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AN UPPER ESTIMATE OF INTEGRAL POINTS IN REAL SIMPLICES WITH AN APPLICATION TO SINGULARITY THEORY

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Abstract. Let $\Delta(a_1, a_2, \cdots, a_n)$ be an n-dimensional real simplex with vertices at $(a_1, 0, \cdots, 0), (0, a_2, \cdots, 0), \cdots, (0, 0, \cdots, a_n)$. Let $P_{(a_1,a_2,\cdots,a_n)}$ be the number of positive integral points lying in $\Delta(a_1, a_2, \cdots, a_n)$. In this paper we prove that $n!P_{(a_1,a_2,\cdots,a_n)} \leq (a_1 - 1)(a_2 - 1)\cdots(a_n - 1)$. As a consequence we have proved the Durfee conjecture for isolated weighted homogeneous singularities: $n!p_g \leq \mu$, where p_g and μ are the geometric genus and Milnor number of the singularity, respectively.

1. Introduction

Let $\Delta(a_1, a_2, \ldots, a_n)$ be an n-dimensional simplex described by

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1, \quad x_1, x_2, \ldots, x_n \geq 0, \tag{1.1}$$

where $a_1 \geq a_2 \geq \cdots \geq a_n \geq 1$ are positive real numbers. Define $P_{(a_1,a_2,\ldots,a_n)}$ and $Q_{(a_1,a_2,\ldots,a_n)}$ to be the number of positive and nonnegative integral solutions of (1.1), respectively (i.e. the number of positive and nonnegative integral points in simplex $\Delta(a_1, a_2, \ldots, a_n)$). If we let $a = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$, then $P_{(a_1,a_2,\ldots,a_n)}$ and $Q_{(a_1,a_2,\ldots,a_n)}$ are related by the following formulas:

$$Q_{(a_1,a_2,\ldots,a_n)} = P_{(a_1(1+a),a_2(1+a),\ldots,a_n(1+a))} \tag{1.2}$$

$$P_{(a_1,a_2,\ldots,a_n)} = Q_{(a_1(1-a),a_2(1-a),\ldots,a_n(1-a))}. \tag{1.3}$$

Hence, the study of $P_{(a_1,a_2,\ldots,a_n)}$ and the study of $Q_{(a_1,a_2,\ldots,a_n)}$ are equivalent. The computation of $Q_{(a_1,a_2,\ldots,a_n)}$ has received attention from many distinguished mathematicians. Hardy and Littlewood wrote several papers on the subject that have applications to problems of Diophantine approximation ([Ha–Li 1], [Ha–Li 2], [Ha–Li 3]). D. C. Spencer followed up the efforts of Hardy and Littlewood and wrote two papers on the estimation of $Q_{(a_1,a_2,\ldots,a_n)}$ as well ([Sp 1], [Sp 2]). Their results, however, are asymptotic in nature and are not useful in the applications described below. In recent years, tremendous effort has been put into finding exact formulas for $Q_{(a_1,a_2,\ldots,a_n)}$ and $P_{(a_1,a_2,\ldots,a_n)}$ where $a_1, a_2, \ldots, a_n$ are positive integers (see [Mo], [Po], [Ca–Sh], [Br–Ve], [Di–Ro], [Ka–Kh]). However, since these results are limited to integral simplices, they have no known application to number theory. Furthermore, the exact formulas involve generalized Dedekind sums or other complicated terms

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[Ba], and therefore it is difficult to determine the order of magnitude of $P_{(a_1,a_2,\ldots,a_n)}$. Ideally, we would like to get a formula for $P_{(a_1,a_2,\ldots,a_n)}$ in terms of a polynomial in $a_1, a_2, \ldots, a_n$, where $a_1, a_2, \ldots, a_n$ are not limited to integers, but can be any positive real numbers. Although such an exact formula may not exist, a relatively sharp upper estimate would suffice for the purpose of many applications in number theory and singularity theory. Barvinok and Pommersheim [Ba-Po] wrote an excellent article on topics related to lattice points in rational polyhedra. Currently the research area of lattice points in simplices is extremely active. For more information, we refer the readers to the collection "Integer Points in Polyhedra – Geometry, Number Theory, Algebra, Optimization," a Snowbird Conference Proceedings recently published by the AMS (Contemporary Mathematics, vol. 374, 2005).

