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European Option Prices and Hedge under Incomplete Information: an Interdisciplinary Research with Yau-Yau Algorithm

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In this paper, we study European option pricing and hedging in an incomplete information market where the observable risky asset price is driven by latent diffusion factors that enter the drift. Investors observe only prices and infer the hidden state via nonlinear filtering. Building on the Yau–Yau nonlinear filter and its pathwise robust DMZ formulation, we construct a filtered market under the observable filtration, prove the existence of an equivalent risk–neutral measure, and show that the filtered market is complete. This yields a generalized Black–Scholes PDE for the option price, coupling the risk–neutral asset dynamics with the generator of the latent factor, and a feedback hedging strategy expressed in terms of the PDE solution; in the original market this strategy is locally risk–minimizing in the sense of Föllmer–Schweizer. Numerical experiments on nonlinear toy models illustrate the practical implementation of the filtering–pricing scheme and its suitability for real–time applications.

KEY WORDS AND PHRASES. Nonlinear Filtering, Incomplete Information, Black–Scholes Equation, Hedging.

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1. Introduction

In this paper, we consider European options in incomplete information financial markets, which give the right to buy one share of a risky assets S with claim Z at time T. Our focus is on market environments in which asset prices are observable, while economically fundamental drivers of price dynamics remain latent and must be inferred from market data. We model the observable risky asset price process $S(t)$ as a Black–Scholes SDEs affected by some unobservable state process $\theta(t)$, which represents latent economic forces shaping investors' expectations. Economically, $\theta(t)$ may be interpreted as a time-varying risk premium, an aggregate investor sentiment factor, a hidden asset price, or a latent macroeconomic business-cycle component. These variables play a central role in real-world price formation but are not directly traded or observed; instead, they influence prices indirectly through trading activity. In the incomplete information financial markets, market participants are restricted to using publicly available information—most notably historical price data—and operate under a reduced filtration generated by observable quantities. The basic question under the incomplete information is: what is the "proper price" at time $t = 0$ of the European options under a reduced filtration generated by observable quantities?

Classical option pricing theory provides a rigorous benchmark for pricing and hedging under the paradigm of complete information (Karagozoglu, 2022; Merton, 1998; Smith Jr, 1976). In the seminal works of Black, Scholes and Merton (Black and Scholes, 1973; Merton et al., 1971), the standard model for stock prices is the geometric Brownian motion. Most of the theoretical development has relied on the assumption that all relevant state variables are fully observable (Ball and Roma, 1994; Bergman, 1995; Carr and Sun, 2007; Chang et al., 1998; Chen et al., 2002; Cox et al., 1979; Eisenberg and Jarrow, 1994; Feunou and Okou, 2019). This complete-information assumption is closely aligned with the efficient market hypothesis in financial economics, in that asset prices are assumed to fully and instantaneously reflect all available information. In this idealized framework, markets are dynamically complete and every European option Z can be perfectly replicated by a self-financing trading strategy. The classical Black–Scholes option pricing theory shows that, reflecting the economic principle that, after adjusting for the time value of money, asset prices exhibit no predictable drift and all profit opportunities from publicly available information are eliminated.

However, in real-world settings, market participants encounter challenges of incomplete information: key parameters or state variables are often observable only through noisy data or indirect indicators (Bergemann and Bonatti, 2019; Rondina, 2008). Since the investors can only use incomplete information for the pricing, this lack of full observability makes the market incomplete (Kallianpur and Karandikar, 2012), and thus perfect replication of contingent claims is no longer possible (Davis, 1997). Consequently, one must turn to probabilistic and statistical tools to extract latent information and define consistent pricing and hedging strategies under uncertainty (Benth, 2003; Kallianpur and Karandikar, 2012).

To address these challenges from incomplete information, filtering theory provides a powerful tool (Frey et al., 2019; Jazwinski, 2007). Since prices are observable but their fundamental drivers are not, rational investors must form conditional beliefs about the latent state $\theta(t)$ based on observed price histories. Mathematically, this corresponds to computing the conditional distribution (filtered estimates) of the hidden signal process given the observation filtration $\mathscr{F}_t^O$. Unlike the classical Black–Scholes setting, where model coefficients are deterministic or fully observable, the effective market coefficients here become random and filter-dependent. Within this framework, the filtered estimate of the unobservable state serves as a dynamically updated endogenous market price of risk, linking observed prices to investors' evolving beliefs. This filtered market model allows one to define option prices and hedging strategies in terms of conditional expectations with respect to the observable filtration, thereby generalizing the classical risk-neutral valuation paradigm to incomplete-information environments (Elliott and Siu, 2023; Siu and Elliott, 2022). Moreover, an alternative approach based on local risk minimization (Ceci et al., 2015; Lee and Zeng, 2010; Schweizer, 1994) naturally emerges in this context: since perfect replication is infeasible, hedging strategies are constructed by minimizing the conditional variance of the hedging error under the physical measure, with the filter providing the necessary real-time estimates of unobservable risk factors.

Early applications in incomplete information financial markets largely relied on linear filtering techniques, most notably the Kalman–Bucy filter (Kalman and Bucy, 1961), due to their analytical tractability. However, linear filters are inherently constrained by Gaussian and linearity assumptions, which are often violated in real financial markets. When asset returns exhibit nonlinear dynamics, heavy tails, regime switching, or stochastic volatility, linear filters tend to produce biased state estimates and unstable hedging strategies. To relax these restrictive assumptions, a variety of nonlinear filtering methods have been developed, including extended and modified Kalman filters (Sayadi and Shamsollahi, 2008; Song and Speyer, 1985; Yucelen and Calise, 2010) and particle filtering methods (Djuric et al., 2003; Gustafsson, 2010). While these approaches are more flexible, their performance deteriorates markedly in strongly nonlinear or high-dimensional settings, and their real-time implementation remains computationally challenging in practical financial applications. A more principled mathematical framework for nonlinear filtering is provided by stochastic partial differential equation (SPDE) approaches, developed since the 1960s. Within this framework, the evolution of the (unnormalized or normalized) posterior density of the hidden state is governed by the Duncan–Mortensen–Zakai (DMZ) equation and the Kushner–Stratonovich (KS) equation (Julier and Uhlmann, 2004; Rozovskii, 1972; Yau et al., 2024; Zakai, 1969). These equations provide a theoretically rigorous description of belief dynamics under partial observation and establish a natural link between stochastic control, filtering, and dynamic asset pricing. Several numerical approaches have been proposed to approximate these filtering SPDEs, including ensemble-based methods and finite-difference discretizations (Bensoussan et al., 1990; Gordon et al., 1993; Itô, 1996; Nagase, 1995). While filtering-based pricing and hedging have been widely studied, much of the existing work lacks mathematical rigor: convergence proofs, stability guarantees, and quantitative error analyses are often absent. Moreover, financial applications require a pathwise robust algorithms since one can only observe one sample path from the historic data. But numerical schemes directly based on the Zakai equation typically do not possess pathwise robustness, making them difficult to justify for real-time financial applications. These limitations highlight the need for new numerical and analytical filtering–pricing schemes that are specifically tailored to financially relevant nonlinear and incomplete-information models.

To address many of the above gaps, Yau, Yau and their collaborators developed a new approach to solving the general nonlinear filtering problem—known as the Yau–Yau filter, see e.g. Yau and Yau (Yau and Yau, 1998; 2000), Luo and Yau (Luo and Yau, 2013a;b), and the references therein. A key advantage of the Yau–Yau formulation is its mathematical tractability: it transforms the nonlinear filtering problem into a deterministic parabolic PDE that can be solved efficiently with some numerical schemes for PDEs. The Yau–Yau framework provides a "real-time, memory-less" algorithm for solving the general nonlinear filtering problem by connecting it to solving a forward Kolmogorov equation and the DMZ equation in an off-line/on-line decomposition (Dong et al., 2020; Luo and Yau, 2013a). Based on these works, Li, Wang, Yau, and Zhang consider the high-dimensional nonlinear filtering problems by tensor train decomposition method in Li et al. (2019) Recently, Chen and Yau (Chen et al., 2024) combine the Yau-Yau nonlinear filter with deep learning framework, and extended in very high dimension case.

The application of the Yau–Yau nonlinear filter to incomplete information markets can offer a conceptually rigorous and computationally efficient framework for option pricing and hedging under partial observation. The Yau–Yau approach bridges stochastic filtering with well-established numerical PDE techniques, thereby achieving both theoretical clarity and practical tractability. In financial contexts, this feature enables real-time and pathwise robust estimation of unobservable market factors—such as stochastic volatility or hidden nonlinear drift—using stable and accurate numerical solvers, which is very crucial in real financial application. Moreover, the Yau–Yau filter's ability to capture nonlinear and non-Gaussian dynamics enhances the precision of conditional state estimation, resulting in more reliable pricing kernels and dynamically consistent hedging ratios. These advantages are particularly relevant when the incomplete information market models violate the linear, Gaussian assumptions underpinning linear filters. Collectively, these features position the Yau–Yau nonlinear filter as a powerful analytical and numerical tool for consistent valuation and robust risk management in modern incomplete information markets.

In this paper, we study the pricing of European options and the construction of hedging strategies in incomplete information financial markets based on the Yau–Yau nonlinear filter. Based on the filtering estimates $\hat{\theta}(t)$ and regularity estimates of the pathwise robust DMZ equation, we construct a complete filtered market with a risk-neutral measure $\mathbb{Q}^N$. Under $\mathbb{Q}^N$, the discounted price process $\hat{V}(t)$ of the European options is a martingale. Thus we can use the martingale representation theorem to prove that in the filtered market, the European contingent claim Z can be perfectly replicated by a self-financing trading strategy process H, leading to the result stated in Theorem 4.2. Combining with Kallianpur-Striebel formula from the filter theory, and the martingale property of the discounted price process $\hat{V}(t)$, we drive the the general Black-Scholes equation (5.5) associated with the price formula of the European option in Theorem 5.1. The another goal of this paper is to investigate the associated with hedging strategy for the European options in the filtered market. Based on the the Black-Scholes equation (5.5), we obtain the hedging strategy in the filtered market (see Theorem 6.1). Furthermore, we show that without filtered estimates for $\theta(t)$, this strategy is still a locally risk-minimizing strategy in the original incomplete information market in Theorem 6.2.

The rest of the paper is organized as follows. In Section 2, we give the setting of the incomplete information financial markets with some motivated examples, and connect it to a filter framework. In Section 3, we review the DMZ equation and the Yau-Yau nonlinear filter. Then based on the filtering estimates $\hat{\theta}(t)$, we construct a complete filtered market with a risk-neutral measure $\mathbb{Q}^N$ in Section 4. In Section 5, we derive the general Black-Scholes equation for the pricing of the European option in the filtered market. In Section 6, we consider the hedging strategy of the European options. In Section 7, we present some numerical examples. Finally, we end with some conclusions and discussions in Section 8.

2. The Options Pricing Models with Incomplete Information

We begin by introducing the mathematical structure of the financial model under incomplete information. As a general set-up, we consider an underlying probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$, and we assume that the full information flow is described by the filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$. However, the investors can only use incomplete information in the financial market, which is given by a smaller filtration (partial available information flow) $\mathbb{F}^O = (\mathscr{F}_t^O)_{t\in[0,T]}$, so that for each $t \in [0,T]$, $\mathscr{F}_t^O \subsetneqq \mathscr{F}_t$. We fix a finite horizon time $T > 0$, that is, the trader is allowed to invest in the time interval $[0,T]$.

Assume that the financial market has one riskless asset $M(t)$ with compounded interest rate r_t, and m risky assets $S = (S^1, S^2, \cdots, S^m)^\top$, which is given by the Black–Scholes SDE

$$dS(t) = diag(S(t))[r(\theta(t),t)dt + \sigma_t dB_t], \quad S(0) = S, \tag{2.1}$$

or for each $1 \leq i \leq m$,

$$dS^i(t) = S^i(t)(r(\theta(t),t)dt + \sum_{j=1}^{m} \sigma_t^{ij} dB_t^j), \quad S_i(0) = S_i > 0, \tag{2.2}$$

where $r(\theta(t),t) \in \mathbb{R}^m$ is the expected instantaneous rate of returns, B_t is a $(\Omega, \mathscr{F}, (\mathscr{F}_t^O)_{t\geq 0}, \mathbb{P})$ standard m-dimension Brownian motion, $\sigma_t \in \mathbb{R}^{m\times m}$ is the volatility matrix, and $diag(S(t))$ denotes the diagonal matrix with elements of $S(t)$ on its diagonal. Since the risky prices are publicly available, we assume that the agents can reasonably observe at least the asset prices, then $S(t)$ is a $\mathbb{F}^O$-adapted process. Without loss of generality, we take the initial riskless asset $M(0) = 1$. Then the price of riskless asset $M(t) = e^{\int_0^t r_s ds} M(0) = e^{\int_0^t r_s ds}$.

We assume that the financial market returns $r(\theta(t),t)$ are influenced by some unobservable market factors $\theta(t)$, which is modeled by the following SDE

$$d\theta(t) = b(\theta(t),t)dt + G(\theta(t),t)dW_t, \tag{2.3}$$

where $b(\theta(t),t)$is the drift term, $G_t \in \mathbb{R}^{n\times r}$ is the diffusion matrix, and W_t is an $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ standard r-dimensional Brownian motion, which is independent with B_t. Then the unobservable process $\theta(t)$ of (2.3) is a Markov process with generator

$$\mathscr{L}_t g(\theta) = b(\theta,t)\cdot\nabla g(\theta) + \frac{1}{2}\sum_{i,j=1}^{n}\Gamma^{ij}(\theta,t)\frac{\partial^2 g}{\partial\theta_i\partial\theta_j}(\theta), \tag{2.4}$$

where $\Gamma = GG^\top$.

