Nonexistence of finite-dimensional estimation algebras on closed smooth manifolds

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Abstract—**Estimation algebras have been extensively studied in Euclidean space, where finite-dimensional estimation algebras form the foundation of the Kalman and Benes filters, and have contributed to the discovery of many other finite-dimensional filters. This work extends the theory of estimation algebras to filtering problems on Riemannian manifolds in continuous time. Our main result demonstrates that, with non-constant observation functions, the estimation algebra associated with the system on closed Riemannian manifolds is infinite-dimensional.**

Index Terms—**Finite dimensional filter, Riemannian manifold, Estimation algebra**

I. Introduction

The filtering problem, which involves estimating the state of a stochastic dynamical system under noisy observations, has been a major research focus since the 1960s when Kalman and Bucy introduced linear filtering theory for systems with Gaussian initial conditions [1], [2]. This foundational work spurred interest in nonlinear filtering. One important point of view on the continuous time filtering problem is given by the Duncan-Mortensen-Zakai (DMZ) equation [3]–[5], which gives a stochastic partial differential equation for the probability density function of the state conditioned on observations.

In the 1970s, Brockett [6], Clark [7], and Mitter [8] independently advanced the field by developing finite-dimensional filters using the estimation algebra method. The estimation algebra can be unconditionally defined for continuous-time filtering systems as a Lie subalgebra of differential operators on the state space, and it enables the construction of finite-dimensional recursive filters, provided the estimation algebra is finite-dimensional. The estimation algebra approach recasts previously discovered finite-dimensional filters, such as the Kalman-Bucy filter and Beneş filters, in a common framework. Yau and his collaborators studied the structure of estimation algebras in detail during the 1990s and 2000s [9]–[11]. They identified a key invariant, the linear rank, the dimension of the vector subspace of the estimation algebra spanned by homogeneous linear functions, and classified all finite-dimensional estimation algebras of maximal rank, when the linear rank

is equal to the dimension of the state space [12]. Since the 2000s, subsequent progress has focused on non-maximal rank estimation algebras in Euclidean space. Wu and Yau classified finite-dimensional estimation algebras (FDEA) with state dimension 2 and rank 1 in 2006 [13]. Shi and Yau extended this work by proving the Mitter conjecture, which states that the observation process must be linear in the state, for state dimension 3 and rank 2 [14]. Further, they constructed new classes of finite-dimensional filters (FDFs) with various state dimensions and ranks. Jiao and Yau recently identified a novel class of FDFs not of Yau-type, broadening the understanding of non-maximal rank estimation algebras across different state space dimensions [15]. And a time-varying extension of FDEA appeared in [16]. However, the extension of FDEA to manifold cases remains an open problem.

Suppose that (M, g) is a Riemannian manifold with a Riemannian metric g. The filtering problem of M considered here is based on the signal observation model:

$$
\begin{aligned}
dX_t &= V_0(X_t)\, dt + \sum_{\alpha=1}^{l} V_\alpha(X_t) \circ dW_t^\alpha \\
dY_t &= h(X_t)\, dt + dV_t
\end{aligned}
\tag{1}
$$

where $X_t \in M$, $Y_t \in \mathbb{R}^m$, the $V_0, V_1, \cdots, V_l$ are all smooth vector fields of M, i.e. each $V_i : M \to TM$ satisfies $V_i(x) \in T_x M \cong \mathbb{R}^n$ for all $x \in M$, $h : M \to \mathbb{R}^m$ is the smooth function, W_t^α is 1 dimensional standard Brownian motion on $\mathbb{R}$, V_t is m dimensional standard Brownian motion and the symbol $\circ$ represents the Stratonovich integral.

The system model in (1) represents a common assumption in recent theoretical works on manifold-based filtering, such as feedback particle filters [17]. Even in practical applications involving pure angular measurements, researchers have demonstrated that transforming angular observations into their corresponding trigonometric forms enhances algorithmic performance, as seen in pseudo-linear Kalman filtering approaches [18]. These developments in both theory and application validate the mathematical formulation presented in (1).

Our main result, established in Theorem 9 of Section 4, demonstrates the nonexistence of finite-dimensional estimation algebras when the state equations are defined on a closed compact Riemannian manifold, with the stochastic component arising from Brownian motion on the manifold. We prove the following result:

Theorem 1 (Main Theorem)**.** *Let M be a closed smooth manifold, and let L be a diffusion process on M with a second-order term that is non-degenerate everywhere (in the*

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sense of Definition 1). Let h be the observation function of a signal-observation model as in Equation (1) *and assume that h is not a constant function. Then the estimation algebra (see Definition 8) attached to the signal-observation model is infinite-dimensional.*

More generally, we show the following result which shows that in many cases when M is non-compact, the estimation algebra is still infinite-dimensional, especially if M has non-trivial topology.