According to Granville [Gr], finding an upper polynomial estimate of $P_{(a_1,a_2,\ldots,a_n)}$ is an extremely important subject in number theory. It could be applied to finding large gaps between primes, to Waring's problem, to primality testing and factoring algorithms, and to bounds for the least prime k-th power residues and non-residues (mod n). Given a set $\mathcal{P}$ of primes $p_1 < p_2 < \cdots < p_n < y$, number theorists are interested in counting the number of integers $m \le y^u$ where $m = p_1^{l_1} p_2^{l_2} \cdots p_n^{l_n}$ for all $u \ge 2$. This is equivalent to counting the number of $(l_1, l_2, \ldots, l_n) \in \mathbb{Z}^n_{\ge 0}$ such that $l_1 p_1 + l_2 p_2 + \cdots + l_n p_n \le \log y^u$, which is also equivalent to counting the number of $(l_1, l_2, \ldots, l_n) \in \mathbb{Z}^n_{\ge 0}$ such that

$$\frac{l_1}{a_1} + \frac{l_2}{a_2} + \cdots + \frac{l_n}{a_n} \le 1, \quad \text{where} \quad a_i = \frac{\log y^u}{\log p_i}. \tag{1.4}$$

Observe that the a_i's are not integral in general. For more information about applications of $P_{(a_1,a_2,\ldots,a_n)}$ and $Q_{(a_1,a_2,\ldots,a_n)}$, see Carl Pomerance's ICM 1994 lecture at Zürich [Pom 1] and his lecture notes [Pom 2].

The current method for counting $P_{(a_1,a_2,\ldots,a_n)}$ is the polynomial estimate (1.6) provided by number theorists. Attach a unit cube to the right of and above each lattice point of $\Delta(a_1, a_2, \ldots, a_n)$. Then

$$\begin{aligned} Q_{(a_1,a_2,\ldots,a_n)} &= \sum \text{volume of the unit cube attached to each lattice point} \\ &\le \text{volume of} \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^n \frac{x_1 - 1}{a_i} \le 1 \right\} \\ &= \frac{1}{n!}(a_1 a_2 \cdots a_n)(1 + \sum_{i=1}^n \frac{1}{a_i})^n. \qquad (1.5) \end{aligned}$$

In view of (1.2), (1.5) can be rewritten as

$$P_{(a_1,a_2,\ldots,a_n)} \le \frac{1}{n!}\, a_1 a_2 \cdots a_n. \tag{1.6}$$

The estimate of $P_{(a_1,a_2,\ldots,a_n)}$ given by (1.6) is interesting. However, it is not strong enough to be useful, particularly when many of the a_i's are small [Gr]. The purpose of this paper is to prove the following upper bound.

Theorem 1.1. *Let* $P_n = P_{(a_1,a_2,\ldots,a_n)} = \#\{(x_1,\ldots,x_n) \in \mathbb{Z}_+^n : \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1\}$, *where* $a_1 \geq a_2 \geq \cdots \geq a_n \geq 1$ *are real numbers. If* $n \geq 3$, *then*

$$n! \cdot P_n \leq (a_1-1)(a_2-1)\cdots(a_n-1). \tag{1.7}$$

Equality in (1.7) holds if and only if $a_n = 1$.

Several mathematicians have attempted to prove Theorem 1.1 for separate cases of n. In fact, Theorem 1.1 was proven for $n=3$ by Xu and Yau [Xu–Ya 1], for $n=4$ and 5 by Xu, Lin and Yau [Xu–Ya 2] [Li–Ya 1] [Li–Ya 2], and for $n=6$ by Wang and Yau [Wa–Ya].

In geometry and in singularity theory, Theorem 1.1 is connected with the Durfee conjecture. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a complex analytic function with an isolated critical point at the origin, and let M be a resolution of $V = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : f(z_1, z_2, \ldots, z_n) = 0\}$. The Milnor number of the singularity $(V, 0)$ is

$$\mu = \dim \mathbb{C}\{z_1, z_2, \ldots, z_n\}/(f_{z_1}, f_{z_2}, \ldots, f_{z_n}). \tag{1.8}$$

The geometric genus of the singularity $(V, 0)$ is

$$p_g = \dim H^{n-2}(M, \mathcal{O}). \tag{1.9}$$

Both μ and p_g are important invariants of the singularity $(V, 0)$. As a corollary of Theorem 1.1, we have proven the following Durfee conjecture [Du] asked in 1978.

