



Splitting-up method for Zakai equation with α -Stable Lévy noise: Convergence analysis and applications to nonlinear filtering problems[☆]

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ABSTRACT

It is well known that Zakai equation in the nonlinear filtering (NLF) governs the evolution of unnormalized density function of the states conditioned on the observation history. In this paper, the NLF is investigated in a jump-diffusive Lévy stochastic system where the state space model is based on heavy-tailed non-Gaussian α -stable Lévy process, and the observations are driven by mutually independent jump and diffusion processes. In order to deal with the NLF problem efficiently, we approximate its corresponding Zakai equation by splitting it up into two stochastic processes on discretized time intervals, which actually correspond to the prediction and updating steps, respectively, in the NLF setting. The main contribution of this paper is that theoretically the strong and weak convergence results of splitting-up Zakai equation are generalized to the NLF models with non-Gaussian α -stable Lévy states and mixed type observations. To demonstrate our theoretical analysis, correspondingly, we extend the Yau-Yau algorithm, originally proposed by the classical NLF problems under Gaussian noises to the heavy-tailed non-Gaussian α -stable Lévy process. Furthermore, the performance of our proposed algorithm is illustrated by presenting some challenging example of highly nonlinear tracking cubic sensor. The experimental results indicate that the proposed filtering algorithm yields much more efficient and accurate performance, vastly superior than that of the sequential importance resampling (SIR) particle filter.

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1. Introduction

The non-Gaussian α -stable Lévy processes, which exhibits extreme values much more frequently than that of a Gaussian model, play a significant role in many practically real-world scenarios. A principal reason for the attention that α -stable Lévy noise has been received in applications is that such noise retains important properties including self-similarity, infinite divisibility, and heavy tail. In financial applications, examples exhibiting such abrupt changes include fluctuations in stock prices or insurance gains/losses, as demonstrated in seminal studies such as Fama (1965). Further applications can be found in signal processing

(Lombardi & Godsill, 2006), image analysis (Achim et al., 2006), and a wide range of engineering fields, such as communications, referring to Azzaoui and Clavier (2010) for statistical modeling of channels, de Freitas et al. (2017) for capacity bounds, Liebherr et al. (2012) for delay bounds in networks with α -stable noise, and Shevlyakov and Kim (2006) for signal detection. In numerous applications, such as biology, physics, target tracking and weather forecast, the nonlinear filtering (NLF) problems in stochastic systems are driven by Wiener process which have been studied extensively. In Kushner (1967), Kushner for the first time derived the evolutionary equation for the posterior distribution of the states conditioned on the observation history, which is called the Kushner–Stratonovich (K–S) equation. Since the K–S equation is a non-local stochastic partial differential equation (SPDE), generally, it cannot be solved analytically, or even numerically, due to its computational complexity. Under some suitable change of probability measure, the unnormalized conditional distribution of the states satisfies a so-called Zakai equation (Zakai, 1969), and it is preferred for computations.

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The NLF problem with Lévy processes is of great significance in modern signal processing, credit risk models, mathematical finance, insurance, and beyond. More attentions have been paid to the filtering problem involving heavy-tailed α -stable Lévy process, especially their corresponding Zakai equation and K-S equation. The study of NLF problems driven by Lévy processes has emerged as a highly popular topic in theoretical research. In Brandis and Proske (2004), the Zakai equation of the NLF problem with mixed observation processes, modeled by a Brownian motion and a generalized Cox process was studied. Later on, the derivation has been extended to the signal processes with jumps (Mandrekar et al., 2011). Ceci and Colaneri (2012), Ceci and Colaneri (2014) considered the NLF problem of a jump diffusion signal process and a correlated jump diffusion observation process in the scalar case. Specifically, the K-S equation and the Zakai equation have been derived by the innovation method in Ceci and Colaneri (2012) and by the change of measure in Ceci and Colaneri (2014), respectively. Qiao and Duan (2015) combined Itô-Lévy diffusion signal processes (Papa & Sritharan, 2009) with mixed observations (Brandis & Proske, 2004). Fernando and Hausenblas (Fernando & Hausenblas, 2017) investigated the NLF model with correlated point processes for the signal and observation, and the well-posedness of corresponding Zakai equation was established by providing sufficient conditions.

There have been a few theoretical studies on the NLF problems driven by the Lévy process. Unfortunately, the literature reflects a dearth of numerical studies with notable achievements. In Frey et al. (2013), they studied Galerkin approximations applied to the Zakai equation of stochastic filtering and solved it numerically by using Hermite polynomials, whose idea is originated from the Galerkin approximations by Germani and Piccioni (1987), Ahmed and Radaideh (1997), and the splitting-up algorithm by Bensoussan et al. (1990). Liu (2007) in his dissertation discussed for the case of pure Poisson point observation process. He derived a numerical recursive scheme by mimicking the S^3 -algorithm in Lototsky et al. (1997).

While for stochastic systems driven by Wiener process, one of the extensively studied numerical approximations for the Zakai equation is the splitting-up method. The splitting-up way for Zakai equation is not unique, and the convergence of various splitting-up ways is a prerequisite for the design of the corresponding implementable algorithms. In terms of convergence analysis and error estimates, most of these studies are concerned with approximating the solution of the undiscretized Zakai equation, or the temporally discretized splitting-up stochastic differential equation (SDE) and PDE system to the exact solution of the Zakai equation. For instance, the authors in Bensoussan et al. (1990) obtained the strong convergence of the undiscretized solution of the splitting system to the exact solution of the Zakai equation. In Gyöngy and Krylov (2003), the half order convergence was proved for the numerical solution of the temporally discretized splitting-up SDE-PDE system. For spatial semi-discretization by using the spectral methods to solve the Zakai equation, error estimates in Dong et al. (2021), Luo and Yau (2013) were obtained, and a strong convergence result without error estimate was obtained in Frey et al. (2013) by using splitting-up method. In Zhang et al. (2022), the authors have derived a half order error estimate for the spatial semi-discretization and full discretization of the Zakai equation with the correlated Gaussian noise system. In Bao et al. (2018), the authors constructed a first order finite difference algorithm to solve the NLF problems for the backward doubly SDE system with Gaussian noise.

In this paper, our goal is to approximately handle the stochastic dynamical system whose states are driven by the heavy-tailed α -stable Lévy noise, while the mixed-type observation processes

consist of the diffusion process and point-process. Actually, this kind of model has not yet seen the development of any filtering algorithm to our best knowledge. In order to deal with this NLF problem efficiently, we combined Galerkin spectral approximation with splitting-up method to solve the corresponding Zakai equation. The main contributions of this work are as follows.

- ★ The corresponding Zakai equation is decomposed into two processes on discretized time intervals, where the one evolves from a deterministic second-order PDE, and the other one is modeled by a fractional order SPDE. These two splitting-up processes propagate alternatively.
- ★ We have derived the weak and strong convergence results for the semi-discretization of the Zakai equation by using the splitting-up technique. To the best of our knowledge, this is the first convergence analysis result for a semi-discretization of the numerical solution for the Zakai equation based on the splitting-up algorithm.
- ★ We extend the Yau-Yau algorithm to the stochastic system disturbed by heavily-tailed Lévy noise, which consists of two computational parts corresponding to the two splitting-up processes. And we divide the filtering algorithm into the on- and off-line computations. Specifically, the Legendre Galerkin spectral method is applied to solve the off-line part of the splitting-up Zakai equation, and the on-line computation is updated by using the Euler-Maruyama (EM) method when the new observation arrives at. While the EM method is fast to implement, it can become quite unstable if the time step is relatively large. This difficulty can be overcome with the splitting-up method.
- ★ Furthermore, demonstrated by some numerical experiments and compared with the SIR particle filter, our filtering algorithm has good performance on both efficiency and accuracy in the NLF problem.

The paper is organized as follows: In Section 2, some notations and Zakai equation associated with the stochastic model we considered are presented. Section 3 is devoted to the appropriate splitting-up approximation for Zakai equation and the necessary apriori estimates in suitable functional spaces. In Section 4, the main results including weak and strong convergence are shown rigorously. The simulations for NLF problem by the Legendre Galerkin spectral approximation and extended on- and off-line filtering algorithm are investigated in Section 5. Finally, we arrive at the conclusions.

2. Preliminary

2.1. Notations

1. Define $L^p_{ul}(\mathbb{R}^d)$, $1 \leq p < \infty$ as the uniformly local L^p -integrable space consisting of functions such that $\|f\|_{L^p_{ul}} := \sup_{x \in \mathbb{R}^d} \left(\int_{|x-y| < 1} |f(y)|^p dy \right)^{1/p} < \infty$. $C_c^k(\mathbb{R}^d)$ stands for the space of all k -times differentiable functions on $\mathbb{R}^d$ with compact support.
2. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of all rapidly decreasing real-valued C^∞ functions on $\mathbb{R}^d$. The space of all tempered distributions on $\mathbb{R}^d$ denoted as $\mathcal{S}'(\mathbb{R}^d)$ is the dual space to $\mathcal{S}(\mathbb{R}^d)$. And further, let $\hat{f}$ and $\check{f}$ (or $\mathcal{F}^{-1}$) be the Fourier transform and Fourier inverse transform of $f \in \mathcal{S}'(\mathbb{R}^d)$, respectively. That is, $\hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) dx$, and $\check{f}(x) = (\mathcal{F}^{-1}f)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\xi \cdot x} \hat{f}(\xi) d\xi$.
3. We introduce the Sobolev space $\mathbb{H}^\gamma(\mathbb{R}^d)$, which is the completion of $\mathcal{S}'(\mathbb{R}^d)$ with respect to the norm $\|f\|_{\mathbb{H}^\gamma(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} \langle \xi \rangle^{2\gamma} |\hat{f}(\xi)|^2 d\xi \right)^{1/2}$, for any $\gamma \in \mathbb{R}$, where $\langle \xi \rangle := (1 +$

$|\xi|^2)^{1/2}$, i.e. $\mathbb{H}^\gamma(\mathbb{R}^d) := \{f \in S'(\mathbb{R}^d) : \|f\|_{\mathbb{H}^\gamma(\mathbb{R}^d)} < \infty\}$, for $\forall \gamma \in \mathbb{R}$. In particular, $\mathbb{H}^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. Obviously, $\mathbb{H}^\gamma(\mathbb{R}^d)$ is a Hilbert space. It is not hard to obtain that $\forall \gamma \geq 0$, $\mathbb{H}^\gamma(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : |\xi|^\gamma \hat{f} \in L^2(\mathbb{R}^d)\}$. And further, if one considers $\gamma \in (0, 1)$, then we have

$$\mathbb{H}^\gamma(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \frac{|f(x) - f(y)|}{|x - y|^{\frac{d}{2} + \gamma}} \in L^2(\mathbb{R}^d \times \mathbb{R}^d)\}.$$

5. In the sequel, we use ω to represent randomness, the probability measure $\mathbb{P}^\omega$, and the expectation under $\mathbb{P}^\omega$ is denoted as $\mathbb{E}^\omega$. For fixed $T > 0$, the Hilbert space $L^2_\omega(0, T; X)$, which is the space of all stochastic processes $\phi(\cdot, t, \omega)$ in a separable Banach space X such that $\mathbb{E}^\omega \int_0^T \|\phi(x)\|_X^2(t, \omega) dt < \infty$. $L^2(\Omega, \mathcal{F}, \mathbb{P}^\omega; X)$ is the space with $\mathbb{E}^\omega \|\phi(x)\|_X^2(t) < \infty$, for a.e. $t \in [0, T]$. $L^2(\Omega, \mathcal{F}, \mathbb{P}^\omega; C(0, T; X))$ is the space of functions taking values in X such that $\mathbb{E}^\omega \sup_{0 \leq t \leq T} \|\phi(x)\|_X^2(t) < \infty$.
6. The notation for $q(t - 0)$ means the left limit of $q(t)$, i.e., $\lim_{s \nearrow t} q(s) := q(t - 0)$. And $X_{t-} := \lim_{s \nearrow t} X(s)$.

2.2. Symmetric α -stable Lévy processes

We recall the following definition about non-Gaussian Lévy process.

Definition 1. A process $L = (L_t)_{t \geq 0}$ with $L_0 = 0$ almost surely (a.s.) is a d -dimensional Lévy process if L is right-continuous with left limit, and it has independent and stationary increments.