We now illustrate several representative scenarios where the observation process is the market price (or its derived quantities), while the signal process captures hidden economic or behavioral components that influence the dynamics. We first consider an incomplete information market where the investor observes only the asset price process $S(t)$, is given by

$$dS(t) = S(t)[(1+\theta(t))dt + \sigma_1 dB_t], \quad S_0 > 0,$$

where $\sigma > 0$ is known and constant, B_t is a Brownian motion, and $\theta(t)$ denotes the unobservable time-periodic factor, which is given by

$$d\theta(t) = \big(\mu_0 + \mu_1\sin(\omega_1 t)\big)dt + \sigma_2 dW_t, \quad \theta(t) = 1,$$

where W_t is a Brownian motion independent of B_t.

In some situation, the influence of the unobservable factor $\theta(t)$ to $S(t)$. For this purpose, we assume that asset price process $S(t)$ satisfies

$$dS(t) = S(t)[(1+\theta(t))^3 dt + \sigma_1 dB_t], \quad S_0 > 0,$$

where $\sigma_1 > 0$ is known and constant, B_t is a Brownian motion, and $\theta(t)$ is given by

$$d\theta(t) = \big(\mu_0\theta(t) + \mu_1\sin(\omega_1 t)\big)dt + \sigma_2 dW_t, \quad \theta(t) = 1,$$

where W_t is a Brownian motion independent of B_t.

Across these settings, the asset price $S(t)$ evolves as an observation process driven by a hidden signal process $\theta(t)$. In the incomplete information markets, the investors can only use access the information from the filtration (partial information flow) $\mathbb{F}^O = (\mathscr{F}_t^O)_{t\in[0,T]}$ generated by $(S(t))$, and they relies the posterior conditional distribution of $\theta(t)$ by using the observation from $S(t)$. Since nonlinear filter theory is a framework of dynamical Bayesian inference, we can use nonlinear filter to estimates the unobservable factors $\theta(t)$, and involve the filtered estimates $\hat{\theta}(t)$ in the filter-based market. In order to formulate the incomplete information market within a nonlinear filtering framework, we view the unobservable market factors $\theta(t)$ as the **signal process** (2.3).

Since the price process $S(t)$ (2.1) is a degenerate diffusion process, we consider the log return process

$$R(t) = \log(S(t)) = (\log(S^1(t)), \cdots, \log(S^m(t)))^\top$$

as the **observation process** in the filtering framework. Then by Itô's formula, the log return process $R(t)$ of stock index is given by

$$dR(t) = \mu(\theta(t),t)dt + \sigma_t dB_t, \tag{2.5}$$

where $R(t)$ is the log return of stock index, $\mu(\theta(t),t) = (\mu^1(\theta(t),t), \cdots, \mu^m(\theta(t),t))$ with each component typically given by $\mu^i(\theta(t),t) := r^i(\theta(t),t) - \frac{1}{2}\sum_{j=1}^m(\sigma_t^{ij})^2$, and B_t is a standard Brownian motion. Then the observation process has the Markov generator

$$\mathscr{A}_t f(r) = \mu(\theta,t)\cdot\nabla f(r) + \frac{1}{2}\sum_{i,j=1}^m A_t^{ij}\frac{\partial^2 f(r)}{\partial r_i \partial r_j},$$

where $A_t := \sigma_t\sigma_t^\top$. The observation process (2.5) states that the instantaneous index return $R(t)$ is driven by the time-varying regression coefficients $\theta(t)$, with Gaussian observation noise. We will use the nonlinear filter theory to recursively estimate the regression coefficients $\theta(t)$ from observed data $\{R(s) : 0 \le s \le t\}$.

The following assumptions are imposed for the drift $r(\theta(t),t)$ and diffusion σ_t:

(A1) The drift term $b(\theta,t)$, observation term $r(\theta,t)$, and diffusion matrices $G(\theta,t)$, σ_t are assumed to be C^2 in space and C^1 in time t.

(A2) The drift term $b(\theta,t)$, and diffusion matrix $G(\theta,t)$ have sublinear growth at infinity.

(A3) The matrices $A_t := \sigma_t\sigma_t^\top$ and $\Gamma = GG^\top$ satisfy the uniformly elliptic condition, and $\Gamma = GG^\top$ is bounded from above on $\Omega_T = [0,T]\times\mathbb{R}^n$.

Under above assumptions, the stochastic differential equation (2.1) for the risky asset price $S(t)$ has a unique solution, see e.g. Klebaner (2012).

Now we define the strategy in incomplete information financial markets as follows.

Definition 2.1. *A strategy over the trading interval* $[0,T]$ *is a* $\mathbb{F}^O$*-progressively measurable* $m+1$*-dimensional process* $H(t) = (H^0(t), H^1(t), \cdots, H^m(t))$ *with the the integrability condition*

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T |H^0(s)|ds + \sum_{i=1}^{m} \int_0^T |H^i(s)|^2 ds\right] < \infty, \tag{2.6}$$

whose general component $H^i(t)$ *stands for the number of units of the i-th asset held by an investor at time* t*. We denote* $\Theta(\mathbb{F}^O)$ *as the collection of strategy.*

With the strategy process $H(t) = (H^0(t), H^1(t), \cdots, H^m(t))$, the associated portfolio is denoted as

$$V^H(t) := H^0(t)M_t + \sum_{i=1}^{m} H^i(t)S^i(t).$$

Note that the integrability condition 2.6 so that the value process $V^H(t)$ is square integrable.

The strategy H is said to be self-financing if it satisfies the condition (2.6) and its associated portfolio V^H satisfies

$$dV^H(t) = \sum_{i=0}^{m} H^i(s)dS^i_t. \tag{2.7}$$

Moreover, the strategy H is admissible if $V^H(t) > 0$ for every $t \in [0,T]$, The market is arbitrage-free if for every admissible strategy H on $[0,T]$ with $V^H(0) = 0$ is such that $\mathbb{P}(V^H(T) > 0) = 0$.

Let (Z,T) be the European option with the payoff $Z := h(S(T), \hat{\theta}(T))$ at maturity date T. Here, the payoff Z is a non-negative $\mathscr{F}_T$-measurable random variable, and we assume that its terminal payoff function h is dependent on the price $S(T)$ and the filtered estimated of some macroeconomic factor $\hat{\theta}(T)$. The incorporation of the auxiliary variable θ into the terminal payoff function $h(x,\theta)$ is motivated by the valuation of hybrid derivatives, where the payout depends on the interplay between the underlying asset x and an external benchmark θ. A classical example for this situation is the Inflation-Indexed Call Option, see Example 1 in Section 7 for details.

Definition 2.2. *We say that a European option* (Z,T) *is attainable if there exists a strategy* $H \in \Theta(\mathbb{F}^O)$ *such that* $V^H(T) = Z$*. Such a strategy* H *is said to replicate the option* (Z,T)*.*

Now we introduce the notations of cost and associated risk process from Ceci et al. (2015); Schweizer (2001).

Definition 2.3. *For any* $\mathbb{F}^O$*-admissible strategy* $H \in \Theta(\mathbb{F}^O)$ *(i.e.* $V^H(t) > 0$ *for every* $t \in [0,T]$*), we define the associated* $\mathbb{F}^O$*-adapted cost process*

$$C^H(t) = \hat{V}^H(t) - \sum_{i=1}^{m} \int_0^t H^i(s) d\hat{S}^i(s), \quad t \in [0,T].$$

Base on the cost process, we define the risk process in the incomplete information setting.

Definition 2.4. *For any* $\mathbb{F}^O$*-admissible strategy, the associated risk process is defined as*

$$R^H(t) = \mathbb{E}^{\mathbb{P}}\left[|C^H(T) - C^H(t)|^2 | \mathscr{F}_t \right].$$

Note that in the complete market, the cost process of a self-financing strategy $C^H(t) \equiv \hat{V}^H(0)$. However, in the incomplete information market, the perfect replication of a given option by a self-financing admissible strategy is not guaranteed. Thus we need extend the notation of self-financing to define the "good" $\mathbb{F}^O$-admissible strategies for the incomplete information market as following.

Definition 2.5. *An* $\mathbb{F}^O$*-admissible strategy* H *is called mean self-financing if the associated cost process* $C^H(t)$ *is an* $(\mathbb{F}^O, \mathbb{P})$*-martingale.*

Then, we introduce the following definition of risk-minimizing strategy under partial information $\mathbb{F}^O = (\mathscr{F}_t^O)_{t\in[0,T]}$.

Definition 2.6. *Let* (Z,T) *be a European option with the payoff* $Z \in L^2(\mathscr{F})$*. An* $\mathbb{F}^O$ *mean self-financing strategy* H *is risk minimizing of the option* (Z,T) *if* $V^H(T) = Z$*, and for any other strategy* $\bar{H} \in \Theta(\mathbb{F}^O)$*, we have* $R^H(t) \leq R^{\bar{H}}(t)$ *for every* $t \in [0,T]$*.*

3. Preliminaries: Stochastic Filtering

3.1. The DMZ equations

Let $\mathscr{F}_t^O$ denote the filtration generated by $\{R(s) : s \leq t\}$. Our aim is to compute the conditional probability distribution of $\theta(t)$ given the entire observation until now, $(R(s))_{s\in[0,t]}$. That is, we want to compute conditional expectation $\mathbb{E}\left[\varphi(\theta(t)) \,\middle|\, \mathscr{F}_t^O\right]$ for all continuous and bounded test functions φ.

We assume that the SDE (2.5) satisfies Novikov's condition

$$\mathbb{E}^{\mathbb{P}}\big[\exp(\frac{1}{2}\int_0^T \|\mu(\theta(s),s)\sigma_s^{-1}\|^2 ds)\big] < \infty.$$

Now we define an equivalent measure $\mathbb{Q}$ on $\mathscr{F}_T$ via

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathscr{F}_T} = \Lambda_T^{-1} := \exp\left(-\int_0^T \mu(\theta(s),s)^T\sigma_s^{-1}dB_s - \frac{1}{2}\int_0^T \|\mu(\theta(s),s)\sigma_s^{-1}\|^2 ds\right).$$

Then by Girsanov theorem, under the measure $\mathbb{Q}$, the drift $\mu(\theta(t),t)$ is removed from (2.5), and the observation process $R(t)$ is a n-dimensional Brownian motion with covariance matrices $\mathbb{E}(dR(t)dR(t)^T) = A(t)dt$, where $A(t) = \sigma_t\sigma_t^T$. We also introduce the stochastic process via

$$d\bar{R}_t = \sigma_t^{-1}dR(t),$$

so that $\bar{R}_t$ is the standard Brownian motion $\bar{R}_t$ under $\mathbb{Q}$. Since W_t is independent with B_t, under the new probability $\mathbb{Q}$, the signal process $\theta(t)$ still satisfies the original SDE (2.3). Moreover, $d\mathbb{P}/d\mathbb{Q}$ is given by

$$\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathscr{F}_t} = \Lambda_t := \exp\left(\int_0^t \mu(\theta(s),s)^T\sigma_s^{-1}\,d\bar{R}_s - \frac{1}{2}\int_0^t \|\mu(\theta(s),s)\sigma_s^{-1}\|^2 ds\right). \tag{3.1}$$

Under $\mathbb{Q}$, the exponential martingale satisfies

$$d\Lambda_t = \Lambda_t\mu(\theta(t),t)^T\sigma_t^{-1}d\bar{R}_t, \qquad \Lambda_0 = 1. \tag{3.2}$$

Here, Λ_t is a $(\mathscr{F}_t^O)_{t\geq 0}$-martingale under $\mathbb{Q}$ independent with $\theta(t)$.

