = e = 0 \Longrightarrow \omega_{21}, \omega_{42}$ are constant. Next we calculate

$$[D_1, x_1 D_1 + x_4 D_4] - D_1 = x_4 \omega_{41} \in E \Longrightarrow d x_4^2 + d_0 x_4 \in E. \tag{A.152}$$

Hence $d = 0$, otherwise it is contradictory to p_1 that has least quadratic rank. Therefore, ω_{41} is constant.

Case [3] $r(p_1) = 1$.

Case [3.1] $p_1 = x_1^2 \in E$.

Then $p_2 := p_0 - p_1 = x_2^2 + x_4^2$. Then we calculate

$$\frac{1}{2}[L_0, p_1] - \frac{1}{2} = x_1 D_1 \in E, \tag{A.153}$$

$$\frac{1}{2}[L_0, p_2] - 1 = x_2 D_2 + x_4 D_4 \in E. \tag{A.154}$$

Then

$$\begin{aligned}[x_1 D_1, x_2 D_2 + x_4 D_4] =& x_1 x_2 \omega_{21} + x_1 x_4 \omega_{41} \\ =& a_1 x_1^2 x_2 + a_2 x_1 x_2^2 + d x_1 x_4^2 + pol_2(x_1, x_2, x_4) \in E.\end{aligned} \tag{A.155}$$

It follows that $a_1 = a_2 = d = 0 \Longrightarrow \omega_{21}, \omega_{41}$ are constant. Next we calculate

$$[D_2, x_2D_2 + x_4D_4] - D_2 = x_4\omega_{42} = ex_4^2 + e_0x_4 \in E. \tag{A.156}$$

In the following, by using operator $x_2D_2 + x_4D_4 \in E$, it follows

$$[x_2D_2 + x_4D_4, x_4\omega_{42}] = E_{2,4}(x_4\omega_{42}) = 2ex_4^2 + e_0x_4 \in E. \tag{A.157}$$

Next, we derive $ex_4^2 \in E$. If $e \neq 0$, then $x_4^2 \in E \Longrightarrow x_2^2 \in E$. It follows

$$\frac{1}{4}[[L_0, x_2^2], [L_0, x_4^2]] = x_2x_4\omega_{42} \in E. \tag{A.158}$$

Then ω_{42} is a constant, a contradiction! Therefore, we obtain $e = 0$, i.e., ω_{42} is a constant.
Case [3.2] $p_1 = x_4^2 \in E$.
By using the same arguments of Case [2.1], we can prove $\omega_{21}, \omega_{41}, \omega_{42}$ are constants.
Up to now, we proved ω_{ij}'s are constants for $i, j \in \{1, 2, 4\}$ for Case (VI). Furthermore, we recall expression of $A_1, A_2 \in E$, it is easily to get $\omega_{41} = \omega_{42} = 0$ due to linear rank condition. ▮

A.22 Proof of Theorem 5.17

Similarly by some basic computation similarly to previous cases, we get

$$\begin{aligned} K_1 :=& [L_0, K_0] \\ =& D_1^2 - \sum_{i\neq 1}\alpha_iD_i - \frac{1}{2}\sum_{i\neq 1}\frac{\partial\alpha_i}{\partial x_i} + \frac{1}{2}E_1(\eta) \\ &+ D_2^2 - \sum_{i\neq 2}\beta_iD_i - \frac{1}{2}\sum_{i\neq 2}\frac{\partial\beta_i}{\partial x_i} + \frac{1}{2}E_2(\eta) \\ &+ D_4^2 - \sum_{i\neq 4}\gamma_iD_i - \frac{1}{2}\sum_{i\neq 4}\frac{\partial\gamma_i}{\partial x_i} + \frac{1}{2}E_4(\eta) \\ =& D_1^2 + D_2^2 + D_4^2 - (\beta_1 + \gamma_1)D_1 - (\alpha_2 + \gamma_2)D_2 \\ &- (\alpha_3 + \beta_3 + \gamma_3)D_3 - (\alpha_4 + \beta_4)D_4, \quad \text{mod } U_0 \in E, \end{aligned} \tag{A.159}$$

where $\alpha_i := x_1\omega_{i1}, \beta_i := x_2\omega_{i2}$ and $\gamma_i := x_4\omega_{i4}$.

$$\begin{aligned} K_2 :=& [K_1, K_0] \\ =& 2D_1^2 + 2D_2^2 + 2D_4^2 + (E_{1,2,4}(\beta_1 + \gamma_1) - 3(\beta_1 + \gamma_1))D_1 \\ &+ (E_{1,2,4}(\alpha_2 + \gamma_2) - 3(\alpha_2 + \gamma_2))D_2 + (E_{1,2,4}(\alpha_4 + \beta_4) - 3(\alpha_4 + \beta_4))D_4 \\ &+ E_{1,2,4}(\alpha_3 + \beta_3 + \gamma_3)D_3, \quad \text{mod } U_0 \in E. \end{aligned} \tag{A.160}$$

