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Lie Algebras and their Representations Arising from Isolated Singularities: Computer Method in Calculating the Lie Algebras and their Cohomology

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Introduction

Let $(V, 0)$ be an isolated hypersurface singularity in $(\boldsymbol{C}^n, 0)$ defined by the zero of a holomorphic function f. The moduli algebra $A(V)$ of V is $\boldsymbol{C}\{z_1, z_2, \cdots, z_n\} \Big/ \left(f, \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}\right)$. Recall that the natural mapping

$$(0.1)\qquad \begin{array}{ccc} \left\{\begin{array}{l}\text{isolated hypersurface singularities}\\ \text{of dimension } n\end{array}\right\} & \longrightarrow & \left\{\begin{array}{l}\text{commutative local}\\ \text{Artinian algebras}\end{array}\right\} \\ (V, 0) & \longrightarrow & A(V) \end{array}$$

is one to one (cf. [9], [2]). In [19], a connection between the set of isolated hypersurface singularities $(V, 0)$ and the set of finite dimensional Lie algebras $L(V)$ was established. Namely $L(V)$ is the algebra of derivations of $A(V)$. Since $A(V)$ is a finite dimensional complex vector space and $L(V)$ is contained in the endomorphism algebra of $A(V)$, $L(V)$ is a finite dimensional Lie algebra. In [20] and [22], the second author has proved that $L(V)$ is a solvable Lie algebra for $n \leq 5$. It is known that the problem of classification of solvable Lie algebras was basically reduced to the problem of classification of nilpotent Lie algebras (cf. [8]). The above construction provides us a new way in studying solvable and nilpotent Lie algebras. For instance, new examples and phenomena of solvable or nilpotent Lie algebras and their representations can be derived via isolated singularities. In Chapter 1, we shall prove that the one parameter family of inequivalent finite dimensional representations of a fixed Lie algebra $L(\tilde{E}_6)$ in [19] and [20] is not obtainable by the action of the automorphism groups of $L(\tilde{E}_6)$ on a representation. We believe that in general a natural representation of a Lie algebra on its moduli algebra determines the complex structure of the singularity. More generally if we consider a

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family of isolated singularities, then we expect the following: either we shall obtain a new family of solvable Lie algebras or we shall have a one parameter family of inequivalent finite dimensional representations of a fixed Lie algebra.

The injectivity of the map (0.1) raises the following natural question. What kind of information does one need from the moduli algebra in order to determine the topological type of the singularity. This question has been studied by many others including Lê and Ramanujan [7], Pham [14], Teissier [17], [18], and Zariski [23], [24]. Zariski shows that two irreducible plane curves are topological equivalent if and only if their associated numerical invariants so called Puiseux characteristic are the same (cf. also Pham [14]). In 1968, Milnor [8] introduced his famous topological invariant Milnor number. In [20], many numerical invariants were introduced, namely, dim $L(V)$; dimension of the maximal nilpotent subalgebra $\mathfrak{g}(V)$ of $L(V)$, dimension of a maximal torus of $\mathfrak{g}(V)$; generalized Cartan matrix $C(V)$; type and nilpotency of the singularity. It was shown by an example in [19] that dim $L(V)$ is not a topological invariant. However there is no evidence that the other numerical invariants are not topological. In Chapter 2, we shall first recall the construction of a generalized Cartan matrix associated to isolated hypersurface singularities (cf. [20]). Since rational double points play a distinguished role in many ways, it is worthwhile to study them more closely than those given in [20]. We shall write down the multiplication table of $\mathfrak{g}(V)$, compute the algebra of derivatives and maximal torus of $\mathfrak{g}(V)$. We shall also find root space decomposition of $\mathfrak{g}(V)$ and generalized Cartan matrix $C(V)$. We remark that the comutations of Der $\mathfrak{g}(V)$ is by no means easy. Such explicit computation will be useful in studying cohomology of $\mathfrak{g}(V)$.