For any bounded measurable smooth test function φ, using the abstract Bayes formula (Kallianpur-Striebel formula), the associated conditional expectation (filtered estimate) is given by

$$\mathbb{E}^{\mathbb{P}}\big[\varphi(\theta(t))\,\big|\mathscr{F}_t^O\big] = \frac{\mathbb{E}^{\mathbb{Q}}\big[\varphi(\theta(t))\Lambda_t\,\big|\mathscr{F}_t^O\big]}{\mathbb{E}^{\mathbb{Q}}\big[\Lambda_t\big|\mathscr{F}_t^O\big]}.$$

Now we define the unnormalized estimate $\rho_t(\varphi) = \mathbb{E}^{\mathbb{Q}}\big[\varphi(\theta(t))\Lambda_t\,\big|\mathscr{F}_t^O\big]$. By applying the Itô's formula to $\varphi(\theta(t))\Lambda_t$, we obtain

$$\begin{aligned} d\big(\varphi(\theta(t))\Lambda_t\big) &= \Lambda_t\,d\varphi(\theta(t)) + \varphi(\theta(t))\,d\Lambda_t + d\langle\varphi(\theta(t)),\Lambda_t\rangle \\ &= \Lambda_t\Big((\mathscr{L}\varphi)(\theta(t))\,dt + \nabla\varphi(\theta(t))G_t dW_t\Big) \\ &\quad + \varphi(\theta(t))\Lambda_t\mu(\theta(t),t)\sigma_t^{-1}d\bar{R}_t + d\langle\varphi(\theta(t)),\Lambda_t\rangle. \end{aligned} \tag{3.3}$$

Taking the conditional expectation $\mathbb{E}^{\mathbb{Q}}[\cdot|\mathscr{F}_t^O]$ to (3.3), we obtain $\mathbb{E}^{\mathbb{Q}}[d\langle\varphi(\theta(t)),\Lambda_t\rangle|\mathscr{F}_t^O] = 0$, and

(3.4)

$$d\mathbb{E}^{\mathbb{Q}}[\varphi(\theta(t))\Lambda_t\,|\mathscr{F}_t^O] = d\rho_t(\varphi) = \rho_t(\mathscr{L}\varphi(\theta(t)))\,dt + \rho_t\big(\varphi(\theta(t))\mu(\theta(t),t)A_t^{-1}\big)dR(t).$$

If there exists a density $p(t,\cdot)$ so that

$$\rho_t(\varphi) = \mathbb{E}^{\mathbb{Q}}[\varphi(\theta(t))\Lambda_t\,|\mathscr{F}_t^O] = \int_{\mathbb{R}^n} p(\theta,t)\varphi(\theta)d\theta,$$

then the (unnormalized) density p of $\theta(t)$ satisfies the Duncan–Mortensen–Zakai (DMZ) equation

$$dp(t,\theta) = \big(\mathscr{L}^* p\big)(t,\theta)dt + p(t,\theta)\mu(\theta,t)A_t^{-1}dR(t), \quad p(0,\theta) = p_0(\theta). \tag{3.5}$$

Here, $p_0(\theta)$ is the density of the initial states θ_0, $\mathscr{L}^*$ is the L^2-adjoint of $\mathscr{L}$:

$$\begin{aligned}\big(\mathscr{L}^*\psi\big)(\theta) &= -\nabla\cdot\big(b(\theta,t)\psi(\theta)\big) + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial\theta_i\partial\theta_j}(\Gamma_{ij}(\theta,t)\psi(\theta))\\ &:= -\nabla\cdot\big(b(\theta,t)\psi(\theta)\big) + \frac{1}{2}(\mathscr{L}_2^*\psi)(\theta).\end{aligned} \tag{3.6}$$

Moreover, the normalized posterior density is given by

$$\rho(t,\theta) = \frac{1}{C_t}p(t,\theta),$$

where $C_t = \int_{\mathbb{R}^m} p(t,\theta)d\theta$ is the normalized constant. The density $\rho(t,x)$ is called the normalized filter of θ. It is an infinite-dimensional state variable, evolving in the space of probability measures.

Using the density $p(t,\theta)$, the filtered estimate of $\theta(t)$ is given by

$$\hat{\theta}(t);= \mathbb{E}^{\mathbb{P}}[\theta(t)|\mathscr{F}_t^O] = \int_{\mathbb{R}^m}\theta\rho(t,\theta)d\theta = C_t^{-1}\int_{\mathbb{R}^m}\theta p(t,\theta)d\theta.$$

Thus under the probability $\mathbb{P}$, the filtered estimate $\hat{\theta}(t)$ is a $(\mathscr{F}_t^O)_{t\geq 0}$-adapted process with density $\rho(t,\theta)$, so that for every test function φ, $\mathbb{E}[\varphi(\theta(t))|\mathscr{F}_t^O] = \varphi(\hat{\theta})$.

3.2. Yau–Yau Nonlinear Filter

Within nonlinear filtering formulation, the DMZ equation (3.5) for the unnormalized posterior density $p(t,\theta)$ is a stochastic partial differential equation and thus an infinite-dimensional object. However, the DMZ equation cannot be solved analytically in general. Many efforts have been made to develop efficient numerical methods. In this subsection, we introduce the Yau–Yau algorithm (Yau and Yau, 1998; 2000) precisely for this financial setting. By passing from the stochastic DMZ SPDE to its pathwise-robust formulation (a deterministic parabolic PDE with coefficients depending on the observed path), Yau–Yau approach gives a numerically stable, real-time and memory-less update scheme, and provides an efficient bridge from the Black–Scholes log-return data stream to pathwise robust filtered estimates $\hat{\theta}(t)$, which then feed directly into the filtered market construction and the subsequent pricing/hedging results.

Specifically, for each given sample of the observation process denoted by $R(t)$, we define $K(\theta,t) := b^T(\theta,t)A_t^{-1}R(t)$, and introduce an invertible exponential transformation

$$p(\theta,t) = \exp(b^T(\theta,t)A_t^{-1}R(t))u(\theta,t).$$

Then the DMZ equation is transformed into a deterministic partial differential equation with stochastic coefficients

$$\begin{cases} \partial_t u(\theta,t) + \partial_t(b^T A_t^{-1})^T R(t)u(\theta,t) \\ \qquad\qquad = \exp(-b^T A_t^{-1}R(t))(\mathscr{L}^* - \frac{1}{2}b^T A_t^{-1}b)\cdot\exp(-b^T A_t^{-1}R(t))u(\theta,t), \\ u(0,\theta) = p(0,x). \end{cases} \tag{3.7}$$

Equation (3.7) is also known as the pathwise robust DMZ equation. Equivalently, the pathwise robust DMZ equation (3.7) can be rewritten as

$$\begin{cases} \partial_t u(\theta,t) = \frac{1}{2}(\mathscr{L}_2^* u)(\theta) + F(\theta,t)\cdot\nabla u(\theta,t) + J(\theta,t)u(\theta,t), \\ u(0,\theta) = p(0,x), \end{cases} \tag{3.8}$$

where

$$F(\theta,t) = \left[\sum_{j=1}^{n}\partial_{\theta_j}\Gamma_{ij}(\theta,t) + \Gamma_{ij}(\theta,t)\frac{\partial K}{\partial\theta_j} - b_i(\theta,t)\right]_{i=1}^{n}, \tag{3.9}$$

$$\begin{aligned} J(\theta,t) = & -\frac{\partial}{\partial t}(b^T A_t^{-1})^T y_t + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial\theta_i\partial\theta_j}\Gamma_{ij} + \sum_{i,j=1}^{n}\frac{\partial}{\partial\theta_i}\Gamma_{ij}\frac{\partial K}{\partial\theta_j} \\ & + \frac{1}{2}\sum_{i,j=1}^{n}\Gamma_{ij}\left[\frac{\partial^2 K}{\partial\theta_i\partial\theta_j} + \frac{\partial K}{\partial\theta_i}\frac{\partial K}{\partial\theta_j}\right] \\ & - \sum_{i=1}^{n}\frac{\partial f_i}{\partial\theta_i} - \sum_{i=1}^{n}f_i\frac{\partial K}{\partial\theta_i} - \frac{1}{2}(b^T A_t^{-1}b). \end{aligned} \tag{3.10}$$

We also denote

$$\mathscr{D}f = \left[\sum_{j=1}^{n} \Gamma_{ij}\partial_{\theta_j} f\right]_{i=1}^{n}. \tag{3.11}$$

The boundedness of the drift term f (contained in the operator $\mathscr{L}$) and observation term b can be replaced by some mild growth conditions in this case.

For the existence and uniqueness of the non-negative weak solution to the pathwise-robust DMZ equation (3.7), we assume that the following conditions for the equation (3.7) throughout this paper:

(B1) boundedness of $\|\frac{d}{dt}\Gamma\|_\infty < \infty$.

(B2) The initial density function $u(0,x) \in H^1(\mathbb{R}^n)$ is assumed to decay fast enough, satisfying

$$\int_{\mathbb{R}^n} e^{\sqrt{1+|x|^2}} u(0,x)\, dx < \infty.$$

(B3) The operator $\mathscr{L}^*$ defined in (3.6) is assumed to be a strong elliptic operator, bounded from above on $Q_T = \mathbb{R}^n \times [0,T]$.

We also assume that there exists a positive function $g(x)$ such that for each $t \in [0,T]$ and $\tilde{g} := g + \log|D_\omega J|$,

(B4) $\|\mathscr{D}g + \frac{1}{2}\nabla\Gamma - F\|^2 + 2\lambda_1 J \le C$;

(B5) $\mathscr{L}_2^* g + 2\mathscr{D}g \cdot \nabla g + 2[\nabla\Gamma - F] \cdot \nabla g + \frac{1}{2}\nabla^2\Gamma - \mathrm{div}F + J \le C$;

(B6) $\mathscr{L}_2^* \tilde{g} + 2\mathscr{D}\tilde{g} \cdot \nabla\tilde{g} + 2[\nabla\Gamma - F] \cdot \nabla\tilde{g} + \frac{1}{2}\nabla^2\Gamma - \mathrm{div}F + J \le C$;

(B7) $\int_{\mathbb{R}^n} e^{2\tilde{g}} p^2(\theta) d\theta < \infty$, and $\int_{\mathbb{R}^n} e^{2g}\mathscr{D}p(\theta) \cdot \nabla p(\theta) d\theta < \infty$,

where $\mathscr{D}$ and $J(x,t)$ is defined in (3.11) and (3.9), respectively.

For uniqueness, we assume further that there exist some $c_1, c_2 > 0$, so that

$$\sup_{0\le t\le T} \int_{\mathbb{R}^n} e^{c_1|\theta|} u^2(\theta,t) d\theta < \infty, \quad \int_{Q_T} |\nabla u(\theta,t)|^2 d\theta dt < \infty$$

and

$$2J(\theta,t) - \frac{1}{4\lambda_1}[c_1\mathscr{D}|\theta| - (F(\theta,t) + \tilde{F}(\theta,t))]^2 \le c_2$$

for all $(x,t) \in Q_T$, where λ_1 is the smallest eigenvalue of the matrix Γ, and

$$\tilde{F}(x,t) = \left[\frac{1}{2} \sum_{j=1}^{n} \Gamma_{ij}(\theta,t) + \Gamma_{ij}(\theta,t) \frac{\partial K}{\partial \theta_j} - b_i(\theta,t) \right]_{i=1}^{n}.$$

Theorem 3.1. *Under the assumptions $(A1) - -(A3)$ and $(B1) - -(B7)$, the pathwise-robust DMZ equation (3.7) has a unique non-negative weak solution $u \in H^{1,1}(Q_T)$.*

Remark 3.1. *We remark that the regularity and ellipticity assumptions imposed in* (A1)–(A4) *and* (B1)–(B7) *are compatible with many classical specifications used in mathematical finance. For instance, in the standard Black–Scholes model with constant interest rate $r > 0$ and constant (nondegenerate) volatility matrix $\sigma \in \mathbb{R}^{m \times m}$ (invertible), one has $A_t = \sigma\sigma^\top$ uniformly elliptic, hence* (A4) *holds automatically. Moreover, the toy specifications adopted in Section 7 fall into this class, and the observation covariance A_t is constant. In such constant-coefficient settings, the pathwise-robust transformation of the DMZ equation simplifies considerably and the structural conditions* (B1)–(B7) *reduce to standard coercivity/growth requirements ensuring the above well-posedness result (see, e.g., Luo and Yau (2013a); Yau and Yau (1998; 2000) and references therein).*

Now we introduce the Yau-Yau algorithm for a general nonlinear filtering problem, see Yau et al. (2024) [Chapter 7].

Fix a given terminal time T, and consider the uniform partition $0 = \tau_0 < \tau_1 < \cdots < \tau_k = T$of the time interval $[0,T]$, with $\tau_i - \tau_{i-1} = \frac{T}{k}$. At each time interval $[\tau_{i-1}, \tau_i]$, let us define an auxiliary function, $\tilde{u}_i(\theta,t)$ to be the solution of

(3.12)
$$\begin{cases} \partial_t \tilde{u}_i(\theta,t) + \partial_t (b^T A_t^{-1})^T R(\tau_{i-1}) \tilde{u}_i(\theta,t) \\ = \exp(-b^T A_t^{-1} R(\tau_{i-1}))(\mathscr{L}^* - \frac{1}{2} b^T A_t^{-1} b) \cdot \exp(-b^T A_t^{-1} R(\tau_{i-1})) \tilde{u}_i(\theta,t), \ t \in (\tau_{i-1}, \tau_i], \\ \tilde{u}_i(\theta, \tau_{i-1}) = \tilde{u}_{i-1}(\theta, \tau_{i-1}). \end{cases}$$

which is an approximation to the pathwise robust DMZ equation (3.7) with the value of the log return process $R(t)$ frozen at $t = \tau_{i-1}$.

Next, an important observation is that with another exponential transformation,

$$u_i(\theta,t) = \exp(b^\top(\theta,t) A_t^{-1} R(\tau_{i-1})) \tilde{u}_i(\theta,t), \ t \in [\tau_{i-1}, \tau_i], \tag{3.13}$$

the evolution of the functions $\{u_i(\theta,t) : i \geq 1\}$ satisfies the following observation independent equation:

$$\partial_t u_i(\theta,t) = \left(\mathscr{L}^* - \frac{1}{2} b^\top A_t^{-1} b\right) u_i(\theta,t), \tag{3.14}$$

for $t \in [\tau_{i-1}, \tau_i]$, with initial value given by

$$\begin{aligned} u_i(\theta,\tau_{i-1}) &= \exp(b^\top(\theta,t)A_t^{-1}R(\tau_{i-1}))\tilde{u}_{i-1}(\theta,\tau_{i-1}) \\ &= \exp(b^\top A_t^{-1}(R(\tau_{i-1}) - R(\tau_{i-2})))u_{i-1}(\theta,\tau_{i-1}). \end{aligned} \tag{3.15}$$

Because the Kolmogorov-type partial differential equation (3.14) is independent of the observations, we can study and solve it off-line in advance, and when the value of the log return process is obtained, the remaining on-line algorithm is to update the initial value through (3.15), which is less computationally-expensive in comparison with solving the original equation (3.7).