Theorem 2 (Main Theorem II). *Let M be a possibly non-compact manifold without boundary, and let L be a diffusion process on M with second order term that is non-degenerate everywhere. Let h be the observation function, and let x and y be distinct critical points of some component h^i of h on M connected by a gradient flow $\gamma\colon \mathbb{R} \to M$, i.e. γ satisfies the differential equation*

$$\gamma'(t) = (\mathrm{grad}h)(\gamma(t))$$

and approaches x at $-\infty$ and y at $+\infty$. Then the estimation algebra attached to the signal-observation model is infinite-dimensional.

A few previous works have explored the equations for filtering stochastic processes on Riemannian manifolds [19]–[21], although to our knowledge, the estimation algebra approach has not been explicitly considered previously. Despite being studied much less compared to their Euclidean state space counterpart, many filtering problems are most naturally phrased on Riemannian manifolds. This is especially true for Lie groups and their homogeneous spaces, which can be used to give a uniform description of configuration spaces. For example, subspace tracking has been explored using Stiefel and Grassmann manifolds [22], [23], offering robust solutions in these domains. Additionally, the manifold of symmetric positive definite matrices has proven instrumental for covariance tracking [24], [25], further highlighting the importance of manifold-based filtering techniques in tackling complex real-world challenges.

In addition to our original findings, it is important to note that while our conclusion has not been previously explored within the context of filtering, related advancements in finite-dimensional representations of stochastic partial differential equations on manifolds—where solutions are represented with finite-dimensional parameters—have been made in recent years [26], [27]. Our conclusion indicates that in the case of compact Riemannian manifolds, a finite-dimensional filter similar to the Kalman filter does not exist. This suggests that developing numerical algorithms suitable for infinite-dimensional filtering is crucial for addressing filtering problems on closed Riemannian manifolds [17], [28], [29].

II. FILTERING EQUATIONS ON A RIEMANNIAN MANIFOLD

A. Preliminaries and notation

Let (M, g) be a compact Riemannian manifold of dimension n, and let ω be the canonical measure $d\omega = \sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n$. In our setup, all functions and operators will be over the real numbers. Let $C^\infty(M) = \Gamma(M)$ be the

smooth functions on M, which we view as the (real-valued) smooth sections of the trivial line bundle. The space of vector fields is the global sections $\Gamma(M, TM)$ of the tangent bundle. Any vector field defines a linear operator on smooth functions by differentiation, which we denote $X(f)$ for $X \in \Gamma(M, TM), f \in \Gamma(M)$. The smooth differential operators $D^\infty(M)$ are the noncommutative algebra over $C^\infty(M)$ of operators $C^\infty(M) \to C^\infty(M)$ generated by vector fields. The differential operators are filtered by order, denoted $\mathrm{ord}(D)$ for any differential operator D, where the differential operators of order 0 are the smooth functions acting on smooth functions by multiplication. Differential operators are graded-commutative in the sense that for $D_1, D_2 \in D^\infty(M)$, $\mathrm{ord}(D_1 + D_2) \leq \max(\mathrm{ord}(D_1), \mathrm{ord}(D_2))$, $\mathrm{ord}(D_1 D_2) \leq \mathrm{ord}(D_1) + \mathrm{ord}(D_2)$, and $\mathrm{ord}(D_1 D_2 - D_2 D_1) \leq \mathrm{ord}(D_1) + \mathrm{ord}(D_2) - 1$ if D_1 and D_2 do not commute. In the course of what follows, we will also use the commutator on differential operators $[D_1, D_2] = D_1 D_2 - D_2 D_1$ which satisfies the usual property of a Poisson bracket.

The Riemannian metric g induces a pairing $\langle \cdot, \cdot \rangle_g \colon \Gamma(M, TM) \times \Gamma(M, TM) \to \Gamma(M)$ between vector fields and induces an isomorphism between the tangent bundle and the cotangent bundle. This allows us to transfer the natural 1-form df for any smooth function to a vector field to a vector field $\mathrm{grad}_g f \in \Gamma(M, TM)$. For any vector field, X, we can characterize the divergence $\mathrm{div}_g X \in \Gamma(M)$ by

$$\int (\mathrm{div}X)fd\omega = \int \langle X, (\mathrm{grad}f)\rangle d\omega = \int X(f)d\omega.$$

The Laplace-Beltrami operator is $\Delta_{M,g} f = \mathrm{div}_g \mathrm{grad}_g f$, and it is a fundamental example of a smooth second-order elliptic differential operator on M and is self-adjoint. We will omit the g in the gradient operator or divergence operator when the underlying Riemannian structure is clear.

Let $T^{(n,m)}M$ be the vector bundle of (r, s)-tensors so that in particular the fiber over $x \in M$ is canonically identified with $T_x^{\otimes n}M \otimes (T^*)_x^{\otimes m}M$. We use the notation $T^n M = T^{(n,0)}M$. The Riemannian metric extends to a pairing $T_x^{(n,m)}M \otimes T_x^{(n,m)}M \to \mathbb{R}$ and a pairing on sections as $\langle \cdot, \cdot \rangle \colon \Gamma(M, T^n M \otimes T^{*m}M) \times \Gamma(M, T^n M \otimes T^{*m}M) \to \Gamma(M)$. The covariant derivative gives a linear operator $\nabla \colon \Gamma(M, T^{(n,m)}M) \to \Gamma(M, T^{(n,m+1)}M)$ that extends the operator d sending a function to a 1-form.