Deformation of the singularity $(V, 0)$ is related to the deformation of the associated Lie algebra $\mathfrak{g}(V)$. It is well known that the Lie algebra cohomology plays an important role in deformation of Lie algebra. For instance, it was shown that $\mathfrak{g}(V)$ is rigid of $H^2(\mathfrak{g}(V), \mathfrak{g}(V))=0$, and a neighborhood of $\mathfrak{g}(V)$ can be parametrized in the real or complex case by the zeros of an analytic map from $H^2(\mathfrak{g}(V), \mathfrak{g}(V))$ to $H^3(\mathfrak{g}(V), \mathfrak{g}(V))$ (cf. [13]). The theory of cohomology groups of $\mathfrak{g}$ with coefficient in $\mathfrak{g}$ module C, implicitly in the work of Elie Cartan, was first explicitly formulated by Chevalley-Eilenberg (Trans. Amer. Math. Soc., **63** (1948), 85–124). For the past two years, they have received special attention. We were told by Professor Zuckerman that Physicists are particularly interested in them. In any event, the dimensions of the Lie algebra cohomology groups are interesting new invariants of the singularity $(V, 0)$. However any explicit computation of these Lie algebra cohomology groups are extremely difficult, if not impossible by hand. Therefore we have developed in Chapter 3 a computation method so that the computer can do this complex calculation. We actually write down the algorithm for computing cohomology of any finite dimensional Lie algebra L with coefficient in L-module W. Our scheme goes as follows. We first observe that by equation (3.14) in Chapter 3 it is enough to compute the ranks of the linear maps $\delta_0, \delta_1, \cdots, \delta_{n-1}$, which are the coboundary operator in the cochain complex. To compute the images of the δ_k, we have to make use of Proposition 3.20 in Chapter 3. In fact, we also write down the algorithms computing the image of δ_k and its rank. Doubtless, the readers may find that most of the explicit computation of the Lie algebra $L(V)$ is extremely time consuming. Hence it will be convenient to let the computer do the calculation for us. For this purpose we also have developed an argorithm so that this kind of calculation can be done in computer too. For more details, we refer the readers to Chapter 3.

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Chapter 1. A continuous family of finite dimensional representations of a Lie algebra

In this chapter, we first construct a one parameter family of inequivalent representations of a Lie algebra $L(\widetilde{E}_6)$. This family of representations is not obtainable by the action of the automorphism group of $L(\widetilde{E}_6)$ on a representation.

Let $L(\widetilde{E}_6)$ be a 10-dimensional complex Lie algebra spanned by $\langle e_1, e_2, e_3, \cdots, e_{10}\rangle$ with the following multiplication table.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
e_1	0	0	0	0	$-e_3$	0	0	e_9	0	$-e_1$
e_2	0	0	0	e_6	0	0	$-e_3$	0	0	$-e_2$
e_3	0	0	0	0	0	0	0	0	0	$-2e_3$
e_4	0	$-e_6$	0	0	0	0	e_9	0	0	$-e_4$
e_5	e_3	0	0	0	0	0	0	$-e_6$	0	$-e_5$
e_6	0	0	0	0	0	0	0	0	0	$-2e_6$
e_7	0	e_3	0	$-e_9$	0	0	0	0	0	$-e_7$
e_8	$-e_9$	0	0	0	e_6	0	0	0	0	$-e_8$
e_9	0	0	0	0	0	0	0	0	0	$-2e_9$
e_{10}	e_1	e_2	$2e_3$	e_4	e_5	$2e_6$	e_7	e_8	$2e_9$	0

Proposition 1.1. *For any $t \in C$, let*

$$\rho_t(e_1)=\begin{bmatrix}0&1&0&0&0&0&0&0\\0&0&0&0&0&t&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}\qquad \rho_t(e_2)=\begin{bmatrix}0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0\\0&0&0&0&t&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}$$

$$\rho_t(e_3)=\begin{bmatrix}0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}\qquad \rho_t(e_4)=\begin{bmatrix}0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&t&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}$$

$$\rho_t(e_5)=\begin{bmatrix}0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&t&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}\qquad \rho_t(e_6)=\begin{bmatrix}0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}$$

$$\rho_t(e_7)=\begin{bmatrix}0&1&0&0&0&0&0&0\\0&0&0&0&0&t&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}\qquad \rho_t(e_8)=\begin{bmatrix}0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&t&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}$$

$$\rho_t(e_9)=\begin{bmatrix}0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}\qquad \rho_t(e_{10})=\begin{bmatrix}3&0&0&0&0&0&0&0\\0&2&0&0&0&0&0&0\\0&0&2&0&0&0&0&0\\0&0&0&2&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0\end{bmatrix}.$$

Then ρ_t gives a matrix representation of $L(\tilde{E}_6)$. Moreover all these repre-

sentations are all inequivalent.