The two-stage algorithm framework , which is given by (3.14) and (3.15), is now referred to as the **Yau-Yau nonlinear filtering algorithm**. The on-line and off-line algorithm in Yau-Yau algorithm are summarized in Algorithm 1 and 2, respectively. In real financial application, for discrete market data sampled at $\{t_k\}$, the natural on-line input is the log-return increment

$$\Delta R_k := R(t_k) - R(t_{k-1}) = \log\left(\frac{S_{t_k}}{S_{t_{k-1}}}\right),$$

which matches exactly the partition-based update through $R(\tau_{i-1}) - R(\tau_{i-2})$ in (3.15).

Algorithm 2 Off-line algorithm

```
1: Initialization: Given $u_1(x,\tau_0)$ in On-Line Algorithm.
2: for $i = 1$ to $k$ do
3:     Solving (3.7) by Galerkin spectral method/finite difference method and get $u_i(\theta,t)$
4:     Normalize
           $u_i(\theta,\tau_i) = \frac{u_i(\theta,\tau_i)}{\int_{\mathbb{R}^n} u_i(\theta,\tau_i)dx}.$
5:     Obtain the unnormalized conditional probability density
           $p(\theta,\tau_i) = \exp\left[-b^\top(\theta,\tau_i)R(\tau_{i-1})\right] u_i(\theta,\tau_i)$
       at each $\tau_i$.
6: end for
```

Algorithm 1 On-line algorithm

```
Initialization: Fix $T$, $\Delta t$, $y_0 = 0$ and $\sigma_0$. Let $k = \frac{T}{\Delta t}$ and $\{0 = \tau_0 < \tau_1 < \cdots < \tau_k = T\}$. Let $u_1(x,0)$ be the normalization of $\sigma_0$, i.e.
    $$u_1(x,0) = \frac{p_0(x)}{\int_{\mathbb{R}^d} p_0(x)dx}.$$
By the Off-Line Algorithm, obtain $u_1(x,\tau_1)$;
At time $\tau_1$, the new observation $R(\tau_1)$ comes and let
    $$u_2(x,\tau_1) = \exp\left[b^\top(x,\tau_1)A_t^{-1}R(\tau_i)\right] u_1(x,\tau_1).$$
for $i = 2$ to $k$ do
    Obtain $u_{i-1}(x,\tau_{i-1})$ from the Off-Line Algorithm.
    Renew the initial value of the partial differential equation satisfied by $u_i(x,t)$,
        $$u_i(x,\tau_{i-1}) = \exp\left[b^\top(x,\tau_{i-1})A_t^{-1}(R(\tau_{i-1}) - R(\tau_{i-2}))\right] u_{i-1}(x,\tau_{i-1}).$$
end for
```

4. The Filtered Market and Risk-Neutral Measure

In this section, we construct a filtered market by using the Yau-Yau filtering. Furthermore, we show the existence of equivalent martingale measure (a.k.a. risk-neutral measure) under partial observation for the filtered market.

Recall that I_t is the innovation process which is given by

$$dI_t = [\sigma_t^{-1}dR(t) - \sigma_t^{-1}\mathbb{E}[\mu(\theta(t),t)|\mathscr{F}_t^O]dt = \sigma_t^{-1}(\mu(\theta(t),t) - \mu(\hat{\theta}(t),t))dt + dB_t.$$

In filter theory, I_t is a $(\mathscr{F}_t^O)_{t\geq 0}$-Brownian motion under $\mathbb{P}$ (see e.g. Yau et al. (2024)). Thus after the filtered estimate, the natural driving noise under the observation filtration is the innovation process I_t, which replaces B_t in the partially observed dynamics. Moreover, the Black–Scholes SDE (2.1) under the observation filtration $\mathscr{F}_t^O$ and $\mathbb{P}$ can be rewritten as

$$dS(t) = diag(S(t))[r(\hat{\theta}(t),t)\,dt + \sigma_t dI_t], \tag{4.1}$$

where $\hat{\theta}(t)$ is the filtered estimate of $\theta(t)$. Using the existence and regularity of the pathwise robust DMZ equation (Theorem 3.1), we obtain that the filtered estimate

$$\hat{\theta}(t) := \left(\int_{\mathbb{R}^m} p(t,\theta)d\theta\right)^{-1} \int_{\mathbb{R}^m} \theta p(t,\theta)d\theta,$$

is progressively measurable, Lipschitz continuous in t, and pathwise robust. Combining with the regularity assumption of $r(\theta,t)$, it follows that the drift $r(\hat{\theta}(t),t)$ is progressively measurable, and Lipschitz continuous in t. Thus the SDE (4.1) has a unique strong solution.

Definition 4.1. *Under the observable filtration $\mathbb{F}^O$, the filtered market is composed of one riskless asset $M(t)$, and m risky assets $S = (S^1, S^2, \cdots, S^m)^T$, which are given by the SDE* (4.1).

Then under $(\mathbb{F}^O, \mathbb{P})$, the observation process is given by

$$dR(t) = \mu(\hat{\theta}(t),t)dt + \sigma_t dI_t, \tag{4.2}$$

where

$$\mu(\hat{\theta}(t),t) = \left[r^i(\hat{\theta}(t),t) - \frac{1}{2}\sum_{j=1}^{m}(\sigma_t^{ij})^2 \right]_{i=1}^{m}.$$

The filtered market can be understood as the effective market perceived by an agent who only has access to the observable filtration $\mathbb{F}^O = (\mathscr{F}_t^O)_{t\in[0,T]}$ generated by prices. Under partial observation, the drift is not directly observable and must be replaced by its belief-based counterpart. The nonlinear filter aggregates all price information up to time t into the posterior distribution of the latent factor $\theta(t)$, and in particular into the filtered estimate $\hat{\theta}(t)$. Consequently, the projected dynamics (4.1) describe the agent's subjective law of motion for tradable assets given her information, where the innovation process I_t represents the new information flow revealed by price increments after subtracting the predictable component implied by current beliefs. In this sense, the filtered market converts an incomplete-information environment into an belief-driven market model.

Now we construct the risk-neutral measure $\mathbb{Q}^N$ for the filtered market under the the partially observed filtration $\mathbb{F}^O$. The first fundament theorem in financial math implies that the financial market is arbitrage free if and only if there is a equivalent measure.

Lemma 4.1. *The filtered market is arbitrage free with respect to the observable filtration $\mathbb{F}^O$.*

Proof. It is enough to show that there exists a risk-neutral measure $\mathbb{Q}^N$ in the filtered market. Let $\mathbf{r}_t$ be the m- dimensional vector with all its components equals to r_t, and $\hat{c}_t = \sigma_t^{-1}(r(\hat{\theta}(t),t) - r_t)$. Using the assumptions $(A1) - (A3)$, and the time-regularity of

$\hat{\theta}(t)$, the drift $\hat{c}_t$ satisfies the Novikov's condition

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \|\hat{c}_t\|^2 dt\right)\right] < \infty. \tag{4.3}$$

Then the $(\mathscr{F}_t^O)_{t\in[0,T]}$-adapted process

$$\exp\left(-\int_0^t \hat{c}_s^\top dI_s - \frac{1}{2}\int_0^t \|\hat{c}_s\|^2 ds\right) = \Lambda_t^N, \tag{4.4}$$

is an $(\mathbb{F}^O,\mathbb{P})$-martingale. The risk-neutral measure $\mathbb{Q}^N$ relative to the observation filtration is defined by the density process

$$\frac{d\mathbb{Q}^N}{d\mathbb{P}}\Big|_{\mathscr{F}_t^O} = \Lambda_t^N. \tag{4.5}$$

By Girsanov's theorem, the process $B_t^N := I_t + \int_0^t \hat{c}_s ds$ is a $(\Omega,\mathscr{F},\mathbb{F}^O,\mathbb{Q}^N)$-standard Brownian motion, and the risk asset price $S(t)$ under the risk-neutral measure $\mathbb{Q}^N$ and filtration $\mathbb{F}^O$ which satisfies the SDE

$$dS(t) = diag(S(t))(\mathbf{r}_t dt + \sigma_t dB_t^N), \tag{4.6}$$

or for each S_i

$$dS_i(t) = S_i(t)(r_t dt + \sum_{j=1}^m \sigma_t^{ij} dB_t^{N,j}). \tag{4.7}$$

Here and below, we denote $\mathbf{r}_t = (r_t,\cdots,t_t)$ as a m-dimensional vector. It is easy to verify that for each $1 \le i \le m$, the discounted price process $\hat{S}_i(t) := e^{-\int_0^t r_s ds} S_i(t)$ satisfies

$$d\hat{S}_i(t) = \hat{S}_i(t)\sum_{j=1}^m \sigma_t^{ij} dB_t^{N,j}.$$

Thus is a $(\mathscr{F}_t^O)$-martingale under the risk-neutral measure $\mathbb{Q}^N$. □

Remark 4.1. *We remark that if the financial market is arbitrage free, then the market is a efficient market.*

The market is complete under the observation filtration $(\mathscr{F}_t^O)$, if every $(\mathscr{F}_t^O)$-adapted contingent claim $Z \in L^2(\Omega,\mathscr{F}_T^O,\mathbb{Q})$ is attainable. This implies that any contingent claim

Z can be perfectly replicated by a self-financing strategy H under $\mathbb{Q}$. Now we show the completeness of the filtered market.

Theorem 4.2. *The filtered market with respect to the observable filtration $\mathscr{F}_t^O$ is complete.*

Proof. It is enough to show that for every $Z \in L^2(\Omega, \mathscr{F}_T^O, \mathbb{Q}^N)$, the European option (Z, T) is attainable. That is, there exists a strategy H such that $V^H(T) = Z$. Consider the following $(\mathscr{F}_t^O, \mathbb{Q}^N)$-martingale

$$\hat{V}(t) := \mathbb{E}^{\mathbb{Q}^N}\left[e^{-\int_0^T r_s ds} Z \middle| \mathscr{F}_t^O\right].$$

Since $\hat{V}(t)$ is a $\mathbb{Q}^N$ square-integrable martingale so that $\hat{V}(T) \in L^2(\Omega, \mathscr{F}_T^O, \mathbb{Q}^N)$, by the martingale representation theorem, there exists a $(\mathscr{F}_t^O)$-predictable process $(h(t))_{t\in[0,T]} \in L^2([0,T] \times \Omega, m_{[0,T]} \times \mathbb{Q}^N)$, so that

$$\hat{V}(t) = \hat{V}(0) + \sum_{i=1}^m \int_0^t h^i(s) dB_s^{N,i}, \quad \mathbb{Q}^N \text{ a.s. for all } t \in [0,T], \tag{4.8}$$

where the uniqueness of $h(\cdot)$ is meant up to a $[0,T] \times \mathbb{Q}^N$-null set. Moreover, by the martingale property of $\hat{V}(t)$,

$$\hat{V}(0) := \mathbb{E}^{\mathbb{Q}^N}[e^{-\int_0^T r_s ds} Z | \mathscr{F}_0^O] = e^{-\int_0^T r_s ds} \mathbb{E}^{\mathbb{Q}^N}[Z] \geq 0. \tag{4.9}$$

Let $\hat{S}(t) = S(t)/M(t)$ be the discounted price process. Now we define the self-financing strategy

$$(H^1(t), \cdots, H^m(t))^\top = \left(\sigma_t^\top \mathrm{diag}(\hat{S}_t)\right)^{-1} h(t), \quad t \in [0,T].$$

Since the discounted price process satisfies $d\hat{S}(t) = \hat{S}(t)\sigma_t dB_t^N$ under $\mathbb{Q}^N$, the stochastic integral (4.8) can be rewritten as

$$\hat{V}(t) = \hat{V}(0) + \sum_{i=1}^m \int_0^t H^i(s) d\hat{S}^i(s). \tag{4.10}$$

Let

$$H^0(t) = V(t) - \sum_{i=1}^m H^i(t)\hat{S}(t) = V(0) + \sum_{i=1}^m \left(\int_0^t H^i(s) d\hat{S}^i(s) - H^i(t)\hat{S}^i(t)\right), \quad t \in [0,T].$$

Recall that the strategy H is self-financing if and only if

$$\hat{V}^H(t) = \hat{V}^H(0) + \sum_{i=1}^{m} \int_0^t H^i(s) d\hat{S}^i(s).$$

Then H is a self-financing strategy, and the European option (Z,T) is attainable. Since $Z \in L^2(\Omega, \mathscr{F}_T^O, \mathbb{Q}^N)$ is arbitrary, the filtered market is complete. $\square$

Remark 4.2. *The measure $\mathbb{Q}^N$ is the pricing measure relative to the investor's information $\mathbb{F}^O$. Under $\mathbb{Q}^N$ the discounted asset prices become $(\mathbb{F}^O, \mathbb{Q}^N)$-martingales. Economically, the exponential martingale $(\Lambda_t^N)_{t\in[0,T]}$ converts the agent's conditional expected excess return (which depends on $\hat{\theta}(t)$) into a risk adjustment. Thus $\mathbb{Q}^N$ does not represent a "true" physical probability; rather, it is the unique no-arbitrage pricing rule consistent with the agent's information and with dynamic trading in S and the money market account.*

Remark 4.3. *The above result show that in the filtered market, for every European option (Z,T), there exists a $\mathscr{F}_t^O$ self-financing strategy $H = (H^0, H^1, \cdots, H^m)^\top$, so that the discounted portfolio of H*

$$\hat{V}^H(T) = \hat{V}(0) + \sum_{i=1}^{m} \int_0^T H^i(s) d\hat{S}^i(s). \tag{4.11}$$

This equality show that a replicating portfolio is uniquely determined by the initial wealth $V(0)$ and the investment in the risky asset.

