

Criteria for Right-Left Equivalence and Right Equivalence of Holomorphic Functions With Isolated Critical Points

STEPHEN S.-T. YAU¹

1. Introduction. Let \mathcal{O}_{n+1} denote the ring of germs at the origin of holomorphic functions $(\mathbb{C}^{n+1}, 0) \rightarrow \mathbb{C}$. If $(V, 0)$ is a germ at the origin of a hypersurface in \mathbb{C}^{n+1} , let $I(V)$ be the ideal of functions in \mathcal{O}_{n+1} vanishing on V , and let f be a generator of $I(V)$. It is well known that $V - \{0\}$ is nonsingular if and only if the \mathbb{C} -vector space

$$A(V) = \mathcal{O}_{n+1} / ((f) + \Delta(f))$$

is finite dimensional, where $\Delta(f)$ is the ideal in \mathcal{O}_{n+1} generated by all first partial derivatives of f . $A(V)$, provided with the obvious \mathbb{C} -algebra structure, is called the moduli algebra of V . In [1] the following theorem was proved.

THEOREM (MATHER-YAU). *Suppose $(V, 0)$ and $(W, 0)$ are germs of hypersurfaces in \mathbb{C}^{n+1} , and $V - \{0\}$ is nonsingular. Then $(V, 0)$ is biholomorphically equivalent to $(W, 0)$ if and only if $A(V)$ is isomorphic to $A(W)$ as a \mathbb{C} -algebra.*

It is also well known that 0 is an isolated critical point if and only if the \mathbb{C} -vector space

$$Q(f) = \mathcal{O}_{n+1} / \Delta(f)$$

is finite dimensional. $Q(f)$, provided with the obvious \mathbb{C} -algebra structure, is called the Milnor algebra of f . We can give $Q(f)$ a $\mathbb{C}\{t\}$ algebra structure by defining $a(t) \cdot \bar{u} = \overline{a(f)}u$ for any $a(t) \in \mathbb{C}\{t\}$ and any $u \in \mathcal{O}_{n+1}$. (Here \bar{u} is the image of u in $Q(f)$.) Abusing notation, we write u for \bar{u} later on. By a theorem of Briançon and Skoda (cf. [0]), we know that $f^{n+1} \in \Delta(f)$. Therefore $Q(f)$ has a natural $\mathbb{C}\{t\}/(t^{n+1})$ algebra structure.

The purpose of this note is to point out that the same technique used in [1] will also yield the following theorems.

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THEOREM 1. *Let $f, g: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be germs at the origin of holomorphic functions. Then (i) f is right-left equivalent to g , i.e., there exist $\phi: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$ and $\psi: (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ germs of biholomorphisms such that $f = \psi \circ g \circ \phi$ if and only if (ii) $Q(f)$ is mixed isomorphic to $Q(g)$ as a $\mathbb{C}\{t\}/(t^{n+1})$ algebra, i.e., there exist isomorphisms as \mathbb{C} -algebras $\sigma: Q(f) \rightarrow Q(g)$, $\tau: \mathbb{C}\{t\}/(t^{n+1}) \rightarrow \mathbb{C}\{t\}/(t^{n+1})$ such that $\sigma(a(t) \cdot h) = \tau(a(t)) \cdot \sigma(h)$ for any $a(t) \in \mathbb{C}\{t\}/(t^{n+1})$ and any $h \in Q(f)$.*

As a corollary of the proof of Theorem 1, we have

THEOREM 2. *Let $f, g: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be germs at the origin of holomorphic functions. Then f is right equivalent to g if and only if $Q(f)$ is isomorphic to $Q(g)$ as a $\mathbb{C}\{t\}/(t^{n+1})$ algebra.*

Actually, if we replace τ by the identity and $a(t)$ by t in the proof of Theorem 1, we will obtain a proof of Theorem 2. Hence we need only give a proof of Theorem 1. We thank R. Ephraim and T. Suwa for posing the problems to us.

Recently Theorem 2 was also observed independently by John Scherk, in a note presented by B. Malgrange, in a slightly different form.