The characteristic function of L_t is given by $\mathbb{E}[\exp\{i\langle z, L_t \rangle\}] = \exp\{t\Psi(z)\}$, $z \in \mathbb{R}^d$, where the function $\Psi : \mathbb{R}^d \rightarrow \mathbb{C}$ is called the characteristic exponent of the Lévy process L . And for any $\circ_1, \circ_2 \in \mathbb{R}^d$, $\langle \circ_1, \circ_2 \rangle := \circ_1^T \circ_2$ stands for the inner product in the Euclidean space $\mathbb{R}^d$. By the Lévy-Khintchine formula, there exists a nonnegative-definite $d \times d$ matrix $\mathbf{Q}$, a so-called Lévy measure ν on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$, $\int_{\mathbb{R}^d \setminus \{0\}} (|u|^2 \wedge 1) \nu(du) < \infty$, and $\tilde{\gamma} \in \mathbb{R}^d$ such that

$$\begin{aligned} \Psi(z) = & i\langle z, \tilde{\gamma} \rangle - \frac{1}{2} \langle z, \mathbf{Q}z \rangle \\ & + \int_{\mathbb{R}^d \setminus \{0\}} (e^{i\langle z, u \rangle} - 1 - i\langle z, u \rangle \mathbf{1}_{|u| \leq 1}) \nu(du). \end{aligned}$$

For an d -dimensional symmetric α -stable Process denoted as L_t^α , for $\alpha \in (0, 2)$, with any $t \geq 0$, the diffusion matrix $\mathbf{Q} = 0$, the drift vector $\tilde{\gamma} = 0$, and the corresponding Lévy measure denoted as ν^α is given by $\nu^\alpha(du) = C(d, \alpha) |u|^{d+\alpha} du$, with $C(d, \alpha) := \frac{\alpha \Gamma(\frac{d+\alpha}{2})}{2^{1-\alpha} \pi^{d/2} \Gamma(1 - \frac{\alpha}{2})}$, and Γ is the Gamma function. The infinitesimal generator of L_t^α is defined in the sense of the Cauchy principal value, denoted by $\mathcal{L}_t^\alpha$ as following

$$\begin{aligned} \mathcal{L}_t^\alpha \phi(x) &= \int_{\mathbb{R}^d \setminus \{0\}} [\phi(x + u) - \phi(x) - \langle \nabla \phi(x), u \mathbf{1}_{|u| \leq 1} \rangle] \nu^\alpha(du) \\ &:= \mathbf{P.V.} \int_{\mathbb{R}^d \setminus \{0\}} [\phi(x + u) - \phi(x)] \nu^\alpha(du), \end{aligned}$$

for any $\phi \in C_c^2(\mathbb{R}^d)$, where $\mathbf{P.V.}$ stands for the Cauchy principal value, and the last two equality follows that the limit of the integral $\lim_{\epsilon \rightarrow 0} \int_{\{|u| \in \mathbb{R}^d : |x-u| > \epsilon\}} \langle \nabla \phi(x), u \mathbf{1}_{|u| \leq 1} \rangle \nu^\alpha(du)$ equals to 0 by the fact that $\langle \nabla \phi(x), u \mathbf{1}_{|u| \leq 1} \rangle$ is an odd function and ν^α is symmetric and rotation invariant.

Let $b(x, \xi)$ be a continuous function on $\mathbb{R}^d \times \mathbb{R}^d$, and define the pseudo-differential operator

$$b(x, D)\phi(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\xi \cdot x} b(x, \xi) \hat{\phi}(\xi) d\xi, \quad (1)$$

for any $\phi \in \mathcal{S}(\mathbb{R}^d)$, where $D := \frac{1}{i}(\partial_{x_1}, \dots, \partial_{x_d})$. And $b(x, \xi)$ is called the symbol of $b(x, D)$. If $\alpha \in (0, 2)$, there is an equivalent form of the $\mathcal{L}_t^\alpha$ as

$$\mathcal{L}_t^\alpha \phi(x) := -C(d, \alpha) \Lambda^\alpha \phi(x), \quad (2)$$

where

$$\Lambda^\alpha \phi(x) := \mathbf{P.V.} \int_{\mathbb{R}^d} \frac{\phi(x) - \phi(y)}{|x - y|^{d+\alpha}} dy, \quad (3)$$

with $0 < \alpha < 2$, is called the fractional Laplacian operator, whose symbol is given by letting $b(x, \xi) := |\xi|^\alpha$.

Remark 1. By Fourier transform and rotation invariant property of ν^α for $\alpha \in (0, 2)$, it is further obtained that

$$\Lambda^\alpha \phi(x) := C_0 \mathcal{F}^{-1}(|\xi|^\alpha \hat{\phi}(\xi))(x), \quad (4)$$

for every $\phi \in C_c^\infty$, where C_0 is a positive constant only depending on d and α . Thus, it yields that

$$\mathcal{L}_t^\alpha \phi(x) = -\tilde{C} \mathcal{F}^{-1}(|\xi|^\alpha \hat{\phi}(\xi))(x), \quad (5)$$

$$\text{with } \tilde{C} := \frac{\Gamma(\frac{1+\alpha}{2})\Gamma(\frac{d}{2})}{\pi^{1/2}\Gamma(\frac{d+\alpha}{2})}.$$

2.3. Models and Zakai equation

The NLF problem we study consists of an unobserved signal process X_t driven by the multiplicative symmetric α -stable process, and the mixed observation processes Z_t and N_t , where Z_t is a nonlinear continuous function of X_t with additional Gaussian noise and N_t is a doubly stochastic Poisson process, whose intensity $\lambda_t := \lambda(X_t)$ is a positive, continuous and bounded function of X_t . Precisely, in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the NLF is modeled by the following SDE:

$$\begin{cases} dX_t = f(X_t, t)dt + dL_t^\alpha, \\ dZ_t = h(X_t, t)dt + dW_t, \\ dN_t = \lambda(X_t)dt + d\tilde{N}_t, \end{cases} \quad (6)$$

where $X_t \in \mathbb{R}^d$, $Z_t \in \mathbb{R}^m$, the drift function $f(X_t, t) : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^d$ and the observation function $h(X_t, t) : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^m$ are measurable. $\{W_t\}$ is the standard Brownian motion, taking values in $\mathbb{R}^m$. L_t^α is the multiplicative symmetric α -stable process with the Lévy measure ν^α , independent of $\{W_t\}$. The increment $N_t - N_s$ with $0 \leq s \leq t$ is a Poisson random variable with distribution $\mathbb{P}(N_t - N_s = k) = \frac{(O_t - O_s)^k}{k!} e^{-(O_t - O_s)(t-s)}$, for $k = 1, 2, \dots$, where $O_u = \int_0^u \lambda(X_s) ds = \lambda(X_u) - \lambda(X_0)$, for any $u \in [0, t]$.

Moreover, let the filtration $\mathcal{F}_t := \sigma(Z_s, N_s : 0 \leq s \leq t) \subset \mathcal{F}$, $t \in [0, T]$. The initial X_0 is a square-integrable random variable with density π_0 , which is $\mathcal{F}_0$ -measurable, and independent of L_t^α , W_t , and N_t . Let $Z_0 = N_0 = 0$ for convenience. Additionally, the jumping times of the Poisson process N_t denoted by $\tau_1 < \tau_2 < \dots < \tau_{n_0}$ are stopping times with respect to the filtration $\mathcal{F}_t$, where n_0 is an integer-valued random variable. By the Doob-Meyer decomposition, N_t is a $(\mathbb{P}, \mathcal{F}_t)$ -martingale.

The aim of NLF is to find suitable ways for computing $\pi_t(dx)$ in a recursive way, where π_t is the conditional distribution of the state X_t given the observation history. That is,

$$\pi_t(\phi) := \mathbb{E}(\phi(X_t) | \mathcal{F}_t), \quad (7)$$

for any $\phi \in L^\infty(\mathbb{R}^d)$. In this subsection, we shall present the Zakai equation of (6) and state its solution's regularity. Let us assume that

$$(\text{As-1}) \quad \mathbb{E}|X_0|^2 < \infty.$$

(As-2) We assume that $f = (f_1, f_2, \dots, f_d)^T$ satisfy that $f_j \in L^1_{ul}$, $\partial_{x_i} f_j \in L^p_{ul}$, with $p > \max\{1, d/\alpha\}$, $1 \leq i, j \leq d$.

(As-3) The functions h satisfies that $h^2 \in L^p_{ul}$. The intensity function $\lambda : \mathbb{R}^d \rightarrow [\bar{\omega}_1, \bar{\omega}_2] \subset \mathbb{R}_+$ is continuous. And further, $(\lambda - 1)^2 \in L^p_{ul}$.

For any $t \in [0, T]$, with respect to the filtration $\mathcal{F}_t$,

$$\Pi_t := \prod_{\tau_n \leq t} \lambda(X_{\tau_n-}) \exp \left(\int_0^t h^T(X_s, s) dW_s + \frac{1}{2} \int_0^t h^T(X_s, s) h(X_s, s) ds - \int_0^t (\lambda(X_s) - 1) ds \right),$$

is a nonnegative square integrable martingale under the assumptions (As-1)-(As-3), according to the Novikov criterion (Protter). Define the probability measure $\mathbb{P}^\diamond$ by virtue of the Radon-Nikodym derivative $\frac{d\mathbb{P}^\diamond}{d\mathbb{P}} = \Pi_T^{-1}$, then by the Kallianpur-Stribel formula (Bain & Crisan, 2009), we have

$$\pi_t(\phi) = \frac{\mathbb{E}^\diamond[\phi(X_t) \Pi_t | \mathcal{F}_t]}{\mathbb{E}^\diamond[\Pi_t | \mathcal{F}_t]} =: \frac{(q_t, \phi)}{(q_t, 1)}, \quad (8)$$

for any $\phi \in L^\infty(\mathbb{R}^d)$, where q_t is the unnormalized conditional density function of the states. By setting $V := \mathbb{H}^{\frac{\alpha}{2}}(\mathbb{R}^d)$, with $\alpha \in (0, 2)$, $H := \mathbb{H}^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, we further denote $(\cdot, \cdot)_H$ and $(\cdot, \cdot)_V$ as the inner products in H and V , respectively. Correspondingly, $\|\cdot\|_H$ and $\|\cdot\|_V$ are defined as the norms on H and V , respectively. By the definitions of the fractional Laplacian operator Λ^α and the Hilbert space $\mathbb{H}^\gamma(\mathbb{R}^d)$, it can be obtained that

$$(\circ_1, \circ_2)_V := (\Lambda^{\alpha/2} \circ_1, \Lambda^{\alpha/2} \circ_2)_H + (\circ_1, \circ_2)_H, \\ \|\circ\|_V^2 := (\circ, \circ)_V,$$

for any $\circ, \circ_1, \circ_2 \in V$. Clearly, $\|\phi\|_V^2 := \|\Lambda^{\alpha/2} \phi\|_H^2 + \|\phi\|_H^2$. Let $V' := \mathbb{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$ represent the dual space of V , the duality pairing between V' and V is given by $\langle T, \phi \rangle_{V', V} := (\Lambda^{-\alpha/2} T, \Lambda^{\alpha/2} \phi)_H$, for any $T \in \mathbb{H}^{-\frac{\alpha}{2}}(\mathbb{R}^d)$ and $\phi \in V$. Obviously, for any $\phi \in V \subset H \subset V'$, there holds $\|\phi\|_{V'} \leq \|\phi\|_H \leq \|\phi\|_V$. Plenty of theoretical results for the stochastic dynamical systems with Lévy noises are to derive the evolution equation of the Zakai equation (Frey et al., 2013; Qiao & Duan, 2015; Wang & Duan, 2017; Xu, 2010). Accordingly, the Zakai equation satisfied by q_t for the system (6) is deduced from the result (Qiao & Duan, 2015) and thus given by

$$dq_t = \mathcal{L}^* q_t dt + q_t h^T dZ_t + (\lambda - 1) q_t - d\tilde{N}_t, \quad (9)$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L} \in L(V; V')$, with $L(V; V')$ being the space of all linear bounded operators from V into V' with the usual supremum norm, i.e.

$$\mathcal{L}\phi := f^T \nabla \phi + \mathcal{L}_t^\alpha \phi.$$

Therefore, originated from the expression (2), we know that in the sense of the Cauchy principal value, the adjoint operator $\mathcal{L}^*$ is given by

$$\mathcal{L}^* \phi(x) = -\nabla \cdot (f \phi) + \mathcal{L}_t^\alpha \phi(x)$$

$$\stackrel{\text{p.v.}}{=} -\sum_{i=1}^d \frac{\partial(f_i(x)\phi(x))}{\partial x_i} + \int_{\mathbb{R}^d} [\phi(x+z) - \phi(x)] v^\alpha(dz)$$

$$\stackrel{(2)}{=} -\nabla \cdot (f \phi) - C(d, \alpha) \Lambda^\alpha \phi(x). \quad (10)$$

The Girsanov theorem asserts that, under $\mathbb{P}^\diamond$, Z_t in (9) is a standard Brownian motion, and $\tilde{N}_t$ is a compensated Poisson process with intensity being one, and that X_t, Z_t and $\tilde{N}_t$ are mutually independent. Moreover, $\tilde{N}_t = N_t - t$ is a $\mathcal{F}_t$ -martingale under $\mathbb{P}^\diamond$. The generalized weak solution for the Zakai Eq. (9) will be defined in the Hilbert space $L^2_\omega(0, T; V)$ as follows.

Definition 2. We call $u(t, x) \in L^2_\omega(0, T; V)$ as a weak solution of Cauchy problem (9), provided with $\pi_0 \in H$, if for any $\phi \in C^\infty_c(\mathbb{R}^d)$, the below identity holds almost everywhere for $t \in [0, T]$ with respect to Lebesgue measure and almost surely with respect to the probability measure:

$$(u(t, x), \phi(x))_H = (\pi_0, \phi(x))_H + \int_0^t \langle \mathcal{L}^* u(s, x), \phi(x) \rangle_{V', V} ds \\ + \int_0^t (u(s, x) h^T, \phi(x))_H dZ_s \\ + \int_0^t ((\lambda(x) - 1) u(s, x), \phi(x))_H d\tilde{N}_s. \quad (11)$$

And there exists a positive number M depending on the coefficients and T, d, γ_0 , such that

$$\mathbb{E}^\diamond \sup_{0 \leq t \leq T} \|q(t)\|_H^2 + \mathbb{E}^\diamond \int_0^T \|q(t)\|_V^2 dt \leq M \|\pi_0\|_H^2.$$

Remark 2. Clearly, if (11) is valid, then $u(\cdot, x) \in C([0, T]; H)$; thus it holds for all $t \in [0, T]$.

It is a classical result of the existence and uniqueness of the solution to (9), which can be deduced by Frey et al. (2013), Wang and Duan (2017), Xu (2010).