Now we consider a European contingent claim $V(t)$ with a replicating attainable strategy H. Since the discounted portfolio

$$\hat{V}(t) = \mathbb{E}^{\mathbb{Q}^N}\left[e^{-\int_0^T r_s ds} Z \middle| \mathscr{F}_t^O\right] = \hat{V}(0) + \sum_{i=1}^{m} \int_0^t H^i(s) d\hat{S}^i(s), \tag{4.12}$$

the conditional price of the European contingent claim at time t is

$$V(t) = e^{\int_0^t r_s ds} \hat{V}(t) = \mathbb{E}^{\mathbb{Q}^N}\left[e^{-\int_t^T r_s ds} Z \middle| \mathscr{F}_t^O\right]. \tag{4.13}$$

Note that the conditional price of the European option (Z,T) at time $t < T$ is only depending on the risky asset price $S(t)$, and filter estimates $\hat{\theta}(t)$. Since the processes $S(t)$ and $\hat{\theta}(t)$ are adapted with respect to the filtration $(\mathscr{F}_t^O)_{t\geq 0}$, the conditional price of the European contingent claim is also equal to

$$V(t) = \mathbb{E}^{\mathbb{Q}^N}\left[e^{-\int_t^T r_s ds} Z \middle| \hat{\theta}(t), S(t)\right], \tag{4.14}$$

where $\mathbb{E}^{\mathbb{Q}^N}[\cdot|\hat{\theta}(t), S(t)]$ is the conditional expectation under the risk-neutral probability measure $\mathbb{Q}^N$. We denote the price of the European option (Z,T) at time t with $\hat{\theta}(t) = \theta, S(t) = x$ by

$$V(t,s,\theta) := \mathbb{E}^{\mathbb{Q}^N}\left[e^{-\int_t^T r_s ds} Z \middle| \hat{\theta}(t) = \theta, S(t) = x\right]. \tag{4.15}$$

5. Black–Scholes PDEs for Derivative Pricing

In this section, we derive the Black–Scholes PDE for the price of the European option $V(t,s,\theta)$ under partial observation via the nonlinear filtering theory.

Under the risk-neutral measure $\mathbb{Q}^N$, the observation process $R(t)$ is governed by the SDE

$$dR^i(t) = (r_t - \frac{1}{2}\sum_{j=1}^m (\sigma_t^{ij})^2)dt + \sum_{j=1}^m \sigma_t^{ij} dB_t^{N,j}, \quad 1 \le i \le m.$$

Recall that the observation process $R(t)$ under the probability $\mathbb{Q}$ satisfies $dR(t) = \sigma_t d\bar{R}_t$. Then

$$\begin{aligned} dB_t^N :=& dI_t + \hat{c}_t dt \\ =& \sigma_t^{-1}[\mu(\theta(t),t) - \mu(\hat{\theta}(t),t)]dt + dB_t + \sigma_t^{-1}[\mu(\hat{\theta}(t),t) - \mathbf{r}_t + \frac{1}{2}diag(A_t))]dt \\ =& d\bar{R}_t - \sigma_t^{-1}(\mathbf{r}_t - \frac{1}{2}diag(A_t))dt. \end{aligned} \tag{5.1}$$

Now we consider the exponential martingale

$$\begin{aligned} \frac{d\mathbb{Q}^N}{d\mathbb{Q}}\Big|_{\mathscr{F}_t^O} :=& \lambda_t \\ =& \exp\left(\int_0^t (\mathbf{r}_s - \frac{1}{2}diag(A_s))^\top \sigma_s^{-1} d\bar{R}_s - \frac{1}{2}\int_0^t \|(\mathbf{r}_s - \frac{1}{2}diag(A_s))^\top \sigma_s^{-1}\|^2 ds\right). \end{aligned}$$

Then under $\mathbb{Q}$, the exponential martingale λ_t satisfies

$$d\lambda_t = \lambda_t(\mathbf{r}_t - \frac{1}{2}diag(A_t))^\top \sigma_t^{-1} d\bar{R}_t, \quad \lambda_0 = 1. \tag{5.2}$$

Then using the Kallianpur-Striebel formula, for every smooth test function φ, the estimate of θ under the Risk–Neutral Measure $\mathbb{Q}^N$ is given by

$$\mathbb{E}^{\mathbb{Q}^N}\left[\varphi(\hat{\theta}(t))\right] = \mathbb{E}^{\mathbb{Q}^N}\left[\varphi(\theta(t))\middle|\mathscr{F}_t^O\right] = \frac{\mathbb{E}^{\mathbb{Q}}\left[\varphi(\theta(t))\lambda_t\middle|\mathscr{F}_t^O\right]}{\mathbb{E}^{\mathbb{Q}}\left[\lambda_t\middle|\mathscr{F}_t^O\right]}.$$

By similar argument in Section 3, we can use the Yau-Yau nonlinear filter to obtain the Bayes estimate $\hat{\theta}$ with pathwise robust under $\mathbb{Q}^N$, which satisfies

(5.3)

$$d\mathbb{E}^{\mathbb{Q}}\left[\varphi(\theta(t))\lambda_t\middle|\mathscr{F}_t^O\right] := d\rho_t^N(\varphi) = \rho_t^N(\mathscr{L}\varphi(\theta))dt + \rho_t^N\big(\varphi(\theta)[r(\theta,t)-r_t]\big)^\top A_t^{-1}dR(t).$$

Now we give the price formula of the European option (Z,T) with the payoff $Z = h(S(T),\theta(T))$ in the filter market $(S(t),\hat{\theta}(t))$. we have the following results.

Theorem 5.1. *Let $h(\cdot):\mathbb{R}^n\times\mathbb{R}^m\mapsto\mathbb{R}^+$ be a continuous payoff function so that $h(x,\theta)\leq C(1+|x|^\alpha)$ for some $C,\alpha>0$. Let*

$$V(t,x,\theta) = \mathbb{E}^{\mathbb{Q}^N}[e^{-\int_t^T r_s ds}h(S(T),\hat{\theta}(T))\big|S(t)=x,\hat{\theta}(t)=\theta] \tag{5.4}$$

be the price function of the option $Z = h(S(T),\theta(T))$. The the price V of the European option (Z,T) satisfies the following generalized Black-Scholes PDE

$$\begin{aligned}&\partial_t V(t,x,\theta) + \mathscr{B}_t^N V(t,x,\theta) + \mathscr{L}_t V(t,x,\theta) - r_t V(t,x,\theta) = 0,\\ &\quad t\in[0,T), x\in(0,\infty)^m, \theta\in\mathbb{R}^n,\end{aligned} \tag{5.5}$$

with terminal condition

$$V(T,x,\theta) = h(x,\theta)$$

where

$$\mathscr{B}_t^N V(x) = r_t x\cdot\nabla_x V(x) + \frac{1}{2}\sum_{i,j=1}^m x_i x_j A_t^{ij}\frac{\partial^2 V}{\partial x_i\partial x_j}, \tag{5.6}$$

and

$$\mathscr{L}_t V(\theta) = b(\theta,t)\cdot\nabla_\theta V(\theta) + \frac{1}{2}\sum_{i,j=1}^n \Gamma^{ij}(\theta,t)\frac{\partial^2}{\partial\theta_i\partial\theta_j}V(\theta). \tag{5.7}$$

Proof. We first assume that V is of class $C^{1,2,2}([0,T)\times\mathbb{R}^m\times\mathbb{R}^n)$. Then the discount price $\hat{V}\in C^{1,2,2}([0,T)\times\mathbb{R}^n\times\mathbb{R}^m)$.

Using the Kallianpur-Striebel formula, we have

$$\hat{V}(t,S(t),\hat{\theta}(t)) = \mathbb{E}^{\mathbb{Q}^N}\big[\hat{V}(t,S(t),\theta(t))\big|\mathscr{F}_t^O\big] = \frac{\mathbb{E}^{\mathbb{Q}}\big[\hat{V}(t,S(t),\theta(t))\lambda_t\big|\mathscr{F}_t^O\big]}{\mathbb{E}^{\mathbb{Q}}\big[\lambda_t\big|\mathscr{F}_t^O\big]}. \tag{5.8}$$

Then after using Itô's formula applied to $(\hat{V}(t,S(t),\theta(t))\lambda_t)$ under the probability measure $\mathbb{Q}$, we obtain

(5.9)

$$\begin{aligned}
&d(\hat{V}(t,S(t),\theta(t))\lambda_t)\\
=&\lambda_t d\hat{V}(t,S(t),\theta(t))dt + \hat{V}(t,S(t),\theta(t))d\lambda_t + d\langle\hat{V}(t,S(t),\theta(t)),\lambda_t\rangle\\
=&-\lambda_t r_t e^{-\int_0^t r_s ds}V(t,S(t),\theta(t)) + \lambda_t e^{-\int_0^t r_s ds}dV(t,S(t),\theta(t)) + \hat{V}(t,S(t),\theta(t))d\lambda_t\\
&+ d\langle\hat{V}(t,S(t),\theta(t)),\lambda_t\rangle\\
=&\lambda_t e^{-\int_0^t r_s ds}(-r_t V(t,S(t),\theta(t)) + \partial_t V(t,S(t),\theta(t)) + \mathscr{L}_t V(t,S(t),\theta(t)))dt\\
&+ \lambda_t e^{-\int_0^t r_s ds}\left(\frac{1}{2}\left(\sum_{i=1}^m S_i(t)A_t^{ii}\partial_{s_i} + \sum_{i,j=1}^m S_i(t)S_j(t)A_t^{ij}\frac{\partial^2}{\partial s_i\partial s_j}\right)V(t,S(t),\theta(t))\right)dt\\
&+ \lambda_t e^{-\int_0^t r_s ds}(\nabla_s V(t,S(t),\theta(t))diag(S(t))\sigma_t d\bar{R}_t + \nabla_\theta V(t,S(t),\theta(t))G_t(\theta(t),t)dW_t)\\
&+ e^{-\int_0^t r_s ds}V(t,S(t),\theta(t))\lambda_t(\mathbf{r}_t - \frac{1}{2}diag(A_t))^\top\sigma_t^{-1}d\bar{R}_t\\
&+ e^{-\int_0^t r_s ds}\langle\nabla_s V(t,S(t),\theta(t))diag(S(t))\sigma_t d\bar{R}_t, \lambda_t(\mathbf{r}_t - \frac{1}{2}diag(A_t))^\top\sigma_t^{-1}d\bar{R}_t\rangle.
\end{aligned}$$

Taking the conditional expectation $\mathbb{E}^{\mathbb{Q}}[\cdot|\mathscr{F}_t^O]$, we obtain

$$\begin{aligned}
&d\mathbb{E}^{\mathbb{Q}}\big[\hat{V}(t,S(t),\theta(t))\lambda_t\big|\mathscr{F}_t^O\big]\\
=&e^{-\int_0^t r_s ds}\mathbb{E}^{\mathbb{Q}}\big[(-r_t V(t,S(t),\theta(t))\lambda_t + \partial_t V(t,S(t),\theta(t))\lambda_t + \mathscr{L}_t V(t,S(t),\theta(t)))\,\lambda_t\big|\mathscr{F}_t^O\big]dt\\
&+ e^{-\int_0^t r_s ds}\mathbb{E}^{\mathbb{Q}}\Bigg[\Bigg(\frac{1}{2}\Bigg(\sum_{i=1}^m S_i(t)A_t^{ii}\partial_{s_i}\\
&\qquad + \sum_{i,j=1}^m S_i(t)S_j(t)A_t^{ij}\frac{\partial^2}{\partial s_i\partial s_j}\Bigg)V(t,S(t),\theta(t))\Bigg)\lambda_t\Big|\mathscr{F}_t^O\Bigg]dt\\
&+ e^{-\int_0^t r_s ds}\mathbb{E}^{\mathbb{Q}}\big[\nabla_S V(t,S(t),\theta(t)))diag(S(t))(\mathbf{r}_t - \frac{1}{2}diag(A_t))^\top\lambda_t\big|\mathscr{F}_t^O\big]dt\\
&+ e^{-\int_0^t r_s ds}\mathbb{E}^{\mathbb{Q}}\big[(\nabla_S V(t,S(t),\theta(t))diag(S(t))\\
&\qquad + V(t,S(t),\theta(t))(\mathbf{r}_t - \frac{1}{2}diag(A_t))^\top A_t^{-1})\lambda_t\big|\mathscr{F}_t^O\big]dR(t).
\end{aligned}$$

