The Applications of Yau-Yau Algorithm on McKean-Vlasov Filtering Problem*

Dedicated to Professor Peter Caines on the occasion of his 80th Birthday

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DOI:
Received: x x 20xx / Revised: x x 20xx

Abstract The McKean-Vlasov filtering problem is a special kind of filtering problem, with the state and/or observation processes governed by McKean-Vlasov stochastic differential equations, which has extensive applications in various scenarios. In this paper, we will propose a novel numerical algorithm to solve the McKean-Vlasov filtering problem based on the Hermite spectral method under the framework of Yau-Yau algorithm. As the first approach to numerically solving the Duncan-Mortensen-Zakai equation associated with the McKean-Vlasov filtering problem, our proposed algorithm can provide accurate estimations of the conditional expectation and conditional probability density of the state process with a reasonable online computational complexity. The efficiency of our proposed algorithm is verified both theoretically and numerically in this paper.

Keywords Nonlinear filtering, McKean-Vlasov equation, Duncan-Mortensen-Zakai equation, Yau-Yau algorithm, Hermite spectral method.

1 Introduction

Filtering is a subject about the sequential estimation of a stochastic process, (referred to as 'state process'), with a series of noisy observations. As an important part of modern control theory, filtering has wide applications in various areas such as aerospace industrial [33],

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*This work is supported by National Natural Science Foundation of China (NSFC) under Grant 123B2020 for Zeju Sun, and by Tsinghua University Education Foundation fund under Grant 042202008 for Stephen Shing-Toung Yau.
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autonomous driving [27], weather forecasting [10], game theory [25] and so on. The design of efficient filtering algorithms stimulates general interests of mathematicians and practitioners. In many applications, the real-time capability of a filtering algorithm is also in high demand, that is, it is appealing that an accurate estimation can be obtained through the algorithm, as soon as each new observation comes [19].

Mathematically, based on historical observations, the best estimate of the state process is the conditional expectation, in the sense that we can obtain the least expected mean square error. Therefore, the studies of filtering algorithms are developed based on the computation and approximation of conditional expectations and conditional distributions [2].

In 1960s, the notable Kalman-Bucy filter was first proposed [16, 17], which described the exact evolution system of conditional expectations and covariance matrices for linear Gaussian case. For general nonlinear filtering problems, however, the evolution of conditional expectation cannot be expressed explicitly by a finite dimensional evolution system, and thus, researchers and practitioners need to construct nonlinear filtering algorithms with different kinds of surrogate dynamical systems to approximate the evolution of conditional expectation efficiently. These nonlinear filtering algorithms include (i) nonlinear Kalman filters based on linearization such as unscented Kalman filter [15] and ensemble Kalman filter [9]; (ii) Monte-Carlo based algorithms such as resampling particle filter [1] and feedback particle filter [24, 29]; and (iii) algorithms based on projection and moment matching such as projection filters [7] and density parametrization filters [28].

In the late 1960s, Duncan [6], Mortensen [23] and Zakai [32] independently derived the evolution equation of the unnormalized conditional probability distribution for general nonlinear filtering systems, which is now referred to as the DMZ equation. From then on, studies on the properties and numerical solutions of the DMZ equations, a second-order linear stochastic partial differential equation driven by the observation process, had become an important problem in the subject of nonlinear filtering [11, 14], because the desired conditional expectation can be obtained by taking normalized integrals of the solution to the DMZ equation.

In standard filtering systems, the state and observation processes are often modeled by a couple of stochastic differential equations driven by Brownian motions [2]. Nowadays, as the application scenarios of state estimation theory become broader, many filtering problems, which has entered into the horizon of researchers, are not in the standard setting, for example the filtering systems with jumps [4], inverse filtering problems [18] and so on. In this paper, we will consider filtering problem with state and observation processes described by a special kind of stochastic differential equation, the McKean-Vlasov equation. McKean-Vlasov equation was first proposed to model particle systems under mean-field interactions [12], as well as the evolution of biological populations [22]. Later on, McKean-Vlasov equations also have wide applications in mean-field control and mean-field game theories [13]. Recently, the McKean-Vlasov filtering problem has also find its applications in the setting of mean-field games with partially observed major player [25].

In comparison with standard stochastic differential equations, the coefficients in McKean-Vlasov equations not only depend on the stochastic process itself, but also depend on the distribution or conditional distribution of the solution. Such dependence makes it harder to analyze and solve McKean-Vlasov equations. In terms of the state estimation or filtering theory of McKean-Vlasov equation, the corresponding DMZ equation satisfied by the conditional probability distribution is derived in [26], which is a nonlinear stochastic partial differential equation. To the best of the authors' knowledge, there is still a lack of literature which focus on the properties and solutions of the DMZ equation of McKean-Vlasov filtering problems.

At the beginning of this century, the second author and his collaborators proposed a two-stage algorithm to solve the DMZ equation and also the standard nonlinear filtering problems [19, 30, 31]. The proposed algorithm is now referred to as Yau-Yau nonlinear filtering algorithm. With a series of exponential transformations, the original DMZ equation is converted into a deterministic partial differential equation independent of the observations. Hence, in the two-stage Yau-Yau algorithm, the heavy computational burden of numerically solving partial differential equations can be done offline, and the online computation only includes the calculation of exponential transformations. Such kind of design guarantees that Yau-Yau algorithm has the potential of solve the general nonlinear filtering problems in a real-time manner. Recent study also shows that with the help of deep learning, Yau-Yau algorithm is very efficient in solving high-dimensional nonlinear filtering problems, and has the capability of overcoming the curse of dimensionality [3].

In this paper, we will generalize the idea of Yau-Yau algorithm to McKean-Vlasov filtering problems, and propose a novel filtering algorithm to solve McKean-Vlasov filtering problems in a real-time and memoryless manner. This proposed algorithm can be regarded as the first attempt on the numerical algorithms of McKean-Vlasov filtering problems as well as the corresponding nonlinear DMZ equations. The major generalization lies in the offline algorithm, where a nonlinear parabolic partial differential equation is required to solve. We propose a novel two-step Hermite-Galerkin method [20] to obtain the numerical solution of the equation. The effectiveness of our proposed algorithm is verified both theoretically and numerically in this paper.

The organization of this paper is as follows. In Section 2, we will briefly summarize the setting of McKean-Vlasov filtering problem as well as the derivation of the corresponding DMZ equation. In Section 3, the Yau-Yau algorithm for McKean-Vlasov filtering problem will be proposed, and the convergence analysis of our proposed algorithm will be given in Section 4. Numerical results will be illustrated in Section 5 and Section 6 is a conclusion.

2 Preliminaries

In this paper, we will consider the following McKean-Vlasov filtering system,

$$\begin{cases} dx(t) = f[t, x(t), \mu_t]dt + \sigma[t, x(t), \mu_t]dw(t) \\ dy(t) = h(t, x(t))dt + dv(t) \end{cases} \tag{1}$$

where the evolution of the state process $\{x(t) : t \geq 0\} \subset \mathbb{R}^n$ is governed by a stochastic differential equation of McKean-Vlasov type, with the distribution of the initial random variable

$x(0)$ to be μ_0, i.e., $x(0) \sim \mu_0$; μ_t is the probability distribution of the state process $x(t)$; $\{y(t) : t \geq 0\} \subset \mathbb{R}^m$ is the observation process with initial value $y(0) = 0$; $\{w_t : t \geq 0\} \subset \mathbb{R}^p$, $\{v(t) : t \geq 0\} \subset \mathbb{R}^m$ are mutually independent standard Brownian motions of corresponding dimensions.