Theorem 1.2 (Durfee conjecture)**.** *Let* $(V, 0)$ *be an isolated singularity defined by a weighted homogeneous polynomial* $f(z_1, z_2, \ldots, z_n)$. *Then* $n! \cdot p_g \leq \mu$ *and equality holds if and only if* $\mu = 0$.

The importance of the Durfee conjecture is that it gives a necessary condition for a singularity to be a hypersurface. It also gives an obstruction to embedding a strongly pseudo-convex $(2n-1)$-dimensional *CR*-manifold in $\mathbb{C}^{n+1}$.

The connection between the Durfee conjecture and the upper polynomial estimate of $P_{(a_1,a_2,\ldots,a_n)}$ in real simplices is as follows. A polynomial $f(z_1, z_2, \ldots, z_n)$ is weighted homogeneous of the type $(w_1, w_2, \ldots, w_n)$, where $w_1, w_2, \ldots, w_n$ are fixed positive rational numbers, if f can be expressed as a linear combination of monomials $z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n}$ for which $\frac{i_1}{w_1} + \frac{i_2}{w_2} + \cdots + \frac{i_n}{w_n} = 1$. If $f(z_1, z_2, \ldots, z_n)$ is a weighted homogeneous polynomial of type $(a_1, a_2, \ldots, a_n)$ with an isolated singularity at the origin, then Milnor and Orlik [Mi–Or] have proven that $\mu = (a_1-1)(a_2-1)\cdots(a_n-1)$. On the other hand, Merle and Teissier [Me–Te] showed that p_g is exactly the number P_n of positive integral points appearing in Theorem 1.1 (the Rough Estimate GLY conjecture). Therefore by proving Theorem 1.1, we have proven the Durfee conjecture.

Theorem 1.1 is the Rough Upper Estimate in the GLY conjecture [Li–Ya 3] [Wa–Ya]. The Durfee conjecture in Theorem 1.2 is not sharp although it has been an open question for more than a quarter of century. In 1995, the first author formulated the Yau conjecture (see Section 2 below), which is sharper than the Durfee conjecture. More importantly, it gives an intrinsic characterization of homogeneous singularities. In order to prove the Yau conjecture, Lin and Yau [Li–Ya 3] [Wa–Ya], independently Granville, formulated the Sharp Upper Estimate GLY conjecture (see Section 2 below). The Sharp Estimate GLY conjecture is true only if a_n is sufficiently large. Consequently, when we use induction to prove the Sharp Estimate by slicing an n-dimensional simplex along the x_n-axis into several $(n-1)$-dimensional simplices, we cannot apply the lower-dimensional Sharp Estimate conjecture to every level. The Rough Estimate is necessary in order to complete the proof of the Sharp Estimate. Hence, proving the Rough Estimate GLY conjecture is a critical first step to proving the complete GLY conjecture. We hope to address the Sharp Estimate GLY conjecture in the future paper.

2. The GLY conjecture on number of integral points and the Yau conjecture in singularity theory

The following is a thirty-year-old problem in singularity theory.

Problem: *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a complex analytic function with isolated critical point at the origin. Find an intrinsic characterization for f to be a homogeneous polynomial.*

In 1971, Saito [Sa] gave an intrinsic characterization for f to be a weighted homogeneous polynomial.

Theorem 2.1 (Saito [Sa]). *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a complex analytic function with isolated critical point at the origin. Then f is a weighted homogeneous polynomial after biholomorphic change of coordinates if and only if $\mu = \tau$, where*
$\mu = \dim \mathbb{C}\{z_1, z_2, \cdots, z_n\}/(f_{z_1}, f_{z_2}, \cdots, f_{z_n})$ *and*
$\tau = \dim \mathbb{C}\{z_1, z_2, \cdots, z_n\}/(f, f_{z_1}, f_{z_2}, \cdots, f_{z_n})$.

In order to characterize homogeneous polynomials with isolated singularity, the first author made the following conjecture in 1995.

Conjecture 2.1 (Yau Conjecture). *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin. Let μ, p_g, and ν be the Milnor number, geometric genus and multiplicity of the singularity $V = \{z : f(z) = 0\}$, then*

$$\mu - p(\nu) \geq n! p_g, \tag{2.1}$$

where $p(\nu) = (\nu-1)^n - \nu(\nu-1)\cdots(\nu-n+1)$, and equality holds if and only if f is a homogeneous polynomial.