We also have

$$d\mathbb{E}^{\mathbb{Q}}\big[\lambda_t\big|\mathscr{F}_t^O\big] = \mathbb{E}^{\mathbb{Q}}\big[(\mathbf{r}_t - \frac{1}{2}diag(A_t))^\top A_t^{-1}\lambda_t\big|\mathscr{F}_t^O\big]dR(t).$$

Since $d\langle R(t)\rangle = A_t dt$ under the probability measure $\mathbb{Q}$, the quadratic variation of $\mathbb{E}^{\mathbb{Q}}\big[\lambda_t\big|\mathscr{F}_t^O\big]$ is given by

$$d\langle \mathbb{E}^{\mathbb{Q}}[\lambda_t|\mathscr{F}_t^O]\rangle = \left(\mathbb{E}^{\mathbb{Q}}\big[(\mathbf{r}_t - \frac{1}{2}diag(A_t))^\top A_t^{-1}\lambda_t\big|\mathscr{F}_t^O\big]\right)^2 A_t dt,$$

and

$$\begin{aligned}
&d\langle \mathbb{E}^{\mathbb{Q}}\big[\hat{V}(t,S(t),\theta(t))\lambda_t\big|\mathscr{F}_t^O\big], \mathbb{E}^{\mathbb{Q}}[\lambda_t|\mathscr{F}_t^O]\rangle \\
=&e^{-\int_0^t r_s ds}\mathbb{E}^{\mathbb{Q}}\big[(\nabla_s V(t,S(t),\theta(t))S(t) + V(t,S(t),\theta(t))(\mathbf{r}_t - \frac{1}{2}diag(A_t))^\top A_t^{-1})\lambda_t\big|\mathscr{F}_t^O\big] \\
&\times \mathbb{E}^{\mathbb{Q}}\big[(\mathbf{r}_t - \frac{1}{2}diag(A_t))^\top \lambda_t\big|\mathscr{F}_t^O\big]dt.
\end{aligned}$$

Since the term $r_t - \frac{1}{2}diag(A_t)$ is deterministic, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\big[\hat{V}(t,S(t),\theta(t))(\mathbf{r}_t - \frac{1}{2}diag(A_t))^\top \lambda_t\big|\mathscr{F}_t^O\big] \\
&\quad = (\mathbf{r}_t - \frac{1}{2}diag(A_t))^\top \mathbb{E}^{\mathbb{Q}}\big[\hat{V}(t,S(t),\theta(t))\lambda_t\big|\mathscr{F}_t^O\big].
\end{aligned}$$

Then we apply Itô's formula to compute (5.8), and obtain

$$\begin{aligned}
&d\hat{V}(t,S(t),\hat{\theta}(t)) \\
=&d\left(\frac{\mathbb{E}^{\mathbb{Q}}\big[\hat{V}(t,S(t),\theta(t))\lambda_t\big|\mathscr{F}_t^O\big]}{\mathbb{E}^{\mathbb{Q}}\big[\lambda_t\big|\mathscr{F}_t^O\big]}\right) \\
=&\frac{d\mathbb{E}^{\mathbb{Q}}\big[\hat{V}(t,S(t),\theta(t))\lambda_t\big|\mathscr{F}_t^O\big]}{\mathbb{E}^{\mathbb{Q}}\big[\lambda_t\big|\mathscr{F}_t^O\big]} - \frac{\mathbb{E}^{\mathbb{Q}}\big[\hat{V}(t,S(t),\theta(t))\lambda_t\big|\mathscr{F}_t^O\big]d\mathbb{E}^{\mathbb{Q}}\big[\lambda_t\big|\mathscr{F}_t^O\big]}{(\mathbb{E}^{\mathbb{Q}}\big[\lambda_t\big|\mathscr{F}_t^O\big])^2} \\
&+\frac{\mathbb{E}^{\mathbb{Q}}\big[\hat{V}(t,S(t),\theta(t))\lambda_t\big|\mathscr{F}_t^O\big]d\langle\mathbb{E}^{\mathbb{Q}}\big[\lambda_t\big|\mathscr{F}_t^O\big]\rangle}{(\mathbb{E}^{\mathbb{Q}}\big[\lambda_t\big|\mathscr{F}_t^O\big])^3} \\
&-\frac{d\langle\mathbb{E}^{\mathbb{Q}}\big[\hat{V}(t,S(t),\theta(t))\lambda_t\big|\mathscr{F}_t^O\big],\mathbb{E}^{\mathbb{Q}}\big[\lambda_t\big|\mathscr{F}_t^O\big]\rangle}{(\mathbb{E}^{\mathbb{Q}}\big[\lambda_t\big|\mathscr{F}_t^O\big])^2} \\
=&e^{-\int_0^t r_s ds}\big(\partial_t V(t,S(t),\theta(t)) - \mathbf{r}_t V(t,S(t),\hat{\theta}(t)) + \mathscr{L}V(t,S(t),\hat{\theta}(t))\big)dt \\
&+e^{-\int_0^t r_s ds}\left(\left(\sum_{i=1}^m S_i(t)A_t^{ii}\partial_{s_i} + \frac{1}{2}\sum_{i,j=1}^m S_i(t)S_j(t)A_t^{ij}\frac{\partial^2}{\partial s_i\partial s_j}\right)V(t,S(t),\hat{\theta}(t))\right)dt \\
&+e^{-\int_0^t r_s ds}\big[\nabla_S V(t,S(t),\hat{\theta}(t))S(t)\big]dR(t).
\end{aligned}$$

Combining with (5.1), we have

$$\begin{aligned}
&d\hat{V}(t,S(t),\hat{\theta}(t))\\
=&e^{-\int_0^t r_s ds}[-r_t V(t,S(t),\hat{\theta}(t)) + \partial_t V(t,S(t),\theta(t)) + \mathscr{L}_t V(t,S(t),\hat{\theta}(t))]dt\\
&+ e^{-\int_0^t r_s ds}\left(\left(\sum_{i=1}^m S_i(t) r_t \partial_{s_i} + \frac{1}{2}\sum_{i,j=1}^m S_i(t)S_j(t)A_t^{ij}\frac{\partial^2}{\partial s_i \partial s_j}\right)V(t,S(t),\hat{\theta}(t))\right)dt\\
&+ e^{-\int_0^t r_s ds}\nabla_S V(t,S(t),\hat{\theta}(t))S(t)\sigma_t^{-1}dB_t^N.
\end{aligned} \tag{5.10}$$

Since $\hat{V}(t,S(t),\hat{\theta}(t))$ is a $(\mathscr{F}_t^O)_{t\geq 0}$ martingale under $\mathbb{Q}^N$, the finite variation part should vanish, i.e.

(5.11)

$$\partial_t V(t,x,\theta) + \mathscr{B}_t^N V(t,x,\theta) + \mathscr{L}_t V(t,x,\theta) - r_t V(t,x,\theta) = 0, \quad V(T,x,\theta) = h(x,\theta).$$

In the computation above we obtained the Black-Scholes PDE (5.5) under the assumption that V is of class $C^{1,2,2}([0,T) \times \mathbb{R}^m \times \mathbb{R}^n)$, which is still to be proved.

In order to overcome the degenerate property of the generator $\mathscr{A}_t$, we rewritten the value V by the logarithm of the prices $r_i = \log s_i$, and denote $(e^{r_1} \cdots, e^{r_m})$ by e^r. Then

$$\varphi(t,r,\theta) := V(t,e^r,\theta) = V(t,x,\theta), \tag{5.12}$$

and

$$\frac{\partial \varphi}{\partial r_i} = \frac{1}{r_i}\frac{\partial V}{\partial x_i}, \quad \frac{\partial^2 \varphi}{\partial r_i \partial r_j} = \frac{1}{r_i r_j}\left(\frac{\partial^2 V}{\partial x_i \partial x_j} - \delta_{ij}\frac{\partial V}{\partial x_i}\right). \tag{5.13}$$

Substituting into the (5.6) yields that $V(t,s,\theta)$ is a classical solution of (5.5), if and only if $\varphi(t,r,\theta)$ satisfies the transformed parabolic equation

$$\partial_t \varphi + \mathbf{r}_t \nabla_r \varphi + \mathscr{B}_t^N \varphi + \mathscr{L}_t \varphi - \mathbf{r}_t \varphi = 0, \quad \text{on } [0,T) \times \mathbb{R}^m \times \mathbb{R}^n, \tag{5.14}$$

with terminal condition $u(T,x,\theta) = h(e^x)$.

Since the operator $L\varphi := \mathbf{r}_t \nabla_r \varphi + \frac{1}{2}\sigma_t^2 \Delta_r \varphi + \mathscr{L}_t \varphi - \mathbf{r}_t \varphi$ is uniformly elliptic, and the terminal condition $h(e^x)$ is continuous and has polynomial growth, by standard well-posedness of parabolic equation result(see e.g. Baldi (2017)[Theorem 10.6]), the transformed parabolic equation (5.14) has a unique solution $\varphi \in C^{1,2,2}([0,T) \times \mathbb{R}^m \times \mathbb{R}^n)$ with polynomial growth. Then using the back-transformation, we preserve the regularity claims of φ also holds for V. Thus the Black–Scholes equation (5.5) has a classical solution $V \in C^{1,2,2}([0,T] \times \mathbb{R}_+^m \times \mathbb{R}^n)$. □

When $t = 0$, the exponential martingales $\lambda_0 = \Lambda_0 = 1$. Recall that the initial density for the DMZ equation is a prior given, and the filter estimates are invariant with different probability measure at time $t = 0$, i.e.

$$\mathbb{E}^{\mathbb{P}}[\theta(0)|\mathscr{F}_t^O] = \mathbb{E}^{\mathbb{Q}^N}[\theta(0)|\mathscr{F}_t^O] = \mathbb{E}^{\mathbb{Q}}[\theta(0)|\mathscr{F}_t^O] = \hat{\theta}(0).$$

By Theorem 5.1, the price of the European option at $t = 0$ is $V(0, S(0), \hat{\theta}(0))$, where V is the solution of (5.5).

Now we consider case that the European option with the payoff $Z = h(S(T))$, which only depends with $S(T)$. As same as the classical result in Black-Scholes theory, it is enough to consider the value $S(t)$, and price does not depend on the return $r(\hat{\theta}(t), t)$ appearing in the Black-Scholes SDE (4.1). Thus it is enough to consider the value of $S(t)$, and we have

$$\mathbb{E}^{\mathbb{Q}^N}\left[e^{-\int_t^T r_s ds} h(S(T)) \middle| \hat{\theta}(t) = \theta, S(t) = x\right] = \mathbb{E}^{\mathbb{Q}^N}\left[e^{-\int_t^T r_s ds} h(S(T)) \middle| S(t) = x\right].$$

We denote

$$V(t,x) = \mathbb{E}^{\mathbb{Q}^N}\left[e^{-\int_t^T r_s ds} h(S(T)) \middle| S(t) = x\right]$$

as the price of the European option (Z, T) with the payoff $Z = h(S(T))$. For this situation, we have the following result for the price of the European option.

Proposition 5.1. *Let $h(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^+$ be a continuous payoff function so that $h(x) \leq C(1 + |x|^\alpha)$ for some $C, \alpha > 0$. The the price $V \in C^{1,2}([0,T] \times \mathbb{R}_+^m)$, and $V(t,x)$ satisfies the Black-Scholes PDE*

$$\begin{cases} \partial_t V(t,x) + \mathscr{B}_t^N V(t,x) - r_t V(t,x) = 0, & t \in [0,T), x \in (0,\infty)^m, \\ V(T,x) = h(x). \end{cases} \tag{5.15}$$

Proof. Since the filtered market is complete with respect to $(\mathscr{F}_t^O)_{t \in [0,T]}$, we can use the classical argument (see e.g. Baldi (2017); Benth (2003)) in Black-Scholes theory to show this statement. □

6. Hedging Strategy

In this section, we consider the hedging strategy based on the Black-Scholes PDE in the filtered market (5.5) (or equation (5.15)). Moreover, this strategy is still a locally risk-minimizing strategy in the original incomplete information market.

Theorem 6.1. *Let* $V(t,x,\theta)$ *is the price of the European option* (Z,T) *in the filtered market, which is given by* (5.5)*. Then we find the replicating strategy* H *for the European option* (Z,T)*, defined by*

$$H^i(t) = \partial_{s_i} V(t,S(t),\hat{\theta}), \quad i = 1,\cdots,m, \tag{6.1}$$

and

$$H^0(t) = \hat{V}(t,S(t),\hat{\theta}(t)) - \sum_{i=1}^{m} H^i(t)\hat{S}^i(t). \tag{6.2}$$

Proof. Note that the price V is a classical solution to the Black–Scholes PDE (5.5). After applying the Itô's formula to $V(t,x,\theta)$, the finite variation part should vanish, and we obtain

$$\begin{aligned} d\hat{V}(t,S(t),\hat{\theta}(t)) =& e^{-\int_0^t r_s ds} \nabla_s V(t,S(t)\hat{\theta}) S(t)\sigma_t dB_t^N \\ =& \nabla_s V(t,S(t),\hat{\theta})\hat{S}(t)\sigma_t dB_t^N \\ =& \sum_{i=1}^{m} \partial_{s_i} V(t,S(t),\hat{\theta}) d\hat{S}^i(t). \end{aligned}$$

Recall that in complete market, the strategy H is a self-financing if and only if

$$\hat{V}(t) = V^H(0) + \sum_{i=1}^{m} \int_0^t H^i(s) d\hat{S}^i(s).$$

Thus the replicating strategy for the European option (Z,T) is given by (6.1) and (6.2), which is self-financing and admissible. $\square$

Remark 6.1. *If the European option* (Z,t) *has the payoff* $Z = h(S(T))$*, then by equation* (5.15)*, the replicating strategy of* (Z,t) *in the filtered market is defined by*

$$H^i(t) = \partial_{s_i} V(S(t),t), \quad H^0(t) = \hat{V}(t) - \nabla_S V(S(t),t). \tag{6.3}$$

Thus the partial derivatives of the solution V, the Greeks, are considered of major importance in finance.