Theorem 1 (Frey et al., 2013; Wang & Duan, 2017; Xu, 2010). Under (As-1)-(As-3), for each initial condition $\pi_0 \in H$, there exists a unique weak solution $q(x, t) \in L^2_\omega(0, T; V) \cap L^2(\mathcal{Q}, \mathcal{F}, \mathbb{P}^\diamond; C(0, T; H))$ of (9).

3. The splitting-up method for Zakai equation

Let us make the transformation $\tilde{q}(x, t) := q(x, t)e^{-\mu t}$ in (9), where μ is a positive constant to be determined later. To be brief, we still denote $\tilde{q}$ as q in the sequel. It satisfies

$$\begin{cases} dq_t = \mathcal{L}^* q_t dt - \mu q_t dt + q_t h^T dZ_t + (\lambda - 1) q_t - d\tilde{N}_t, \\ q(0) = \pi_0. \end{cases} \quad (12)$$

The parameter $\mu > 0$ will be clarified in Lemma 1, which is essential in the apriori estimates for the coercivity condition in Lemma 1.

Let $\chi(\xi) : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth function such that $\chi = 1$ for $|\xi| \geq 2$ and $\chi = 0$ for $|\xi| \leq 1$, and define its corresponding pseudo-differential operator as $\chi(D)$. We shall apply the splitting-up algorithm directly to (12), instead of (9). Let us divide $[0, T]$ into M_T equal-distant subintervals, $0 < \kappa < 2\kappa < \dots < M_T \kappa = T$, with $\kappa := T/M_T$. For conciseness of notations, we shall drop the argument x in $q_{1\kappa}, q_{2\kappa}$ and move the subscript t in $q_{1\kappa}, q_{2\kappa}$ into the parenthesis, if necessary. Thus, let us define two processes $q_{1\kappa}$ and $q_{2\kappa}$ recursively in each time interval $[r\kappa, (r+1)\kappa)$, $r = 0, 1, \dots, M_T - 1$, by

$$\begin{cases} dq_{1\kappa} = \mathcal{A}^* q_{1\kappa} dt - \frac{\mu}{2} q_{1\kappa} dt, \\ q_{1\kappa}(r\kappa) = \begin{cases} \pi_0, & \text{if } r = 0, \\ q_{2\kappa}(r\kappa - 0), & \text{if } r = 1, 2, \dots, \end{cases} \end{cases} \quad (13)$$

where

$$\begin{aligned} \mathcal{A}^*(\circ) &:= \mathcal{L}^*(\circ) + \varsigma \Lambda^\alpha(\circ) \\ &= \mathcal{L}^*(\circ) + \varsigma \Lambda^\alpha(\chi(D)\circ) + \varsigma \Lambda^\alpha((1 - \chi(D))\circ), \end{aligned} \quad (14)$$

and

$$\begin{cases} dq_{2\kappa} = \left(-\varsigma \Lambda^\alpha q_{2\kappa} - \frac{\mu}{2} q_{2\kappa} \right) dt \\ \quad + q_{2\kappa} h^T dZ_t + (\lambda - 1) q_{2\kappa}(t-) d\tilde{N}_t, \\ q_{2\kappa}(r\kappa) = q_{1\kappa}((r+1)\kappa - 0), \quad \text{for } r = 0, 1, \dots \end{cases} \quad (15)$$

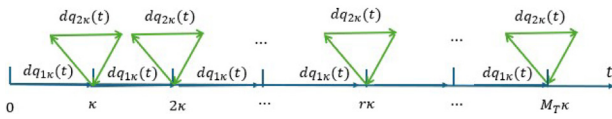


Fig. 1. The two processes $q_{1\kappa}$ and $q_{2\kappa}$ evolve recursively in all time interval $[r\kappa, (r+1)\kappa)$, for $r = 0, 1, \dots, M_T - 1$.

respectively, for some small constant $\varsigma > 0$ to be determined later.

As shown in Fig. 1, for a start point, once $q_{1\kappa}(0) = \pi_0$ is given, $q_{1\kappa}$ and $q_{2\kappa}$ in $[0, T)$ are defined completely and recursively by (13) and (15), respectively. In each time interval $[r\kappa, (r+1)\kappa)$, $q_{1\kappa}$ is deterministic, while $q_{2\kappa}$ is a stochastic process adapted to the filtration $\mathcal{F}_t$. The processes $q_{1\kappa}$ and $q_{2\kappa}$ are right-continuous at the time instants $t = \kappa, 2\kappa, \dots, M_T\kappa$. Furthermore, $q_{1\kappa}$ and $q_{2\kappa}$ are $\mathcal{F}_{T\kappa-}$ - and $\mathcal{F}_{t-}$ -measurable, for all $t \in [r\kappa, (r+1)\kappa)$, respectively, where $\mathcal{F}_{T\kappa-} := \sigma(Z_s, N_s : 0 \leq s < r\kappa)$. In each time subinterval $[r\kappa, (r+1)\kappa)$, $r = 0, 1, \dots, M_T - 1$, the existence result for (13) and (15) can be obtained similarly as in Theorem 1, and we present it in the following proposition.

Proposition 1. *Given the time discretization, under assumptions (As-1)-(As-3) and $\pi_0 \in V$, both (13) and (15) have unique solutions in $L^2_\omega(0, T; V) \cap L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; C(0, T; H))$ in each time subinterval $[r\kappa, (r+1)\kappa)$ for $r = 0, 1, \dots, M_T - 1$.*

The existence result for (13) and (15) has been established by Proposition 1, whereas, it is worthy of pointing out that the $L^2_\omega(0, T; V) \cap L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; C(0, T; H))$ norm of $q_{2\kappa}$ may depend on κ . Therefore, one cannot directly let $\kappa \rightarrow 0$ to guarantee that the limit of $q_{2\kappa}$ is still in the same space, even if the limit exists. This is why the apriori estimates in Section 3.1 is the key to the proof of the weak convergence for $q_{1\kappa}$ and $q_{2\kappa}$.

3.1. Apriori estimates

Besides (As-1)-(As-3), we need some further assumptions given in the following Lemma 1.

Lemma 1. *If the assumptions (As-1)-(As-3) hold, then there exist $\mu > 0$, $\varsigma > 0$ such that*

$$(As-4) \quad -2\langle \mathcal{A}^* \phi, \phi \rangle_{V', V} + \mu \|\phi\|_H^2 \geq \beta_1 \|\phi\|_V^2,$$

$$(As-5) \quad \mu \|\phi\|_H^2 + 2\varsigma \langle \mathcal{A}^* \phi, \phi \rangle_{V', V} - \|\phi h^T\|_H^2 - \|(\lambda - 1)\phi\|_H^2 \geq \beta_2 \|\phi\|_V^2,$$

hold for some constants $\beta_1, \beta_2 > 0$, and for all $\phi \in V$.

We shall show that the conditions (As-4) and (As-5) can be satisfied by appropriately chosen $\mu > 0$, and $\varsigma > 0$. In order not to distract readers' attentions, we put the proof of Lemma 1 in the appendix.

Thus, we have the following apriori estimates for $q_{1\kappa}$ and $q_{2\kappa}$ in Lemma 2.

Lemma 2. *For any given time discretization, the processes $q_{1\kappa}, q_{2\kappa} \in L^2_\omega(0, T; V)$ and for all $t \in [0, T)$, $q_{1\kappa}(t), q_{2\kappa}(t) \in L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; H)$, with the upper bounds of the norms independent of κ . Precisely, one has*

$$\mathbb{E}^\diamond \int_0^T \|q_{1\kappa}\|_V^2 dt \leq \beta_1^{-1} \mathbb{E}^\diamond \|\pi_0\|_H^2, \quad (16)$$

$$\mathbb{E}^\diamond \int_0^T \|q_{2\kappa}\|_V^2 dt \leq \beta_2^{-1} \mathbb{E}^\diamond \|\pi_0\|_H^2,$$

and

$$\mathbb{E}^\diamond \|q_{1\kappa}(t)\|_H^2 \leq \mathbb{E}^\diamond \|\pi_0\|_H^2, \quad \mathbb{E}^\diamond \|q_{2\kappa}(t)\|_H^2 \leq \mathbb{E}^\diamond \|\pi_0\|_H^2, \quad (17)$$

for all $t \in [0, T)$.

Proof. For any $t \in [r\kappa, (r+1)\kappa)$, one has

$$\begin{aligned} d\|q_{1\kappa}\|_H^2 &= 2\langle \mathcal{A}^* q_{1\kappa}, q_{1\kappa} \rangle_{V', V} ds - \mu \|q_{1\kappa}\|_H^2 ds \\ &\leq -\beta_1 \|q_{1\kappa}\|_V^2 ds \end{aligned} \quad (18)$$

by (As-4). Integrating (18) from $r\kappa$ to t and taking the expectation under $\mathbb{P}^\diamond$ on both sides, it yields that

$$\mathbb{E}^\diamond \|q_{1\kappa}(t)\|_H^2 - \mathbb{E}^\diamond \|q_{1\kappa}(r\kappa)\|_H^2 \leq -\beta_1 \mathbb{E}^\diamond \int_{r\kappa}^t \|q_{1\kappa}\|_V^2 ds. \quad (19)$$

Since Z_t is a standard Brownian motion and $\tilde{N}_t$ is a standard compensated Poisson process under $\mathbb{P}^\diamond$, by applying the Itô's formula to $\|q_{2\kappa}\|_H^2$, one has

$$\begin{aligned} d\|q_{2\kappa}\|_H^2 &= 2\langle dq_{2\kappa}, q_{2\kappa} \rangle_{V', V} + \langle dq_{2\kappa}, dq_{2\kappa} \rangle_H \\ &= -2\varsigma \langle \mathcal{A}^* q_{2\kappa}, q_{2\kappa} \rangle_{V', V} ds - \mu \|q_{2\kappa}\|_H^2 ds \\ &\quad + 2\langle q_{2\kappa} h^T, q_{2\kappa} \rangle_H dZ_s + 2\langle (\lambda - 1)q_{2\kappa}(s-), q_{2\kappa} \rangle_H d\tilde{N}_s \\ &\quad + \|q_{2\kappa} h^T\|_H^2 ds + \|(\lambda - 1)q_{2\kappa}\|_H^2 ds \\ &\leq -\beta_2 \|q_{2\kappa}\|_V^2 ds + 2\langle q_{2\kappa} h^T, q_{2\kappa} \rangle_H dZ_s \\ &\quad + 2\langle (\lambda - 1)q_{2\kappa}(s-), q_{2\kappa} \rangle_H d\tilde{N}_s, \end{aligned} \quad (20)$$

by (As-5). Then integrating (20) from $r\kappa$ to t and taking expectation under $\mathbb{P}^\diamond$, it leads to

$$\mathbb{E}^\diamond \|q_{2\kappa}(t)\|_H^2 - \mathbb{E}^\diamond \|q_{2\kappa}(r\kappa)\|_H^2 \leq -\beta_2 \mathbb{E}^\diamond \int_{r\kappa}^t \|q_{2\kappa}\|_V^2 ds. \quad (21)$$

Let us take $t = (r+1)\kappa - 0$ in (19) and (21), and adding them up, then we have

$$\begin{aligned} &\mathbb{E}^\diamond \|q_{1\kappa}((r+1)\kappa - 0)\|_H^2 + \mathbb{E}^\diamond \|q_{2\kappa}((r+1)\kappa - 0)\|_H^2 \\ &\quad - \mathbb{E}^\diamond \|q_{1\kappa}(r\kappa)\|_H^2 - \mathbb{E}^\diamond \|q_{2\kappa}(r\kappa)\|_H^2 \\ &\quad + \beta_1 \mathbb{E}^\diamond \int_{r\kappa}^{(r+1)\kappa} \|q_{1\kappa}\|_V^2 ds + \beta_2 \mathbb{E}^\diamond \int_{r\kappa}^{(r+1)\kappa} \|q_{2\kappa}\|_V^2 ds \leq 0. \end{aligned} \quad (22)$$

Recalling the initial condition of $q_{2\kappa}$ in (15), we have

$$\begin{aligned} &\mathbb{E}^\diamond \|q_{2\kappa}((r+1)\kappa - 0)\|_H^2 - \mathbb{E}^\diamond \|q_{1\kappa}(r\kappa)\|_H^2 \\ &\quad + \beta_1 \mathbb{E}^\diamond \int_{r\kappa}^{(r+1)\kappa} \|q_{1\kappa}\|_V^2 ds + \beta_2 \mathbb{E}^\diamond \int_{r\kappa}^{(r+1)\kappa} \|q_{2\kappa}\|_V^2 ds \leq 0, \end{aligned} \quad (23)$$

with $\beta_1, \beta_2 \geq 0$. It is clear to see that

$$\begin{aligned} &\mathbb{E}^\diamond \|q_{2\kappa}(r\kappa)\|_H^2 \stackrel{(15)}{=} \mathbb{E}^\diamond \|q_{1\kappa}((r+1)\kappa - 0)\|_H^2 \\ &\stackrel{(19)}{\leq} \mathbb{E}^\diamond \|q_{1\kappa}(r\kappa)\|_H^2 \stackrel{(13)}{=} \mathbb{E}^\diamond \|q_{2\kappa}(r\kappa - 0)\|_H^2 \\ &\stackrel{(21)}{\leq} \mathbb{E}^\diamond \|q_{2\kappa}((r-1)\kappa)\|_H^2 \stackrel{(15)}{=} \mathbb{E}^\diamond \|q_{1\kappa}(r\kappa - 0)\|_H^2, \end{aligned} \quad (24)$$

for any $r = 1, \dots, M_T - 1$. Namely, we have

$$\begin{aligned} &\mathbb{E}^\diamond \|q_{2\kappa}(T)\|_H^2 \leq \mathbb{E}^\diamond \|q_{1\kappa}(T - 0)\|_H^2 \leq \dots \\ &\leq \mathbb{E}^\diamond \|q_{2\kappa}(r\kappa)\|_H^2 \leq \mathbb{E}^\diamond \|q_{1\kappa}(r\kappa)\|_H^2 \leq \mathbb{E}^\diamond \|q_{2\kappa}(0)\|_H^2 \\ &= \mathbb{E}^\diamond \|q_{1\kappa}(\kappa - 0)\|_H^2 \leq \mathbb{E}^\diamond \|q_{1\kappa}(0)\|_H^2 = \mathbb{E}^\diamond \|\pi_0\|_H^2 \end{aligned} \quad (25)$$

by (24). Thus, it immediately leads to

$$\mathbb{E}^\diamond \|q_{1\kappa}(t)\|_H^2 \leq \mathbb{E}^\diamond \|\pi_0\|_H^2, \quad \mathbb{E}^\diamond \|q_{2\kappa}(t)\|_H^2 \leq \mathbb{E}^\diamond \|\pi_0\|_H^2, \quad (26)$$

by (25), $\mathbb{E}^\diamond \|q_{1\kappa}(t)\|_H^2 \stackrel{(19)}{\leq} \mathbb{E}^\diamond \|q_{1\kappa}(r\kappa)\|_H^2$ and $\mathbb{E}^\diamond \|q_{2\kappa}(t)\|_H^2 \stackrel{(21)}{\leq} \mathbb{E}^\diamond \|q_{2\kappa}(r\kappa)\|_H^2$.