III. BACKGROUND ON FILTERING ON CLOSED MANIFOLDS

We recall the filtering equations, following the exposition of Bain and Crisan [30, Chapter 3]. Let $\mathcal{P}(M)$ denote the set of probability measures on M. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $(\mathcal{F}_t)_{t \geq 0}$ so that $\mathcal{F}_t$ is complete, right-continuous, and contains all $\mathbb{P}$-null sets. Let $\mathbb{E}$ denote expectation with respect to the probability measure $\mathbb{P}$.

A. The Riemannian metric attached to a non-degenerate diffusion process

A general stochastic differential equation in Stratonovich formulation has the form:

$$dX_t = V_0(X_t)\,dt + \sum_{\alpha=1}^{l} V_\alpha(X_t) \circ dW_t^\alpha, \qquad (2)$$

where the $V_0, V_1, \cdots, V_l$ are all smooth vector fields of M, i.e. $V_0, V_1, \cdots, V_l \in \Gamma(M, TM)$, W_t^α are standard 1 dimensional Brownian motion on $\mathbb{R}$, and the symbol $\circ$ represents the Stratonovich integral. We consider diffusion processes X_t generated by $L = \frac{1}{2}\sum_{i=1}^{l} V_\alpha^2 + V_0$, in our setting a smooth second-order elliptic differential operator.

Definition 1. *A non-degenerate diffusion process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a process M^ϕ such that for any smooth function $\phi \in C^\infty(M)$, the process M_t^ϕ defined by*

$$M_t^\phi = \phi(X_t) - \phi(X_0) - \int_0^t L\phi(X_s)ds$$

is an $\mathcal{F}_t$-adapted martingale, and L is a second-order differential operator with smooth coefficients such that the second-order part is everywhere non-degenerate.

We remark that for L an elliptic operator, the existence of diffusion processes can be established as in [31, Section 1.3]. Given an L, we can choose a 'good' Riemannian metric g to give our operator some extra structure.

Lemma 2. *Consider a given diffusion process X_t on M generated by a non-degenerate L. There exists a new Riemannian metric $\tilde{g}$ such that the degree 2 part of L is $\frac{1}{2}\Delta_{M,\tilde{g}}$.*

Proof. In local coordinates $x_1, \cdots, x_n$, we can represent the degree 2 part as

$$\text{degree 2 part of}(L) = \frac{1}{2}\sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j}.$$

Since L is non-degenerate, we may write $A(x) = (a_{i,j}(x))_{i,j=1}^{n} \in \mathbb{R}^{n \times n}$ as a invertible matrix for any $x \in M$. so we can construct the inverse of $A(x)$ as $G(x) = (g_{i,j}(x))_{i,j=1}^{n}$. Then, we can construct a 2-form $\tilde{g}$ as

$$\tilde{g} = \sum_{i,j=1}^{n} g_{ij}dx_i dx_j.$$

The Laplace-Beltrami operator of $\tilde{g}$ is given by

$$\Delta_{M,\tilde{g}} = \sum_{i,j=1}^{n} \frac{1}{\sqrt{\det(G)}}\frac{\partial}{\partial x_j}\left(\sqrt{\det(G)}a_{i,j}\frac{\partial}{\partial x_i}\right).$$

Here $\det(G)$ is the determinant of the matrix G. We have

$$\frac{1}{2}\Delta_{M,\tilde{g}} = \frac{1}{2}\sum_{i,j=1}^{n} a_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j} + b^i \frac{\partial}{\partial x_i},$$

where

$$b^i = \sum_{j=1}^{n} \frac{1}{\sqrt{G}}\frac{\partial(\sqrt{G}a_{i,j}(x))}{\partial x_j} = \sum_{j,k=1}^{n} a_{jk}\Gamma_{jk}^i,$$

where Γ_{jk}^i are the Christoffel symbols of the metric $\tilde{g}$ in the local coordinates. $\qquad \square$

B. The DMZ equation on Riemannian manifolds

As seen above, for any given non-degenerate diffusion system on a compact smooth manifold without boundary as in Equation (2), we can choose a metric g of M as Lemma 2 such that the generator L of the diffusion process can be decomposed into a Laplacian part $\frac{1}{2}\Delta_M$ and a degree 1 vector field part $F \in \Gamma(M, TM)$, i.e.,

$$L = \frac{1}{2}\Delta_{M,g} + F. \qquad (3)$$

For simplicity, we donate $\Delta_{M,g} = \Delta_M$ in the rest of the paper. The aim of this section is to generalize the construction of the estimation algebra to general M, allowing M to be non-compact, so that we recover the usual theory for $M = \mathbb{R}^n$.