Then

$$\begin{aligned} K_3 := K_2 - 2K_1 =& (E_{1,2,4}(\beta_1 + \gamma_1) - (\beta_1 + \gamma_1))D_1 + (E_{1,2,4}(\alpha_2 + \gamma_2) - (\alpha_2 + \gamma_2))D_2 \\ &+ (E_{1,2,4}(\alpha_4 + \beta_4) - (\alpha_4 + \beta_4))D_4 + (E_{1,2,4}(\alpha_3 + \beta_3 + \gamma_3) \\ &+ 2(\alpha_3 + \beta_3 + \gamma_3))D_3, \quad \text{mod } U_0 \in E \\ =& (E_{1,2,4}(\alpha_3 + \beta_3 + \gamma_3) + 2(\alpha_3 + \beta_3 + \gamma_3))D_3, \quad \text{mod } U_0 \in E. \end{aligned} \tag{A.161}$$

By using the following substitution.

$$\begin{cases} \beta_1 = x_2\omega_{12} \in P_1(x_2), \quad \gamma_1 = x_4\omega_{14} = 0, \\ \alpha_2 = x_1\omega_{21} \in P_1(x_1), \quad \gamma_2 = x_4\omega_{24} = 0, \\ \alpha_4 = x_1\omega_{41} = 0, \quad \beta_4 = x_2\omega_{42} = 0, \\ \alpha_3 = x_1\omega_{31} = b_1x_1^2 + b_2x_1x_2 + b_0x_1, \\ \beta_3 = x_2\omega_{32} = cx_1x_2 + c_2x_2^2 + c_0x_2, \\ \gamma_3 = x_4\omega_{34} = lx_4^2 + l_0x_4, \end{cases} \tag{A.162}$$

K_3 can be simplified as follows:

$$\begin{aligned} M_1 :=& \frac{1}{4}K_3 \\ =& (b_1x_1^2 + (b_2 + c_1)x_1x_2 + c_2x_2^2 + lx_4^2 + pol_1(x_1, x_2, x_4))D_3, \quad \mod U_0 \in E \\ =& (b_1x_1^2 + 2b_2x_1x_2 + c_2x_2^2 + lx_4^2 + pol_1(x_1, x_2, x_4))D_3, \quad \mod U_0 \in E, \end{aligned} \tag{A.163}$$

where we use cyclical condition satisfied by Ω and it derives $b_2 = c_1$.

In the following, we construct infinite sequence

$$\begin{aligned} W_1 :=& \frac{1}{2}[L_0, M_1] = (b_1x_1 + b_2x_2 + const)D_1D_3 + (b_2x_1 + c_2x_2 + const)D_2D_3 \\ & + (lx_4 + const)D_4D_3, \quad \mod U_1 \in E, \\ W_2 :=& \frac{1}{2}Ad_{L_0}^2 M_1 = b_1D_1^2D_3 + 2b_2D_1D_2D_3 + c_2D_2^2D_3 + lD_4^2D_3, \quad \mod U_2 \in E, \\ W_3 :=& \frac{1}{2}[W_2, W_1] = const \cdot D_1^2D_3^2 + const \cdot D_1D_2D_3^2 + const \cdot D_2^2D_3^2 \\ & + l^2D_4^2D_3^2, \quad \mod U_3 \in E, \\ W_4 :=& \frac{1}{2}[W_3, W_1] = const \cdot D_1^2D_3^3 + const \cdot D_1D_2D_3^3 + const \cdot D_2^2D_3^3 \\ & + l^2D_4^2D_3^3, \quad \mod U_4 \in E, \quad \cdots \end{aligned} \tag{A.164}$$

By repeating such procedure, it can be obtained that

$$\begin{aligned} W_m =& const \cdot D_1^2D_3^{m-1} + const \cdot D_1D_2D_3^{m-1} \\ & + const \cdot D_2^2D_3^{m-1} + l^2D_4^2D_3^{m-1}, \quad \mod U_m \in E. \end{aligned} \tag{A.165}$$

Due to finite dimensionality of E, we can deduce $l = 0$. Next we notice that

$$-A_3 = b_1x_1^2 + 2b_2x_1x_2 + c_2x_2^2 + b_0x_1 + c_0x_2 + l_0x_4 \in E \tag{A.166}$$

and

$$\frac{1}{2}\left[[L_0, p_0], -A_3\right] = E_{1,2,4}(-A_3) = 2(b_1x_1^2 + 2b_2x_1x_2 + c_2x_2^2) + b_0x_1 + c_0x_2 + l_0x_4 \in E. \tag{A.167}$$

The above two equations yields that $l_0x_4 \in E \Longrightarrow l_0 = 0$. It means $\omega_{34} = 0$.

Then M_1 can be reduced as follows

$$M_1 :=(b_1x_1^2 + 2b_2x_1x_2 + c_2x_2^2 + pol_1(x_1, x_2))D_3, \quad \text{mod } U_0 \in E. \tag{A.168}$$

And we will derive the following operators:

$$\begin{aligned} W_1 =&\frac{1}{2}[L_0, M_1] \\ =&(b_1x_1 + b_2x_2 + const)D_1D_3 + (b_2x_1 + c_2x_2 + const)D_2D_3, \quad \text{mod } U_1 \in E, \\ Z_1 :=&\frac{1}{2}[D_1, M_1] = (b_1x_1 + b_2x_2 + const)D_3, \quad \text{mod } U_0 \in E, \\ Z_2 :=&\frac{1}{2}[D_2, M_1] = (b_2x_1 + c_2x_2 + const)D_3, \quad \text{mod } U_0 \in E. \end{aligned} \tag{A.169}$$