Eq. (16) follows immediately by summing (23) over r from 0 to $M_T - 1$, it yields

$$\begin{aligned} &\mathbb{E}^\diamond \int_0^T \|q_{1\kappa}\|_V^2 \leq \beta_1^{-1} [\mathbb{E}^\diamond \|\pi_0\|_H^2 - \mathbb{E}^\diamond \|q_{1\kappa}(T)\|_H^2] \\ &\stackrel{(24), (25)}{\leq} \beta_1^{-1} \mathbb{E}^\diamond \|\pi_0\|_H^2, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}^\diamond \int_0^T \|q_{2\kappa}\|_V^2 &\leq \beta_2^{-1} [\mathbb{E}^\diamond \|\pi_0\|_H^2 - \mathbb{E}^\diamond \|q_{2\kappa}(T-0)\|_H^2] \\ &\stackrel{(24),(25)}{\leq} \beta_2^{-1} \mathbb{E}^\diamond \|\pi_0\|_H^2. \quad \blacksquare \end{aligned}$$

The Lemma 2 implies that $\mathbb{E}^\diamond \|q_{2\kappa}(t)\|_H^2$ is uniformly bounded in t , but $\mathbb{E}^\diamond \|q_{2\kappa}(t)\|_V^2$ may go to ∞ , as $\kappa \rightarrow 0$, at some $t \in [0, T]$. In the proof of weak and strong convergence, we need to control the terms like $\mathbb{E}^\diamond \int_{r\kappa}^{(r+1)\kappa} \|q_{2\kappa}(s)\|_V^2 ds < \infty$, as $\kappa \rightarrow 0$. This is an alternative control on the terms like $\mathbb{E}^\diamond \|q_{2\kappa}(t)\|_V^2$ in some sense, for almost all $t \in [0, T]$. Therefore, by introducing Lemma 3 (Luo et al., 2023), it yields the bounded results shown in Lemma 4.

Lemma 3 (Luo et al., 2023, Lemma 3.5). Assume that a sequence of nonnegative functions $\{g_\kappa(t)\} \in C([0, T])$, and $\int_0^T g_\kappa(s) ds \leq C < \infty$ uniformly in κ . For almost all $t \in [0, T]$, there exists some time discretization $0 < 1\kappa < \dots < M_T\kappa = T$, with $M_T := \lceil T/\kappa \rceil$, and $[r\kappa, (r+1)\kappa)$ is the time interval containing t , then as $\kappa \rightarrow 0$, we have

$$\int_{r\kappa}^{(r+1)\kappa} g_\kappa(s) ds \rightarrow 0.$$

Lemma 4. Assume that $q_{1\kappa}$ and $q_{2\kappa}$ are the solutions for (13) and (15), respectively, then we know that

$$\begin{aligned} (i) \quad &\mathbb{E}^\diamond \int_{r\kappa}^{(r+1)\kappa} \|q_{2\kappa}(s)\|_V^2 ds \rightarrow 0, \\ (ii) \quad &\mathbb{E}^\diamond \int_{r\kappa}^{(r+1)\kappa} \|q(s) - q_{1\kappa}(s)\|_V^2 ds \rightarrow 0, \\ &\mathbb{E}^\diamond \int_{r\kappa}^{(r+1)\kappa} \|q(s) - q_{2\kappa}(s)\|_V^2 ds \rightarrow 0, \end{aligned}$$

as $\kappa \rightarrow 0$, for all $r = 0, \dots, M_T - 1$.

Proof. The result (i) follows immediately from Lemma 3 and $\mathbb{E}^\diamond \int_0^T \|q_{2\kappa}(s)\|_V^2 ds \leq \beta_2^{-1} \mathbb{E}^\diamond \|\pi_0\|_H^2$ bounded uniformly in κ , by (16). Also, (ii) can be obtained by Lemma 3, Theorem 1 and (16). $\blacksquare$

4. Convergence

In this section, we shall investigate the weak and strong convergence of $q_{1\kappa}$ and $q_{2\kappa}$ to the solution of the Zakai Eq. (12), as the time step size κ shrinks to zero.

4.1. Weak convergence

From the apriori estimates in Lemma 2, one can extract subsequences from $q_{1\kappa}, q_{2\kappa}$. Without any confusion, we still denote them as $q_{1\kappa}, q_{2\kappa}$, such that

$$q_{1\kappa} \rightharpoonup q_1 \quad \text{and} \quad q_{2\kappa} \rightharpoonup q_2$$

in $L_\omega^2(0, T; V)$, respectively; and for almost all $t \in [0, T]$,

$$q_{1\kappa}(t) \rightharpoonup q_1(t) \quad \text{and} \quad q_{2\kappa}(t) \rightharpoonup q_2(t)$$

in $L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; H)$, where $\rightharpoonup$ denotes the weak convergence in the sense that $f_\kappa \rightharpoonup f$ in some function space $\circ$, as $\kappa \rightarrow 0$, if $\langle f_\kappa, \phi \rangle_{\circ, \circ'} \rightarrow \langle f, \phi \rangle_{\circ, \circ'}$, for any $\phi \in \circ'$.

Lemma 5. $q_{1\kappa}$ and $q_{2\kappa}$ converge weakly to the same function $\theta \in L_\omega^2(0, T; V)$, i.e. $q_1 = q_2 = \theta$. Moreover, for almost all $t \in [0, T]$, $\theta(t) = q_1(t) = q_2(t) \in L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; H)$.

Proof. For any $t \in [r\kappa, (r+1)\kappa)$, integrating (13) from t to $(r+1)\kappa$, it yields that

$$q_{1\kappa}((r+1)\kappa - 0) - q_{1\kappa}(t) - \int_t^{(r+1)\kappa} \left(\mathcal{A}^* q_{1\kappa} - \frac{\mu}{2} q_{1\kappa} \right) ds = 0, \quad (27)$$

while integrating (15) from $r\kappa$ to t , one has

$$\begin{aligned} q_{2\kappa}(t) - q_{2\kappa}(r\kappa) - \int_{r\kappa}^t \left(-\mathcal{L}^\alpha q_{2\kappa} - \frac{\mu}{2} q_{2\kappa} \right) ds \\ = \int_{r\kappa}^t q_{2\kappa} h^T dZ_s + \int_{r\kappa}^t (\lambda - 1) q_{2\kappa}(s-) d\tilde{N}_s. \end{aligned} \quad (28)$$

Summing (27) and (28) up, we have

$$\begin{aligned} q_{2\kappa}(t) - q_{1\kappa}(t) &= \int_t^{(r+1)\kappa} \left(\mathcal{A}^* q_{1\kappa} - \frac{\mu}{2} q_{1\kappa} \right) ds \\ &\quad + \int_{r\kappa}^t \left(-\mathcal{L}^\alpha q_{2\kappa} - \frac{\mu}{2} q_{2\kappa} \right) ds \\ &\quad + \int_{r\kappa}^t q_{2\kappa} h^T dZ_s + \int_{r\kappa}^t (\lambda - 1) q_{2\kappa}(s-) d\tilde{N}_s, \end{aligned} \quad (29)$$

since the initial condition of $q_{2\kappa}$ in (15). Taking the norm $\mathbb{E}^\diamond \|\cdot\|_V^2$, on both sides of (29), it gives

$$\begin{aligned} &\mathbb{E}^\diamond \|q_{2\kappa}(t) - q_{1\kappa}(t)\|_V^2 \\ &\lesssim \mathbb{E}^\diamond \left\| \int_t^{(r+1)\kappa} \left(\mathcal{A}^* q_{1\kappa} - \frac{\mu}{2} q_{1\kappa} \right) ds \right\|_{V'}^2 \\ &\quad + \mathbb{E}^\diamond \left\| \int_{r\kappa}^t \left(-\mathcal{L}^\alpha q_{2\kappa} + \frac{\mu}{2} q_{2\kappa} \right) ds \right\|_{V'}^2 \\ &\quad + \mathbb{E}^\diamond \left\| \int_{r\kappa}^t q_{2\kappa} h^T dZ_s \right\|_{V'}^2 + \mathbb{E}^\diamond \left\| \int_{r\kappa}^t (\lambda - 1) q_{2\kappa}(s-) d\tilde{N}_s \right\|_{V'}^2, \end{aligned} \quad (30)$$

by the triangle inequality. In the following, we shall estimate the right-hand side of (30) term by term:

$$\begin{aligned} &\mathbb{E}^\diamond \left\| \int_t^{(r+1)\kappa} \left(\mathcal{A}^* q_{1\kappa} - \frac{\mu}{2} q_{1\kappa} \right) ds \right\|_{V'}^2 \\ &\leq ((r+1)\kappa - t) \mathbb{E}^\diamond \int_t^{(r+1)\kappa} \left\| \mathcal{A}^* q_{1\kappa} - \frac{\mu}{2} q_{1\kappa} \right\|_{V'}^2 ds \\ &\leq ((r+1)\kappa - t) \mathbb{E}^\diamond \int_t^{(r+1)\kappa} \|q_{1\kappa}\|_V^2 ds, \end{aligned} \quad (31)$$

where the first inequality follows from $\left| \int_a^b \phi(x) dx \right|^2 \leq (b-a) \int_a^b |\phi|^2 dx$, and the last one is due to the fact that $\mathcal{A} \in L(V; V')$, under (As-1)-(As-5), and $\|\cdot\|_{V'}^2 \lesssim \|\cdot\|_H^2 \lesssim \|\cdot\|_V^2$. Argued similarly as (31), the second term can be controlled as

$$\begin{aligned} &\mathbb{E}^\diamond \left\| \int_{r\kappa}^t \left(-\mathcal{L}^\alpha q_{2\kappa} - \frac{\mu}{2} q_{2\kappa} \right) ds \right\|_{V'}^2 \\ &\lesssim (t - r\kappa) \mathbb{E}^\diamond \int_{r\kappa}^t \left(\|\mathcal{L}^\alpha q_{2\kappa}\|_H^2 + \frac{\mu}{2} \|q_{2\kappa}\|_H^2 \right) ds \\ &\lesssim (t - r\kappa) \mathbb{E}^\diamond \int_{r\kappa}^t (\|\mathcal{L}^{\alpha/2} q_{2\kappa}\|_H^2 + \|q_{2\kappa}\|_H^2) ds \\ &\lesssim (t - r\kappa) \mathbb{E}^\diamond \int_{r\kappa}^t \|q_{2\kappa}\|_V^2 ds, \end{aligned} \quad (32)$$

where the first inequality follows from $\left| \int_a^b \phi(x) dx \right|^2 \leq (b-a) \int_a^b |\phi|^2 dx$ and $\|\cdot\|_{V'}^2 \lesssim \|\cdot\|_H^2 \lesssim \|\cdot\|_V^2$, and the second inequality is due to the fact that

$$\|\mathcal{L}^\alpha q_{2\kappa}\|_H^2 \leq \|\mathcal{L}^\alpha [\chi(D) q_{2\kappa}]\|_H^2 + \|\mathcal{L}^\alpha [(1 - \chi(D)) q_{2\kappa}]\|_H^2$$

$$\begin{aligned} &\lesssim \|A^{\alpha/2} \chi(D) A^{\alpha/2} q_{2\kappa}\|_H^2 + \|A^{\alpha/2} (1 - \chi(D)) A^{\alpha/2} q_{2\kappa}\|_H^2 \\ &\lesssim \|A^{\alpha/2} q_{2\kappa}\|_H^2. \end{aligned}$$