2. Proof of Theorem 1, (i) \Rightarrow (ii). Let $\phi: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$ and $\psi: (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ be germs at the origin of biholomorphic mappings such that $g = \psi \circ f \circ \phi$. Write $\phi = (\phi_0, \dots, \phi_n)$, where $\phi_i: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$. Then

$$\frac{\partial g}{\partial z_i} = \left(\frac{\partial \psi}{\partial t} \circ f \circ \phi \right) \sum_{j=0}^n \left(\frac{\partial f}{\partial z_j} \circ \phi \right) \frac{\partial \phi_j}{\partial z_i}.$$

Hence, $\partial g/\partial z_i$ is in the ideal generated by $(\partial f/\partial z_0) \circ \phi, \dots, (\partial f/\partial z_n) \circ \phi$. A similar argument shows that $\partial f/\partial z_i$ is in the ideal generated by $(\partial g/\partial z_0) \circ \phi^{-1}, \dots, (\partial g/\partial z_n) \circ \phi^{-1}$. From this it follows immediately that

$$\phi^* \left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n} \right) \mathcal{O}_{n+1} = \left(\frac{\partial g}{\partial z_0}, \dots, \frac{\partial g}{\partial z_n} \right) \mathcal{O}_{n+1},$$

where $\phi^*: \mathcal{O}_{n+1} \rightarrow \mathcal{O}_{n+1}$ is the \mathbb{C} -algebra isomorphism defined by $\phi^*u = u \circ \phi$. It follows that ϕ^* induces a \mathbb{C} -algebra isomorphism $\sigma: Q(f) \rightarrow Q(g)$. Let $\tau: \mathbb{C}\{t\}/(t^{n+1}) \rightarrow \mathbb{C}\{t\}/(t^{n+1})$ be the \mathbb{C} -algebra isomorphism induced by $(\psi^{-1})^*$. We want to prove $\sigma(a(t) \cdot u) = \tau(a(t)) \cdot \sigma(u)$ for any $a(t) \in \mathbb{C}\{t\}$ and any $u \in Q(f)$. This can be checked easily as follows:

$$\begin{aligned} \sigma(a(t) \cdot u) &= \sigma((a \circ f)u) = (a \circ f \circ \phi)(u \circ \phi) = (a \circ \psi^{-1} \circ g)(u \circ \phi) \\ &= (\psi^{-1})^*(a(t)) \cdot (u \circ \phi) = \tau(a(t)) \cdot (u \circ \phi). \end{aligned}$$

3. Proof of Theorem 1, (ii) \Rightarrow (i). *Reduction to the special case.* Here we show it is enough to prove (i) under the hypothesis that (3)–(6) below hold.

Let $\sigma: Q(f) \rightarrow Q(g)$ be a \mathbb{C} -algebra isomorphism. We let z_0, \dots, z_n be a holomorphic local system of coordinates on \mathbb{C}^{n+1} centered at the origin and $[z_0], \dots, [z_n]$ the images of z_0, \dots, z_n under the projection $\mathcal{O}_{n+1} \rightarrow Q(f)$. Let K

denote the kernel of this projection and m the maximal ideal of \mathcal{O}_{n+1} . Let

$$k = \dim_{\mathbb{C}} \frac{K \cap m + m^2}{m^2}.$$

We may choose a system of coordinates in the following way. First, we choose $z_0, \dots, z_{k-1} \in K \cap m$ which are linearly independent mod m^2 . Then we choose $z_k, \dots, z_n \in m$ such that z_0, \dots, z_n are linearly independent mod m^2 . By the inverse function theorem, z_0, \dots, z_n form a holomorphic local system of coordinates on \mathbb{C}^{n+1} centered at the origin.

Let $K' = \ker(\mathcal{O}_{n+1} \rightarrow Q(g))$. Since $Q(f)$ and $Q(g)$ are isomorphic as \mathbb{C} -algebras, we have

$$k = \dim_{\mathbb{C}} \frac{K' \cap m + m^2}{m^2}.$$

Hence, we may choose elements $w_0, \dots, w_{k-1} \in K' \cap m$ which are linearly independent mod m^2 . For $i \geq k$ we let w_i be an element of m which projects onto $\sigma([z_i])$ under the projection $\mathcal{O}_{n+1} \rightarrow Q(g)$. Since $[z_k], \dots, [z_n]$ are linearly independent modulo the square of the maximal ideal in $Q(f)$, we have that $\sigma([z_k]), \dots, \sigma([z_n])$ are linearly independent modulo the square of the maximal ideal in $Q(g)$. Hence w_k, \dots, w_n are linearly independent modulo $K' \cap m + m^2$ and w_0, \dots, w_n are linearly independent modulo m^2 . By the inverse function theorem, w_0, \dots, w_n therefore form a local system of coordinates for \mathbb{C}^{n+1} centered at the origin.