Theorem 2.1 together with Yau conjecture will give an intrinsic characterization for a complex analytic function to be a homogeneous function after a biholomorphic change of variables. The Yau conjecture was answered affirmatively by Xu and Yau [Xu–Ya 2] for $n = 3$ and Lin and Yau [Li–Ya 4] for $n = 4$. In order to prove the Yau conjecture above, Lin, Yau [Li–Ya 3], and Granville have formulated the following GLY conjecture.

Before we state the GLY conjecture, it is convenient to introduce some notations. Recall the Stirling number of the first kind (see [Co] for more information on Stirling number):

$$s_k^{n-1} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n-1} i_1 i_2 \cdots i_k, \ S_0^{n-1} = 1, \ S_{n-1}^{n-1} = 1 \cdot 2 \cdots (n-1), \tag{2.2}$$

where $i_1, i_2, \cdots, i_k$ are integers. It has the following property:

$$\begin{aligned}
& x(x-1)(x-2)\cdots(x-n+1) \\
= \;& x^n - \left(\sum_{i=1}^{n} i_1\right) x^{n-1} + (-1)^2 \left(\sum_{1\le i_1<i_2\le n-1} i_1 i_2\right) x^{n-2} \\
& + \cdots + (-1)^k \left(\sum_{1\le i_1<i_2<\cdots<i_k\le n-1} i_1 i_2 \cdots i_k\right) x^{n-k} + \cdots + (-1)^{n-1}\left(\prod_{i=1}^{n-1} i\right) x \\
= \;& x^n + (-1)S_1^{n-1}x^{n-1} + (-1)^2 S_2^{n-1} x^{n-2} + \cdots + (-1)^k S_k^{n-1} x^{n-k} \\
& + \cdots + (-1)^{n-1} S_{n-1}^{n-1} x.
\end{aligned}$$

Let $a_1, a_2, \cdots, a_n$ be positive real numbers. We shall denote

$$\begin{aligned}
A_{n-k}^n &= \left(\prod_{i=1}^n a_i\right) \sum_{1\le i_1<i_2<\cdots<i_k\le n} \frac{1}{a_{i_1} a_{i_2}\cdots a_{i_k}}, \\
A_n^n &= \prod_{i=1}^n a_i, \quad A_0^n = 1.
\end{aligned}$$

Observe that A_{n-k}^n is a polynomial in $a_1, a_2, \cdots, a_n$ of degree $n-k$.

Conjecture 2.2 (Granville-Lin-Yau (GLY) conjecture [Li–Ya 3] [Wa–Ya])**.** *Let $P_n = P_{(a_1,a_2,\ldots,a_n)} = \#\{(x_1,\ldots,x_n) \in \mathbb{Z}_+^n : \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \le 1\}$, where $a_1 \ge a_2 \ge \cdots \ge a_n \ge 1$ are real numbers. If $n \ge 3$, then*

(I) Rough (General) Upper Estimate *For all $a_n \ge 1$,*

$$n! \cdot P_n \le (a_1-1)(a_2-1)\cdots(a_n-1). \tag{2.3}$$

Equality holds if and only if $a_n = 1$.

(II) Sharp Upper Estimate *For a_n sufficiently large: there exists an integer $\beta_n(n)$ that depends on n such that when $a_n \ge \beta_n(n)$, then*

$$\begin{aligned}
n!\cdot P_{(a_1,\ldots,a_n)} \;\le\;& A_n^n + (-1)\frac{S_1^{n-1}}{n}A_{n-1}^n + (-1)^2\frac{S_2^{n-1}}{\binom{n-1}{1}}A_{n-2}^{n-1} + (-1)^3\frac{S_3^{n-1}}{\binom{n-1}{2}}A_{n-3}^{n-1} \\
& + \cdots + (-1)^{k+1}\frac{S_{k+1}^{n-1}}{\binom{n-1}{k}}A_{n-k-1}^{n-1} + \cdots + (-1)^{n-1}\frac{S_{n-1}^{n-1}}{\binom{n-1}{n-2}}A_1^{n-1}.
\end{aligned} \tag{2.4}$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n \in \mathbb{Z}_{\ge 0}$.