The third term on the right-hand side of (30) becomes

$$\begin{aligned} &\mathbb{E}^\diamond \left\| \int_{r\kappa}^t q_{2\kappa} h^T dZ_s \right\|_{V'}^2 \lesssim \mathbb{E}^\diamond \left\| \int_{r\kappa}^t q_{2\kappa} h^T dZ_s \right\|_H^2 \\ &= \mathbb{E}^\diamond \int_{r\kappa}^t \|q_{2\kappa} h^T\|_H^2 ds \lesssim \mathbb{E}^\diamond \int_{r\kappa}^t \|q_{2\kappa}\|_V^2 ds, \end{aligned} \quad (33)$$

where the equality follows from the Itô isometry identity, and the last inequality holds by (As-3). Similarly as in (33), the last term of (30) can be estimated as

$$\begin{aligned} &\mathbb{E}^\diamond \left\| \int_{r\kappa}^t (\lambda - 1) q_{2\kappa}(s-) d\tilde{N}_s \right\|_{V'}^2 \\ &\lesssim \mathbb{E}^\diamond \left\| \int_{r\kappa}^t (\lambda - 1) q_{2\kappa}(s-) d\tilde{N}_s \right\|_H^2 \\ &= \mathbb{E}^\diamond \int_{r\kappa}^t \|(\lambda - 1) q_{2\kappa}(s-)\|_H^2 ds \lesssim \mathbb{E}^\diamond \int_{r\kappa}^t \|q_{2\kappa}\|_V^2 ds, \end{aligned} \quad (34)$$

with the last inequality depending on $(\lambda - 1)^2 \in L_u^p$ and by Lemma 7. Substituting (31)–(34) back to (30), for almost all $t \in [0, T]$, one has

$$\begin{aligned} \mathbb{E}^\diamond \|q_{2\kappa}(t) - q_{1\kappa}(t)\|_{V'}^2 &\lesssim \kappa \mathbb{E}^\diamond \int_{r\kappa}^{(r+1)\kappa} \|q_{1\kappa}\|_V^2 + \|q_{2\kappa}\|_V^2 ds \\ &\quad + \mathbb{E}^\diamond \int_{r\kappa}^{(r+1)\kappa} \|q_{2\kappa}\|_V^2 ds \rightarrow 0, \end{aligned} \quad (35)$$

as $\kappa \rightarrow 0$, by Lemma 3 and (16) in Lemma 2. By Fatou's lemma, for almost all $t \in [0, T]$, we have

$$\begin{aligned} 0 &\leq \mathbb{E}^\diamond \|q_{2\kappa}(t) - q_{1\kappa}(t)\|_{V'}^2 \\ &\leq \liminf_{\kappa \rightarrow 0} \mathbb{E}^\diamond \|q_{2\kappa}(t) - q_{1\kappa}(t)\|_{V'}^2 \stackrel{(35)}{\leq} 0. \end{aligned} \quad (36)$$

Therefore, for almost all t , $q_1(t) = q_2(t) = \theta(t) \in L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; V')$. Since for almost all t , $q_{2\kappa}(t) - q_{1\kappa}(t) \rightarrow q_2(t) - q_1(t)$ in $L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; H)$, as well as in $L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; V')$. Thus for almost all t , $q_1(t) = q_2(t) = \theta(t) \in L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; H)$. Similarly as in (36), one has

$$\begin{aligned} 0 &\leq \mathbb{E}^\diamond \int_0^T \|q_2(t) - q_1(t)\|_{V'}^2 dt \\ &\leq \liminf_{\kappa \rightarrow 0} \mathbb{E}^\diamond \int_0^T \|q_{2\kappa}(t) - q_{1\kappa}(t)\|_{V'}^2 dt \stackrel{(35)}{\leq} 0, \end{aligned} \quad (37)$$

by Fatou's lemma. Thus, one deduces that $q_1 = q_2 = \theta \in L_\omega^2(0, T; V')$, and consequently also in $L_\omega^2(0, T; V)$, since $q_{2\kappa} - q_{1\kappa} \rightarrow q_2 - q_1$ in $L_\omega^2(0, T; V)$. ■

Next, our object is to show that the limit θ in the previous lemma is actually the solution of the Zakai Eq. (12), i.e. $\theta = q$ by the uniqueness.

Lemma 6. $\theta = q$, where q is the solution to the Zakai Eq. (12).

Proof. In the time interval $[\ell\kappa, (\ell + 1)\kappa]$ for $\ell \leq r$, on the one hand, taking $t = \ell\kappa$ in (27) and $t = (\ell + 1)\kappa - 0$ in (28), and summing them up, one obtains that

$$\begin{aligned} &q_{2\kappa}((\ell + 1)\kappa - 0) - q_{1\kappa}(\ell\kappa) \\ &\quad - \int_{\ell\kappa}^{(\ell+1)\kappa} \left[\mathcal{A}^* q_{1\kappa} - \varsigma A^\alpha q_{2\kappa} - \frac{\mu}{2} (q_{1\kappa} + q_{2\kappa}) \right] ds \\ &= \int_{\ell\kappa}^{(\ell+1)\kappa} q_{2\kappa} h^T dZ_s + \int_{\ell\kappa}^{(\ell+1)\kappa} (\lambda - 1) q_{2\kappa}(s-) d\tilde{N}_s. \end{aligned} \quad (38)$$

Summing (38) with respect to ℓ from 0 to $r - 1$, it yields that

$$\begin{aligned} &q_{2\kappa}(r\kappa - 0) - q_{1\kappa}(0) \\ &\quad - \int_0^{r\kappa} \left[\mathcal{A}^* q_{1\kappa} - \varsigma A^\alpha q_{2\kappa} - \frac{\mu}{2} (q_{1\kappa} + q_{2\kappa}) \right] ds \\ &= \int_0^{r\kappa} q_{2\kappa} h^T dZ_s + \int_0^{r\kappa} (\lambda - 1) q_{2\kappa}(s-) d\tilde{N}_s. \end{aligned} \quad (39)$$

On the other hand, for $t \in [r\kappa, (r + 1)\kappa]$, integrating (13) from $r\kappa$ to t , one has

$$q_{1\kappa}(t) - q_{1\kappa}(r\kappa) - \int_{r\kappa}^t \left[\mathcal{A}^* q_{1\kappa} - \frac{\mu}{2} q_{1\kappa} \right] ds = 0. \quad (40)$$

Summing (39) and (40) up, we obtain

$$\begin{aligned} &q_{1\kappa}(t) = \pi_0 \\ &\quad + \int_0^t \left[\mathcal{A}^* q_{1\kappa} - \varsigma A^\alpha q_{2\kappa} - \frac{\mu}{2} (q_{1\kappa} + q_{2\kappa}) \right] ds \\ &\quad + \int_0^t q_{2\kappa} h^T dZ_s + \int_0^t (\lambda - 1) q_{2\kappa}(s-) d\tilde{N}_s \\ &\quad - \int_{r\kappa}^t \left(-\varsigma A^\alpha q_{2\kappa} - \frac{\mu}{2} q_{2\kappa} \right) ds - \int_{r\kappa}^t q_{2\kappa} h^T dZ_s \\ &\quad - \int_{r\kappa}^t (\lambda - 1) q_{2\kappa}(s-) d\tilde{N}_s. \end{aligned} \quad (41)$$

Notice that

$$\begin{aligned} &\mathbb{E}^\diamond \left\| \int_{r\kappa}^t \left(-\varsigma A^\alpha q_{2\kappa} - \frac{\mu}{2} q_{2\kappa} \right) ds \right\|_{V'}^2 \\ &\lesssim \kappa \mathbb{E}^\diamond \int_{r\kappa}^t \left\| \varsigma A^\alpha q_{2\kappa} + \frac{\mu}{2} q_{2\kappa} \right\|_H^2 ds \\ &\lesssim \kappa \mathbb{E}^\diamond \int_{r\kappa}^t \|q_{2\kappa}\|_V^2 ds \rightarrow 0, \end{aligned} \quad (42)$$

$$\begin{aligned} &\mathbb{E}^\diamond \left\| \int_{r\kappa}^t q_{2\kappa} h^T dZ_s \right\|_{V'}^2 \lesssim \mathbb{E}^\diamond \left\| \int_{r\kappa}^t q_{2\kappa} h^T dZ_s \right\|_H^2 \\ &= \mathbb{E}^\diamond \int_{r\kappa}^t \|q_{2\kappa} h^T\|_H^2 ds \lesssim \mathbb{E}^\diamond \int_{r\kappa}^t \|q_{2\kappa}\|_V^2 ds \rightarrow 0, \end{aligned} \quad (43)$$

and

$$\begin{aligned} &\mathbb{E}^\diamond \left\| \int_{r\kappa}^t (\lambda - 1) q_{2\kappa} d\tilde{N}_s \right\|_{V'}^2 \lesssim \mathbb{E}^\diamond \left\| \int_{r\kappa}^t (\lambda - 1) q_{2\kappa} d\tilde{N}_s \right\|_H^2 \\ &= \mathbb{E}^\diamond \int_{r\kappa}^t \|(\lambda - 1) q_{2\kappa}\|_H^2 ds \lesssim \mathbb{E}^\diamond \int_{r\kappa}^t \|q_{2\kappa}\|_V^2 ds \rightarrow 0, \end{aligned} \quad (44)$$

as $\kappa \rightarrow 0$, with the same arguments in (32)–(34).

Next, we shall show that for almost all $t \in [0, T]$,

$$\begin{aligned} &q_{1\kappa}(t) \rightarrow \pi_0 + \int_0^t \left[\mathcal{A}^* \theta - \varsigma A^\alpha \theta - \mu \theta \right] ds \\ &\quad + \int_0^t \theta h^T dZ_s + \int_0^t (\lambda - 1) \theta(s-) d\tilde{N}_s \\ &\stackrel{(14)}{=} \pi_0 + \int_0^t \left[\mathcal{L}^* \theta - \mu \theta \right] ds + \int_0^t \theta h^T dZ_s \\ &\quad + \int_0^t (\lambda - 1) \theta(s-) d\tilde{N}_s \end{aligned} \quad (45)$$

in $L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; V')$, as $\kappa \rightarrow 0$. If this is true, combining with the fact that for almost all $t \in [0, T]$, $q_{1\kappa}(t) \rightarrow \theta(t)$ in $L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; V')$, by Lemma 5. It arrives at the conclusion that $\theta(t)$ satisfies (12),

i.e.

$$\begin{aligned}\theta(t) = & \pi_0 + \int_0^t [\mathcal{L}^* \theta - \mu \theta] ds + \int_0^t \theta h^T dZ_s \\ & + \int_0^t (\lambda - 1) \theta(s-) d\tilde{N}_s\end{aligned}$$

by the uniqueness of the limit in $L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; V')$. Therefore, we complete the proof that

$$\theta = q \in L^2_\omega(0, T; V) \cap L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; C(0, T; H))$$

by Theorem 1.

In the sequel, we shall focus on the proof of (45). Letting $\kappa \rightarrow 0$ on the right-hand side of (41), it is clear that for almost all $t \in [0, T]$, we have

$$\begin{aligned}\int_0^t [\mathcal{A}^* q_{1\kappa} - \mathcal{L}^\alpha q_{2\kappa} - \frac{\mu}{2}(q_{1\kappa} + q_{2\kappa})] ds \\ \rightarrow \int_0^t [\mathcal{L}^* \theta - \mu \theta] ds\end{aligned}\quad (46)$$

in $L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; V')$, since for any test function $\phi(x, t) \in L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; V)$, one has

$$\begin{aligned}\mathbb{E}^\diamond \left\langle \int_0^t [\mathcal{A}^* q_{1\kappa} - \mathcal{L}^\alpha q_{2\kappa} - \frac{\mu}{2}(q_{1\kappa} + q_{2\kappa})] ds, \phi(x, t) \right\rangle_{V', V} \\ = \mathbb{E}^\diamond \int_0^t \langle q_{1\kappa}, \mathcal{A}\phi(x, t) \rangle_{V, V'} ds \\ - \mathbb{E}^\diamond \int_0^t \langle q_{2\kappa}, \mathcal{L}^\alpha \phi(x, t) \rangle_{V, V'} ds \\ - \frac{\mu}{2} \mathbb{E}^\diamond \int_0^t \langle q_{1\kappa} + q_{2\kappa}, \phi(x, t) \rangle_{V, V'} ds \\ \rightarrow \mathbb{E}^\diamond \int_0^t \langle \theta, \mathcal{A}\phi(x, t) \rangle_{V, V'} ds \\ - \mathbb{E}^\diamond \int_0^t \langle \theta, \mathcal{L}^\alpha \phi(x, t) \rangle_{V, V'} ds - \mu \mathbb{E}^\diamond \int_0^t \langle \theta, \phi(x, t) \rangle_{V, V'} ds \\ = \mathbb{E}^\diamond \int_0^t \langle \mathcal{A}^* \theta - \mathcal{L}^\alpha \theta - \mu \theta, \phi(x, t) \rangle_{V', V} ds \\ \stackrel{(14)}{=} \mathbb{E}^\diamond \int_0^t \langle \mathcal{L}^* \theta - \mu \theta, \phi(x, t) \rangle_{V', V} ds,\end{aligned}\quad (47)$$

where the convergence follows from $q_{1\kappa}, q_{2\kappa} \rightharpoonup \theta$ in $L^2_\omega(0, T; V)$, for almost all $t \in [0, T]$. We are left to check that for almost all $t \in [0, T]$, the remaining terms on the right-hand side of (41) converge weakly to the corresponding terms in (12), i.e.

$$\int_0^t q_{2\kappa} h^T dZ_s \rightharpoonup \int_0^t \theta h^T dZ_s, \quad (48)$$

and

$$\int_0^t (\lambda - 1) q_{2\kappa}(s-) d\tilde{N}_s \rightharpoonup \int_0^t (\lambda - 1) \theta(s-) d\tilde{N}_s, \quad (49)$$

in $L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; H)$, as $\kappa \rightarrow 0$, respectively. Since (48) and (49) can be shown similarly, we shall only focus on the proof of (48).