Lemma 2.17 implies $b_1 = b_2 = c_2 = 0$. It means that ω_{31}, ω_{32} are constants. ∎

A.23 Proof of Lemma 5.18

Notice $\omega_{12}, \omega_{13}, \omega_{23} \in P_1(x_1, x_2, x_3)$ and $\omega_{4i} \in P_1(x_1, x_2, x_3, x_4)$ for $1 \le i \le 3$. By linear rank condition, Lemma 3.2 shows that for any function $p \in E$, p does not contain term $x_ix_4, 1 \le i \le 3$. It directly implies $\omega_{4i} \in P_1(x_4)$ from $A_i \in E$ for $1 \le i \le 3$. ∎

A.24 Proof of Theorem 5.19

In this proof, we still start with the tool of minimal qudratic structure proposed in Section 3. We assume $p_1 \in E$ has minimal quadratic rank, i.e., $1 \le r(p_1) \le r(p_0) = 4$. Following the discussion of Section 3, minimal rank quadratic polynomial can be following two forms:

$$E \ni p_1 = \begin{cases} \displaystyle\sum_{i=1}^{k_1} x_i^2, & 1 \le k_1 \le 3, \\ \displaystyle\sum_{i=1}^{k_1} x_i^2 + x_4^2, & 0 \le k_1 \le 3. \end{cases} \tag{A.170}$$

Case [1] $r(p_1) = 4$, i.e., $p_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \in E$.

Structure of minimal rank quadratic polynomial shows that (3.7) for any function $p \in E$, $p^{(2)} = \lambda p_1$ for some number λ. Observe $A_i \in E$ for $1 \le i \le 3$ and it implies $A_i^{(2)} = 0$ for $1 \le i \le 3$. Then $a_1 = a_2 = b_1 = b_3 = a_3 + b_2 = l_1 = c_1 - a_3 = c_2 = c_3 = m_1 = b_2 + c_1 = n_1 = 0$ which implies $\omega_{21} \in P_1(x_3), \omega_{31} \in P_1(x_2), \omega_{32} \in P_1(x_1)$ and ω_{4i}'s are constants for $1 \le i \le 3$. Considering cyclical condition satisfied by Ω, it follows

$$\frac{\partial \omega_{21}}{\partial x_3} + \frac{\partial \omega_{13}}{\partial x_2} + \frac{\partial \omega_{32}}{\partial x_1} = 0 \Longrightarrow a_3 - b_2 + c_1 = 0. \tag{A.171}$$

By considering the equations

$$\begin{cases} a_3 - b_2 + c_1 = 0, \\ a_3 + b_2 = 0, \\ c_1 - a_3 = 0, \\ b_2 + c_1 = 0, \end{cases} \tag{A.172}$$

it implies $a_3 = b_2 = c_1 = 0 \Longrightarrow \omega_{ij}$'s are constants for $i, j \in \{1, 2, 3\}$.

Case [2.1] $r(p_1(x)) = 3$, i.e., $p_1 = x_1^2 + x_2^2 + x_3^2 \in E \cap Q$.

It is easy to get $p_2 := p_0 - p_1 = x_4^2 \in E$. It follows that

$$\frac{1}{2}[L_0, p_2] - \frac{1}{2} = x_4 D_4 \in E, \tag{A.173}$$

$$\frac{1}{2}[L_0, p_1] - \frac{3}{2} = \sum_{i=1}^{3} x_i D_i \in E. \tag{A.174}$$

Then

$$\begin{aligned}\left[\sum_{i=1}^{3} x_i D_i, x_4 D_4\right] &= \sum_{i=1}^{3} x_4 x_i \omega_{4i} \\ &= l_1 x_1 x_4^2 + m_1 x_2 x_4^2 + n_1 x_3 x_4^2 + pol_2(x) \in E.\end{aligned} \tag{A.175}$$

Ocone's theorem shows $l_1 = m_1 = n_1 = 0$, i.e., ω_{4i}'s are constants for $1 \leq i \leq 3$. Then $A_i^{(2)} \in P_2(x_1, x_2, x_3)$ for $1 \leq i \leq 3$. Theorem 3.7 implies $A_i^{(2)} = \lambda_i p_1$ for some $\lambda_i \in \mathbb{R}$ and $1 \leq i \leq 3$. It follows that $\lambda_i = 0$. Similarly, by combining cyclical condition satisfied by Ω, we get that ω_{ij}'s are constants for $i, j \in \{1, 2, 3\}$.

Case [2.2] $r(p_1(x)) = 3$, i.e., $p_1 = x_1^2 + x_2^2 + x_4^2 \in E \cap Q$.