Let us denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all probability measures on $\mathbb{R}^n$, then the drift term $f : \mathbb{R} \times \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^n$ and the diffusion term $\sigma : \mathbb{R} \times \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^{n\times p}$ are defined as follows:

$$\begin{aligned} f[t, x, \mu] &= \int_{\mathbb{R}^n} \mathfrak{f}(t, x, z)\mu(dz), \\ \sigma[t, x, \mu] &= \int_{\mathbb{R}^n} \varsigma(t, x, z)\mu(dz), \end{aligned} \tag{2}$$

with $\mathfrak{f} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\varsigma : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times p}$ smooth functions of corresponding dimensions.

Remark 2.1 For practical implementations, there is an important class of McKean-Vlasov equations with diffusion term $\sigma \equiv$ constant and

$$\mathfrak{f}(t, x, z) = \phi(t, x) + z. \tag{3}$$

In this case, the drift term of the McKean-Vlasov equation is given by $f[t, x(t), \mu_t] = \phi(t, x(t)) + Ex(t)$ and the evolution of $x(t)$ is dependent on its probability distribution μ_t through the expectations. This special kind of McKean-Vlasov equation is often used in the model of particles with mean-field interactions [12] as well as the evolution of feedback particle filter [29].

Given a fixed terminal time $T > 0$, for each $0 \leq t \leq T$, let us define by $\mathcal{Y}_t = \sigma\{y(s) : 0 \leq s \leq t\}$ the σ-algebra generated by historical observations. The main goal of McKean-Vlasov filtering problem is to compute the conditional expectations $E[\varphi(x(t))|\mathcal{Y}_t]$ for a given smooth test function $\varphi : \mathbb{R}^n \to \mathbb{R}$ at each time $0 \leq t \leq T$ in a real-time and memoryless manner.

Following the idea in [26], we can derive the DMZ equation of McKean-Vlasov filtering problem, which is satisfied by the unnormalized conditional probability distribution, with the change-of-measure method. Firstly, let us define the exponential martingale $\{z(t) : 0 \leq t \leq T\}$ by

$$z(t) = \exp\left(\int_0^t h^\top(s, x(s))dy(s) - \frac{1}{2}|h(s, x(s))|^2 ds\right), \tag{4}$$

then, under the reference probability measure $\widetilde{P}$ given by

$$\left.\frac{d\widetilde{P}}{dP}\right| = z(t)^{-1} = \exp\left(-\int_0^t h^\top(s, x(s))dy(s) + \frac{1}{2}|h(s, x(s))|^2 ds\right). \tag{5}$$

The observation process $\{y(t) : 0 \leq t \leq T\}$ is a standard Brownian motion, and the unnormalized conditional expectation, which is defined by

$$\rho_t(\varphi) := \widetilde{E}[z(t)\varphi(x(t))|\mathcal{Y}_t],$$

satisfies the following DMZ equation:

$$\rho_t(\varphi) = \rho_0(\varphi) + \int_0^t \rho_s(\mathcal{L}(s)\varphi)ds + \int_0^t \rho(h^\top\varphi)dy(s),\ 0 \leq t \leq T, \tag{6}$$

where

$$\mathcal{L}(t)\varphi = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2\varphi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} f_i(t,x)\frac{\partial \varphi}{\partial x_i}, \tag{7}$$

is the second-order elliptic operator with $a(t,x) = (a_{ij}(t,x))_{1\le i,j\le n} = \sigma[t,x,\mu_t]\sigma[t,x,\mu_t]^\top$ and $f(t,x) = f[t,x,\mu_t]$.

With mild coefficients in the filtering system and regular initial distribution, (which guarantees that equation (8) below possesses a generalized solution in some properly chosen Sobolev-type space), the unnormalized conditional probability distribution $\{\rho_t : 0 \le t \le T\}$ is absolutely continuous with respect to the Lebesgue measure, and the density function $p(t,x)$ is almost surely square-integrable.

With the integration-by-part formula, $p(t,x)$ satisfies the following stochastic partial differential equation given by

$$dp(t,x) = \mathcal{L}^*(t)p(t,x)dt + h^\top(t,x)p(t,x)dy(t), \tag{8}$$

with

$$\mathcal{L}^*(t)(\star) := \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}(t,x)\star) - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}(f_i(t,x)\star) \tag{9}$$

the dual operator of $\mathcal{L}(t)$.

The DMZ equation (8) is a stochastic partial differential equation. However, the stochastic differentiation term in (8) can be eliminated through exponential transformations [5]. Let us introduce a new function $u(t,x) : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ defined by

$$p(t,x) = u(t,x)\exp\left(h^\top(t,x)y(t)\right), \tag{10}$$

then, according to Itô's formula, $u(t,x)$ satisfies the following deterministic partial differential equation with stochastic coefficients:

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) = &- \frac{\partial h^\top}{\partial t}(t,x)y(t)u(t,x)dt - \frac{1}{2}|h(t,x)|^2 u(t,x)dt \\ &+ \exp\left(-h^\top(t,x)y(t)\right)\mathcal{L}^*(t)\left[\exp\left(h^\top(t,x)y(t)\right)u(t,x)\right].\end{aligned} \tag{11}$$

Since the stochastic differentiation term vanishes in (11), it will show more robust properties through time-discretization. Therefore, we would like to call (11) the robust DMZ equation of McKean-Vlasov filtering problem. Because in general, the solution of the DMZ equation (8), together with the solution of the robust DMZ equation (11), does not possesses a closed form, we need to propose time-discretization methods to solve these equations numerically. The generalized Yau-Yau algorithm we would like to propose in the next section will provide a proper time-discretization scheme for the robust DMZ equation (11), and then, solve the McKean-Vlasov filtering problem.

3 Yau-Yau Algorithm for McKean-Vlasov Filtering Problems

3.1 Time-Discretization of the Robust DMZ Equation

The robust DMZ equation for McKean-Vlasov filtering problem is a deterministic partial differential equation with coefficients dependent on the entire trajectory of observations. In real applications, the observations $\{y(t) : 0 \leq t \leq T\}$ can only be obtained or sampled discretely in the time interval $[0, T]$. Therefore, we need to introduce a proper time-discretization scheme and construct an auxiliary equation which is only dependent on the value of the observation process on some discrete time steps.