Remark 6.2. *The strategy in Theorem 6.1 is a Delta hedge computed in the filtered state variables $(S(t), \hat{\theta}(t))$:*

$$H^i(t) = \partial_{s_i} V\big(t, S(t), \hat{\theta}(t)\big), \qquad i = 1, \ldots, m.$$

In the filtered market, this hedge is the replicating strategy associated with the unique no-arbitrage price, hence it achieves exact replication under the idealized assumptions of continuous trading, frictionless markets, and correct model specification. From an economic viewpoint, the hedge ratio depends on beliefs: as new data arrive, the filter updates $\hat{\theta}(t)$, which in turn updates both the perceived drift and the pricing function V, and therefore the hedge $H(t)$. This provides a coherent "learning–pricing–hedging" mechanism: price discovery and risk management are driven by the same posterior belief dynamics.

Although the filtered market is complete, the original market under partial observation is generally incomplete when payoffs depend on the latent factor. Accordingly, when the filtered-market hedge is projected back to the original market, it should be interpreted as an optimal quadratic hedge: Theorem 6.2 shows that the same trading strategy H is locally risk-minimizing in the sense of Föllmer–Schweizer. Thus the strategy remains financially meaningful even when perfect replication is impossible: it minimizes the conditional mean-square hedging error given the investor's information. In order to show this property, we first recall the following the Föllmer-Schweizer decomposition for from Schweizer (2001).

Proposition 6.1. *Let $Z \in L^2(\mathscr{F}_T^O)$ be the payoff of a European option. Then the European option (T, Z) admits an $\mathbb{F}^O$-pseudo optimal strategy $H \in \Theta(\mathbb{F}^O)$ with $V^H(T) = Z$, $\mathbb{P}$-a.s. if and only if Z admits a Föllmer-Schweizer decomposition with respect to the discount prices $\hat{S}(t)$ and $\mathbb{F}^O$:*

$$e^{-\int_0^t r_s ds} Z = U_0 + \sum_{i=1}^{m} \int_0^T H^i(s) d\hat{S}^i(s) + G_T, \quad \mathbb{P}\text{-a.s.},$$

where $U_0 \in L^2(\Omega, \mathscr{F}_0^O, \mathbb{P})$, and the hedging error $(G_t)_{t\in[0,T]}$ is a square integrable $(\mathbb{F}^O, \mathbb{P})$-martingale with $G_0 = 0$, strongly orthogonal to the Brownian motion B_t associated with the price process $S(t)$.

Based on the above Föllmer-Schweizer decomposition, we show that the strategy H in filtered market given by Theorem 6.1 still has some properties in the original incomplete information market.

Theorem 6.2. *Let* (Z,T) *be the European option with the payoff* $Z = h(S(T),\theta(T))$*. Then in the original incomplete information market, the hedging strategy* H *given by* (6.1) *and* (6.2) *is a locally risk-minimizing strategy.*

Proof. Based on the innovation process I_t, we introduce the residual (discounted) hedging error

$$G(t) = \sum_{i=1}^{m} \int_0^t H^i(s)S^i(s)[r(\theta(t),t) - r(\hat{\theta}(t),t)]dt. \tag{6.4}$$

Note that for every random variable $X \in L^2(\Omega,\mathscr{F}_t^O,\mathbb{P})$ and test function φ, the filtered estimate $\hat{\theta}(t)$ satisfies

$$\mathbb{E}^{\mathbb{P}}[X(\varphi(\theta(t)) - \varphi(\hat{\theta}(t)))] = 0.$$

Then $(G(t))_{t\in[0,T]}$ is a square integrable $(\mathbb{F}^O,\mathbb{P})$-martingale with $G(0) = 0$. Moreover, $(G(t))_{t\in[0,T]}$ is orthogonal to the Brownian motion B_t under the probability $\mathbb{P}$.

By decomposition (4.10), we can take $U(0) = \hat{V}(0)$, so that

$$e^{-\int_0^t r_s ds}Z = \hat{V}(0) + \sum_{i=1}^{m} \int_0^T H^i(s)d\hat{S}^i(s) + G(T), \quad \mathbb{Q}^N\text{-a.s..}$$

Since the density process Λ_t^N used to define $\mathbb{Q}^N$ is strictly positive and integrable, the probability measures $\mathbb{Q}^N$ and $\mathbb{P}$ on $\mathscr{F}_T^O$ are equivalent; in particular any event $A \in \mathscr{F}_T^O$ has $\mathbb{Q}^N(A) = 0$ if and only if $\mathbb{P}(A) = 0$. Thus H provides an $\mathbb{F}^O$-predictable strategy and the above equality (6.4) satisfies the required form in the Föllmer-Schweizer decomposition. Hence H minimizes the global mean-square hedging error, and satisfies the local minimization property. □

In applications, the hedge can be implemented by the following pipeline:

1) *Online filtering.* From observed prices (or log-returns) compute the posterior density of $\theta(t)$ and the filtered estimate $\hat{\theta}(t)$ using the Yau–Yau algorithm.
2) *Pricing step.* Solve the generalized Black–Scholes PDE (5.5) on a grid in (t,s,θ), and evaluate $V\big(t,S(t),\hat{\theta}(t)\big)$ in real time.

3) *Hedging step.* Compute the Delta vector $H(t) = \nabla_s V\big(t, S(t), \hat{\theta}(t)\big)$ numerically, and rebalance the portfolio accordingly.

This procedure is pathwise in the sense relevant to finance: all computations are driven by the single realized observation path, and the filter provides a stable state estimate that can be updated recursively.

Our numerical implementation couples (a) the Yau–Yau filtering step—solving the observation-independent PDE (3.14) off-line and updating through (3.15)—with (b) a finite-difference solver for the (generalized) Black–Scholes PDE (5.5). We refer Yau et al. (2024) for the rigorous error estimates and convergence analysis of Yau–Yau filtering, and Morton and Mayers (2005) for the finite-difference solver for the (generalized) Black–Scholes PDE.

In practice, portfolios are rebalanced at discrete times, leading to a nonzero hedging error even in complete models; the local risk-minimization perspective is therefore particularly relevant. Transaction costs (bid–ask spreads, brokerage fees) further penalize frequent rebalancing; common remedies include band/“no-trade” regions, cost-adjusted objectives, or volatility/Delta adjustments that trade off tracking error against turnover. Moreover, model risk and parameter uncertainty (e.g. estimation of σ_t) can be incorporated by recalibration, robustification, or by using the filtered density rather than only $\hat{\theta}(t)$ when computing V and its Greeks. These considerations do not change the theoretical form of the hedge, but they guide how the strategy is discretized and stabilized in real trading systems.

7. Numerical Experiments

In this section, we provide some numerical experiments associated with the above theoretic results.

Example 1: Nonlinear Growth Saturation with Periodic Macro Factor In this example, we consider an incomplete information market where the investor observes only the asset price process, while the drift (expected return) is driven by an unobservable, time-periodic macroeconomic factor $\theta(t)$. This setting captures business-cycle effects and learning behavior in financial markets. Moreover, we assume that the effect from the unobservable factor $\theta(t)$ is nonlinear.

Moreover precisely, we consider the price process

$$dS(t) = S(t)\big[(\theta(t)^3 + \alpha)dt + \sigma_1 dB_t\big]. \tag{7.1}$$

Table 1: 50-Run Statistical Results of 1D Filtering Monte Carlo Simulations

Method	**Mean MSE**	**SD of MSE**	**Median MSE**	**Mean Time(s)**	**SD of Time (s)**
Yau-Yau Filter	0.245	0.121	0.207	0.113	0.004
Particle Filter	0.243	0.114	0.211	0.656	0.012
EKF	0.488	0.435	0.321	0.019	0.000

Then the associated observation process is

$$dR(t) = (\theta(t)^3 + \alpha - \sigma_1^2/2)dt + \sigma_1 dB_t. \tag{7.2}$$

The hidden signal process evolves as

$$d\theta(t) = (\mu\cos(\omega\theta(t)) + \beta)dt + \sigma_2 dW_t, \quad \theta(t) = 1, \tag{7.3}$$

where the constants $\omega, \sigma_1, \sigma_2$. The nonlinear drift $\mu(\theta(t))$ captures a saturation effect.

Let (Z,T) be the European option with the payoff $Z = h(S(T), \hat{\theta}(T))$ at maturity data T. The price of the European option $V(t,x,\theta)$ in the filter market $(S(t), \hat{\theta}(t))$ satisfies the following generalized Black-Scholes PDE

$$\begin{cases} \frac{\partial V}{\partial t} + r_t x \frac{\partial V}{\partial x} + \frac{x^2\sigma_2^2}{2}\frac{\partial^2 V}{\partial x^2} + (\mu\cos(\omega\theta) + \beta)\frac{\partial V}{\partial \theta} + \frac{\sigma_2^2}{2}\frac{\partial^2 V}{\partial \theta^2} - r_t V = 0, \\ \qquad t \in [0,T), x \in (0,\infty), \theta \in \mathbb{R}; \\ V(T,x,\theta) = h(x,\theta). \end{cases} \tag{7.4}$$

In particular, we choose $\alpha = 1/2$, $\beta = 0$, $\sigma_1 = 1$, $\mu = 1$, $\omega = 1$ and $\sigma_1 = 1$. Figure 1 and Table 1 shows the outcomes of the Yau-Yau Filter applied to the example 1.

The numerical results demonstrate that the Yau-Yau filter achieves an optimal balance between accuracy and computational efficiency. It attains a mean MSE of 0.245, comparable to the particle filter (0.243) and significantly lower than the Extended Kalman Filter (EKF, 0.488). More importantly, the Yau-Yau filter shows superior stability with the smallest standard deviation in MSE (0.121). In terms of computational performance, the Yau-Yau filter is approximately 5.8 times faster than the particle filter (0.113s vs. 0.656s) with minimal time variance (SD=0.004s), while maintaining comparable accuracy. Although the EKF is slightly faster (0.019s), its estimation accuracy is substantially inferior. These results highlight the Yau-Yau filter's dual advantage: achieving particle-filter-level accuracy with near-EKF-level efficiency.

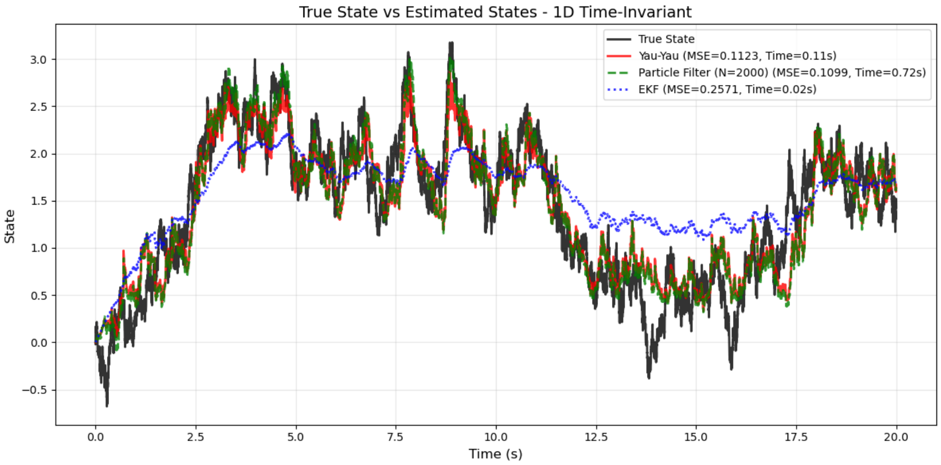

Figure 1: Example 1

Now we consider the Inflation-Indexed Call Option (see e.g. Kallianpur and Karandikar (2012)) as a canonical example for the Black-Scholes PDE. In this context, θ represents a macroeconomic indicator (e.g., the Consumer Price Index). To hedge against purchasing power risk, the strike price is not fixed but formulated as a floating function $K(\theta)$, leading to the terminal condition:

$$h(x,\theta) = \max(x - K(\theta), 0). \tag{7.5}$$

With $h(S(T),\theta(T)) = (S(T) - (100 + 5\hat{\theta}(T)))^{+}$, we solve the Black-Scholes PDE (7.4) by finite difference method. The numerical result is shown in Figure 2.