Without loss of generality, we can extract a subsequence $[\int_0^t q_{2\kappa} h^T dZ_s] \rightharpoonup \eta$ in $L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; H)$, by the fact that

$$\begin{aligned}\mathbb{E}^\diamond \left\| \int_0^t q_{2\kappa} h^T dZ_s \right\|_H^2 &= \mathbb{E}^\diamond \int_0^t \|q_{2\kappa} h^T\|_H^2 ds \\ &\leq \mathbb{E}^\diamond \int_0^T \|q_{2\kappa}\|_V^2 ds \stackrel{(16)}{\lesssim} \mathbb{E}^\diamond |\pi_0|_H^2.\end{aligned}$$

To show that $\int_0^t [\theta h^T] dZ_s = \eta$, it is sufficient to check that, for arbitrary $\psi(x) \in H$ and $\zeta \in L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond)$, one has

$$\mathbb{E}^\diamond \left[\left(\psi, \int_0^t [\theta h^T] dZ_s \right)_H \zeta \right] = \mathbb{E}^\diamond [(\psi, \eta)_H \zeta]. \quad (50)$$

Since both η and $\int_0^t [\theta h^T] dZ_s$ are in $L^2(\Omega, \mathcal{F}_t^Z, \mathbb{P}^\diamond)$ with $\mathcal{F}_t^Z := \sigma(Z_s : 0 \leq s \leq t) \subset \mathcal{F}$, we just need to check that (50) holds for arbitrary $\zeta \in L^2(\Omega, \mathcal{F}_t^Z, \mathbb{P}^\diamond)$. It is well known that $\mathcal{Z}^1(t) := \exp \left[\int_0^t \varrho(s) dZ_s - \frac{1}{2} \int_0^t |\varrho(s)|^2 ds \right]$ generates a dense subspace of $L^2(\Omega, \mathcal{F}_t^Z, \mathbb{P}^\diamond)$, where $\varrho \in L^\infty(0, t; \mathbb{R}^m)$. Thus by Itô formula, we have

$$\begin{aligned}\mathbb{E}^\diamond \left[\left(\psi, \int_0^t [q_{2\kappa} h^T] dZ_s \right)_H \mathcal{Z}^1(t) \right] \\ = \mathbb{E}^\diamond \int_0^t ((\psi h^T) \varrho(s) \mathcal{Z}^1(s), q_{2\kappa})_H ds \\ \rightarrow \mathbb{E}^\diamond \int_0^t ((\psi h^T) \varrho(s) \mathcal{Z}^1(s), \theta)_H ds \\ = \mathbb{E}^\diamond \left[\left(\psi, \int_0^t [\theta h^T] dZ_s \right)_H \mathcal{Z}^1(t) \right]\end{aligned}\quad (51)$$

as $\kappa \rightarrow 0$, where the convergence follows from $q_{2\kappa} \rightharpoonup \theta$ in $L^2_\omega(0, T; V)$, as well as in $L^2_\omega(0, T; H)$. The term (49) can be dealt with similarly, just replacing ζ with $\mathcal{Z}^1(t) := \prod_{\tau_n \leq t} (\hat{\varrho}(\tau_n) + 1) \exp(-\int_0^t \hat{\varrho}(s) ds)$, where $\hat{\varrho}(s) \in L^\infty(0, t; \mathbb{R})$. Therefore, (45) follows immediately by substituting (46)–(51). Meanwhile, for almost all $t \in [0, T]$, $q_{1\kappa}(t) \rightharpoonup \theta(t)$ in $L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; V')$, by Lemma 5.

Thus, we arrive at the conclusion that $\theta(t)$ satisfies (12), i.e.

$$\begin{aligned}\theta(t) = & \pi_0 + \int_0^t [\mathcal{L}^* \theta - \mu \theta] ds + \int_0^t \theta h^T dZ_s \\ & + \int_0^t (\lambda - 1) \theta(s-) d\tilde{N}_s,\end{aligned}$$

by the uniqueness of the limit in $L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; V')$, since $\theta = q \in L^2_\omega(0, T; V) \cap L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; C(0, T; H))$ by Theorem 1. ■

4.2. Strong convergence

Theorem 2. Under (As-1)–(As-5), we have

$$q_{1\kappa}, q_{2\kappa} \rightarrow q \text{ in } L^2_\omega(0, T; V),$$

and for almost all $t \in [0, T]$,

$$q_{1\kappa}(t), q_{2\kappa}(t) \rightarrow q(t) \text{ in } L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; H),$$

$$q_{1\kappa}(T - 0), q_{2\kappa}(T - 0) \rightarrow q(T) \text{ in } L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; H).$$

Proof. For any $\phi(x, t)$, let us first define

$$\begin{aligned}\bar{\phi}(\phi)(t) := & -2\mathcal{L} \langle \mathcal{A}^\alpha \phi, \phi \rangle_{V', V} - \mu \|\phi\|_H^2 + \|\phi h^T\|_H^2 \\ & + \|(\lambda - 1)\phi(t-)\|_H^2,\end{aligned}\quad (52)$$

then by (As-5), we have $\bar{\phi}(\phi)(t) \leq -\beta_2 \|\phi\|_V^2$. For both (18) and (20), integrating with respect to s from $\ell\kappa$ to $(\ell + 1)\kappa$, with any $0 \leq \ell \leq r$; and integrating (18) from $(\ell + 1)\kappa$ to $(\ell + 2)\kappa$, for any $0 \leq \ell \leq r - 1$, then by taking the expectation under $\mathbb{P}^\diamond$, we have

$$\mathbb{E}^\diamond \|q_{1\kappa}((\ell + 1)\kappa - 0)\|_H^2 - \mathbb{E}^\diamond \|q_{1\kappa}(\ell\kappa)\|_H^2 \quad (53)$$

$$- \mathbb{E}^\diamond \int_{\ell\kappa}^{(\ell+1)\kappa} [2\langle \mathcal{A}^* q_{1\kappa}, q_{1\kappa} \rangle_{V', V} - \mu \|q_{1\kappa}\|_H^2] ds = 0,$$

$$\mathbb{E}^\diamond \|q_{1\kappa}((\ell + 2)\kappa - 0)\|_H^2 - \mathbb{E}^\diamond \|q_{1\kappa}((\ell + 1)\kappa)\|_H^2 \quad (54)$$

$$- \mathbb{E}^\diamond \int_{(\ell+1)\kappa}^{(\ell+2)\kappa} [2\langle \mathcal{A}^* q_{1\kappa}, q_{1\kappa} \rangle_{V', V} - \mu \|q_{1\kappa}\|_H^2] ds = 0,$$

and

$$\begin{aligned} & \mathbb{E}^\diamond \|q_{2\kappa}((\ell+1)\kappa - 0)\|_H^2 - \mathbb{E}^\diamond \|q_{2\kappa}(\ell\kappa)\|_H^2 \\ & - \mathbb{E}^\diamond \int_{\ell\kappa}^{(\ell+1)\kappa} \bar{\sigma}(q_{2\kappa})(s)ds = 0. \end{aligned} \quad (55)$$

Summing (53) and (55) up, it yields that

$$\begin{aligned} & \mathbb{E}^\diamond \|q_{1\kappa}((\ell+1)\kappa)\|_H^2 - \mathbb{E}^\diamond \|q_{1\kappa}(\ell\kappa)\|_H^2 \\ & \stackrel{(13),(15)}{=} \mathbb{E}^\diamond \int_{\ell\kappa}^{(\ell+1)\kappa} [2\langle \mathcal{A}^* q_{1\kappa}, q_{1\kappa} \rangle_{V',V} - \mu \|q_{1\kappa}\|_H^2] \\ & + \mathbb{E}^\diamond \int_{\ell\kappa}^{(\ell+1)\kappa} \bar{\sigma}(q_{2\kappa})(s)ds. \end{aligned} \quad (56)$$

And by summing (54) and (55) up, it implies that

$$\begin{aligned} & \mathbb{E}^\diamond \|q_{2\kappa}((\ell+1)\kappa)\|_H^2 - \mathbb{E}^\diamond \|q_{2\kappa}(\ell\kappa)\|_H^2 \\ & \stackrel{(13),(15)}{=} \mathbb{E}^\diamond \int_{(\ell+1)\kappa}^{(\ell+2)\kappa} [2\langle \mathcal{A}^* q_{1\kappa}, q_{1\kappa} \rangle_{V',V} - \mu \|q_{1\kappa}\|_H^2] ds \\ & + \mathbb{E}^\diamond \int_{\ell\kappa}^{(\ell+1)\kappa} \bar{\sigma}(q_{2\kappa})(s)ds. \end{aligned} \quad (57)$$

Thus, for any $t \in [r\kappa, (r+1)\kappa)$, we have

$$\begin{aligned} & \mathbb{E}^\diamond \|q_{1\kappa}(t)\|_H^2 - \mathbb{E}^\diamond \|\pi_0\|_H^2 \\ & \stackrel{(53),(56)}{=} \mathbb{E}^\diamond \int_0^t [2\langle \mathcal{A}^* q_{1\kappa}, q_{1\kappa} \rangle_{V',V} - \mu \|q_{1\kappa}\|_H^2] ds \\ & + \mathbb{E}^\diamond \int_0^{r\kappa} \bar{\sigma}(q_{2\kappa})(s)ds, \end{aligned} \quad (58)$$

and

$$\begin{aligned} & \mathbb{E}^\diamond \|q_{2\kappa}(t)\|_H^2 - \mathbb{E}^\diamond \|\pi_0\|_H^2 \\ & \stackrel{(55),(57)}{=} \mathbb{E}^\diamond \int_0^{(r+1)\kappa} [2\langle \mathcal{A}^* q_{1\kappa}, q_{1\kappa} \rangle_{V',V} - \mu \|q_{1\kappa}\|_H^2] ds \\ & + \mathbb{E}^\diamond \int_0^t \bar{\sigma}(q_{2\kappa})(s)ds, \end{aligned} \quad (59)$$

respectively.

Our strategy is to show that, for almost all $t \in [0, T)$, both

$$\begin{aligned} \mathcal{Y}_\kappa(t) &:= \mathbb{E}^\diamond \|q(t) - q_{1\kappa}(t)\|_H^2 \\ & - \mathbb{E}^\diamond \int_0^t [2\langle \mathcal{A}^*(q - q_{1\kappa}), q - q_{1\kappa} \rangle_{V',V} \\ & - \mu \|q - q_{1\kappa}\|_H^2] ds \\ & - \mathbb{E}^\diamond \int_0^{r\kappa} \bar{\sigma}(q - q_{2\kappa})(s)ds, \end{aligned} \quad (60)$$

and

$$\begin{aligned} \tilde{\mathcal{Y}}_\kappa(t) &:= \mathbb{E}^\diamond \|q(t) - q_{2\kappa}(t)\|_H^2 \\ & - \mathbb{E}^\diamond \int_0^{(r+1)\kappa} [2\langle \mathcal{A}^*(q - q_{1\kappa}), q - q_{1\kappa} \rangle_{V',V} \\ & - \mu \|q - q_{1\kappa}\|_H^2] ds \\ & - \mathbb{E}^\diamond \int_0^t \bar{\sigma}(q - q_{2\kappa})(s)ds, \end{aligned} \quad (61)$$

converge to zero, as $\kappa \rightarrow 0$, for $t \in [r\kappa, (r+1)\kappa)$. If the claim is true, then one can deduce that

$$\begin{aligned} & \mathbb{E}^\diamond \|q(t) - q_{1\kappa}(t)\|_H^2 + \beta_1 \mathbb{E}^\diamond \int_0^t \|q - q_{1\kappa}\|_V^2 ds \\ & + \beta_2 \mathbb{E}^\diamond \int_0^t \|q - q_{2\kappa}\|_V^2 ds \end{aligned} \quad (62)$$

$$\stackrel{(As-4),(As-5)}{\leq} \mathcal{Y}_\kappa(t) - \mathbb{E}^\diamond \int_{r\kappa}^t \bar{\sigma}(q - q_{2\kappa})(s)ds \rightarrow 0,$$

and

$$\begin{aligned} & \mathbb{E}^\diamond \|q(t) - q_{2\kappa}(t)\|_H^2 + \beta_1 \mathbb{E}^\diamond \int_0^t \|q - q_{1\kappa}\|_V^2 ds \\ & + \beta_2 \mathbb{E}^\diamond \int_0^t \|q - q_{2\kappa}\|_V^2 ds \end{aligned} \quad (63)$$

$$\stackrel{(As-4),(As-5)}{\leq} \tilde{\mathcal{Y}}_\kappa(t) - \beta_1 \mathbb{E}^\diamond \int_t^{(r+1)\kappa} \|q - q_{1\kappa}\|_V^2 ds \rightarrow 0,$$

as $\kappa \rightarrow 0$, since $\mathcal{Y}_\kappa(t) \rightarrow 0$, $\tilde{\mathcal{Y}}_\kappa(t) \rightarrow 0$, and $\mathbb{E}^\diamond \int_t^{(r+1)\kappa} \|q - q_{1\kappa}\|_V^2 ds \rightarrow 0$ by Lemma 3. Therefore, (62)–(63) imply that

$$q_{1\kappa}, q_{2\kappa} \rightarrow q \quad \text{in } L^2_\omega(0, T; V),$$

and for almost all $t \in [0, T)$,

$$q_{1\kappa}(t) \rightarrow q(t), \quad q_{1\kappa}(T-0) \rightarrow q(T),$$

$$q_{2\kappa}(t) \rightarrow q(t), \quad q_{2\kappa}(T-0) \rightarrow q(T),$$

in $L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; H)$.