Let us define a uniform partition of the time interval $0 = \tau_0 < \tau_1 < \cdots < \tau_K = T$, with the time discretization step size $\Delta t = \frac{T}{K}$, and $\tau_k - \tau_{k-1} = \Delta t$, for all $1 \leq k \leq K$. Assume that the practical observations which we can obtain are $\{y(\tau_k) : 0 \leq k \leq K\}$, then if the observation terms are frozen in (11) at the left endpoint of each time interval $[\tau_{k-1}, \tau_k]$, we obtain a set of functions $\{u_k(t, x) : [\tau_{k-1}, \tau_k] \times \mathbb{R}^n \to \mathbb{R}\}_{0 \leq k \leq K}$, which satisfies

$$\begin{cases} \dfrac{\partial u_k}{\partial t}(t,x) = -\dfrac{\partial h^\top}{\partial t}(t,x)y(\tau_{k-1})u_k(t,x)dt - \dfrac{1}{2}|h(t,x)|^2 u_k(t,x)dt \\ \qquad\qquad + \exp\left(-h^\top(t,x)y(\tau_{k-1})\right)\mathcal{L}^*(t)\left[\exp\left(h^\top(t,x)y(\tau_{k-1})\right)u_k(t,x)\right], \\ u_k(\tau_{k-1},x) = u_{k-1}(\tau_{k-1},x),\ 1 \leq k \leq K. \end{cases} \tag{12}$$

The relationship between the solution of equation (12) and the robust DMZ equation (11) has been discussed for standard time-varying nonlinear filtering systems in [19], and it has been proved that for standard time-varying filtering systems, the corresponding solutions $u_k(t, x)$ of (12) will converge to the solution of the robust DMZ equation in L^1-sense under mild conditions, as the time-discretization step size $\Delta t \to 0$.

Here in the McKean-Vlasov case, the discussions in [19] also hold true, because the form of auxiliary equation (12) is identical to the corresponding equation of standard time-varying filtering problems, although due to the dependence on μ_t, the elliptic operator $\mathcal{L}^*(t)$ cannot be written explicitly. Therefore, similar to the standard time-varying cases, it can be proved that under mild conditions, the solutions $\{u_k(t, x) : 1 \leq k \leq K\}$ is a good approximator of the solution $u(t, x)$ of the robust DMZ equation (11), for sufficiently small Δt. The remaining task in solving McKean-Vlasov filtering problem is converted into numerically solving (12) with relatively less online computation cost.

3.2 Yau-Yau Algorithm Framework for McKean-Vlasov Filtering Problems

The idea of Yau-Yau algorithm for solving McKean-Vlasov filtering problems is to separate the observations from the auxiliary equation (12) and obtain a observation-independent equation which can be solved offline. Notice that for $\widetilde{u}_k(t, x)$ defined by another exponential transformation,

$$\widetilde{u}_k(t,x) = \exp\left(h^\top(t,x)y(\tau_{k-1})\right)u_k(t,x),\ t \in [\tau_{k-1}, \tau_k], \tag{13}$$

we have

$$\frac{\partial \widetilde{u}_k}{\partial t}(t,x) = \left(\mathcal{L}^*(t) - \frac{1}{2}|h(t,x)|^2\right)\widetilde{u}_k(t,x),\ t \in [\tau_{k-1}, \tau_k], \tag{14}$$

with initial value

$$\begin{aligned}\widetilde{u}_k(\tau_{k-1},x) &= \exp\left(h^\top(\tau_{k-1},x)y(\tau_{k-1})\right)u_k(\tau_{k-1},x)\\ &= \exp\left(h^\top(\tau_{k-1},x)y(\tau_{k-1})\right)u_{k-1}(\tau_{k-1},x)\\ &= \exp\left(h^\top(\tau_{k-1},x)(y(\tau_{k-1})-y(\tau_{k-2})))\right)\widetilde{u}_{k-1}(\tau_{k-1},x),\end{aligned} \tag{15}$$

for $2 \leq k \leq K$, and $\widetilde{u}_1(0,x) = u_1(0,x) = \mu_0(x)$, for $k = 1$.

With this exponential transformation (13), the task of solving the auxiliary equation (12) can be divided into two steps.

(i) Solve the observation-independent partial differential equation (14) at each time interval $[\tau_{k-1}, \tau_k]$, for $1 \leq k \leq K$.

(ii) Compute the exponential transformation (15) at the beginning of each time interval.

Because (14) is independent of the observation process, with a suitable representation method for the initial values, we can solve it offline and reduce the online computational burden into only computing the exponential transformation. This two-stage nonlinear filtering algorithm, after proposed at the beginning of this century by the second author and his collaborator, is referred to as Yau-Yau algorithm [3].

In comparison with standard time-varying nonlinear filtering problems, the online computation procedure (ii) remains the same, while the second-order differential operator $\mathcal{L}^*(t)$ in McKean-Vlasov filtering problems is not explicitly expressed, because the coefficients $a(t,x)$ and $f(t,x)$ are dependent with the probability distribution μ_t. Fortunately, the measure term μ_t is also independent of the observations, and we can compute or approximate μ_t, as well as solve the observation-independent partial differential equation (14) offline.

In fact, with sufficient regularity assumptions on the coefficients of the state equation, the probability distribution μ_t is absolutely continuous with respect to the Lebesgue measure, and possesses a density function denoted by $\rho(t,x)$. The probability density function $\rho(t,x)$ satisfies the following nonlinear Fokker-Planck equation [12]

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[a_{ij}[t,x,\mu_t]\rho\right] - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(f_i[t,x,\mu_t]\rho\right), \ (t,x) \in [0,T]\times\mathbb{R}^n, \tag{16}$$

with

$$\begin{aligned} f[t,x,\mu_t] &= \int_{\mathbb{R}^n} \mathfrak{f}(t,x,z)\mu_t(dz) = \int_{\mathbb{R}^n} \mathfrak{f}(t,x,z)\rho(t,z)dz,\\ \sigma[t,x,\mu_t] &= \int_{\mathbb{R}^n} \varsigma(t,x,z)\mu_t(dz) = \int_{\mathbb{R}^n} \varsigma(t,x,z)\rho(t,z)dz,\end{aligned} \tag{17}$$

and $a[t,x,\mu_t] = \sigma[t,x,\mu_t]\sigma[t,x,\mu_t]^\top$.

In order to generalize the Yau-Yau algorithm and solve the McKean-Vlasov filtering problems, we first need to solve the nonlinear Fokker-Planck equation (16), and obtain a good approximation of the probability distribution μ_t as well as its density function $\rho(t,x)$. Then, with the information of μ_t, we are ready to solve the parabolic partial differential equation (14) satisfied by $\widetilde{u}_k$ and complete the offline procedure of the Yau-Yau algorithm.

3.3 Hermite Spectral Method and the Practical Implementation of Yau-Yau Algorithm

In the framework of generalized Yau-Yau algorithm for McKean-Vlasov filtering problems, the solutions of the nonlinear Fokker-Planck equation (14) and the time-varying parabolic equation (16) cannot by written down explicitly in a closed form, and we need to consider the numerical solutions. In this paper, we would like to apply Hermite spectral method [20] to solve (14) and (16) in the offline procedure and conduct Yau-Yau algorithm for practical implementations.

Let $\{\psi_k\}_{k=1}^{\infty} \subset \mathcal{L}^2(\mathbb{R}^n)$ be the set of Hermite basis functions in $\mathcal{L}^2(\mathbb{R}^n)$. As is introduced in [21], the expressions of each ψ_k are listed as follows.