In the case that L is $\Delta_M/2$, then the diffusion process X_t is called Brownian motion on the Riemannian manifold M.

Remark 3 (Relation to continuous filtering literature in Euclidean space). When considering $M = \mathbb{R}^n$, condition (3) will allow us to write the diffusion process X_t as a stochastic integral. For arbitrary manifolds, we may consider this condition as giving a stochastic integral condition on X_t. We will sometimes find it convenient to consider the Stratonovich integrals rather than Itô integrals in order to use the chain rule. At the same time, on manifolds, these two types of stochastic integrals can still be freely converted between each other, which can be easily seen as in [31, Chapter 1] by taking an isometric embedding into Euclidean space and taking smooth extensions of functions on the manifold.

The diffusion process X_t gives the equations of motion for the state process. We will also consider an observation process which satisfies its own evolution equation. To simplify the exposition, let $h\colon M \to \mathbb{R}^m$ be a smooth function for compact M or at most linear growth for $M = \mathbb{R}^n$, which in our setting guarantees that certain boundedness conditions (e.g. Novikov's condition) automatically hold. We let $h^i\colon M \to \mathbb{R}$ be the ith coordinate function of h.

Let W_t be an $\mathcal{F}_t$-adapted standard Brownian motion on $\mathbb{R}^m$ and Y_t the process satisfying the evolution equation

$$Y_t = Y_0 + \int_0^t h(X_s)dt + V_t. \qquad (4)$$

The process Y_t determines an augmented filtration $\mathcal{Y}_t \subset \mathcal{F}_t$. In this setting, there is an optional $\mathcal{Y}_t$-adapted probability-valued measurable process π_t that solves the filtering problem, so that for any Borel-measurable function $\phi\colon M \to \mathbb{R}$,

$$\pi_t \phi = \mathbb{E}\left[\phi(X_t)|\mathcal{Y}_t\right].$$

Theorem 4 (Theorem 3.24 in [30], or in [32])**.** *Consider a filtering problem on a compact Riemannian manifold, where the system equation satisfies (2) and the observation equation satisfies (4). Its unnormalized density function satisfies the following DMZ equation:*

$$\rho_t(\varphi) = \pi_0(\varphi) + \int_0^t \rho_s(A\varphi)ds + \sum_{i=1}^{m}\int_0^t \rho_s(\varphi h^i)dY_s^i, \quad (5)$$

which holds $\tilde{\mathbb{P}}$-a.s., for all times $t \geq 0$ for any smooth test function φ.

Definition 5 ($H^k(M)$ on Riemannian Manifolds)**.** *Let (M, g) be a smooth closed Riemannian manifold of dimension n. For $k \in \mathbb{N}$, we define the reflexive Sobolev space $H^k(M)$ as:*

$$H^k(M) = \left\{ f \in L^2(M) : \nabla^j f \in L^2(M, T^{*j}M) \text{in weak sense,} \right.$$
$$\left. \text{for all } 1 \leq j \leq k \right\}$$

where $\nabla^j f$ denotes the j-th covariant derivative of f with respect to the Levi-Civita connection associated with the metric g. The Sobolev norm for $f \in H^k(M)$ is defined as:

$$\|f\|_{H^k(M)} = \left(\sum_{j=0}^n \|\nabla^j f\|_{L^2(M)}^2 \right)^{1/2}$$

Moreover, $H^k(M)$ is a Hilbert space with the inner product:

$$\langle f, g \rangle_{H^n(M)} = \sum_{j=0}^n \int_M \langle \nabla^j f, \nabla^j g \rangle \, d\omega$$

where $d\omega$ is the canonical measure and $\langle \cdot, \cdot \rangle$ is the inner product induced by the metric g on tensor fields.

More importantly, we have the following theorem.

Theorem 6 (Theorem 3.1 in [32])**.** *Assume that the conditional law π_0 is absolutely continuous with respect to the canonical measure ω, and that $p_0 := d\pi_0/d\omega$ belongs to $C^\infty(M)$ for some integer $m > 0$. Then for every $t > 0$ the conditional distribution π_t, given the observations Y_s for $0 \leq s \leq t$ has a density with respect to w. Moreover, the density of π_t we denote as $p = (p(t,x)) \in C^1([0,\infty), H^{m+1}(M))$ and*

$$p(t,x) = \sigma(t,x) \left(\int_M \sigma(t,x) dw \right)^{-1},$$

where σ is the unique generalized solution of the problem,

$$d\sigma(t,x) = L^*\sigma(t,x)dt + \sum_{i=1}^m L_i\sigma(t,x)dY_t^i,$$
$$\sigma(0,x) = \pi_0, \tag{6}$$

where L^* is adjoint operator of L and L_i is the zero degree differential operators of multiplication by h^i for any $1 \leq i \leq m$.

Next we will transform the equation (6) from Ito form to Stranovich form as follows,

$$d\sigma(t,x) = L_0\sigma(t,x)dt + \sum_{i=1}^m L_i\sigma(t,x) \circ dY_t^i,$$
$$\sigma(0,x) = \pi_0 \tag{7}$$

where $L_0 := L^* - \frac{1}{2}h^\top h$.