Proof. It is a trivial matter to check that ρ_t is a representation for all $t \in \mathbf{C}$.

Suppose $t_1 \neq t_2$. If ρ_{t_1} were equivalent to ρ_{t_2}, then there would exist a nonsingular matrix Q such that

$$Q\rho_{t_1}(e_1)Q^{-1}=\rho_{t_2}(e_1) \tag{1.1}$$

$$Q\rho_{t_1}(e_2)Q^{-1}=\rho_{t_2}(e_2) \tag{1.2}$$

$$Q\rho_{t_1}(e_4)Q^{-1}=\rho_{t_2}(e_4). \tag{1.3}$$

(1.1) implies

$$Q\rho_{t_1}(e_1)^2=\rho_{t_2}(e_1)^2Q. \tag{1.4}$$

Since $t_1 \neq t_2$, we shall assume without loss of generality that $t_1 \neq 0$. By (1.1) we have

$$q_{21}=q_{31}=q_{41}=q_{51}=q_{61}=q_{71}=q_{81}=0 \tag{1.5}$$

$$q_{44}=q_{77}. \tag{1.6}$$

(1.2) implies

$$q_{11}=q_{44}. \tag{1.7}$$

(1.3) implies

$$q_{11}=q_{33} \tag{1.8}$$

$$t_1q_{33}=t_2q_{77}. \tag{1.9}$$

(1.6), (1.7) and (1.8) imply

$$q_{33}=q_{77}. \tag{1.10}$$

(1.9) and (1.10) give

$$q_{33}=q_{77}=0.$$

(1.8) implies

$$q_{11}=0. \tag{1.11}$$

In view of (1.5) and (1.11), Q cannot be nonsingular. Q.E.D.

Proposition 1.2. *The family of finite dimensional representations of* $L(\tilde{E}_6)$ *in Proposition* 1.1 *is not obtainable by letting the automorphism* *group of the Lie algebra acting on a representation.*

Proof. Suppose that there exist $s \in C$ and a one parameter family $A(t)$ of automorphisms of the Lie algebra $L(\tilde{E}_6)$ such that $\rho_s^{A(t)}$ is equivalent to ρ_t for all t. Choose t_0 such that $\rho_s^{A(t_0)}$ is equivalent to ρ_0. This implies that ρ_s is equivalent to $\rho_0^{A(t_0)^{-1}}$. Therefore $\rho_0^{A(t_0)^{-1}A(t)}$ is equivalent to ρ_t for all t.

In order to prove the proposition, it suffices to prove that ρ_0^A is not equivalent to ρ_t for any $t \neq 0$, and for any automorphism A of the Lie algebra $L(\tilde{E}_6)$. Suppose on the contrary that there exist a $t \neq 0$ and an automorphism A of the Lie algebra $L(\tilde{E}_6)$ such that is equivalent to ρ_t. Then there exists a 8×8 nonsingular matrix Q such that

$$
\begin{aligned}
& Q^{-1}\rho_0^A(e_i)Q = \rho_t(e_i) && \text{for all } 1 \leq i \leq 10 \\
\Rightarrow\ & Q^{-1}\rho_0(Ae_i)Q = \rho_t(e_i) && \text{for all } 1 \leq i \leq 10 \\
\Rightarrow\ & \rho_0(a_{i1}e_1 + a_{i2}e_2 + \cdots + a_{i,10}e_{10})Q = Q\rho_t(e_i) && \text{for all } 1 \leq i \leq 10
\end{aligned}
$$