- For space dimension $n = 1$, the $(k-1)$-th Hermite basis function ψ_k in $\mathcal{L}^2(\mathbb{R}^n)$ is defined by
$$\psi_k^{(1)}(x) = \left(\frac{1}{2^{k-1}(k-1)!\sqrt{\pi}}\right)^{\frac{1}{2}} h_{k-1}(x)e^{-\frac{1}{2}x^2}, \ k \geq 1, \tag{18}$$
with $h_k(x)$ the k-th Hermite polynomial defined recursively as follows:
$$h_0 \equiv 1, \ h_1(x) = 2x, \ h_{k+1}(x) = 2xh_k(x) - 2kh_{k-1}(x). \tag{19}$$

- For space dimension $n \geq 2$, the Hermite basis functions $\psi_{\mathbf{k}}^{(n)}(x)$ is defined by the tensor product of one-dimensional Hermite basis functions $\psi_k^{(1)}$ with
$$\psi_{\mathbf{k}}^{(n)}(x) = \prod_{j=1}^{n} \psi_{k_j}^{(1)}(x), \tag{20}$$
with $\mathbf{k} = (k_1, \cdots, k_n)$ a multi-index.

With the above definition, the functions $\{\psi_{\mathbf{k}}^{(n)}(x)\}$ form an orthonormal basis in $\mathcal{L}^2(\mathbb{R}^n)$. In order to keep the simplicity of notations, we omit the superscripts in $\psi_{\mathbf{k}}^{(n)}$, renumber the multi-indexed subscripts, and still denote by $\{\psi_k\}_{k=1}^{\infty}$ the Hermite basis functions in $\mathcal{L}^2(\mathbb{R}^n)$.

For a given positive integer $N \in \mathbb{N}$, assume that the projection of $\rho(t,x)$ onto the finite dimensional subspace $\mathrm{Span}(\{\psi_k\}_{k=1}^N)$ is a good approximator to the solution $\rho(t,x)$ of (16), then according to the spectral method, we can choose an element
$$\rho_N(t,x) := \sum_{k=1}^{N} \lambda_k(t)\psi_k(x), \ (t,x) \in [0,T] \times \mathbb{R}^n, \tag{21}$$
in the finite-dimensional subspace spanned by $\{\psi_i\}_{i=1}^N$ as an approximation to $\rho(t,x)$, with the parameters $\lambda = [\lambda_1, \cdots, \lambda_N]^\top$ to be determined later.

With the approximated density function $\rho_N(t,x)$, we can further define
$$\begin{aligned} f_N(t,x) &:= \int_{\mathbb{R}^n} f(t,x,z)\rho_N(t,z)dz = \sum_{k=1}^{N} \lambda_k(t) \int_{\mathbb{R}^n} f(t,x,z)\psi_k(z)dz = \sum_{k=1}^{N} \lambda_k(t)F^k(t,x), \\ \sigma_N(t,x) &:= \int_{\mathbb{R}^n} \sigma(t,x,z)\rho_N(t,z)dz = \sum_{k=1}^{N} \lambda_k(t) \int_{\mathbb{R}^n} \sigma(t,x,z)\psi_k(z)dz = \sum_{k=1}^{N} \lambda_k(t)\Sigma^k(t,x), \end{aligned} \tag{22}$$

and $a_N(t,x) = \sigma_N(t,x)\sigma_N(t,x)^\top$ as the approximated coefficients in equation (16).

The parameters λ in (21) can be determined through the standard Galerkin approach. In fact, we may assume that $\rho_N(t,x)$ satisfies the nonlinear Fokker-Planck equation (16) on the finite-dimensional subspace spanned by $\{\psi_i\}_{i=1}^N$, in the sense that

$$\begin{aligned}\frac{\partial}{\partial t}\langle \rho_N(t,\cdot), \psi_k\rangle = &\left\langle \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j}\left(a_N^{ij}(t,\cdot)\rho_N(t,\cdot)\right), \psi_k \right\rangle \\ &- \left\langle \sum_{i=1}^n \frac{\partial}{\partial x_i}(f_N^i(t,\cdot)\rho_N(t,\cdot)), \psi_k \right\rangle\end{aligned} \tag{23}$$

for all $1 \le k \le N$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner products in $L^2(\mathbb{R}^n)$.

According to the integration-by-part formula, we can obtain the evolution equation of the parameters λ,

$$\begin{aligned}\frac{d}{dt}\lambda_k(t) = &\sum_{l_1,l_2,l_3=1}^N \lambda_{l_1}(t)\lambda_{l_2}(t)\lambda_{l_3}(t)\left\langle \sum_{i,j,s=1}^n \Sigma_{is}^{l_1}(t,\cdot)\Sigma_{js}^{l_2}(t,\cdot)\psi_{l_3}, \frac{\partial^2}{\partial x_i \partial x_j}\psi_k \right\rangle \\ &- \sum_{l_1,l_2=1}^N \lambda_{l_1}(t)\lambda_{l_2}(t)\left\langle \sum_{i=1}^n F_i^{l_1}(t,\cdot)\psi_{l_2}, \frac{\partial \psi_k}{\partial x_i}\right\rangle,\end{aligned} \tag{24}$$

where the vector-value functions F^k and matrix-value functions Σ^k are defined in (22).

With the approximated density function $\rho_N(t,x)$, we can define the approximated second-order elliptic operator $\mathcal{L}_N^*(t)$ by

$$\mathcal{L}_N^*(t)(\varphi) = \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j}(a_N^{ij}(t,x)\varphi) + \sum_{i=1}^n \frac{\partial}{\partial x_i}(f_N^i(t,x)\varphi), \tag{25}$$

and we would like again use Galerkin method to solve

$$\frac{\partial \widetilde{u}_k^N}{\partial t}(t,x) = \left(\mathcal{L}_N^*(t) - \frac{1}{2}|h(t,x)|^2\right)\widetilde{u}_k^N(t,x),\ t \in [\tau_{k-1}, \tau_k]. \tag{26}$$

Let us define

$$\widetilde{u}_k^{N,N'}(t,x) = \sum_{i=1}^{N'} \alpha_i^k(t)\psi_i(x) \tag{27}$$

to be the solution of the approximated equation (26) in the finite-dimensional subspace spanned by $\{\psi_i\}_{i=1}^{N'}$, in the sense that,

$$\frac{\partial}{\partial t}\langle \widetilde{u}_k^{N,N'}(t,\cdot), \psi_k\rangle = \left\langle \left(\mathcal{L}_N^*(t) - \frac{1}{2}|h(t,x)|^2\right)\widetilde{u}_k^{N,N'}(t,\cdot), \psi_k \right\rangle,\ t \in [\tau_{k-1}, \tau_k], \tag{28}$$

for all $1 \le k \le K$.