Example 2: Financial markets with multi risk assets and periodic Macro Factors We consider the price process with two risk assets with log return process

$$dR_1(t) = \left[\theta_1(t) - \frac{1}{2}\sigma_{S1}^2\right] dt + \sigma_{S1} dV_1(t),$$
$$dR_2(t) = \left[\theta_2(t) - \frac{1}{2}\sigma_{S2}^2\right] dt + \sigma_{S2} dV_2(t).$$

We also assume that the hidden signal process evolves as

$$d\theta_1(t) = (\mu_0 + \mu_1 \sin(\omega t))dt + \sigma_\theta dW_1(t),$$
$$d\theta_2(t) = (\mu_0 + \mu_1 \cos(\omega t))dt + \sigma_\theta dW_2(t).$$

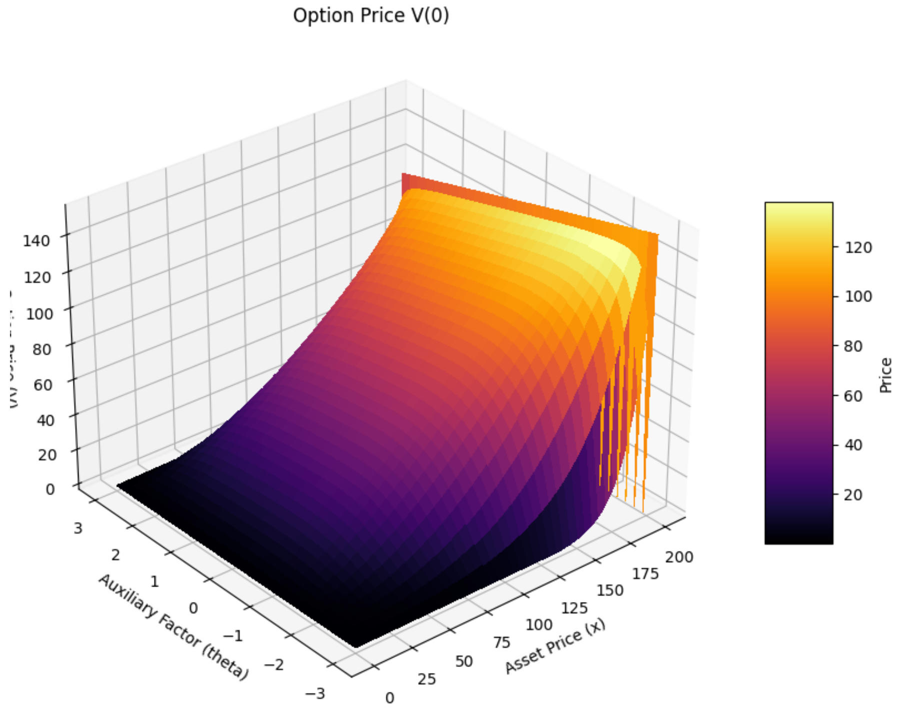

Figure 2: Black-Scholes PDEs

Let (Z,T) be the European option with the payoff $Z = h(S(T),\hat{\theta}(T))$ at maturity data T. The price of the European option $V(t,s,\theta)$ in the filter market $(S(t),\hat{\theta}(t))$ satisfies the following generalized Black-Scholes PDE

(7.6)

$$\begin{cases} \dfrac{\partial V(t,s,\theta)}{\partial t} + r_t s_1 \dfrac{\partial V(t,s,\theta)}{\partial s_1} + r_t s_2 \dfrac{\partial V(t,s,\theta)}{\partial s_2} + \dfrac{s_1^2\sigma_{S_1}^2}{2}\dfrac{\partial^2 V(t,s,\theta)}{\partial s_1^2} \\ \quad + \dfrac{s_2^2\sigma_{S_2}^2}{2}\dfrac{\partial^2 V(t,s,\theta)}{\partial s_2^2} + (\mu_0 + \mu_1 \sin(\omega t))\dfrac{\partial V(t,s,\theta)}{\partial \theta_1} + (\mu_0 + \mu_1 \cos(\omega t))\dfrac{\partial V(t,s,\theta)}{\partial \theta_2} \\ \quad + \dfrac{\sigma_\theta^2}{2}\dfrac{\partial^2 V(t,s,\theta)}{\partial \theta_1^2} + \dfrac{\sigma_\theta^2}{2}\dfrac{\partial^2 V(t,s,\theta)}{\partial \theta_2^2} - r_t V(t,s,\theta) = 0, \\ V(T,x,\theta) = h(x,\theta). \end{cases}$$

In particular, we choose the parameters

$$\mu_0 = 0.05, \quad \mu_1 = 0.05, \quad \omega = 0.02, \quad \sigma_\theta = 0.7, \quad \sigma_{S1} = 1, \quad \sigma_{S2} = 1.$$

Figure 3 shows the outcomes of the Yau-Yau Filter applied to the example 2. The Table 2 is the numerical results in this example. The two-dimensional nonlinear filtering results

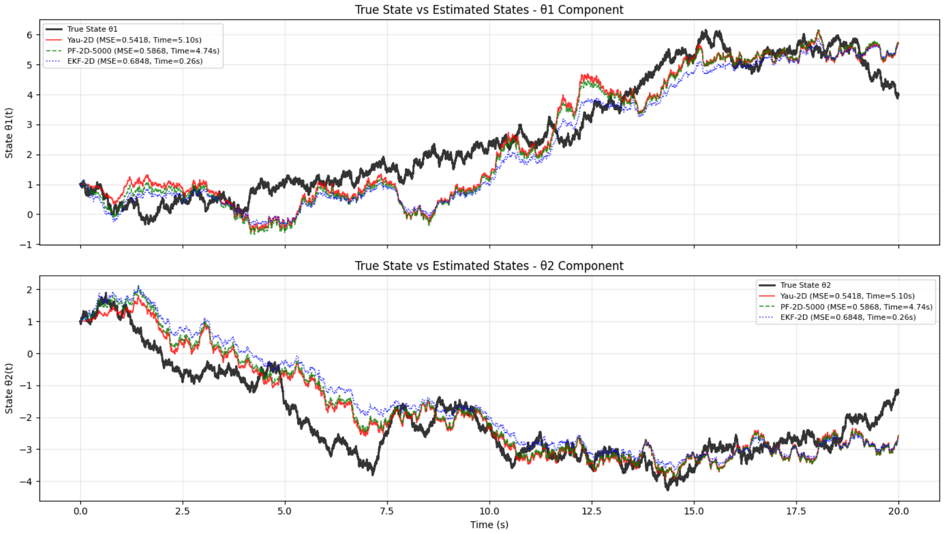

Figure 3: Example 3

demonstrate that the Yau-Yau filter maintains its performance advantages over traditional methods in higher-dimensional settings. As shown in Table 2, the Yau-Yau filter achieves the best estimation accuracy with a mean MSE of 0.693, slightly outperforming the particle filter (0.698) and significantly surpassing the Extended Kalman Filter (EKF, 0.750). Notably, the Yau-Yau filter exhibits the smallest standard deviation in MSE (0.190), indicating superior stability and consistency across Monte Carlo runs. Although computationally more demanding than the EKF (5.093s vs. 0.265s), it provides a substantial improvement in mean MSE while maintaining comparable computational efficiency to the particle filter (5.093s vs. 4.716s). This optimal accuracy-efficiency trade-off positions the Yau-Yau filter as a highly effective solution for multidimensional nonlinear filtering problems where precision is paramount.

We note that the absence of a large MSE gap in Tables 1–2 should not be interpreted as a lack of advantage of the Yau–Yau filter. In the low-dimensional settings considered here, particle filtering can often be viewed as a near-optimal numerical benchmark for the nonlinear filtering problem: when the number of particles is sufficiently large and resampling is well-tuned, the particle approximation is close to the target posterior and thus provides a natural “gold standard” for MSE comparisons. From this perspective, the fact that the Yau–Yau filter achieves comparable MSE while remaining competitive in running time is already a meaningful success: it indicates that a deterministic PDE-based, pathwise robust scheme can approximate the benchmark Monte Carlo posterior accuracy without relying on large particle ensembles.

More importantly, MSE and wall-clock time do not fully capture the operational advantages of the Yau–Yau approach. The Yau–Yau algorithm admits an off-line/on-line decomposition: the heavy computation is shifted to an observation-independent off-line PDE solve, while the on-line update is explicit, memory-less, and requires storing only a small set of precomputed objects. This yields a real-time filtering pipeline with low memory footprint, which is particularly attractive in streaming market applications where one observes a single price path and must update estimates rapidly. These features become increasingly valuable as the latent dimension grows.

When the latent dimension $n = \dim(\theta)$ becomes large (e.g., $n > 3$), a direct grid-based discretization of the off-line Kolmogorov-type PDE (3.14) and the pricing obstacle/PDE becomes computationally prohibitive due to the curse of dimensionality. This limitation is not specific to the Yau–Yau filter itself, but rather reflects the intrinsic complexity of PDE-based approaches in high-dimensional state spaces. That said, the Yau–Yau framework is compatible with several dimension-reduction and surrogate strategies, such as tensor decompositions and learning-based surrogates. In particular, the off-line/on-line decomposition of the Yau–Yau algorithm makes it natural to combine off-line heavy computation (including TT solvers or deep surrogates) with light on-line exponential updates (3.15), which is especially appealing for real-time financial applications with streaming observations. We leave a systematic high-dimensional implementation and empirical benchmarking (e.g., $n \gg 1$ latent factors) to future work.

Table 2: 50-Run Statistical Results of 2D Filtering Monte Carlo Simulations

Method	**Mean MSE**	**SD of MSE**	**Median MSE**	**Mean Time (s)**	**SD of Time (s)**
Yau-Yau Filter	0.693	0.190	0.641	5.093	0.280
Particle Filter	0.698	0.200	0.632	4.716	0.172
EKF	0.750	0.255	0.689	0.265	0.011

8. Conclusion

In this paper, we investigated the pricing and hedging of European options in an incomplete information market where the risky asset price is driven by latent economic factors that are not directly observable. Agents only observe the asset price (or log–return) process and must infer the hidden state through a nonlinear filtering procedure. Within this setting, we combined stochastic filtering theory, in particular the Yau–Yau nonlinear filter, with the Black–Scholes framework to obtain a unified filtering–pricing–hedging methodology under partial observation.

On the theoretical side, our first contribution is the construction of a filtered market under the observable filtration generated by asset prices and filter estimates. Using the pathwise robust DMZ formulation and the associated regularity results, we proved the existence of an equivalent risk–neutral measure $\mathbb{Q}^N$ with respect to the observable filtration and showed that the resulting filtered market is arbitrage–free and complete. This allows us to apply a martingale representation theorem under $\mathbb{Q}^N$ and to establish that every square–integrable European payoff $Z = h(S(T), \theta(T))$ is attainable in the filtered market. Our second main theoretical contribution is the derivation of a generalized Black–Scholes PDE that governs the option price $V(t, s, \theta)$, where the spatial variables include both the observable asset price and the (filtered) latent factor. The PDE couples the risk–neutral Black–Scholes generator with the generator of the hidden Markov signal and admits a classical solution under suitable regularity and ellipticity assumptions. Third, we characterized the associated hedging strategy in feedback form through the gradient of the pricing function with respect to the asset price and showed that, when projected back to the original incomplete information market, this strategy is locally risk–minimizing in the sense of Föllmer–Schweizer.

On the numerical side, the paper exploits the Yau–Yau nonlinear filtering algorithm as a pathwise robust and computationally efficient tool to approximate the DMZ equation. By decomposing the problem into an off–line Kolmogorov–type PDE and an on–line exponential update driven by incoming observations, the Yau–Yau framework yields a real–time, memory–less filtering scheme suitable for financial applications. We then coupled this filter with finite–difference solvers for the generalized Black–Scholes PDE and demonstrated the resulting filtering–pricing pipeline on several nonlinear toy models. These examples feature periodic and saturating hidden drifts, as well as multi–asset settings, illustrating how latent macroeconomic cycles and nonlinear growth effects can be incorporated into incomplete information option pricing. The numerical experiments confirm that the proposed approach delivers stable filter estimates and consistent option prices, and that the resulting hedging strategies can be implemented in a pathwise manner along a single observed price trajectory.

Despite these advances, the present work has several limitations. From a modeling perspective, we restricted attention to diffusion–type hidden factors with continuous sample paths and to markets driven by Brownian noise. Extensions to jump–diffusion signals, Lévy–type price dynamics, or models with stochastic volatility and multiple interacting latent factors remain open and are likely to require substantial refinements of both the filtering and pricing components. From an analytical viewpoint, our results rely on strong regularity and uniform ellipticity assumptions on the coefficients (b,Γ) and on the observation structure, which may not hold under severe model misspecification or in highly degenerate settings. On the numerical side, while the Yau–Yau filter with PDE–based solvers is effective in the low–dimensional examples considered here, the curse of dimensionality remains a serious challenge for higher–dimensional state spaces and large portfolios, and the convergence analysis of the fully discrete filtering–pricing scheme is still incomplete.

These limitations point to several promising directions for future research. One natural extension is to develop high–dimensional variants of the Yau–Yau filter by combining tensor–train techniques, low–rank approximations, or deep learning surrogates for the off–line Kolmogorov PDE, and to provide rigorous error bounds for the resulting schemes. Another important direction is to generalize the present framework to American or barrier–type options, where optimal stopping under partial observation leads to coupled obstacle problems for filtering–pricing equations. From a financial perspective, it would be interesting to calibrate the model to real market data and to study empirically how the filtered latent factors relate to implied risk premia, investor sentiment, or macroeconomic cycles. Finally, incorporating transaction costs, model ambiguity, and robust control into the filter–based hedging framework could lead to more realistic and robust risk–management strategies in incomplete information markets.

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Declaration of Competing Interests

The authors declare no competing interests.

Code and Data Availability

Data sharing is not applicable to this article. The code is standard and available from the corresponding author upon reasonable request.

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