We now recall an exponential transform originally introduced by Davis [33] adapted to our setting, which has been extensively analyzed in the case that M is Euclidean space. Define a new unnormalized density by

$$u(t,x) = \exp\left(-\sum_{i=1}^m h^i(x)Y_t^i \right) \sigma(t,x).$$

Proposition 7. *The density $u \colon \mathbb{R} \times M \to \mathbb{R}$ satisfies the following partial differential equation.*

$$\frac{\partial u}{\partial t} = L_0 u + \sum_{i=1}^m Y_t^i[L_0, L_i]u + \frac{1}{2}\sum_{i,j=1}^m Y_t^i Y_t^j[[L_0, L_i], L_j]u,$$
$$u(0,x) = \pi_0. \tag{8}$$

Proof. We begin by applying the chain rule for Stratonovich integrals to equation (7). This yields:

$$du(t,x) = d(\exp\left(-\sum_{i=1}^m h^i(x)Y_t^i \right))\sigma(t,x)$$
$$+ \exp\left(-\sum_{i=1}^m h^i(x)Y_t^i \right) d\sigma(t,x)$$
$$= -\sum_{j=1}^m h^j(x)\exp\left(-\sum_{i=1}^m h^i(x)Y_t^i \right)\sigma(t,x) \circ dY_t^j$$
$$+ \exp\left(-\sum_{i=1}^m h^i(x)Y_t^i \right) L_0\sigma(t,x)dt$$
$$+ \sum_{i=1}^m h^i(x)\exp\left(-\sum_{i=1}^m h^i(x)Y_t^i \right)\sigma(t,x) \circ dY_t^i$$
$$= \exp\left(-\sum_{i=1}^m h^i(x)Y_t^i \right) L_0\sigma(t,x)dt$$
$$= \left[\exp\left(-\sum_{i=1}^m h^i(x)Y_t^i \right) L_0 \exp\left(\sum_{i=1}^m h^i(x)Y_t^i \right) \right] u(t,x)dt. \tag{9}$$

Notably, the stochastic terms cancel out, indicating that $u(t,x)$ satisfies a partial differential equation. Our task now is to explicitly calculate the operator

$$\exp\left(-\sum_{i=1}^m h^i(x)Y_t^i \right) L_0 \exp\left(\sum_{i=1}^m h^i(x)Y_t^i \right).$$

Let $K(t,x) = \sum_{i=1}^m h^i(x)Y_t^i$. Applying the Baker-Campbell-Hausdorff formula, we obtain:

$$e^{-K(t,x)}L_0 e^{K(t,x)} = L_0 + [L_0, K(t,x)]$$
$$+ \frac{1}{2!}[[L_0, K(t,x)], K(t,x)] + \cdots$$
$$+ \frac{1}{n!}[\underbrace{[\ldots[L_0, K(t,x)], \ldots K(t,x)]]}_{n \text{ nested commutators}} + \cdots \tag{10}$$

Now, consider the order of these terms. We have:

$$\text{ord}([[L_0, K(t,x)], K(t,x)]) \leq \text{ord}([L_0, K(t,x)]) - 1$$
$$\leq \text{ord}(L_0) - 2 = 0. \tag{11}$$

From equation (11), we can deduce that all terms on the right-hand side of equation (10) are zero starting from the fourth term. Therefore, we can simplify our expression to:

$$\frac{\partial u}{\partial t} = \left(L_0 + [L_0, K(t,x)] + \frac{1}{2!}[[L_0, K(t,x)], K(t,x)] \right) u \tag{12}$$

Finally, we substitute $K(t,x) = \sum_{i=1}^m h^i(x) Y_t^i$ back into equation (12), which completes the proof. $\qquad\square$

The above proposition forms the basis of estimation algebra techniques, due to the method of Wei and Norman [34]. In particular, the above equations define a linear PDE with stochastic coefficients, and Wei and Norman show that if the Lie algebra generated by the coefficient operators form a finite-dimensional Lie algebra, the resulting PDE can be integrated explicitly as a product of exponentials.

Definition 8. *The estimation algebra $\mathcal{E}$ is the Lie algebra inside the algebra of differential operators generated by the operators $L_0, L_1, \ldots, L_m$, where $L_0 = L^* - \frac{1}{2} h^\top h$ and L_i are multiplication by h^i.*

IV. NONEXISTENCE OF FINITE-DIMENSIONAL ESTIMATION ALGEBRAS

We are now ready to give the proof of Theorem 1. With the preliminaries established in the previous section, it suffices to prove the following theorem.