Hence,

$$\frac{d}{dt}\alpha_j^k(t) = \sum_{i=1}^{N'} \alpha_i^k(t)\left\langle \left(\mathcal{L}_N^*(t) - \frac{1}{2}|h(t,\cdot)|^2\right)\psi_i, \psi_j \right\rangle,\ 1 \le j \le N'. \tag{29}$$

In the online computation procedure, we also project the results of exponential transformations onto the finite dimensional subspace $\mathrm{Span}(\{\psi_i\}_{i=1}^{N'})$, and we have

$$\widetilde{u}_k^{N,N'}(\tau_{k-1},x)=\sum_{i=1}^{N'}\alpha_i^k(\tau_{k-1})\psi_i(x) \tag{30}$$

with

$$\alpha_i^k(\tau_{k-1})=\sum_{j=1}^{N'}\alpha_j^{k-1}(\tau_{k-1})\left\langle\exp\left(h^\top(\tau_{k-1},\cdot)(y(\tau_{k-1})-y(\tau_{k-2}))\right)\psi_j,\psi_i\right\rangle. \tag{31}$$

Recursively, the propagation of the parameters α is given by

$$\begin{cases}\dfrac{d}{dt}\alpha_i^k(t)=\displaystyle\sum_{j=1}^{N'}\alpha_j^k(t)\left\langle\left(\mathcal{L}_N^*(t)-\frac{1}{2}|h(t,\cdot)|^2\right)\psi_j,\psi_i\right\rangle,\ t\in[\tau_{k-1},\tau_k]\\ \alpha_i^k(\tau_{k-1})=\displaystyle\sum_{j=1}^{N'}\alpha_j^{k-1}(\tau_{k-1})\left\langle\exp\left(h^\top(\tau_{k-1},\cdot)(y(\tau_{k-1})-y(\tau_{k-2}))\right)\psi_j,\psi_i\right\rangle.\end{cases} \tag{32}$$

for all $1\le i\le N'$, $1\le k\le K$.

Finally, with the two exponential transformations (10) and (13), we can use $\widetilde{u}_k^{N,N'}(\tau_{k-1},x)$ as an approximator to the unnormalized conditional probability density function $p(\tau_{k-1},x)$, and therefore, for a given test function φ, the solution of McKean-Vlasov filtering problem $E[\varphi(x(t))|\mathcal{Y}_t]$ can be approximated by

$$\widehat{\varphi}(\tau_{k-1}):=\frac{\int_{\mathbb{R}^n}\varphi(x)\widetilde{u}_k^{N,N'}(\tau_{k-1},x)dx}{\int_{\mathbb{R}^n}\widetilde{u}_k^{N,N'}(\tau_{k-1},x)dx}=\frac{\sum_{j=1}^{N'}\alpha_j^k(\tau_{k-1})\int_{\mathbb{R}^n}\varphi(x)\psi_j(x)dx}{\sum_{j=1}^{N'}\alpha_j^k(\tau_{k-1})\int_{\mathbb{R}^n}\psi_j(x)dx}. \tag{33}$$

The entire procedure of the generalized Yau-Yau algorithm for McKean-Vlasov filtering problem is summarized in Algorithm 1.

4 Convergence Analysis

In this section, we will provide some convergence analysis of our proposed Yau-Yau algorithm for McKean-Vlasov filtering problems. For the simplicity of notations, the convergence analysis will be given for the case where the state and observation processes are both one-dimensional. We would like to remark that it is straightforward to generalize the results in this section to multi-dimensional cases.

Because the convergence results of Hermite spectral method and Yau-Yau algorithm for standard nonlinear filtering problems methods have already been discussed in details previously in [20, 21] and [30, 31], respectively, we will only focus on the specialty of McKean-Vlasov filtering problems in this paper.

Based on Algorithm 1, the major differences between the Yau-Yau algorithm for McKean-Vlasov filtering problems and standard time-varying ones lie in the offline steps. In Offline Step 1, the nonlinear Fokker-Planck equation (16) of the state process should be numerically solved,

Algorithm 1 Yau-Yau Algorithm for McKean-Vlasov Filtering Problem

Input: McKean-Vlasov filtering system (1); Terminal time $T > 0$; Time discretization step $\Delta t > 0$; The number of the time step $K = \frac{T}{\Delta t}$; The number of Hermite basis functions N and N'.

Offline Step 1 (completed in the preparation stage of the algorithm):
Utilize Hermite spectral method to solve the nonlinear Fokker-Planck equation (16) of the McKean-Vlasov stochastic differential equation, and obtain an approximated density function $\rho_N(t, x)$.

Offline Step 2 (completed in the preparation stage of the algorithm):
With the approximated elliptic operator $\mathcal{L}_N^*(t)$, solve the observation-independent parabolic partial differential equation (26).

Online Step (processing during the application stage of the algorithm):
for $k = 1$ to K **do**

1. Compute the exponential transformations (10) and (13) when the new observation is obtained.
2. Solve the ordinary differential equation (32) and obtain the approximated unnormalized conditional probability density function $\widetilde{u}_k^{N,N'}(\tau_{k-1}, x)$.
3. Compute the approximated conditional expectation $\widehat{\varphi}(\tau_{k-1})$ according to (33).

end for

and we need to show that utilizing Hermite spectral method, we can approximate the exact solution, $\rho(t, x)$, well. In Offline Step 2, the observation-independent equation (26) is solved with the approximated operator $\mathcal{L}_N^*(t)$, and we need to show that the solution $\widetilde{u}_k^N(t, x)$ of (26) can approximate the solution $\widetilde{u}_k(t, x)$ of (14) well, as long as $\mathcal{L}_N^*(t)$ is close to $\mathcal{L}^*(t)$.

To address the problem in the convergence analysis of Offline Step 1, we need the following approximation theory of Hermite functions in [20].

Lemma 4.1 (Theorem 2.1 in [20]) *Let $u : \mathbb{R} \to \mathbb{R}$ be a function with sufficient regularity. Assume that u, together with its high-order derivatives, vanishes rapid enough at infinity. Consider the projection u_N of u onto the N-dimensional vector space spanned by the Hermite functions $\{\psi_i\}_{i=1}^N$. Then, for sufficiently large N, u_N and its derivative can approximate u and its derivative well in the sense that*

$$\lim_{N\to\infty} \|u - u_N\|_2 = 0, \quad \lim_{N\to\infty} \left\| \frac{d}{dx}u - \frac{d}{dx}u_N \right\|_2 = 0, \tag{34}$$

where $\|\cdot\|_2$ denotes the L^2-norm.

The proof of Lemma 4.1, as well as the exact convergence rate and the required regularity conditions for u, can be found in [20] and references therein. Especially, it is common to obtain super-linear convergence rate for smooth enough functions.

With the above Hermite approximation lemma (Lemma 4.1), we can derive the following convergence result for the nonlinear Fokker-Planck equation (16).

Theorem 4.2 *For a given terminal time $T > 0$, assume that the solution $\rho(t,x)$ of the nonlinear Fokker-Planck equation* (16)*, as well as its high-order derivatives, is square integrable with respect to $x \in \mathbb{R}$, for each $t \in [0,T]$. Also, assume that for every smooth test function with compact support $\varphi \in C_0^\infty(\mathbb{R})$,*

$$\|(\mathcal{L}^*(t) - \mathcal{L}_N^*(t))(\varphi)\|_2 \leq M(\varphi)\|\rho(t,\cdot) - \rho_N(t,\cdot)\|_2, \tag{35}$$

where $\mathcal{L}^*(t)$ and $\mathcal{L}_N^*(t)$ are the operators defined in (9) *and* (25)*; $M(\varphi) > 0$ is a function of φ. Then, the approximated density function $\rho_N(t,x)$ can approximate $\rho(t,x)$ well for sufficiently large N, in the sense that*

$$\lim_{N\to\infty} \max_{t\in[0,T]} \|\rho(t,\cdot) - \rho_N(t,\cdot)\|_2 = 0, \tag{36}$$

and the convergence rate is the same as that of $\frac{\partial}{\partial x}\rho_N(t,x)$ to $\frac{\partial}{\partial x}\rho(t,x)$ in Lemma 4.1.