We define a germ $h: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$ of a biholomorphic mapping by $z_i \circ h = w_i$. Letting \bar{w}_i denote the image of w_i under the projection $\mathcal{O}_{n+1} \rightarrow Q(g)$, we have $\sigma([z_i]) = \bar{w}_i$, $i = 0, \dots, n$. (For $0 \leq i \leq k - 1$, $[z_i] = 0$ and $\bar{w}_i = 0$.) Hence, the diagram

$$\begin{array}{ccc} \mathcal{O}_{n+1} & \xrightarrow{h^*} & \mathcal{O}_{n+1} \\ \downarrow & & \downarrow \\ Q(f) & \xrightarrow{\sigma} & Q(g) \end{array}$$

commutes. In other words,

$$\begin{aligned} (1) \quad \left(\frac{\partial g}{\partial z_0}, \dots, \frac{\partial g}{\partial z_n} \right)_{\mathcal{O}_{n+1}} &= h^* \left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n} \right)_{\mathcal{O}_{n+1}} \\ &= \left(\frac{\partial f}{\partial z_0} \circ h, \dots, \frac{\partial f}{\partial z_n} \circ h \right)_{\mathcal{O}_{n+1}} \\ &= \left(\frac{\partial(f \circ h)}{\partial z_0}, \dots, \frac{\partial(f \circ h)}{\partial z_n} \right)_{\mathcal{O}_{n+1}}. \end{aligned}$$

The last equality comes from the chain rule

$$\frac{\partial(f \circ h)}{\partial z_i} = \sum_j \left(\frac{\partial f}{\partial z_j} \circ h \right) \frac{\partial h_j}{\partial z_i}$$

and the fact that $(\partial h_j / \partial z_i)$ is invertible.

Since $Q(f)$ is mixed isomorphic to $Q(g)$ as a $\mathbf{C}\{t\}/(t^{n+1})$ algebra, there exists

$$a(t) = \sum_{i=1}^n a_i t^i \in \mathbf{C}\{t\}/(t^{n+1}),$$

where $a_1 \neq 0$, such that

$$f \circ h - a(g) \in \left(\frac{\partial g}{\partial z_0}, \dots, \frac{\partial g}{\partial z_n} \right) \mathcal{O}_{n+1},$$

i.e.,

$$(2) \quad f \circ h - a(g) = \sum_i \alpha_i \frac{\partial g}{\partial z_i}, \quad \alpha_0, \dots, \alpha_n \in \mathcal{O}_{n+1}.$$

We will show that $\alpha_i(0) = 0$ for $i = 0, \dots, n$. If not, say $\alpha_j(0) \neq 0$. Let

$$U = \left\{ \frac{\partial g}{\partial z_0} = \dots = \frac{\partial g}{\partial z_{j-1}} = \frac{\partial g}{\partial z_{j+1}} = \dots = \frac{\partial g}{\partial z_n} = 0 \right\}.$$

Since U is defined by n equations in $n+1$ variables, and $U \cap \{\partial g / \partial z_j = 0\} = \{0\}$, it follows that the germ of U at 0 is the germ of a curve. Let $\beta: (\mathbf{C}, 0) \rightarrow (U, 0)$ be the normalization map-germ. Then $(\partial g / \partial z_i) \circ \beta = 0$ for $i \neq j$ and $(\partial g / \partial z_j) \circ \beta \neq 0$.

Let $O(u)$ denote the order of u when $u \in \mathcal{O}_1$. We have

$$\frac{d}{dt}(a(g \circ \beta)) = \left(\frac{da}{dt}(g \circ \beta) \right) \left(\frac{\partial g}{\partial z_j} \circ \beta \right) \frac{d\beta_j}{dt}.$$

Consequently, $O((\partial g / \partial z_j) \circ \beta) < O(a(g \circ \beta))$. Since $f \circ h \circ \beta = a(g) \circ \beta + (\alpha_j(\partial g / \partial z_j)) \circ \beta$, and $\alpha_j(0) \neq 0$, we then obtain $O(f \circ h \circ \beta) = O((\partial g / \partial z_j) \circ \beta)$.