Theorem 9. *Suppose that M is a closed Riemannian manifold and $h \colon M \to \mathbb{R}^m$ is smooth and non-constant. Then the estimation algebra $\mathcal{E}$ is infinite-dimensional.*

Proof. Suppose h^j is non-constant for some j. Let H_i be defined by

$$
\begin{aligned}
H_0 &:= h^j \\
H_{i+1} &:= [[L_0, H_i], H_i] = \langle \mathrm{grad} H_i, \mathrm{grad} H_i \rangle
\end{aligned}
\tag{13}
$$

We can define a non-linear operator $Q : f \in C^\infty(M) \to Q(f) = \langle \mathrm{grad} f, \mathrm{grad} f \rangle \in C^\infty(M)$. We claim that if h is nonconstant and smooth, then the sequence $\{h, Qh, Q^2 h, \ldots\}$ is linearly independent.

Lemma 10. *For the operator Q on a compact Riemannian manifold M:*

1) *$Qf = 0$ if and only if f is constant.*
2) *There are no functions f, g such that $\langle \mathrm{grad} f, \mathrm{grad} g \rangle = 1$ everywhere on M.*

Proof of Lemma 10. If f is non-constant, then df is not identically zero. Consequently, $\mathrm{grad} f$ cannot be identically zero, which implies $\langle \mathrm{grad} f, \mathrm{grad} f \rangle$ is not identically zero. Conversely, if f is constant, then $\mathrm{grad} f = 0$, and thus $Qf = 0$. So, we have proved the first one.

Since M is compact, every $f \in C^\infty(M)$ has at least one critical point. Let x_0 be a critical point of f, i.e., $\mathrm{grad} f(x_0) = 0$. Then, $0 = \langle \mathrm{grad} f(x_0), \mathrm{grad} g(x_0) \rangle \neq 1$. $\qquad\square$

By using Lemma 10, $Q^i f$ can not be an identically zero function for any $i \geq 1$ for any non-constant function f. We note that for any smooth function f, $Qf \geq 0$, is equal to 0 at critical points of f, and Qf must be nonzero. In particular, the critical points of f are also critical points of Qf, so inductively, if x is a critical point of f, then x is a critical point of $Q^i(f)$ for all i and $Q^i(f) = 0$ for all $i \geq 1$. Since M is compact, Qf must have at least one critical point that

is not a critical point of f itself, for example, the maximum value of Qf. Let x_i be a sequence so that $Q^i h(x_i) \neq 0$ and x_i is a critical point of $Q^i h$. By construction $Q^i h(x_i) \neq 0$ and $Q^j h(x_i) = 0$ for all $j > i$. Thus, we observe that the matrix $A_{ij} = Q^i h(x_j)$ is upper triangular and nonzero on the diagonal, hence nonsingular. Since each $H_i = Q^i h$ lives in the estimation algebra, this shows that the image of $\mathcal{E}$ under the map $f \mapsto (f(x_1), \ldots, f(x_n)) \in \mathbb{R}^n$ spans the image for any n, so $\mathcal{E}$ must be infinite-dimensional. $\qquad\square$

Remark 11. In Euclidean space, Ocone showed that any functions h in a finite-dimensional estimation algebra are polynomials of order ≤ 2. If h is a polynomial of degree ≤ 2, then $Q^i h$ is still a polynomial of degree ≤ 2, but the polynomial degree increases arbitrarily for h a polynomial of degree ≥ 3. This observation was used by Ocone to rule out these polynomial cases in the course of proving that any observation function on Euclidean space must be a polynomial of degree ≤ 2 [35, Theorem 5.3]. The result of [36] provides a criterion for the nonexistence of FDEA in $\mathbb{R}^n$. Our work extends this conclusion to manifolds. While the criterion are analogous, our proof relies on critical point analysis rather than stochastic analysis.

With a bit more work, the idea above generalizes to Theorem 2.

Theorem 12. *Let M be a Riemannian manifold (not necessarily compact) and let $h \colon M \to \mathbb{R}$ be an observation function such that h has at least two distinct critical points x and y that are connected by a gradient flow along h. Then the estimation algebra generated by L_0 and h is infinite-dimensional.*

Remark 13. Examples of functions h that satisfy the setup above arise in Morse theory when there is at least one nonzero cohomology class in H^i for some $i > 0$, with any coefficients, such as the fundamental class on a compact manifold with $\mathbb{Z}/2$ coefficients. From a Morse-theoretic perspective, the above theorem can be interpreted as saying that finite-dimensional estimation algebras can only arise in situations where the observation function cannot "see" any nontrivial topology.

Proof. Let $A_h(f)$ be the operator $\langle \mathrm{grad} h, \mathrm{grad} f \rangle$. We claim that the sequence $\{A_h(h), A_h^2(h), A_h^3(h), \ldots\}$ is linearly independent.