Let us go back to show that $\mathcal{Y}_\kappa(t) \rightarrow 0$, as $\kappa \rightarrow 0$. Denote $\mathcal{Y}_\kappa(t) := \mathcal{Y}_\kappa^1(t) + \mathcal{Y}_\kappa^2(t) + \mathcal{Y}_\kappa^3(t)$, where $\mathcal{Y}_\kappa^1(t)$ contains all the terms with $\|q\|_H^2$ and $\|\mathcal{A}^{\alpha/2} q\|_H^2$ in (60), $\mathcal{Y}_\kappa^2(t)$ contains all the terms with $\|q_{1\kappa}\|_H^2$, $\|\mathcal{A}^{\alpha/2} q_{1\kappa}\|_H^2$, $\|q_{2\kappa}\|_H^2$ and $\|\mathcal{A}^{\alpha/2} q_{2\kappa}\|_H^2$, and the remaining terms are in $\mathcal{Y}_\kappa^3(t)$. We shall analyze each term $\mathcal{Y}_\kappa^i(t)$, $i = 1, 2, 3$, one-by-one. Firstly, let us consider $\mathcal{Y}_\kappa^1(t)$:

$$\begin{aligned} \mathcal{Y}_\kappa^1(t) &= \mathbb{E}^\diamond \|q\|_H^2 - \mathbb{E}^\diamond \int_0^t [2\langle \mathcal{A}^* q, q \rangle_{V',V} - \mu \|q\|_H^2] ds \\ & - \mathbb{E}^\diamond \int_0^t \bar{\sigma}(q)(s)ds + \mathbb{E}^\diamond \int_{r\kappa}^t \bar{\sigma}(q)(s)ds \\ & \stackrel{(12),(14)}{=} \mathbb{E}^\diamond \|\pi_0\|_H^2 + \mathbb{E}^\diamond \int_{r\kappa}^t \bar{\sigma}(q)(s)ds \rightarrow \mathbb{E}^\diamond \|\pi_0\|_H^2, \end{aligned} \quad (64)$$

as $\kappa \rightarrow 0$, by $\mathbb{E}^\diamond \int_{r\kappa}^t \bar{\sigma}(q)(s)ds \rightarrow 0$, as $\kappa \rightarrow 0$ due to Lemma 3. Secondly, we examine the limit $\mathcal{Y}_\kappa^2(t)$:

$$\begin{aligned} \mathcal{Y}_\kappa^2(t) &= -2\mathbb{E}^\diamond \langle q(t), q_{1\kappa}(t) \rangle_H \\ & - \mathbb{E}^\diamond \int_0^t [-2\langle \mathcal{A}^* q, q_{1\kappa} \rangle_{V',V} - 2\langle \mathcal{A}^* q_{1\kappa}, q \rangle_{V',V} \\ & + 2\mu \langle q, q_{1\kappa} \rangle_H] ds \\ & - \mathbb{E}^\diamond \int_0^t [2\zeta \langle \mathcal{A}^\alpha q, q_{2\kappa} \rangle_{V',V} + 2\zeta \langle \mathcal{A}^\alpha q_{2\kappa}, q \rangle_{V',V} \\ & + 2\mu \langle q, q_{2\kappa} \rangle_H - 2\langle qh^T, q_{2\kappa}h \rangle_H \\ & - 2((\lambda - 1)q(s-), (\lambda - 1)q_{2\kappa}(s-))_H] ds \\ & + \mathbb{E}^\diamond \int_{r\kappa}^t [2\zeta \langle \mathcal{A}^\alpha q, q_{2\kappa} \rangle_{V',V} + 2\zeta \langle \mathcal{A}^\alpha q_{2\kappa}, q \rangle_{V',V} \\ & + 2\mu \langle q, q_{2\kappa} \rangle_H - 2\langle qh^T, q_{2\kappa}h \rangle_H \\ & - 2((\lambda - 1)q(s-), (\lambda - 1)q_{2\kappa}(s-))_H] ds \\ & \rightarrow -2\mathbb{E}^\diamond \|q(t)\|_H^2 \\ & + 2\mathbb{E}^\diamond \int_0^t [2\langle \mathcal{A}^* q, q \rangle_{V',V} - \mu \|q\|_H^2] ds \\ & + 2\mathbb{E}^\diamond \int_0^t \bar{\sigma}(q)(s)ds \stackrel{(12)}{=} -2\mathbb{E}^\diamond \|\pi_0\|_H^2, \end{aligned} \quad (65)$$

as $\kappa \rightarrow 0$, by weak convergence in Lemmas 5 and 6, and

$$\begin{aligned} \mathbb{E}^\diamond \int_{r\kappa}^t & [2\zeta \langle \Lambda^\alpha q, q_{2\kappa} \rangle_{V',V} + 2\zeta \langle \Lambda^\alpha q_{2\kappa}, q \rangle_{V',V} \\ & + 2\mu(q, q_{2\kappa})_H - 2(qh^T, q_{2\kappa}h)_H \\ & - 2((\lambda - 1)q(s-), (\lambda - 1)q_{2\kappa}(s-))_H] ds \rightarrow 0 \end{aligned}$$

by Lemma 3, due to $q \in L^2_\omega(0, T; V)$, $q_{2\kappa} \in L^2_\omega(0, T; V)$, and dominated convergence theorem under assumption (As-5). At last, one has

$$\begin{aligned} \Upsilon_\kappa^3(t) &:= \mathbb{E}^\diamond \|q_{1\kappa}(t)\|_H^2 \\ &- \mathbb{E}^\diamond \int_0^t [2\langle \mathcal{A}^* q_{1\kappa}, q_{1\kappa} \rangle_{V',V} - \mu \|q_{1\kappa}\|_H^2] ds \\ &- \mathbb{E}^\diamond \int_0^{r\kappa} \mathfrak{G}(q_{2\kappa})(s) ds \stackrel{(58)}{=} \mathbb{E}^\diamond \|\pi_0\|_H^2. \end{aligned} \quad (66)$$

Therefore, we arrive at the conclusion that for almost all $t \in [0, T)$, $\Upsilon_\kappa(t) \rightarrow 0$, as $\kappa \rightarrow 0$, by (64)–(66).

Similarly, one can also obtain that for almost all $t \in [0, T)$, both $q_{2\kappa}(t) \rightarrow q(t)$ and $q_{2\kappa}(T - 0) \rightarrow q(T)$ in $L^2(\Omega, \mathcal{F}, \mathbb{P}^\diamond; H)$, by showing that $\tilde{\Upsilon}_\kappa(t)$ defined in (61) tends to 0, as $\kappa \rightarrow 0$. ■

5. Numerical solution for the Zakai equation

5.1. Legendre Galerkin approximation and extended on- and off-line algorithm

The Galerkin approximation for a SPDE essentially projects the equation to a finite-dimensional subspace. Considering that using N_t as the driver instead of $\tilde{N}_t$ will be more convenient in the design of the numerical simulation, let us rewrite the Zakai Eq. (9) as

$$dq_t = (\mathcal{L}^* - \mathcal{C})q_t dt + \mathcal{B}q_t dZ_t + \mathcal{C}q_t dN_t, \quad (67)$$

with $\mathcal{B}(\circ) := h^T \circ$, and $\mathcal{C}(\circ) := (\lambda - 1)\circ$. Formally the Galerkin approximation is defined as follows: Let $\{e_1, e_2, \dots\}$ be a basis of the Hilbert-space H , and H_n be the linear subspace spanned by $\{e_1, \dots, e_n\}$, and P_n denotes the projection from H to H_n . We define the projection of the operator $\mathcal{L}^*$ by $(\mathcal{L}^*)^{(n)} := P_n \mathcal{L}^* P_n$, the operators $\mathcal{B}^{(n)}$ and $\mathcal{C}^{(n)}$ are defined analogously. The weak form of the Galerkin approximation (67) becomes

$$\begin{cases} d(q_t^{(n)}, \phi)_H = ((\mathcal{L}^{*(n)} - \mathcal{C}^{(n)})q_t^{(n)}, \phi)_H dt \\ \quad + (\mathcal{B}^{(n)}q_t^{(n)}, \phi)_H dZ_t + (\mathcal{C}^{(n)}q_t^{(n)}, \phi)_H dN_t, \\ q_0^{(n)} = P_n \pi_0. \end{cases} \quad (68)$$

The Galerkin approximation involves solving (68) in a finite-dimensional subspace $H_n^d := H_n \bigotimes \dots \bigotimes H_n$. And we define

by P_n^d the projection from the d -dimensional Hilbert space H^d to H_n^d . Similarly as in Dong et al. (2021), the projection from H^d to V_n^d is denoted by $P_n^{d,0}$. The choice of the basis functions has a large impact on the quality of the Galerkin approximation. Let us define the univariate generalized Legendre polynomials as $\zeta_i(x) = a_i(L_i(x) - L_{i+2}(x))$, for $x \in D := [-1, 1]$, and the $L_i(x)$ is the traditional i -th Legendre polynomials for $i = 0, 1, \dots$. Consequently, the linear space spanned by the multivariate generalized Legendre polynomials is

$$\begin{aligned} V_n^d &:= \text{span}\{\zeta_{k_1}(x_1)\zeta_{k_2}(x_2)\cdots\zeta_{k_d}(x_d) : (x_1, \dots, x_d) \in D^d, \\ &0 \leq k_j \leq n - 2, j = 1, 2, \dots, d\} \subset H_n^d, \end{aligned}$$

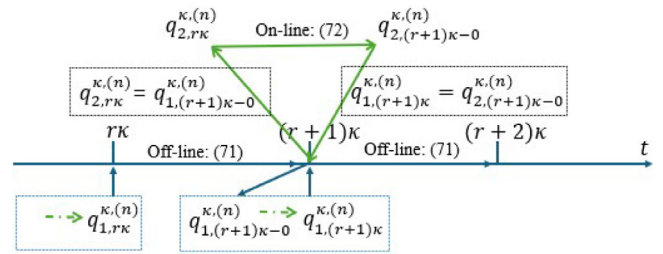


Fig. 2. The framework of the on- and off-line filtering algorithm. On the time interval $[r\kappa, (r+1)\kappa]$, once the initial condition $q_{1,r\kappa}^{(n)}$ is given, the solution $q_{1,t}^{(n)}(x) := q_1^{(n)}(t, x)$ of Eq. (70) can be obtained numerically or analytically. Then, by evaluating at $t = (r+1)\kappa$, we set $q_{1,(r+1)\kappa-0}^{(n)} := q_1^{(n)}((r+1)\kappa, x)$. Similarly, with the initial condition $q_{2,r\kappa}^{(n)} := q_{1,(r+1)\kappa-0}^{(n)}$ in hand, the solution $q_{2,t}^{(n)}(x) := q_{2\kappa}^{(n)}(t, x)$ of Eq. (71) can be solved numerically or analytically. And thus, by evaluating at $t = (r+1)\kappa$, we set $q_{2,(r+1)\kappa-0}^{(n)} := q_{2\kappa}^{(n)}((r+1)\kappa, x)$. Subsequently, by incrementing $r := r+1$, we proceed to the next time interval $[(r+1)\kappa, (r+2)\kappa]$. The initial condition for the off-line computation in the (70) is now available as $q_{1,(r+1)\kappa}^{(n)} := q_{2,(r+1)\kappa-0}^{(n)}$. This recursive procedure is repeated sequentially over successive time intervals, thereby advancing the evolution process in time.

which was first proposed in Shen (1994) to efficiently solve the second- and fourth-order equations. The solution $q_t^{(n)}$ is approximated by:

$$q_t^{(n)}(x) = \sum_{|\vec{k}|_1=0}^{n-2} J_{\vec{k}}(t) \zeta_{\vec{k}}(x), \quad (69)$$

where $\zeta_{\vec{k}}(x) \in V_n^d$ are the generalized Legendre polynomials, $|\vec{k}|_1 = \sum_{j=1}^d k_j$ with $\vec{k} = (k_1, \dots, k_d)$, k_j s are integers, and $J_{\vec{k}}(t) \in \mathbb{R}^{n-1}$ are the coefficients.

In all, in order to solve the SDE system in (68) recursively, we discretize the system in time and split (68) into off-line and on-line computations. On each time interval $\forall t \in [r\kappa, (r+1)\kappa]$, with $r = 0, 1, \dots, M_T - 1$, our aim is to numerically solve the (68) efficiently. Specifically, the establishment of the on- and off-line filtering algorithm is based on the splitting-up processes in which (67) is divided into the computations of the off-line part $q_{1,t}^{(n)}$, and on-line part $q_{2,t}^{(n)}$, respectively, starting from the initial condition $q_{1\kappa,0}^{(n)} = P_n \pi_0$. The flowchart is depicted in Fig. 2.