Remark 4.3 In [30] and [19], it has been proved that under mild conditions, most of the densities of $\rho(t,x)$ will be concentrated in a compact domain $B_R := \{x \in \mathbb{R} : |x| \leq R\}$, for some $R \gg 1$, and we can consider the initial-boundary value problem of (16) in B_R, instead of the initial value problem in the whole space. In this case, the assumption (35) in Theorem 4.2 is easy to satisfy given the definition of $f_N(t,x)$ and $\sigma_N(t,x)$ in (22).

Moreover, according to the expression of $\mathcal{L}^*(t)$ and $\mathcal{L}_N^*(t)$, the function $M(\varphi)$ in (35) can be chosen as the norm of φ in a proper Sobolev space. For a better understanding of readers with different backgrounds, we would like to avoid the introduction of Sobolev spaces in this paper, and interesting readers can refer to textbooks such as [8] for more details of Sobolev spaces.

Proof [Proof of Theorem 4.2] Let us denote by $\widetilde{\rho}_N(t,x)$ the projection of $\rho(t,x)$ onto the space spanned by $\{\psi_i\}_{i=1}^N$, for each $t \in [0,T]$. According to Lemma 4.1, we have

$$\lim_{N\to\infty} \|\rho(t,\cdot) - \widetilde{\rho}_N(t,\cdot)\|_2 = 0. \tag{37}$$

Since

$$\|\rho(t,\cdot) - \rho_N(t,\cdot)\|_2 \leq \|\rho(t,\cdot) - \widetilde{\rho}_N(t,\cdot)\|_2 + \|\widetilde{\rho}_N(t,\cdot) - \rho_N(t,\cdot)\|_2, \tag{38}$$

the remaining task is to estimate the difference $\|\widetilde{\rho}_N(t,\cdot) - \rho_N(t,\cdot)\|_2$.

According to the property of projections, we have

$$\langle \rho(t,\cdot), \psi_i\rangle = \langle \widetilde{\rho}_N(t,\cdot), \psi_i\rangle, \ \forall\ 1 \leq i \leq N, \tag{39}$$

and thus,

$$\frac{d}{dt}\langle \widetilde{\rho}_N(t,\cdot), \psi_i\rangle = \frac{d}{dt}\langle \rho(t,\cdot), \psi_i\rangle = \langle \mathcal{L}^*(t)\rho, \psi_i\rangle \tag{40}$$

Combining the definition of ρ_N, we obtain

$$\frac{d}{dt}\langle \widetilde{\rho}_N(t,\cdot) - \rho_N(t,\cdot), \psi_i\rangle = \langle \mathcal{L}^*(t)(\rho - \rho_N), \psi_i\rangle + \langle (\mathcal{L}^*(t) - \mathcal{L}_N^*(t))\rho_N, \psi_i\rangle \tag{41}$$

for each $1 \leq i \leq N$.

Since $\rho_N(t,x)$ and $\widetilde{\rho}_N(t,x)$ are both elements of the N-dimensional space spanned by $\{\psi_i\}_{i=1}^N$, we can substitute ψ_i into $\widetilde{\rho}_N - \rho_N$, and obtain

$$\begin{aligned}\frac{d}{dt}\|\widetilde{\rho}_N(t,\cdot)-\rho_N(t,\cdot)\|_2^2 &= \frac{d}{dt}\langle\widetilde{\rho}_N-\rho_N,\widetilde{\rho}_N-\rho_N\rangle\\ &= 2\langle\mathcal{L}^*(t)(\rho-\rho_N),(\widetilde{\rho}_N-\rho_N)\rangle + 2\langle(\mathcal{L}^*(t)-\mathcal{L}_N^*(t))\rho_N,(\widetilde{\rho}_N-\rho_N)\rangle\\ &= 2\langle\mathcal{L}^*(t)(\rho-\rho_N),\rho-\rho_N\rangle + 2\langle\mathcal{L}^*(t)(\rho-\rho_N),\widetilde{\rho}_N-\rho\rangle\\ &\quad + 2\langle(\mathcal{L}^*(t)-\mathcal{L}_N^*(t))\rho_N,(\widetilde{\rho}_N-\rho_N)\rangle\end{aligned} \tag{42}$$

Since $\mathcal{L}^*(t)$ is an elliptic operator for all $t\in[0,T]$, there exists a constant $\lambda>0$, such that

$$\langle\mathcal{L}^*(t)(\rho-\rho_N),\rho-\rho_N\rangle \le -\lambda\left\|\frac{\partial}{\partial x}\rho-\frac{\partial}{\partial x}\rho_N\right\|_2^2, \tag{43}$$

and according to Young's inequality,

$$\langle\mathcal{L}^*(t)(\rho-\rho_N),\widetilde{\rho}_N-\rho\rangle \le \frac{\lambda}{2}\left\|\frac{\partial}{\partial x}\rho-\frac{\partial}{\partial x}\rho_N\right\|_2^2 + \frac{C_1}{\lambda}\left\|\frac{\partial}{\partial x}\rho-\frac{\partial}{\partial x}\widetilde{\rho}_N\right\|_2^2 \tag{44}$$

for some constant $C_1>0$.

Also, we can derive from the assumption (35) that

$$\begin{aligned}|\langle(\mathcal{L}^*(t)-\mathcal{L}_N^*(t))\rho_N,\widetilde{\rho}_N-\rho_N\rangle| &\le \|(\mathcal{L}^*(t)-\mathcal{L}_N^*(t))(\rho_N)\|_2\|\widetilde{\rho}_N-\rho_N\|_2\\ &\le M(\rho_N)\|\rho-\rho_N\|_2\|\widetilde{\rho}_N-\rho_N\|_2\\ &\le C_2(\|\widetilde{\rho}_N-\rho_N\|_2^2+\|\rho-\widetilde{\rho}_N\|_2^2)\end{aligned} \tag{45}$$

for some constant $C_2>0$, where we use that fact that ρ_N is a combination of Hermite functions, which are smooth functions with derivatives vanishing rapidly at infinity.

Therefore,

$$\frac{d}{dt}\|\widetilde{\rho}_N-\rho_N\|_2^2 \le 2C_2\|\widetilde{\rho}_N-\rho_N\|_2^2 + 2C_2\|\rho-\widetilde{\rho}_N\|_2^2 + \frac{2C_1}{\lambda}\left\|\frac{\partial}{\partial x}\rho-\frac{\partial}{\partial x}\widetilde{\rho}_N\right\|_2^2 \tag{46}$$

According to Gronwall's inequality, we have

$$\max_{t\in[0,T]}\|\widetilde{\rho}_N(t,\cdot)-\rho_N(t,\cdot)\|_2^2 \le \frac{e^{2C_2T}-1}{2C_2}\left(2C_2\|\rho-\widetilde{\rho}_N\|_2^2+\frac{2C_1}{\lambda}\left\|\frac{\partial}{\partial x}\rho-\frac{\partial}{\partial x}\widetilde{\rho}_N\right\|_2^2\right) \tag{47}$$

and thus

$$\lim_{N\to\infty}\max_{t\in[0,T]}\|\rho(t,\cdot)-\rho_N(t,\cdot)\|_2^2 = \lim_{N\to\infty}\max_{t\in[0,T]}\|\widetilde{\rho}_N(t,\cdot)-\rho_N(t,\cdot)\|_2^2 = 0. \tag{48}$$

To address the problem in Offline Step 2, we have the following estimation result for general parabolic partial differential equations with turbulence in coefficients.