Let $\gamma \colon \mathbb{R} \to M$ be a parametrization of the gradient flow so that $\gamma(t)$ satisfies the differential equation

$$
\begin{aligned}
\gamma'(t) &= (\mathrm{grad} h)(\gamma(t)) \\
\lim_{t \to a} \gamma(t) &= x \\
\lim_{t \to b} \gamma(t) &= y.
\end{aligned}
\tag{14}
$$

We may assume that there are no critical points along the interior of the path $\gamma(t)$. Therefore, $A_h(h)(\gamma(t)) > 0$ for all t and approaches 0 at $\pm\infty$. Therefore, it must have a maximum. We also observe that

$$
\begin{aligned}
\frac{d}{dt} A_h^n h(\gamma(t)) &= d(A_h^n(h))(\gamma'(t)) \\
&= \langle \mathrm{grad} A_h^n(h), \mathrm{grad} h \rangle \\
&= A_h^{n+1}(h)(\gamma(t)).
\end{aligned}
\tag{15}
$$

If the estimation algebra were finite-dimensional, then there would be a linear dependency of the form $\sum_{i=1}^{N} c_i A_h^i(h) = 0$ for sufficiently large n and at least one nonzero coefficients $c_N \neq 0$. This would imply that $f(t) = A_h(h)(\gamma(t))$ satisfies a nontrivial linear ordinary differential equation in t

$$c_0 f(t) + c_1 f'(t) + c_2 f''(t) + \cdots + c_N f^{(N)}(t) = 0, \quad \forall t \in \mathbb{R}.$$

with constant coefficients and $c_N \neq 0$. But every such solution is a linear combination of the form $C_i P_i(t) e^{\lambda_i t}$ for some complex C_i, λ_i and polynomials P_i. However, no linear combination can satisfy $\lim_{t \to \pm\infty} f(t) = 0$ at both $\pm\infty$. This is a contradiction, so $\mathcal{E}$ cannot be finite-dimensional. $\qquad\square$

Many researchers have focused on extending linear Gaussian filtering techniques to handle nonlinear filter problem of Euclidean space, including methods such as the Extended Kalman Filter, Unscented Kalman Filter, and Ensemble Kalman Filter. All of these algorithms are based on the Kalman filter framework. However, our proof results reveal that a true "Kalman filter" does not exist on closed Riemannian manifolds. Although several effective Kalman-type filtering methods have been extended to manifold settings, this finding indicates that alternative approaches [28], [29], such as Particle Filters and methods for solving the DMZ equation, offer significant potential for addressing filtering problems on manifolds.

V. Conclusion

In this paper, we have explored the problem of finite-dimensional filtering on closed Riemannian manifolds. Our main result establishes the nonexistence of finite-dimensional estimation algebras for stochastic systems defined on compact manifolds. This finding extends prior work on finite-dimensional filters, particularly in Euclidean spaces. Our conclusion can be extended to general non-compact manifolds under appropriate conditions. Our work demonstrates that a finite-dimensional estimation algebra structure can only exist when the observation function is unable to detect the nontrivial topological structure of the manifold.

References

[1] R. E. Kalman, "A new approach to linear filtering and prediction problem," *Journal of Fluids Engineering*, vol. 82, pp. 35–45, 1960.

[2] R. E. Kalman and R. S. Bucy, "New results in linear filtering and prediction theory," *Journal of Fluids Engineering*, vol. 83, pp. 95–108, 1961.

[3] T. E. Duncan, "Probability densities for diffusion processes with applications to nonlinear filtering theory and detection theory," Stanford University, Stanford, CA, USA, Technical Report TR-7001-4, May 1967.

[4] R. E. Mortensen, "Optimal control of continuous time stochastic systems," Ph.D. dissertation, University of California, Berkeley, Berkeley, CA, USA, August 1966, electronic Research Laboratory.

[5] M. Zakai, "On the optimal filtering of diffusion processes," *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, vol. 11, no. 3, pp. 230–243, 1969.

[6] R. W. Brockett and J. M. C. Clark, "The geometry of the conditional density function," in *Analysis and Optimization of Stochastic Systems*. New York: Academic Press, 1980, pp. 299–309.

[7] R. W. Brockett, "Nonlinear systems and nonlinear estimation theory," in *Stochastic Systems: The Mathematics of Filering and Identification and Applications*. Dordrecht: Reidel, 1981, pp. 479–504.

[8] S. K. Mitter, "On the analogy between mathematical problems of nonlinear filtering and quantum physics," *Ricerche Automat.*, vol. 10, pp. 163–216, 1979.

[9] S. S.-T. Yau, "Recent results on nonlinear filtering: New class of finite dimensional filters," in *Proc. 29th Conf. Dec. Contr. at Honolulu, Hawaii*, 1990, pp. 231–233.

[10] S. S.-T. Yau and A. Rasoulian, "Classification of four-dimensional estimation algebras," *IEEE Trans. on Automat. Contr.*, vol. 44, pp. 2312–2318, 1999.

[11] X. Wu, S. S.-T. Yau, and G. Q. Hu, "Finite dimensional filters with nonlinear drift XII: Linear and constant structure of ω," in *Stochastic Theory and Control, LNCIS 280, Springer-Verlag*, Berlin Heidelberg, 2002, pp. 7507–7518.

[12] S. S.-T. Yau, "Complete classification of finite-dimensional estimation algebras of maximal rank," *International Journal of Contr.*, vol. 76, pp. 657–677, 2003.