The GLY conjecture was proven by Xu and Yau for $n = 3$ [Xu–Ya 1] and $n = 4$ [Xu–Ya 3], Lin and Yau for $n = 5$ [Li–Ya 2] and Wang and Yau for $3 \le n \le 6$ [Wa–Ya]. In fact, Wang and Yau's method in [Wa–Ya] can be used to prove the GLY conjecture for any fixed n. It has been checked that the GLY conjecture is true for $n \le 10$. However, it takes a long time (several weeks for $n = 10$) for computer to do the computation. The purpose of this paper is to give a proof of the Rough Estimate GLY conjecture for all n.

3. Three lemmas

Before proving Theorem 1.1, we need to establish three technical lemmas.

Lemma 3.1. *Given any positive real number β where $0 < \beta < 1$, let $a > 1$ be any number such that $\beta = a - \lfloor a \rfloor$, where $\lfloor a \rfloor$ denotes the greatest positive integer less than or equal to a. If $n \geq 3$, then*

$$a - 1 > (n+1) \sum_{k=0}^{\lfloor a \rfloor - 1} \frac{(k+\beta)^n}{a^n}. \tag{3.1}$$

Proof. We shall prove (3.1) by induction on $\lfloor a \rfloor$. Consider the expression

$$a - 1 - (n+1) \sum_{k=0}^{\lfloor a \rfloor - 1} \frac{(k+\beta)^n}{a^n}. \tag{3.2}$$

For $\lfloor a \rfloor = 1$, we have $a = 1 + \beta$ and $\lfloor a \rfloor - 1 = 0$, therefore (3.2) becomes

$$\begin{aligned} a - 1 - (n+1) \sum_{k=0}^{\lfloor a \rfloor - 1} \frac{(k+\beta)^n}{a^n} &= 1 + \beta - 1 - (n+1) \frac{\beta^n}{(1+\beta)^n} \\ &= \beta \left[1 - \frac{n\beta^{n-1} + \beta^{n-1}}{(1+\beta)^n} \right]. \end{aligned} \tag{3.3}$$

For $n \geq 2$, $(1+\beta)^n = \beta^n + n\beta^{n-1} + \cdots + 1 > n\beta^{n-1} + \beta^{n-1}$, so the right-hand-side of (3.3) is positive. To finish the proof, we need to show that if the statement of Lemma 3.1 is true for a, then it is also true for a. By the induction hypothesis, we have

$$a - 1 > (n+1) \sum_{k=0}^{\lfloor a \rfloor - 1} \frac{(k+\beta)^n}{a^n}. \tag{3.4}$$

Then

$$\begin{aligned} (a+1) - 1 - (n+1) \sum_{k=0}^{\lfloor a+1 \rfloor - 1} \frac{(k+\beta)^n}{(a+1)^n} &= a - \frac{n+1}{(a+1)^n} \left(\sum_{k=0}^{\lfloor a \rfloor - 1} (k+\beta)^n + a^n \right) \\ &> a - \frac{n+1}{(a+1)^n} \left(\frac{a^n(a-1)}{n+1} + a^n \right) \\ &= a \left[1 - \frac{a^n + na^{n-1}}{(a+1)^n} \right] \\ &> 0. \end{aligned} \tag{3.5}$$

The last inequality in (3.5) comes from $(a+1)^n = a^n + na^{n-1} + \cdots + 1 > a^n + na^{n-1}$. $\square$

Lemma 3.2. *Let $m \geq 2$ and $n \geq 3$ be positive integers, then*

$$m - 1 > (n+1) \sum_{k=1}^{m-1} \frac{k^n}{m^n}. \tag{3.6}$$

Proof. It is easy to see that Lemma 3.2 is true for $m = 2$. The proof of general $m \geq 2$ follows easily by induction. The argument is identical to that of Lemma 3.1.

Proposition 3.1. *Given any positive real number β where $0 \le \beta < 1$, let $a > 1$ be any number such that $\beta = a - \lfloor a \rfloor$, where $\lfloor a \rfloor$ denotes the greatest positive integer less than or equal to a. If $n \ge 3$, then*

$$a - 1 > (n+1) \sum_{k=0}^{\lfloor a \rfloor - 1} \frac{(k+\beta)^n}{a^n}. \tag{3.7}$$

Proof. Immediate consequence of Lemma 3.1 and Lemma 3.2.