$$
= \begin{bmatrix}
3a_{i,10} & a_{i,1}+a_{i,7} & a_{i,4}+a_{i,8} & a_{i,2}+a_{i,5} & a_{i,9} & a_{i,6} & a_{i,3} & 0 \\
0 & 2a_{i,10} & 0 & 0 & a_{i8} & 0 & a_{i2} & 0 \\
0 & 0 & 2a_{i,10} & 0 & a_{i7} & a_{i5} & 0 & 0 \\
0 & 0 & 0 & 2a_{i,10} & 0 & a_{i4} & a_{i1} & 0 \\
0 & 0 & 0 & 0 & a_{i,10} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_{i,10} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_{i,10} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$\rho_0(a_{i1}e_1 + a_{i2}e_2 + \cdots + a_{i,10}e_{10})Q =$

$$
\begin{bmatrix}
* & * & * & * & * & * & * & * \\
\begin{array}{l} 2a_{i,10}q_{21} \\ +a_{i8}q_{52} \\ +a_{i2}q_{71} \end{array} & \begin{array}{l} 2a_{i,10}q_{22} \\ +a_{i8}q_{52} \\ +a_{i2}q_{72} \end{array} & \begin{array}{l} 2a_{i,10}q_{23} \\ +a_{i8}q_{5} \\ +a_{i2}q_{73} \end{array} & \begin{array}{l} 2a_{i,10}q_{24} \\ +a_{i8}q_{54} \\ +a_{i2}q_{74} \end{array} & * & * & * & * \\
\begin{array}{l} a_{i,10}q_{31} \\ +a_{i7}q_{51} \\ +a_{i5}q_{61} \end{array} & \begin{array}{l} a_{i,10}q_{32} \\ +a_{i7}q_{52} \\ +a_{i5}q_{62} \end{array} & \begin{array}{l} a_{i,10}q_{33} \\ +a_{i7}q_{53} \\ +a_{i5}q_{63} \end{array} & \begin{array}{l} a_{i,10}q_{34} \\ +a_{i7}q_{54} \\ +a_{i5}q_{64} \end{array} & * & * & * & * \\
\begin{array}{l} 2a_{i,10}q_{41} \\ +a_{i4}q_{61} \\ +a_{i1}q_{71} \end{array} & \begin{array}{l} 2a_{i,10}q_{42} \\ +a_{i4}q_{62} \\ +a_{i1}q_{72} \end{array} & \begin{array}{l} 2a_{i,10}q_{43} \\ +a_{i4}q_{63} \\ +a_{i1}q_{73} \end{array} & \begin{array}{l} 2a_{i,10}q_{44} \\ +a_{i4}q_{64} \\ +a_{i1}q_{74} \end{array} & * & * & * & * \\
a_{i,10}q_{51} & a_{i,10}q_{52} & a_{i,10}q_{53} & a_{i,10}q_{54} & a_{i,10}q_{55} & a_{i,10}q_{56} & a_{i,10}q_{57} & a_{i,10}q_{58} \\
a_{i,10}q_{61} & a_{i,10}q_{62} & a_{i,10}q_{63} & a_{i,10}q_{64} & a_{i,10}q_{65} & a_{i,10}q_{66} & a_{i,10}q_{67} & a_{i,10}q_{68} \\
a_{i,10}q_{71} & a_{i,10}q_{72} & a_{i,10}q_{73} & a_{i,10}q_{74} & a_{i,10}q_{75} & a_{i,10}q_{76} & a_{i,10}q_{77} & a_{i,10}q_{78} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$Q\rho_t(e_{10}) = \begin{bmatrix} 3q_{11} & 2q_{12} & 2q_{13} & 2q_{14} & q_{15} & q_{16} & q_{17} & 0 \\ 3q_{21} & 2q_{22} & 2q_{23} & 2q_{24} & q_{25} & q_{26} & q_{27} & 0 \\ 3q_{31} & 2q_{32} & 2q_{33} & 2q_{34} & q_{35} & q_{36} & q_{37} & 0 \\ 3q_{41} & 2q_{42} & 2q_{43} & 2q_{44} & q_{45} & q_{46} & q_{47} & 0 \\ 3q_{51} & 2q_{52} & 2q_{53} & 2q_{54} & q_{55} & q_{56} & q_{57} & 0 \\ 3q_{61} & 2q_{62} & 2q_{63} & 2q_{64} & q_{65} & q_{66} & q_{67} & 0 \\ 3q_{71} & 2q_{72} & 2q_{73} & 2q_{74} & q_{75} & q_{76} & q_{77} & 0 \\ 3q_{81} & 2q_{82} & 2q_{83} & 2q_{84} & q_{85} & q_{86} & q_{87} & 0 \end{bmatrix}$$