First we observe $p_2 := p_0 - p_1 = x_3^2 \in E$. It follows

$$\frac{1}{2}[L_0, p_2] - \frac{1}{2} = x_3 D_3 \in E, \tag{A.176}$$

$$\frac{1}{2}[L_0, p_1] - \frac{3}{2} = \sum_{i \neq 3} x_i D_i \in E. \tag{A.177}$$

Then

$$\begin{aligned}\left[\sum_{i \neq 3} x_i D_i, x_3 D_3\right] = \sum_{i \neq 3} x_3 x_i \omega_{3i} &= b_1 x_1^2 x_3 + (b_2 + c_1) x_1 x_2 x_3 + b_3 x_1 x_3^2 \\ &\quad + c_2 x_2^2 x_3 + c_3 x_2 x_3^2 - n_1 x_4^2 x_3 + pol_2(x) \in E.\end{aligned} \tag{A.178}$$

It leads to $b_1 = b_2 + c_1 = b_3 = c_2 = c_3 = n_1 = 0$, i.e., ω_{43} is constant and $\omega_{31} \in P_1(x_2), \omega_{32} \in P_1(x_1)$. Furthermore,

$$[D_1, x_3 D_3] - b_0 x_3 = x_3 \omega_{31} - b_0 x_3 = b_2 x_2 x_3 \in E. \tag{A.179}$$

If $b_2 \neq 0$, then $r([D_1, x_3 D_3] - b_0 x_3) = 2 < r(p_1)$ which is a contradiction. Therefore, we get $b_2 = 0 \Longrightarrow c_1 = 0$, i.e., ω_{31}, ω_{32} are constants.

Equation (A.171) implies $a_3 = 0$. Additionally, A_1, A_2 can be rewritten as

$$A_1^{(2)} = a_1 x_1 x_2 + a_2 x_2^2 + l_1 x_4^2, \tag{A.180}$$

$$A_2^{(2)} = -a_1 x_1^2 - a_2 x_1 x_2 + m_1 x_4^2. \tag{A.181}$$

Observe that $A_1^{(2)}, A_2^{(2)}$ are only dependent on x_1, x_2, x_4. Then $A_i^{(2)} = \lambda_i p_1$ for some λ_i and $1 \leq i \leq 2$. It implies $\lambda_1 = \lambda_2 = 0 \Longrightarrow a_1 = a_2 = l_1 = m_1 = 0 \Longrightarrow \omega_{21}, \omega_{41}, \omega_{42}$ are constants.

Case [3.1] $r(p_1(x)) = 2$, i.e., $p_1 = x_1^2 + x_2^2 \in E \cap Q$.
Similarly, we first get $p_2 = p_0 - p_1 = x_3^2 + x_4^2 \in E$. It follows that

$$\frac{1}{2}[L_0, p_1] - 1 = x_1 D_1 + x_2 D_2 \in E, \tag{A.182}$$

$$\frac{1}{2}[L_0, p_2] - 1 = x_3 D_3 + x_4 D_4 \in E. \tag{A.183}$$

Then

$$\begin{aligned}[x_1D_1 + x_2D_2, x_3D_3 + x_4D_4] =& x_1x_3\omega_{31} + x_1x_4\omega_{41} + x_2x_3\omega_{32} + x_2x_4\omega_{42} \\ =& b_1x_1^2x_3 + b_3x_1x_3^2 + l_1x_1x_4^2 + c_2x_2^2x_3 + c_3x_2x_3^2 \\ &+ m_1x_2x_4^2 + (b_2 + c_1)x_1x_2x_3 + pol_2(x) \in E.\end{aligned} \tag{A.184}$$

Therefore, we get $b_1 = b_3 = l_1 = c_2 = c_3 = m_1 = b_2 + c_1 = 0$, i.e., $\omega_{31} \in P_1(x_2), \omega_{32} \in P_1(x_1)$ and ω_{41}, ω_{42} are constants. Observing that $A_3^{(2)} = n_1x_4^2$, it follows $n_1 = 0$ since least quadratic rank is two. It means ω_{43} is a constant.

Next we calculate

$$\begin{cases} E \ni [D_1, x_1D_1 + x_2D_2] \Longrightarrow E \ni x_2\omega_{21} = a_1x_1x_2 + a_2x_2^2 + a_3x_2x_3 + a_0x_2, \\ E \ni [D_2, x_1D_1 + x_2D_2] \Longrightarrow E \ni x_1\omega_{21} = a_1x_1^2 + a_2x_1x_2 + a_3x_1x_3 + a_0x_1, \\ E \ni [D_3, x_1D_1 + x_2D_2] \Longrightarrow E \ni x_1\omega_{31} + x_2\omega_{32}, \\ E \ni [D_1, x_3D_3 + x_4D_4] \Longrightarrow E \ni x_3\omega_{31} + x_4\omega_{41} = b_2x_2x_3 + b_0x_3 + l_0x_4, \\ E \ni [D_2, x_3D_3 + x_4D_4] \Longrightarrow E \ni x_3\omega_{32} + x_4\omega_{42} = c_1x_1x_3 + c_0x_3 + m_0x_4, \\ E \ni [D_3, x_3D_3 + x_4D_4] \Longrightarrow E \ni x_4\omega_{43}. \end{cases} \tag{A.185}$$

In the following, we claim $b_2 = c_1 = 0$. Otherwise, if $b_2 \neq 0$,

$$\xi := \frac{1}{b_2}[D_1, x_3D_3 + x_4D_4] - \frac{b_0}{b_2}x_3 = x_2x_3 + \frac{l_0}{b_2}x_4 \in E, \tag{A.186}$$

which implies

$$[Ad_{L_0}\xi, \xi] - \left(\frac{l_0}{b_2}\right)^2 = \sum_{i=1}^{n}\left(\frac{\partial \xi}{\partial x_i}\right)^2 - \left(\frac{l_0}{b_2}\right)^2 = x_2^2 + x_3^2 \in E. \tag{A.187}$$