Lemma 3.3. *Let $a_{j-1}, a_j, \dots, a_{n+1}$ be real numbers and $\beta = a_{n+1} - \lfloor a_{n+1} \rfloor$. Assume that $a_{j-1} > 1$ and $a_j \ge a_{j+1} \ge \cdots \ge a_n \ge a_{n+1} > 1$. If $\frac{a_n}{a_{n+1}}\beta \ge 1$, and*

$$\prod_{i=j}^{n+1} (a_i - 1) > (n+1) \sum_{k=0}^{\lfloor a_{n+1} \rfloor - 1} \left[\frac{(k+\beta)^{j-1}}{a_{n+1}^{j-1}} \prod_{i=j}^{n} \left(\frac{a_i}{a_{n+1}}(k+\beta) - 1 \right) \right], \tag{3.8}$$

then

$$\prod_{i=j-1}^{n+1} (a_i - 1) > (n+1) \sum_{k=0}^{\lfloor a_{n+1} \rfloor - 1} \left[\frac{(k+\beta)^{j-2}}{a_{n+1}^{j-2}} \prod_{i=j-1}^{n} \left(\frac{a_i}{a_{n+1}}(k+\beta) - 1 \right) \right]. \tag{3.9}$$

Proof. For fixed $a_j \ge a_{j+1} \ge \cdots \ge a_n \ge a_{n+1} > 1$, let

$$F(a_{j-1}) = \prod_{i=j-1}^{n+1} (a_i - 1) - (n+1) \sum_{k=0}^{\lfloor a_{n+1} \rfloor - 1} \left[\frac{(k+\beta)^{j-2}}{a_{n+1}^{j-2}} \prod_{i=j-1}^{n} \left(\frac{a_i}{a_{n+1}}(k+\beta) - 1 \right) \right]. \tag{3.10}$$

To prove the lemma, we only need to prove that $F(a_{j-1})$ is a strictly increasing function of a_{j-1} and $F(1) \ge 0$.

If $a_{j-1} = 1$, then $\frac{a_{j-1}}{a_{n+1}}(k+\beta) - 1 < 0$ for $0 \le k \le \lfloor a_{n+1} \rfloor - 1$. Moreover, by the assumptions $a_j \ge a_{j+1} \ge \cdots \ge a_n \ge a_{n+1} > 1$ and $\frac{a_n}{a_{n+1}}\beta \ge 1$, we have $\frac{a_i}{a_{n+1}}\beta \ge 1$ for all i where $j \le i \le n$. Therefore

$$\prod_{i=j}^{n} \left(\frac{a_i}{a_{n+1}}(k+\beta) - 1 \right) \ge 0 \quad \text{for} \quad 0 \le k \le \lfloor a_{n+1} \rfloor - 1, \tag{3.11}$$

and

$$(n+1) \sum_{k=0}^{\lfloor a_{n+1} \rfloor - 1} \left[\frac{(k+\beta)^{j-2}}{a_{n+1}^{j-2}} \prod_{i=j-1}^{n} \left(\frac{a_i}{a_{n+1}}(k+\beta) - 1 \right) \right] \le 0. \tag{3.12}$$

Hence, we have shown that $F(1) \ge 0$. Next, we compute

$$\begin{aligned} \frac{dF}{da_{j-1}} &= \prod_{i=j}^{n+1} (a_i - 1) - (n+1) \sum_{k=0}^{\lfloor a_{n+1} \rfloor - 1} \left[\frac{(k+\beta)^{j-1}}{a_{n+1}^{j-1}} \prod_{i=j}^{n} \left(\frac{a_i}{a_{n+1}}(k+\beta) - 1 \right) \right] \\ &> 0 \quad \text{by} \quad (3.8), \end{aligned} \tag{3.13}$$

so $F(a_{j-1})$ is a strictly increasing function of a_{j-1}. Thus, $F(a_{j-1}) > 0$ for $a_{j-1} > 1$. $\square$

Now we introduce Lemma 3.4, which is slightly different from Lemma 3.3. In Lemma 3.4, we let $\frac{a_n}{a_{n+1}}\beta < 1$ and ignore the layer $k = 0$ on the right-hand-side of (3.9).