Theorem 4.4 *Fix a terminal time $T>0$. Let u and v be the solution of the following parabolic partial differential equations in its divergence form:*

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\frac{\partial}{\partial x}\left(A(t,x)\frac{\partial}{\partial x}u(t,x)\right)+\frac{\partial}{\partial x}(B(t,x)u(t,x))+C(t,x)u(t,x),\\ \frac{\partial v}{\partial t}(t,x) &= \frac{1}{2}\frac{\partial}{\partial x}\left(A'(t,x)\frac{\partial}{\partial x}v(t,x)\right)+\frac{\partial}{\partial x}(B'(t,x)v(t,x))+C'(t,x)v(t,x),\end{aligned} \tag{49}$$

for $(t,x) \in [0,T] \times \mathbb{R}$, with initial values $u(0,x)$ and $v(0,x)$, where $A(t,x) \geq \lambda > 0$ with some constant λ for all $(t,x) \in [0,T] \times \mathbb{R}$ and $C(t,x) \leq 0$.

Assume that the solutions $u(t,x)$ and $v(t,x)$, together with their derivatives, are smooth enough, and uniformly bounded for all $t \in [0,T]$. If there exists a constant $\varepsilon > 0$, such that

$$\begin{aligned} &|A(t,x) - A'(t,x)| < \varepsilon,\ |B(t,x) - B'(t,x)| < \varepsilon,\ |C(t,x) - C'(t,x)| < \varepsilon, \\ &\left|\frac{\partial}{\partial x}A(t,x) - \frac{\partial}{\partial x}A'(t,x)\right| < \varepsilon,\ \left|\frac{\partial}{\partial x}B(t,x) - \frac{\partial}{\partial x}B'(t,x)\right| < \varepsilon, \\ &\left|\frac{\partial}{\partial x}C(t,x) - \frac{\partial}{\partial x}C'(t,x)\right| < \varepsilon, \end{aligned} \tag{50}$$

for all $(t,x) \in [0,T] \times \mathbb{R}$, then

$$\|u(t,\cdot) - v(t,\cdot)\|_2 \leq C_1 e^{C_2 t} \left(\|u(0,\cdot) - v(0,\cdot)\|_2 + \varepsilon\right). \tag{51}$$

for some constant $C_1, C_2 > 0$ independent of t and ε.

Proof [Proof of Theorem 4.4] According to the parabolic equations (49), the difference between the solutions u and v satisfies

$$\begin{aligned} \frac{\partial}{\partial t}(u-v) =& \frac{1}{2}\frac{\partial}{\partial x}\left(A\frac{\partial}{\partial x}((u-v))\right) + \frac{\partial}{\partial x}(B(u-v)) + C(u-v) \\ &+ \frac{1}{2}\frac{\partial}{\partial x}\left((A-A')\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial x}((B-B')v) + (C-C')v \end{aligned} \tag{52}$$

Since v and its derivatives are uniformly bounded, there exists a constant $K_1 > 0$ such that

$$\left|\frac{1}{2}\frac{\partial}{\partial x}\left((A-A')\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial x}((B-B')v) + (C-C')v\right| < K_1\varepsilon, \tag{53}$$

for all $(t,x) \in [0,T] \times \mathbb{R}$.

Therefore,

$$\begin{aligned} &\frac{d}{dt}\|u(t,\cdot) - v(t,\cdot)\|_2^2 \\ &\quad = 2\left\langle \frac{\partial}{\partial t}(u-v), u-v \right\rangle \\ &\quad \leq -\left\langle A\frac{\partial(u-v)}{\partial x}, \frac{\partial(u-v)}{\partial x} \right\rangle - \left\langle \frac{\partial(u-v)}{\partial x}, B(u-v) \right\rangle \\ &\qquad + \langle C(u-v), u-v\rangle + \langle 2K_1\varepsilon, u-v\rangle \\ &\quad \leq -\lambda\|\nabla(u-v)\|_2^2 + \frac{\lambda}{2}\|\nabla(u-v)\|_2^2 + \frac{K_2}{2\lambda}\|u-v\|_2^2 + K_3\varepsilon \\ &\quad \leq \frac{K_2}{2\lambda}\|u(t,\cdot) - v(t,\cdot)\|_2^2 + K_3\varepsilon, \end{aligned} \tag{54}$$

for some constant $K_2, K_3 > 0$, where we use the elliptic property of $A(t,x)$ and also the Young's inequality.

According to the Gronwall's inequality, we have

$$\|u(t,\cdot) - v(t,\cdot)\|_2^2 \leq K_3\varepsilon\frac{2\lambda(e^{\frac{K_2}{2\lambda}t} - 1)}{K_2} + e^{\frac{K_2}{2\lambda}t}\|u(0,\cdot) - v(0,\cdot)\|_2^2. \tag{55}$$

Up to now, we have addressed all the differences between Yau-Yau algorithm for McKean-Vlasov filtering problems and standard time-varying filtering problems in the convergence analysis. Therefore, previous convergence results of Yau-Yau algorithm [30] and Hermite spectral methods [20, 21] also hold true in this McKean-Vlasov case, which theoretically guarantees the effectiveness of our proposed algorithm in this paper.

5 Numerical Results

In this section, we will use our proposed generalized Yau-Yau algorithm to solve the following McKean-Vlasov filtering problem:

$$\begin{cases} dx(t) = (-0.1x(t) + 0.1Ex(t))\,dt + dw(t) \\ dy(t) = x(t)(1 + 0.25\sin x(t))dt + dv(t), \end{cases} \tag{56}$$

with initial value $x(0) \sim \mathcal{N}(1, 1)$ a normal random variable and $y(0) = 0$. The noises $\{v(t) : t \geq 0\}$ and $\{w(t) : t \geq 0\}$ are mutually independent standard one-dimensional Brownian motions. The state equation in (56) can also be written in the standard form (1), with

$$f[t, x(t), \mu_t] = \int_{\mathbb{R}} \mathfrak{f}(t, x(t), z)\mu_t(z), \tag{57}$$

and

$$\mathfrak{f}(t, x, z) = -0.1x + 0.1z, \ \ (t, x, z) \in \mathbb{R}^3. \tag{58}$$

We set the terminal time of this filtering system to be $T = 50$, and in order to apply Yau-Yau algorithm, the time-discretization step size is set to be $\Delta t = 0.01$. The total time step is $K = \frac{T}{\Delta t} = 5000$. The number of Hermite basis functions we use in Offline Step 1 and Offline Step 2 are both $N = N' = 15$.