Then

$$\frac{1}{2}[L_0, x_2^2 + x_3^2] - 1 = x_2D_2 + x_3D_3 \in E \tag{A.188}$$

and

$$\begin{aligned}[x_1D_1 + x_2D_2, x_2D_2 + x_3D_3] =& x_1x_2\omega_{21} + x_1x_3\omega_{31} + x_2x_3\omega_{32} \\ =& a_1x_1^2x_2 + a_2x_1x_2^2 + (a_3 + b_2 + c_1)x_1x_2x_3 + pol_2(x) \\ =& a_1x_1^2x_2 + a_2x_1x_2^2 + a_3x_1x_2x_3 + pol_2(x) \in E.\end{aligned} \tag{A.189}$$

It implies $a_1 = a_2 = a_3 = 0$. By cyclical condition of Ω, it is direct to get $b_2 = c_1$. Combining $b_2 + c_1 = 0$, it is obvious that $b_2 = c_1 = 0$, contradiction! Hence, $b_2 = 0$, i.e., ω_{31} is a constant. It leads to $c_1 = 0$, i.e., ω_{32} is a constant.

By the equation (A.171), it implies $a_3 = 0$, i.e., $\omega_{21} \in P_1(x_1, x_2)$. Then $E \ni A_1 = a_1x_1x_2 + a_2x_2^2 + pol_1(x)$. Considering $p_1 = x_1^2 + x_2^2 \in E$ with least quadratic rank in E, $A_1^{(2)} = \lambda p_1$ which implies $a_1 = a_2 = 0$, i.e., ω_{21} is a constant.

Case [3.2] $r(p_1(x)) = 2$, i.e., $p_1 = x_1^2 + x_4^2 \in E \cap Q$.

Naturally, we get $p_2 := p_0 - p_1 = x_2^2 + x_3^2 \in E$. Then it follows

$$\frac{1}{2}[L_0, p_1] - 1 = x_1D_1 + x_4D_4 \in E, \tag{A.190}$$

$$\frac{1}{2}[L_0, p_2] - 1 = x_2D_2 + x_3D_3 \in E. \tag{A.191}$$

Then

$$\begin{aligned} E \ni &[x_1D_1 + x_4D_4, x_2D_2 + x_3D_3] \\ &= x_1x_2\omega_{21} + x_1x_3\omega_{31} + x_2x_4\omega_{24} + x_3x_4\omega_{34} \\ &= a_1x_1^2x_2 + a_2x_1x_2^2 + b_1x_1^2x_3 + b_3x_1x_3^2 + (a_3 + b_2)x_1x_2x_3 \\ &\quad - m_1x_2x_4^2 - n_1x_3x_4^2 + pol_1(x). \end{aligned} \tag{A.192}$$

By Ocone's theorem, it implies $a_1 = a_2 = b_1 = b_3 = a_3 + b_2 = m_1 = n_1 = 0$, i.e., $\omega_{21} \in P_1(x_3), \omega_{31} \in P_1(x_2)$ and $\omega_{42},\ \omega_{43}$ are constants. Observe $A_1^{(2)} = l_1x_4^2 \Longrightarrow l_1 = 0$. So ω_{41} is a constant.

Next we calculate

$$\begin{cases} E \ni [D_1, x_1D_1 + x_4D_4] \Longrightarrow E \ni x_4\omega_{41}, \\ E \ni [D_2, x_1D_1 + x_4D_4] \Longrightarrow E \ni x_1\omega_{12} + x_4\omega_{42} = -a_3x_1x_3 - a_0x_1 + m_0x_4, \\ E \ni [D_3, x_1D_1 + x_4D_4] \Longrightarrow E \ni x_1\omega_{13} + x_4\omega_{43} = -b_2x_1x_2 - b_0x_1 + n_0x_4, \\ E \ni [D_1, x_2D_2 + x_3D_3] \Longrightarrow E \ni x_2\omega_{21} + x_3\omega_{31} = a_0x_2 + b_0x_3, \\ E \ni [D_2, x_2D_2 + x_3D_3] \Longrightarrow E \ni x_3\omega_{32} = c_1x_1x_3 + c_2x_2x_3 + c_3x_3^2 + c_0x_3, \\ E \ni [D_3, x_2D_2 + x_3D_3] \Longrightarrow E \ni x_2\omega_{32} = c_1x_1x_2 + c_2x_2^2 + c_3x_3x_2 + c_0x_2. \end{cases} \tag{A.193}$$

In the following, we claim $a_3 = b_2 = 0$. Otherwise, if $a_3 \neq 0$,

$$\xi := \frac{1}{a_3}[x_1D_1 + x_4D_4, D_2] - \frac{a_0}{a_3}x_1 = x_1x_3 - \frac{m_0}{a_3}x_4 \in E, \tag{A.194}$$

which implies

$$[Ad_{L_0}\xi, \xi] - \left(\frac{m_0}{a_3}\right)^2 = \sum_{i=1}^{n}\left(\frac{\partial \xi}{\partial x_i}\right)^2 - \left(\frac{m_0}{a_3}\right)^2 = x_1^2 + x_3^2 \in E. \tag{A.195}$$

Then

$$\frac{1}{2}[L_0, x_1^2 + x_3^2] - 1 = x_1D_1 + x_3D_3 \in E \tag{A.196}$$

and

$$\begin{aligned} E \ni &[x_1D_1 + x_3D_3, x_2D_2 + x_3D_3] \\ &= x_1x_2\omega_{21} + x_1x_3\omega_{31} + x_2x_3\omega_{23} \\ &= -c_1x_1x_2x_3 - c_2x_2^2x_3 - c_3x_2x_3^2 + pol_2(x). \end{aligned} \tag{A.197}$$

It implies that $c_1 = c_2 = c_3 = 0$, i.e., ω_{32} is a constant. Cyclical condition (A.171) shows $a_3 = b_2$. Considering $a_3 + b_2 = 0$, it means $a_3 = b_2 = 0$, a contradiction. Hence $a_3 = 0$, i.e., ω_{21} is a constant. It leads to $b_2 = 0$, i.e., ω_{31} is a constant.