Lemma 3.4. *Let $a_{j-1}, a_j, \dots, a_{n+1}$ be real numbers and $\beta = a_{n+1} - \lfloor a_{n+1} \rfloor$. Assume that $a_{j-1} > 1$ and $a_j \geq a_{j+1} \geq \cdots \geq a_n \geq a_{n+1} > 1$. If $\frac{a_n}{a_{n+1}}\beta < 1$, and*

$$\prod_{i=j}^{n+1}(a_i - 1) > (n+1) \sum_{k=1}^{\lfloor a_{n+1}\rfloor - 1} \left[\frac{(k+\beta)^{j-1}}{a_{n+1}^{j-1}} \prod_{i=j}^{n} \left(\frac{a_i}{a_{n+1}}(k+\beta) - 1 \right) \right], \tag{3.14}$$

then

$$\prod_{i=j-1}^{n+1}(a_i - 1) > (n+1) \sum_{k=1}^{\lfloor a_{n+1}\rfloor - 1} \left[\frac{(k+\beta)^{j-2}}{a_{n+1}^{j-2}} \prod_{i=j-1}^{n} \left(\frac{a_i}{a_{n+1}}(k+\beta) - 1 \right) \right]. \tag{3.15}$$

Proof. The proof of Lemma 3.4 is similar to that of Lemma 3.3. For fixed $a_j \geq a_{j+1} \geq \cdots \geq a_n \geq a_{n+1} > 1$, let

$$G(a_{j-1}) = \prod_{i=j-1}^{n+1}(a_i - 1) - (n+1) \sum_{k=1}^{\lfloor a_{n+1}\rfloor - 1} \left[\frac{(k+\beta)^{j-2}}{a_{n+1}^{j-2}} \prod_{i=j-1}^{n} \left(\frac{a_i}{a_{n+1}}(k+\beta) - 1 \right) \right]. \tag{3.16}$$

It suffices to show that $G(a_{j-1})$ is a strictly increasing function of a_{j-1} and $G(1) \geq 0$.

Letting $a_{j-1} = 1$, it can be seen that $\frac{a_{j-1}}{a_{n+1}}(k+\beta) - 1 < 0$ for $1 \leq k \leq \lfloor a_{n+1} \rfloor - 1$. Furthermore, since $k \geq 1$ and $a_j \geq a_{j+1} \geq \cdots \geq a_n \geq a_{n+1}$, we have

$$\prod_{i=j}^{n} \left(\frac{a_i}{a_{n+1}}(k+\beta) - 1 \right) \geq 0 \quad \text{for} \quad 1 \leq k \leq \lfloor a_{n+1} \rfloor - 1, \tag{3.17}$$

and

$$(n+1) \sum_{k=1}^{\lfloor a_{n+1}\rfloor - 1} \left[\frac{(k+\beta)^{j-2}}{a_{n+1}^{j-2}} \prod_{i=j-1}^{n} \left(\frac{a_i}{a_{n+1}}(k+\beta) - 1 \right) \right] \leq 0. \tag{3.18}$$

Therefore, $G(1) \geq 0$. We then compute

$$\begin{aligned} \frac{dG}{da_{j-1}} &= \prod_{i=j}^{n+1}(a_i - 1) - (n+1) \sum_{k=1}^{\lfloor a_{n+1}\rfloor - 1} \left[\frac{(k+\beta)^{j-1}}{a_{n+1}^{j-1}} \prod_{i=j}^{n} \left(\frac{a_i}{a_{n+1}}(k+\beta) - 1 \right) \right] \\ &> 0 \quad \text{by} \quad (3.14). \end{aligned} \tag{3.19}$$

so $G(a_{j-1})$ is a strictly increasing function of a_{j-1}. We conclude that $G(a_{j-1}) > 0$ for $a_{j-1} > 1$. ☐

4. Main result

The purpose of this section is to prove Theorem 1.1 (the Rough Estimate GLY conjecture).

We first consider the case where $a_n > 1$. Our intention is to show that if Theorem 1.1 is true for n-dimensional simplices, then it must also be true for $(n+1)$-dimensional simplices. From [Xu–Ya 1], Theorem 1.1 was proven for $n = 3$, which shall be our base case in the induction. Let $n \geq 3$ and $P_n = P_{(a_1,a_2,\ldots,a_n)}$ be the number of positive integral solutions satisfying

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1, \tag{4.1}$$

where $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n > 1$. By the induction hypothesis, we have

$$n! \cdot P_n < \prod_{i=1}^{n} (a_i - 1). \tag{4.2}$$

Consider $P_{n+1} = P_{(a_1,a_2,\ldots,a_{n+1})}$, which is the number of positive integral solutions satisfying