Every ideal $J \subset \mathcal{O}_1$ is principal; we write $O(J)$ for the order of its generator. Since

$$\frac{d(f \circ h \circ \beta)}{dt} = \sum_{i=0}^n \left(\frac{\partial(f \circ h)}{\partial z_i} \circ \beta \right) \frac{d\beta_i}{dt},$$

we obtain $O(\{\partial(f \circ h) / \partial z_i\} \circ \beta) < O(f \circ h \circ \beta)$ for at least one i . Hence

$$\begin{aligned} O\left(\frac{\partial(f \circ h)}{\partial z_0} \circ \beta, \dots, \frac{\partial(f \circ h)}{\partial z_n} \circ \beta \right) &< O(f \circ h \circ \beta) = O\left(\frac{\partial g}{\partial z_j} \circ \beta \right) \\ &= O\left(\frac{\partial g}{\partial z_0} \circ \beta, \dots, \frac{\partial g}{\partial z_n} \circ \beta \right) \mathcal{O}_1. \end{aligned}$$

But this contradicts (1).

We obtained this contradiction by assuming some $\alpha_j(0) \neq 0$; therefore $\alpha_i(0) = 0$ for $i = 0, \dots, n$.

By replacing f with $f \circ h$ in (1) and (2), we may therefore assume

$$(3) \quad \left(\frac{\partial g}{\partial z_0}, \dots, \frac{\partial g}{\partial z_n} \right) \mathcal{O}_{n+1} = \left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n} \right) \mathcal{O}_{n+1}$$

and

$$(4) \quad f - a(g) \in m \left(\frac{\partial g}{\partial z_0}, \dots, \frac{\partial g}{\partial z_n} \right) \mathcal{O}_{n+1},$$

where

$$a(t) = \sum_{i=1}^n a_i t^i \in \mathbf{C}\{t\}/(t^{n+1}) \quad \text{with } a_1 \neq 0.$$

There exists

$$b(t) = \sum_{i=1}^n b_i t^i \in \mathbf{C}\{t\}/(t^{n+1})$$

such that $b_1 \neq 0$ and $b(a(t)) = t$. By (4) we have

$$a(g) = f + q,$$

where q is an element in $m(\partial g/\partial z_0, \dots, \partial g/\partial z_n) \mathcal{O}_{n+1}$. Hence,

$$g = b(a(g)) = b(f + q) = b(f) + p,$$

where p is an element in $m(\partial g/\partial z_0, \dots, \partial g/\partial z_n) \mathcal{O}_{n+1}$. In other words,

$$(5) \quad g - b(f) \in m \left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n} \right) \mathcal{O}_{n+1}.$$

Equations (3)–(5) imply

$$(6) \quad f^{-1}m_1 + m \left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n} \right) \mathcal{O}_{n+1} = g^{-1}m_1 + m \left(\frac{\partial g}{\partial z_0}, \dots, \frac{\partial g}{\partial z_n} \right) \mathcal{O}_{n+1}.$$

4. End of the proof that (ii) \Rightarrow (i).

DEFINITION OF \mathcal{RL} . By \mathcal{RL} we mean the set of pairs (ϕ, ψ) such that $\phi: (\mathbf{C}^{n+1}, 0) \rightarrow (\mathbf{C}^{n+1}, 0)$ and $\psi: (\mathbf{C}, 0) \rightarrow (\mathbf{C}, 0)$ are germs of biholomorphic mappings. We define the group structure by $(\phi, \psi)(\phi_1, \psi_1) = (\phi \circ \phi_1, \psi \circ \psi_1)$.

The action of \mathcal{RL} on \mathcal{O}_{n+1} . Given $(\phi, \psi) \in \mathcal{RL}$ and $f \in \mathcal{O}_{n+1}$, we define $g = (\phi, \psi) \cdot f$ to be the unique element $\psi \circ f \circ \phi$ of \mathcal{O}_{n+1} .

Let J^k denote the \mathbf{C} -vector of k -jets of the origin of elements of \mathcal{O}_{n+1} . Let \mathcal{RL}^k denote the Lie group of k -jets at the origin of members of \mathcal{RL} . Since \mathcal{RL} acts on \mathcal{O}_{n+1} , we have that \mathcal{RL}^k acts on J^k . For $f \in \mathcal{O}_{n+1}$, let $f^{(k)} \in J^k$ denote the k -jet of f at the origin. We say f is k -determined with respect to \mathcal{RL} if $g \in \mathcal{O}_{n+1}$ and $g^{(k)} \in \mathcal{RL}^k f^{(k)}$ imply $g \in \mathcal{RL}f$. We say f is finitely determined with respect to \mathcal{RL} if it is k -determined with respect to \mathcal{RL} for some positive integer k .