Case [4.1] $r(p_1(x)) = 1$, i.e., $p_1 = x_1^2 \in E \cap Q$.

Similarly, we define $\widetilde{p}_1 := p_0 - p_1 = x_2^2 + x_3^2 + x_4^2 \in E$. Then it follows that

$$\frac{1}{2}[L_0, p_1] - \frac{1}{2} = x_1 D_1 \in E, \tag{A.198}$$

$$\frac{1}{2}[L_0, \widetilde{p}_1] - \frac{3}{2} = \sum_{i=2}^{4} x_i D_i \in E. \tag{A.199}$$

By bracketing these two operators, we can see that

$$\begin{aligned} E \ni & \left[x_1 D_1, \sum_{i=2}^{4} x_i D_i\right] \\ &= x_1 x_2 \omega_{21} + x_1 x_3 \omega_{31} + x_1 x_4 \omega_{41} \\ &= a_1 x_1^2 x_2 + a_2 x_1 x_2^2 + (a_3 + b_2) x_1 x_2 x_3 + b_1 x_1^2 x_3 + b_3 x_1 x_3^2 + l_1 x_1 x_4^2 + pol_2(x). \end{aligned} \tag{A.200}$$

Ocone's theorem implies that $a_1 = a_2 = a_3 + b_2 = b_1 = b_3 = l_1 = 0$, i.e., which means $\omega_{21} \in P_1(x_3), \omega_{31} \in P_1(x_2), \omega_{41}$ is a constant. Next we use brackets between $D_i, 1 \leq i \leq 3$ and $x_1 D_1, \sum_{i=2}^{4} x_i D_i$ to get more information.

$$E \ni [D_2, x_1 D_1] \Longrightarrow E \ni x_1 \omega_{21} = a_3 x_1 x_3 + a_0 x_1, \tag{A.201}$$

$$E \ni [D_3, x_1 D_1] \Longrightarrow E \ni x_1 \omega_{31} = b_2 x_1 x_2 + b_0 x_1, \tag{A.202}$$

$$E \ni \left[D_1, \sum_{i=2}^{4} x_i D_i\right] = x_2 \omega_{21} + x_3 \omega_{31} + x_4 \omega_{41} = a_0 x_2 + b_0 x_3 + l_0 x_4, \tag{A.203}$$

$$\begin{aligned} & \left[D_2, \sum_{i=2}^{4} x_i D_i\right] \in E \\ \Longrightarrow & E \ni x_3 \omega_{32} + x_4 \omega_{42} = c_1 x_1 x_3 + c_2 x_2 x_3 + c_3 x_3^2 + c_0 x_3 + m_1 x_4^2 + m_0 x_4, \end{aligned} \tag{A.204}$$

$$\begin{aligned} & \left[D_3, \sum_{i=2}^{4} x_i D_i\right] \in E \\ \Longrightarrow & E \ni x_2 \omega_{23} + x_4 \omega_{43} = -c_1 x_1 x_2 - c_2 x_2^2 - c_3 x_2 x_3 + n_1 x_4^2 - c_0 x_2 + n_0 x_4. \end{aligned} \tag{A.205}$$

Next we claim $a_3 = 0$. Otherwise, if $a_3 \neq 0$, (A.201) implies that $x_1 x_3 \in E \Longrightarrow x_1^2 + x_3^2 \in E$. Combining with $p_1 = x_1^2 \in E$, it implies $x_1^2 \in E$ and $x_3^2 \in E$. Lemma 4.1 shows ω_{31} is a constant. Then $b_2 = 0 \Longrightarrow a_3 = 0$, a contradiction! Therefore, $a_3 = b_2 = 0$, i.e., ω_{21}, ω_{31} are constants. In addition, cyclical condition restricts that $c_1 = 0$, i.e., $\omega_{32} \in P_1(x_2, x_3)$.

It has been left to show that $\omega_{32}, \omega_{42}, \omega_{43}$ are constants. In order to do this, we need to explore more detailed polynomial structure in E.