In order to evaluate the performance of our proposed algorithm, we simulate $M = 100$ mutually independent trajectories of the filtering system (56). For each trajectory, the state process $x(t)$ is simulated by 100 samples $x_i(t)$, $1 \leq i \leq 100$, with each sample evolves according to

$$dx_i(t) = (-0.1x_i(t) + 0.1m(t))\,dt + dw_i(t), \tag{59}$$

where $m(t) = \frac{1}{N_1}\sum_{i=1}^{100} x_i(t)$ is the mean of the 100 samples. The *propagation of chaos* theory [12] implies that the evolution equation of each $x_i(t)$ will be close to the state equation in (56), as long as the number of the samples is large enough. In fact, for many application scenarios such as the models in the study of mean field game theory, this is how McKean-Vlasov equation is introduced.

We consider the mean square error (MSE) between the state process $x(t)$ and the approximated $\widehat{x}(t)$ with Yau-Yau algorithm, which is defined as

$$\text{MSE} = \frac{1}{M}\sum_{i=1}^{M} \frac{1}{K+1}\sum_{k=0}^{K} |x^{(i)}(\tau_k) - \widehat{x}^{(i)}(\tau_k)|^2, \tag{60}$$

where the superscript in $x^{(i)}$ and $\widehat{x}^{(i)}$ represents the i-th trajectory. In our experiment, the mean square error we obtain is MSE $= 1.0389$.

In the meanwhile, for a particular trajectory i_0, we can define the mean square error of this trajectory up to time $t = \tau_k$ by

$$\text{MSE}(k) = \frac{1}{k+1} \sum_{j=0}^{k} |x^{(i_0)}(\tau_j) - \widehat{x}^{(i_0)}(\tau_j)|^2, \ 0 \leq k \leq K. \tag{61}$$

The evolution of MSE(k), $0 \leq k \leq K$, of a particular trajectory is shown in Figure 1. The mean square error of our proposed algorithm for this particular trajectory will remain around 0.6, which implies that the estimation error of the algorithm will not accumulate or explode as time goes by.

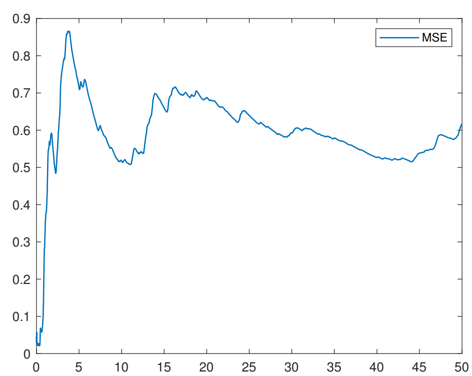

Figure 1: The evolution of the mean square error of a particular trajectory.

The comparison between the real state $x(t)$ and our estimation $\widehat{x}(t)$, the approximated conditional expectation (cept), of this particular trajectory is also illustrated in Figure 2. In this particular trajectory, the solution of our proposed algorithm can approximate the trend of the real state well, which shows the capability of Yau-Yau algorithm to effectively capture the information of the state process from the noisy observations of the McKean-Vlasov filtering system.

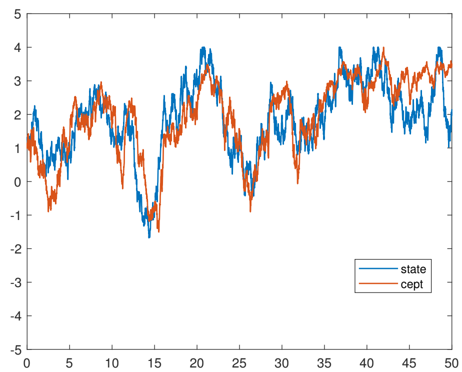

Figure 2: The performance of Yau-Yau algorithm for a particular trajectory of McKean-Vlasov filtering problem, with the blue line representing the real state and the orange one representing the approximated conditional expectation (cept).

In fact, if we compare the evolution of the state process $x(t)$ and the *raw* observations $\{\frac{y(\tau_k)-y(\tau_{k-1})}{\Delta t} : 1 \leq k \leq K\}$, as is shown in Figure 3, the amplitude of the raw observation is much bigger than that of the real state, which implies that the observations we can obtain is heavily noised. After the process of our proposed filtering algorithm, the result amplitude of approximated conditional expectation is comparable with the state process, as is shown in Figure 2, which means that our filtering algorithm has made a good performance in denoising.

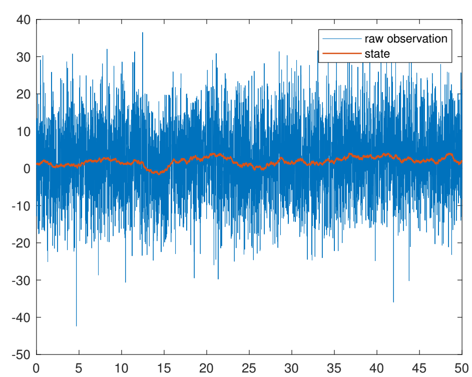

Figure 3: The evolution of the *raw* observation and the real state

We would like to remark that this numerical result shows the potentials of Yau-Yau algorithms in addressing McKean-Vlasov equation filtering problems with economics or sociology backgrounds, in which noise contributes the major part of the data and it is important for algorithms to capture useful information and make accurate prediction on the trend of the state process.

Finally, the above numerical experiments are all conducted on a laptop with Intel(R) Core(TM) i7-9750H CPU @2.60GHz, 6 physical cores, 12 logical processors. The average online computational time for one trajectory of conditional expectations is around 23 seconds. Therefore, for a McKean-Vlasov dynamics of $T = 50$ seconds with time-discretization step $\Delta t = 0.01$ seconds, our proposed algorithm can provide a real-time solution and update the estimation of the state process as soon as new observations are obtained.

6 Conclusion

In this paper, we propose a novel numerical algorithm to solve McKean-Vlasov filtering problems, which have wide applications in various areas such as mean-field control and mean-field game theory. Following the idea of Yau-Yau algorithm framework, the computational procedure of McKean-Vlasov filtering problems can be divided into three steps, and the computationally complicated steps of solving nonlinear Fokker-Planck equations and time-varying parabolic partial differential equations can be done offline.

Because the online computation in our proposed algorithm only contains the exponential transformations and numerical integrations, it can be completed efficiently under a suitable representation. For problems with low or medium-high dimensions, Hermite spectral method has been proved to be a good choice to implement Yau-Yau algorithm for standard filtering problems. In this paper, we also employ Hermite spectral method to solve McKean-Vlasov filtering problem under Yau-Yau algorithm framework.

Our proposed algorithm serves to be the first attempt on the numerical solutions of McKean-Vlasov filtering problems. Both theoretical analysis and numerical results show that our proposed algorithm can successfully solve McKean-Vlasov filtering problem in low dimensions with enough accuracy and reasonable online computational cost.

Some promising future research directions are listed as follows. Firstly, numerical methods of solving the nonlinear Fokker-Planck equation can be further explored. Especially, algorithms which preserve the total density of the solution is appealing, because this kind of algorithms can guarantee that the solution remains to be a probability density through evolution. Secondly, efficient implementations of Yau-Yau algorithm framework for high-dimensional McKean-Vlasov filtering problems are also important for practical use. Finally, apart from partial differential equation based algorithms, Monte-Carlo based algorithms for McKean-Vlasov filtering problems, such as particle filter, also deserve further analysis and discussion.

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