* Off-line computation:

$$\begin{cases} d(q_{1,t}^{(n)}, \phi)_H = ((\mathcal{L}^{*(n)} - \mathcal{C}^{(n)})q_{1,t}^{(n)}, \phi)_H dt, \\ q_{1,0}^{(n)} = P_n^{d,0} \pi_0, \\ \text{or} \\ q_{1,r\kappa}^{(n)} = P_n^{d,0} q_{2,r\kappa-0}^{(n)}, \text{ for } r = 1, \dots, M_T - 1. \end{cases} \quad (70)$$

* On-line computation:

$$\begin{cases} d(q_{2,t}^{(n)}, \phi)_H = (\mathcal{B}^{(n)}q_{2,t}^{(n)}, \phi)_H dZ_t \\ \quad + (\mathcal{C}^{(n)}q_{2,t}^{(n)}, \phi)_H dN_t, \\ q_{2,r\kappa}^{(n)} = P_n^{d,0} q_{1,(r+1)\kappa-0}^{(n)}, \text{ for } r = 0, 1, \dots, M_T - 1. \end{cases} \quad (71)$$

We sequentially apply the forward Euler method to (70) and EM method to (71), respectively, then it yields that

$$\begin{cases} (q_{1,r+1}^{(n)}, \phi)_H = (q_{1,r}^{(n)}, \phi)_H + \kappa ((\mathcal{L}^{*(n)} - \mathcal{C}^{(n)})q_{1,r}^{(n)}, \phi)_H, \\ q_{1,0}^{(n)} = P_n^{d,0} \pi_0, \\ \text{or} \\ q_{1,r}^{(n)} = P_n^{d,0} q_{2,r-0}^{(n)}, \text{ for } r = 1, \dots, M_T - 1. \end{cases} \quad (72)$$

and

$$\begin{cases} (q_{2,r+1}^{(\kappa,n)}, \phi)_H = (q_{2,r}^{(\kappa,n)}, \phi)_H \\ \quad + (B^{(n)} q_{2,r}^{(\kappa,n)}, \phi)_H (Z_{(r+1)\kappa} - Z_{r\kappa}) \\ \quad + (C^{(n)} q_{2,r}^{(\kappa,n)}, \phi)_H (N_{(r+1)\kappa} - N_{r\kappa}), \\ q_{2,r}^{(\kappa,n)} = p_n^{d,0} q_{1,(r+1)\kappa-0}^{(\kappa,n)}, \text{ for } r = 0, 1, \dots, M_T - 1. \end{cases} \quad (73)$$

where $N_{(r+1)\kappa} - N_{r\kappa}$ is the number of jumps of the standard Poisson process N_t within time interval $[r\kappa, (r+1)\kappa]$, $r = 0, \dots, M_T - 1$, and each jump quantity is approximated by $C^{(n)} q_{2,r}^{(\kappa,n)} = (\lambda(X_{r\kappa}) - 1) q_{2,r}^{(\kappa,n)}$. The iterative solutions $q_{1,r+1}^{(\kappa,n)}$ for (72) and $q_{2,r+1}^{(\kappa,n)}$ for (73), respectively are numerical approximations to $q_{1,t}^{(\kappa,n)}$ in (70) and $q_{2,t}^{(\kappa,n)}$ in (71), respectively. It needs to be pointed out that the terms $(\cdot, \cdot)_H$ in (72) and (73) are numerically computed by the Gaussian quadrature formula.

Additionally, we now consider the computational complexity of our filtering algorithm. For each time intervals $[r\kappa, (r+1)\kappa]$, with $r = 0, 1, \dots, M_T - 1$, we need to compute the off-line and on-line parts. From the expression (69), the basis number of the Galerkin Legendre spectral approximation is $(n-1)^d$. In both off-line and on-line computations, the terms $(\cdot, \cdot)_H$ are numerically computed by the Gaussian quadrature formula, if we set the Gaussian quadrature nodes at one spatial dimension as N_{GQ} , then at each time interval, the computational complexity is $\mathcal{O}((n-1)^d * N_{GQ}^d)$ and it inevitably will bear the curse of spatial dimension. Thus, when considering the whole time recursion, the computational complexity is $\mathcal{O}(M_T * (n-1)^d * N_{GQ}^d)$.

5.2. Simulations for cubic sensor

Let us consider $d = m = 1$ and the cubic sensor problem in the spacial region of $[-2.2, 2.2]$,

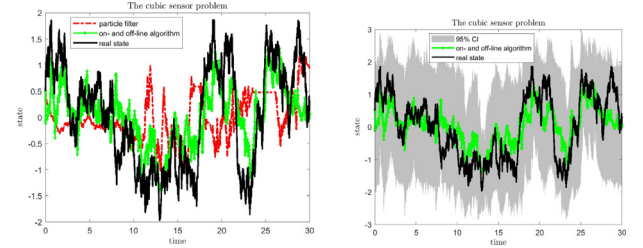
$$\begin{cases} dX_t = X_t(1 - X_t^2)dt + dL_t^\alpha, \\ dZ_t = X_t^3 dt + dW_t, \\ dN_t = \lambda_0 X_t^2 dt + d\tilde{N}_t, \end{cases} \quad (74)$$

where L_t^α and W_t are mutually independent, and W_t is standard Brownian motion. In this simulation, the total experimental time is $T = 30$ s. The initial state X_0 is sampled from $\mathcal{N}(0.5, 1)$. N_t is a doubly stochastic Poisson process with the intensity of $\lambda_0 X_t^2$. Particularly, the corresponding Zakai equation in this case is

$$\begin{aligned} dq_t = & - \frac{\partial[x(1-x^2)q_t]}{\partial x} dt - (\lambda_0 x^2 - 1)q_t dt \\ & + \int_{\mathbb{R}^1 \setminus \{0\}} [q_t(x + \xi) - q_t(x)] \nu^\alpha(d\xi) dt \\ & + x^3 q_t dZ_t + (\lambda_0 x^2 - 1)q_t dN_t. \end{aligned} \quad (75)$$

On each time interval $t \in [r\kappa, (r+1)\kappa]$, for $r = 0, \dots, M_T - 1$, furthermore, our so-called splitting-up Zakai filter is to decompose the Zakai Eq. (75) into off-line and on-line part, and to find their weak solutions $q_{1\kappa,t}^{(n)}, q_{2\kappa,t}^{(n)}$ in the space $V_n := \text{span}\{\zeta_0, \zeta_1, \dots, \zeta_{n-2}\} \subset H_n$, which is concretely illustrated by finding $q_{1\kappa,t}^{(n)}(x) := \sum_{i=0}^{n-2} J_i^{(1)}(t) \zeta_i(x) \in V_n$, and $q_{2\kappa,t}^{(n)} := \sum_{i=0}^{n-2} J_i^{(2)}(t) \zeta_i(x) \in V_n$.

In the numerical simulations, by taking a time stepsize $\kappa = 0.0146$ s, and thus $M_T := T/\kappa = 2^{11}$, and choosing $\lambda_0 = 2$, $\alpha = 3/2$, we can recursively compute the off-line and on-line. And further, we introduce the mean of the squared estimation error (MSE) for one realization, defined by $\text{MSE}(X) = \frac{1}{M_T} \sum_{r=1}^{M_T} (X_{r\kappa} - \hat{X}_{r\kappa})^2$, where M_T is the number of time steps, $X_{r\kappa}$ is the real state X at the instant $r\kappa$ and $\hat{X}_{r\kappa}$ is the estimation of $X_{r\kappa}$. All filters are implemented in MATLAB 2022 A and all examples have been performed Monto Carlo simulations.



(a) The on- and off-line filtering algorithm with $N_L = 30$ and particle filter algorithm with 3000 particles

(b) The on- and off-line filtering algorithm with 95% confidence inference

Fig. 3. The different filtering performance in highly nonlinear tracking problem with cubic sensor.

We trace a sample trajectory by using our on- and off-line filtering algorithm with the number of Legendre polynomial $N_L = 30$, compared with the SIR particle filter algorithm with 3000 particles and depict it in Fig. 3. Illustrated by Fig. 3(a), our algorithm keeps track of the real state very well, and surpasses the tracking performance of the SIR particle filter at nearly every time instant. When solving the Zakai equation by the on- and off-line filtering algorithm, it is easy to get a confidence interval at each time instant, thus we set a 95% confidence inference and depict a confidence tube alongside the state estimate of the on- and off-line filtering algorithm, shown in Fig. 3(b) with the shaded area.

On the other hand, as shown in Table 1, we take 20 runs of Monte Carlo simulations, then the averaged MSE of our proposed algorithm with $N_L = 10$ is 0.7817, while that of the SIR particle filter with 300 particles is 1.8314, and 3000 particles is 1.1932. Meanwhile, the averaged CPU time of our algorithm is 0.7817 s, while that of the SIR particle filter in 300 particles is 5.0868 s, and that in 3000 particles is 24.2724 s. This indicates that the performance of our algorithm is much more efficient and accurate than that of the particle filter. As for our algorithm, the on-line computational time for every estimation of the state is around 0.038×10^{-2} s, which is about 10^2 times less than the update time $\kappa = 0.0146$ s. This strongly declares that our algorithm is an real-time solver for the NLF problem.

By virtue of different numbers of the generalized Legendre basis in our proposed filtering algorithm, there shows the diversity of the filtering performance in both accuracy and efficiency. We take 20 runs of Monte Carlo simulations, and the averaged CPU time and MSE are illustrated in Table 1. Table 1 demonstrates that our proposed algorithm performs fairly well in all different number of generalized Legendre polynomials, and maintains the real-time property as a filtering solver.

6. Conclusions

In this article, we investigate a jump-diffusive Lévy stochastic NLF model where the states are based on heavy-tailed non-Gaussian α -stable Lévy process, while the jump-diffusive observations are involving a mixture of non-Gaussian diffusion process and Poisson point process. After deriving the corresponding Zakai equation, we construct a splitting-up scheme for the Zakai equation in this paper. By additionally introducing a fractional Laplacian operator, i.e. the term $\Lambda^\alpha(\circ)$, this paper rigorously manifests the weak and strong convergence of the Zakai equation. Furthermore, the extended on- and off-line filtering algorithm is proposed in this paper, and the Legendre Galerkin spectral method is established to numerically solve the off-line part of the Zakai equation. Additionally, compared with the SIR particle

Table 1
Performance of our proposed filter and SIR particle filter for the highly nonlinear tracking problem.

Methods	SIR particle filter		On-and off-line filter				
	300	3000	$N_L = 10$	$N_L = 15$	$N_L = 20$	$N_L = 25$	$N_L = 30$
CPU time (s)	5.0868	24.2724	0.337	0.3855	0.4751	0.6155	0.6862
MSE	1.8314	1.1932	0.7817	0.7295	0.7178	0.7026	0.6985

filter method, the proposed algorithm of this paper manifests the good performance on efficiency and accuracy particularly for highly nonlinear cubic NLF problems by presenting the numerical experiments.

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Appendix. The proof of Lemma 1

The result from Zheng and Yao (2009) will be utilized in the proof of Lemma 1, and is therefore presented below.

Lemma 7 (Zheng & Yao, 2009, Theorem 4.2). *If $w \in L^p_{ul}(\mathbb{R}^d)$, $p > \max\{1, \frac{d}{\alpha}\}$, with $\alpha \in (0, 2)$, for any $\epsilon > 0$, there exists $C(\epsilon)$ such that*

$$\left| \int_{\mathbb{R}^d} w |\phi|^2 dx \right| \leq \epsilon \int_{\mathbb{R}^d} |\Lambda^{\alpha/2} \phi|^2 dx + C(\epsilon) \int_{\mathbb{R}^d} |\phi|^2 dx, \quad (\text{A.1})$$

for all $\phi \in V$.

To this end, let us begin the proof of Lemma 1.

Proof. By the direct computations, we have

$$\begin{aligned} & - \langle \mathcal{A}^* \phi, \phi \rangle_{V', V'} \\ &= (\nabla \cdot (f \phi), \phi)_H + C(d, \alpha) (\Lambda^\alpha \phi, \phi)_H - \varsigma (\Lambda^\alpha \phi, \phi)_H \\ & \geq - \left| \int_{\mathbb{R}^d} (\nabla \cdot f) |\phi|^2 dx \right| + (C(d, \alpha) - \varsigma) \int_{\mathbb{R}^d} |\Lambda^{\alpha/2} \phi|^2 dx \\ & \stackrel{(\text{A.1})}{\geq} (C(d, \alpha) - \varsigma - \epsilon_1) \int_{\mathbb{R}^d} |\Lambda^{\alpha/2} \phi|^2 dx - C(\epsilon_1) \int_{\mathbb{R}^d} |\phi|^2 dx, \end{aligned} \quad (\text{A.2})$$

if we choose $\varsigma \leq C(d, \alpha) - \epsilon_1 - \beta_1/2$, and $\mu \geq 2C(\epsilon_1) + \beta_1$. And further,

$$\begin{aligned} & 2\varsigma \langle \Lambda^\alpha \phi, \phi \rangle_{V', V'} + \mu \|\phi\|_H^2 - \|\phi h^T\|_H^2 - \|(\lambda - 1)\phi\|_H^2 \\ &= 2\varsigma \int_{\mathbb{R}^d} |\widehat{\Lambda^{\alpha/2} \phi}(\xi)|^2 d\xi + \mu \|\phi\|_H^2 \\ & \quad - \int_{\mathbb{R}^d} (|h|^2 + |\lambda - 1|^2) |\phi|^2 dx \\ & \stackrel{(\text{A.1})}{\geq} (2\varsigma - \epsilon_2) \int_{\mathbb{R}^d} |\Lambda^{\alpha/2} \phi|^2 dx - (C(\epsilon_2) - \mu) \int_{\mathbb{R}^d} |\phi|^2 dx \\ &= (2\varsigma - \epsilon_2) \|\Lambda^{\alpha/2} \phi\|_H^2 - (C(\epsilon_2) - \mu) \|\phi\|_H^2, \end{aligned} \quad (\text{A.3})$$

if we choose that $\varsigma \geq (\beta_2 + \epsilon_2)/2$, and $\mu \geq C(\epsilon_2) + \beta_2$.

Therefore, we can choose $(\beta_2 + \epsilon_2)/2 \leq \varsigma \leq C(d, \alpha) - \epsilon_1 - \beta_1/2$, and $\mu = \max\{2C(\epsilon_1) + \beta_1, C(\epsilon_2) + \beta_2\}$, then the assumptions (As-4) and (As-5) hold. ■

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