Let $Q_1 := \text{span}\{x_i x_j, x_k, 1 : 2 \leq i, j \leq 4, 1 \leq k \leq 4\}$. $E \cap Q_1$ contains quadratic polynomial due to $\widetilde{p}_1 = x_2^2 + x_3^2 + x_4^2 \in E \cap Q_1$. Denote $p_2 \in E \cap Q_1$ with least positive quadratic rank in

$E \cap Q_1$. Then $1 \le r(p_2) \le 3$. Without loss of generality, we can assume

$$p_2 = \sum_{2 \le i,j \le 4} A_{ij} x_i x_j + d_0 x_4 \in E \cap Q_1. \tag{A.206}$$

By linear rank, $p_2^{(2)}$ does not contain term $x_i x_4, 1 \le i \le 3$. It follows

$$p_2 = \sum_{2 \le i,j \le 3} A_{ij} x_i x_j + d_4 x_4^2 + d_0 x_4 \in E \cap Q_1. \tag{A.207}$$

By an orthogonal transformation fixing x_1 and x_4, we can assume

$$p_2 = d_2 x_2^2 + d_3 x_3^2 + d_4 x_4^2 + d_0 x_4 \in E \cap Q_1. \tag{A.208}$$

Similarly, by using bracket of $\sum_{i=2}^4 x_i D_i$, we can seperate homogeneous quadratic part of p_2,

$$p_2^{(2)}(x) = \left[\sum_{i=2}^4 x_i D_i, p_2(x)\right] - p_2(x) = E_{2,3,4}(p_2(x)) - p_2(x) = \sum_{i=2}^4 d_i x_i^2 \in E. \tag{A.209}$$

Again we use method of Vandermonde matrix, we have

$$E \ni p_2^{(2)}(x) = \begin{cases} \sum_{i=2}^{k_1} x_i^2, & \text{if } d_4 = 0, 2 \le k_1 \le 3, \\ \sum_{i=2}^{k_1} x_i^2 + x_4^2, & \text{if } d_4 \ne 0, 1 \le k_1 \le 3. \end{cases} \tag{A.210}$$

Notice previous orthogonal transformation keeps quadratic function $p_0, p_1, \widetilde{p}_1$ and linear rank structure unchange. This is quite important.

In terms of rank of p_2, we discuss the following cases.

Case [4.1.1] $p_2 = x_2^2 \in E$.

Then $\widetilde{p}_2 := \widetilde{p}_1 - p_2 = x_3^2 + x_4^2$. It follows that

$$\frac{1}{2}[L_0, p_2] - \frac{1}{2} = x_2 D_2 \in E \tag{A.211}$$

and

$$\frac{1}{2}[L_0, \widetilde{p}_2] - 1 = x_3 D_3 + x_4 D_4 \in E. \tag{A.212}$$

Then

$$\begin{aligned} E \ni &[x_2 D_2, x_3 D_3 + x_4 D_4] \\ &= x_2 x_3 \omega_{32} + x_2 x_4 \omega_{42} \\ &= c_2 x_2^2 x_3 + c_3 x_2 x_3^2 + m_1 x_2 x_4^2 + c_0 x_2 x_3 + m_0 x_2 x_4. \end{aligned} \tag{A.213}$$

It follows $c_2 = c_3 = m_1 = 0$, i.e., ω_{32}, ω_{42} are constants. (A.205) shows that

$$\xi := n_1 x_4^2 + n_0 x_4 \in E \Longrightarrow \xi^{(2)} = \left[\frac{1}{2}[L_0, \widetilde{p}_2], \xi\right] - \xi = n_1 x_4^2 \in E. \tag{A.214}$$

If $n_1 \neq 0$, then $x_4^2 \in E \Longrightarrow x_3^2 \in E$. By Lemma 4.1, we obtain ω_{43} is a constant, a contradiction. Therefore, $n_1 = 0$ and ω_{43} is a constant.

Case [4.1.2] $p_2 = x_4^2 \in E$.

Then $\widetilde{p}_2 := \widetilde{p}_1 - p_2 = x_2^2 + x_3^2$. It follows that

$$\frac{1}{2}[L_0, p_2] - \frac{1}{2} = x_4 D_4 \in E \tag{A.215}$$

and

$$\frac{1}{2}[L_0, \widetilde{p}_2] - 1 = x_2 D_2 + x_3 D_3 \in E. \tag{A.216}$$

Then

$$\begin{aligned} E \ni &[x_2 D_2 + x_3 D_3, x_4 D_4] \\ &= x_2 x_4 \omega_{42} + x_3 x_4 \omega_{43} = m_1 x_2 x_4^2 + n_1 x_3 x_4^2 + m_0 x_2 x_4 + n_0 x_3 x_4. \end{aligned} \tag{A.217}$$

It follows $m_1 = n_1 = 0$, i.e., ω_{42}, ω_{43} are constants.

Notice (A.204) and (A.205) become

$$M_1 := c_2 x_2 x_3 + c_3 x_3^2 + m_0 x_4 \in E, \tag{A.218}$$

$$M_2 := c_2 x_2^2 + c_3 x_2 x_3 - n_0 x_4 \in E, \tag{A.219}$$

and

$$\frac{1}{2}\left[\frac{1}{2} Ad_{L_0} \widetilde{p}_2, M_1\right] = c_2 x_2 x_3 + c_3 x_3^2 \in E, \tag{A.220}$$

$$\frac{1}{2}\left[\frac{1}{2} Ad_{L_0} \widetilde{p}_2, M_2\right] = c_2 x_2^2 + c_3 x_2 x_3 \in E. \tag{A.221}$$

Combining $\widetilde{p}_2 = x_2^2 + x_3^2 \in E$ and Lemma 4.2, we obtain ω_{32} is a constant.

Case [4.1.3] $p_2 = x_2^2 + x_3^2 \in E$.