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_{n+1}}{a_{n+1}} \leq 1, \tag{4.3}$$

where $a_1, a_2, \ldots, a_{n+1}$ are positive real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_{n+1} > 1$. We slice the $(n+1)$-dimensional simplex described by (4.3) along the x_{n+1} axis into $\lfloor a_{n+1} \rfloor$ similar n-dimensional simplices described by (4.1). Specifically, the n-dimensional simplex at $x_{n+1} = \lfloor a_{n+1} \rfloor - k$, where $\lfloor a_{n+1} - 1 \rfloor \geq k \geq 0$, is

$$\frac{x_1}{\frac{a_1}{a_{n+1}}(k+\beta)} + \frac{x_2}{\frac{a_2}{a_{n+1}}(k+\beta)} + \cdots + \frac{x_n}{\frac{a_n}{a_{n+1}}(k+\beta)} \leq 1, \tag{4.4}$$

where $\beta = a_{n+1} - \lfloor a_{n+1} \rfloor$. We are going to consider two cases.

Case 1: $\dfrac{a_n}{a_{n+1}}\beta \geq 1$

When we sum up the number of lattice points in each n-dimensional simplex described by (4.4), we have the following estimate according to the induction hypothesis in (4.2).

$$n! \cdot P_{n+1} < \sum_{k=0}^{\lfloor a_{n+1} \rfloor - 1} \left[\prod_{i=1}^{n} \left(\frac{a_i}{a_{n+1}}(k+\beta) - 1 \right) \right]. \tag{4.5}$$

Our goal in this case is to show that

$$(n+1) \sum_{k=0}^{\lfloor a_{n+1} \rfloor - 1} \left[\prod_{i=1}^{n} \left(\frac{a_i}{a_{n+1}}(k+\beta) - 1 \right) \right] < \prod_{i=1}^{n+1} (a_i - 1). \tag{4.6}$$

From Proposition 3.1, we have

$$(n+1) \sum_{k=0}^{\lfloor a_{n+1} \rfloor - 1} \frac{(k+\beta)^n}{a_{n+1}^n} < (a_{n+1} - 1). \tag{4.7}$$

If we repeatedly apply Lemma 3.3 to (4.7), then after n times we will have (4.6). The inequality sign in (4.6) indicates that when $a_{n+1} > 1$, Theorem 1.1 is strictly larger than P_{n+1}.

Case 2: $\frac{a_n}{a_{n+1}}\beta < 1$

Examining (4.4) closely, we see that if $\frac{a_n}{a_{n+1}}\beta < 1$, then the number of lattice points is zero in the layer $k = 0$ of the $(n+1)$-dimensional simplex. Therefore, k goes from 1 to $\lfloor a_{n+1} \rfloor - 1$ in this case.

From (4.2), the number of lattice points in (4.3) has the following estimate:

$$n! \cdot P_{n+1} < \sum_{k=1}^{\lfloor a_{n+1} \rfloor - 1} \left[\prod_{i=1}^{n} \left(\frac{a_i}{a_{n+1}}(k+\beta) - 1 \right) \right]. \tag{4.8}$$

Therefore we only need to show that if $\frac{a_n}{a_{n+1}}\beta < 1$, then

$$(n+1) \sum_{k=1}^{\lfloor a_{n+1} \rfloor - 1} \left[\prod_{i=1}^{n} \left(\frac{a_i}{a_{n+1}}(k+\beta) - 1 \right) \right] < \prod_{i=1}^{n+1} (a_i - 1). \tag{4.9}$$

From Proposition 3.1, we have

$$\begin{aligned} (a_{n+1} - 1) \quad &> \quad (n+1) \sum_{k=0}^{\lfloor a_{n+1} \rfloor - 1} \frac{(k+\beta)^n}{a_{n+1}^n} \\ &> \quad (n+1) \sum_{k=1}^{\lfloor a_{n+1} \rfloor - 1} \frac{(k+\beta)^n}{a_{n+1}^n}. \end{aligned} \tag{4.10}$$

Again, we can repeatedly apply Lemma 3.4 to (4.10), and after n times we will have (4.9). Thus far, we have shown that if $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n > 1$, then $n! \cdot P_n < (a_1 - 1)(a_2 - 1) \cdots (a_n - 1)$. Finally, notice that if $a_n = 1$, then the number of lattice points in any n-dimensional simplex becomes zero, so our upper estimate becomes an equality if and only if $a_n = 1$. This completes the proof of Theorem 1.1. $\square$

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