PROPOSITION [2]. Let $(V, 0)$ be a germ of a hypersurface in \mathbf{C}^{n+1} and let f be a generator of $I(V)$. Then the following are equivalent:

- (a) f has isolated critical point at the origin.
- (b) $V - \{0\}$ is nonsingular.
- (c) $Q(f)$ is finite dimensional as a \mathbf{C} -vector space.
- (d) f is finite determined with respect to $\mathfrak{R}\mathcal{L}$.

The hypothesis that f has an isolated critical point at the origin in our theorem and the implication (a) \Rightarrow (d) in the Proposition imply that f is finitely determined with respect to $\mathfrak{R}\mathcal{L}$. Hence it is enough to prove $g^{(k)} \in \mathfrak{R}\mathcal{L}^k f^{(k)}$ for every positive integer k . In what follows, we let k be a fixed positive integer.

Obviously, there is nothing to prove when $f^{(k)} = g^{(k)}$, so suppose $f^{(k)} \neq g^{(k)}$. Let L be the complex line in J^k containing $f^{(k)}$ and $g^{(k)}$, and L_0 the set of $h^{(k)}$ in L such that

$$(7) \quad (h^{-1}m_1)J^k + m\left(\frac{\partial h}{\partial z_0}, \dots, \frac{\partial h}{\partial z_n}\right)J^k = (f^{-1}m_1)J^k + m\left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\right)J^k.$$

The left side depends only on $h^{(k)}$. Consider $w \in \mathbf{C}$ and set $h = (1-w)f + wq$. Then $h^{(k)} \in L$. (3) and (5) imply

$$(h^{-1}m_1)J^k + m\left(\frac{\partial h}{\partial z_0}, \dots, \frac{\partial h}{\partial z_n}\right)J^k \subseteq (f^{-1}m_1)J^k + m\left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\right)J^k.$$

For $w = 0, 1$, we have

$$(h^{-1}m_1)J^k + m\left(\frac{\partial h}{\partial z_0}, \dots, \frac{\partial h}{\partial z_n}\right)J^k = (f^{-1}m_1)J^k + m\left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\right)J^k.$$

It follows that L_0 is L with finitely many points deleted. Since L is a complex line, it follows that L_0 is a connected manifold.

We shall show $g^{(k)} \in \mathfrak{R}\mathcal{L}^k f^{(k)}$ by an application of [3, Lemma 3.1]. For this purpose, we take the action α of [3, Lemma 3.1] to be the action of $\mathfrak{R}\mathcal{L}^k$ on J^k which we have been discussing. We take the submanifold V of [3, Lemma 3.1] to be L_0 . By the previous section L_0 is a connected C^∞ submanifold of J^k .

To verify condition a) of [3, Lemma 3.1], we observe that

$$T(\mathfrak{R}\mathcal{L}^k[h])_{[h]} = (h^{-1}m_1)J^k + m\left(\frac{\partial h}{\partial z_0}, \dots, \frac{\partial h}{\partial z_n}\right)J^k,$$

for any $h \in \mathcal{O}_{n+1}$, where $[h] = h^{(k)}$. This is a special case of the complex analogue of [3, Proposition 7.4]. If $[h] \in L_0$, then (7) holds, and we obtain

$$(8) \quad T(\mathfrak{R}\mathcal{L}^k[h]) = (f^{-1}m_1)J^k + m\left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\right)J^k.$$

Obviously, $T(L_0)_{[h]}$ is the one-dimensional complex subspace of J^k spanned by $g - f$. By (5),

$$g^{(k)} - f^{(k)} \in (f^{-1}m_1)J^k + m\left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\right)J^k.$$

Hence $T(L_0)_{[h]} \subset T(\mathfrak{R}\mathcal{L}^k[h])_{[h]}$, i.e., condition a) of [3, Lemma 3.1] holds.

Condition b) of [3, Lemma 3.1] follows immediately from (8). Hence L_0 is contained in a single orbit of the action $\mathcal{R}\mathcal{L}^k$ on J^k by [3, Lemma 3.1]. In particular, $g^{(k)} \in \mathcal{R}\mathcal{L}^k f^{(k)}$.

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UNIVERSITY OF ILLINOIS AT CHICAGO