Naturally, we have $\widetilde{p}_2 := \widetilde{p}_1 - p_2 = x_4^2 \in E$. By using same argument as Case [4.1.2], ω_{ij}'s are constants for $i, j \in \{2, 3, 4\}$.

Case [4.1.4] $p_2 = x_2^2 + x_4^2 \in E$.

Naturally, we have $\widetilde{p}_2 := \widetilde{p}_1 - p_2 = x_3^2 \in E$. It follows that

$$\frac{1}{2}[L_0, p_2] = x_2 D_2 + x_4 D_4 \in E, \tag{A.222}$$

$$\frac{1}{2}[L_0, \widetilde{p}_2] = x_3 D_3 \in E, \tag{A.223}$$

which results in

$$\begin{aligned} [x_2 D_2 + x_4 D_4, x_3 D_3] =& x_2 x_3 \omega_{32} + x_3 x_4 \omega_{34} \\ =& c_2 x_2^2 x_3 + c_3 x_2 x_3^2 - n_1 x_3 x_4^2 + c_0 x_2 x_3 - n_0 x_3 x_4 \in E. \end{aligned} \tag{A.224}$$

Then $c_2 = c_3 = n_1 = 0$, i.e., ω_{32}, ω_{43} are constants.

Notice (A.204) implies $m_1 x_4^2 + m_0 x_4 \in E \cap Q_1$. Since p_2 has least positive quadratic rank, $m_1 = 0 \Longrightarrow \omega_{42}$ is a constant.

Case [4.1.5] $p_2 = x_2^2 + x_3^2 + x_4^2 \in E$.

(A.204) and (A.205) imply

$$\xi_1 := c_2x_2x_3 + c_3x_3^2 + m_1x_4^2 + m_0x_4, \quad \xi_2 := -c_2x_2^2 - c_3x_2x_3 + n_1x_4^2 + n_0x_4. \tag{A.225}$$

Notice $\xi_1, \xi_2 \in E \cap Q_1$. Then by least quadratic rank property in $E \cap Q_1$, we induce $\xi_i^{(2)} = \lambda_i p_2$ for $1 \le i \le 2$ and some $\lambda_i \in \mathbb{R}$. Thus, we get $\lambda_1 = \lambda_2 = 0 \Longrightarrow c_2 = c_3 = m_1 = n_1 = 0 \Longrightarrow \omega_{ij}$'s are constants for $i, j \in \{2, 3, 4\}$.

Up to now, we prove that Ω is a constant matrix in Case [4.1].

Case [4.2] $r(p_1(x)) = 1$, i.e., $p_1 = x_4^2 \in E \cap Q$.

Naturally, we obtain $\widetilde{p}_1 := p_0 - p_1 = \sum_{i=1}^3 x_i^2 \in E$. It follows that

$$\frac{1}{2}[L_0, p_1] = x_4D_4 \in E, \tag{A.226}$$

$$\frac{1}{2}[L_0, \widetilde{p}_1] = \sum_{i=1}^{3} x_iD_i \in E. \tag{A.227}$$

By bracketing two differential operators, we get

$$\begin{aligned} E \ni \left[\sum_{i=1}^{3} x_iD_i, x_4D_4\right] &= \sum_{i=1}^{3} x_ix_4\omega_{4i} \\ &= l_1x_1x_4^2 + m_1x_2x_4^2 + n_1x_3x_4^2 + l_0x_1x_4 + m_0x_2x_4 + n_0x_3x_4. \end{aligned} \tag{A.228}$$

It implies $l_1 = m_1 = n_1 = 0$, i.e., ω_{4i}'s are constants for $1 \le i \le 3$.

Let $Q_1 :=$ real vector space spanned by $\{x_ix_j, x_k, 1 : 1 \le i, j \le 3, 1 \le k \le 4\}$. $E \cap Q_1$ contains quadratic polynomial due to $\widetilde{p}_1 \in E \cap Q_1$. Denote $p_2 \in E \cap Q_1$ with least positive quadratic rank in $E \cap Q_1$. Then $1 \le r(p_2) \le 3$. Without loss of generality, we can assume

$$p_2 = \sum_{1 \le i,j \le 3} A_{ij}x_ix_j + d_0x_4 \in E \cap Q_1. \tag{A.229}$$

By an orthogonal transformation fixing x_4, we have

$$p_2 = \sum_{i=1}^{k_1} d_ix_i^2 + d_0x_4 \in E, \tag{A.230}$$

where $1 \le k_1 \le 3$ and $d_i \ne 0$. By technique of Vandermonde matrix proposed in [19]., we can assume

$$p_2 = \sum_{i=1}^{k_1} x_i^2. \tag{A.231}$$

Then we only need to discuss three cases.

Case [4.2.1] $p_2 = x_1^2 \in E$.

Case [4.2.2] $p_2 = x_1^2 + x_2^2 \in E$.

Case [4.2.3] $p_2 = x_1^2 + x_2^2 + x_3^2 \in E$.

By using the same argument of Lemma 5.12 in Case (V), we can prove ω_{ij}'s are constants for $i, j \in \{1, 2, 3\}$. Then Ω is a constant matrix. ∎

A.25 Proof of Theorem 5.20

Theorems 5.3, 5.5, 5.7, 5.11, 5.14, 5.17, 5.7 demonstrate Ω possesses partially constant structure. By applying Theorem 4.5, h_i's are affine functions. ∎