[13] X. Wu and S. S.-T. Yau, "Classification of estimation algebras with state dimension 2," *SIAM J. Control Optim.*, vol. 45, pp. 1039–1073, 2006.

[14] J. Shi and S. S. T. Yau, "Finite dimensional estimation algebras with state dimension 3 and rank 2. I: Linear structure of wong matrix," *SIAM Journal on Control and Optimization*, vol. 55, no. 6, pp. 4227–4246, 2017.

[15] X. P. Jiao and S. S.-T. Yau, "New classes of finite dimensional filters with nonmaximal rank estimation algebra on state dimension n and linear rank $n-2$," *SIAM Journal on Control and Optimization*, vol. 58, no. 6, pp. 3413–3427, 2020.

[16] J. Kang, X. Jiao, and S. S.-T. Yau, "Finite dimensional estimation algebra for time-varying filtering system and optimal transport particle filter: A tangent flow point of view," *IEEE Transactions on Aerospace Electronic Systems*, vol. 59, no. 6, pp. 8005–8021, 2023.

[17] C. Zhang, A. Taghvaei, and P. G. Mehta, "Feedback particle filter on riemannian manifolds and matrix lie groups," *IEEE Transactions on Automatic Control*, vol. 63, no. 8, pp. 2465–2480, 2018.

[18] N. H. Nguyen and K. Doğançay, "Improved pseudolinear kalman filter algorithms for bearings-only target tracking," *IEEE Transactions on Signal Processing*, vol. 65, no. 23, pp. 6119–6134, 2017.

[19] T. E. Duncan, "Some filtering results in riemann manifolds," *Information and Control*, vol. 35, no. 3, pp. 182–195, 1977.

[20] S.-K. Ng and P. E. Caines, "Nonlinear filtering in riemannian manifolds," *IMA Journal of Mathematical Control and Information*, vol. 2, no. 1, pp. 25–36, Jan 1985.

[21] V. Solo, "On nonlinear state estimation in a riemannian manifold," in *Proceedings of the 48th IEEE Conference on Decision and Control*. Shanghai, China: IEEE, 2009, pp. 8500–8505.

[22] F. Tompkins and P. J. Wolfe, "Bayesian filtering on the stiefel manifold," in *2nd IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*. IEEE, 2007, pp. 261–264.

[23] A. Srivastava and E. Klassen, "Bayesian and geometric subspace tracking," *Advances in Applied Probability*, vol. 36, no. 1, pp. 43–56, 2004.

[24] H. Snoussi, "Particle filtering on riemannian manifolds. application to covariance matrices tracking," in *Matrix Information Geometry*. Berlin, Heidelberg: Springer, 2013, pp. 427–449.

[25] S. Hauberg, F. Lauze, and K. S. Pedersen, "Unscented kalman filtering on riemannian manifolds," *Journal of Mathematical Imaging and Vision*, vol. 46, no. 1, pp. 103–120, 2013.

[26] F. C. DE VECCHI, "Lie symmetry analysis and geometrical methods for finite and infinite dimensional stochastic differential equations," Ph.D. dissertation, Universita degli Studi di Milano, 2018.

[27] S. Albeverio and F. C. De Vecchi, "Some recent developments on lie symmetry analysis of stochastic differential equations," in *International Conference on Random Transformations and Invariance in Stochastic Dynamics*. Springer, 2019, pp. 1–24.

[28] A. Barrau and S. Bonnabel, "The invariant extended kalman filter as a stable observer," *IEEE Transactions on Automatic Control*, vol. 62, no. 4, pp. 1797–1812, 2017.

[29] R. Mahony, T. Hamel, and J.-M. Pflimlin, "Nonlinear complementary filters on the special orthogonal group," *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1203–1218, 2008.

[30] A. Bain and D. Crisan, *Fundamentals of stochastic filtering*. Springer, 2009, vol. 3.

[31] E. P. Hsu, *Stochastic analysis on manifolds*. American Mathematical Soc., 2002, no. 38.

[32] I. Gyöngy, "Stochastic partial differential equations on manifolds˜ ii. nonlinear filtering," *Potential Analysis*, vol. 6, no. 1, pp. 39–56, 1997.

[33] M. H. Davis, "On a multiplicative functional transformation arising in nonlinear filtering theory," *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, vol. 54, pp. 125–139, 1980.

[34] J. Wei and E. Norman, "On global representations of the solutions of linear differential equations as a product of exponentials," *Proceedings of the American Mathematical Society*, vol. 15, no. 2, pp. 327–334, 1964.

[35] D. Ocone, "Topics in nonlinear filtering theory," Ph.D. dissertation, Massachusetts Institute of Technology, 1980.

[36] D. Ocone and E. Pardoux, "A lie algebraic criterion for non-existence of finite dimensionally computable filters," in *Stochastic Partial Differential Equations and Applications II: Proceedings of a Conference held in Trento, Italy February 1–6, 1988.* Springer, 2006, pp. 197–204.