$$Q\rho_t(e_8) = \begin{bmatrix} 0 & 0 & q_{11} & 0 & q_{12} & 0 & tq_{13} & 0 \\ 0 & 0 & q_{21} & 0 & q_{22} & 0 & tq_{23} & 0 \\ 0 & 0 & q_{31} & 0 & q_{32} & 0 & tq_{33} & 0 \\ 0 & 0 & q_{41} & 0 & q_{42} & 0 & tq_{43} & 0 \\ 0 & 0 & q_{51} & 0 & q_{52} & 0 & tq_{53} & 0 \\ 0 & 0 & q_{61} & 0 & q_{62} & 0 & tq_{63} & 0 \\ 0 & 0 & q_{71} & 0 & q_{72} & 0 & tq_{73} & 0 \\ 0 & 0 & q_{81} & 0 & q_{82} & 0 & tq_{83} & 0 \end{bmatrix}$$

$$\rho_0(a_{10,1}e_1 + a_{10,2}e_2 + \cdots + a_{10,10}e_{10})Q = Q\rho_t(e_{10})$$

$$\begin{aligned} \Rightarrow (a_{10,10}-3)q_{51} = 0 &= (a_{10,10}-2)q_{52} = (a_{10,10}-2)q_{53} = (a_{10,10}-2)q_{54} \\ &= (a_{10,10}-1)q_{55} = (a_{10,10}-1)q_{56} = (a_{10,10}-1)q_{57} = a_{10,10}q_{58} \end{aligned} \tag{1.12}$$

$$\begin{aligned} &(a_{10,10}-3)q_{61} = 0 = (a_{10,10}-2)q_{62} = (a_{10,10}-2)q_{63} = (a_{10,10}-2)q_{64} \\ &(a_{10,10}-1)q_{65} = (a_{10,10}-1)q_{66} = (a_{10,10}-1)q_{67} = a_{10,10}q_{68} \end{aligned} \tag{1.13}$$

$$\begin{aligned} &(a_{10,10}-3)q_{71} = 0 = (a_{10,10}-2)q_{72} = (a_{10,10}-2)q_{73} = (a_{10,10}-2)q_{74} \\ &= (a_{10,10}-1)q_{75} = (a_{10,10}-1)q_{76} = (a_{10,10}-1)q_{77} = a_{10,10}q_{78} \end{aligned} \tag{1.14}$$

$$q_{81} = q_{82} = q_{83} = q_{84} = q_{85} = q_{86} = q_{87} = 0. \tag{1.15}$$

Case 1. $a_{10,10} \notin \{1, 2, 3\}$
Then (1.12), (1.13), (1.14) imply

$$q_{51} = q_{52} = q_{53} = q_{54} = q_{55} = q_{56} = q_{57} = 0$$
$$q_{61} = q_{62} = q_{63} = q_{64} = q_{65} = q_{66} = q_{67} = 0$$
$$q_{71} = q_{72} = q_{73} = q_{74} = q_{75} = q_{76} = q_{77} = 0.$$

It follows that the matrix Q cannot be nonsingular, a contradiction to our original assumption.

Case 2. $a_{10,10} = 3$
Then (1.12), (1.13) and (1.14) imply

$$q_{52}=0=q_{53}=q_{54}=q_{55}=q_{56}=q_{57}$$
$$q_{62}=0=q_{63}=q_{64}=q_{65}=q_{66}=q_{67}$$
$$q_{72}=0=q_{73}=q_{74}=q_{75}=q_{76}=q_{77}.$$

The above equalities together with (1.15) imply that the matrix

$$\begin{bmatrix} q_{51} & q_{52} & q_{53} & q_{54} & q_{55} & q_{56} & q_{57} & q_{58} \\ q_{61} & q_{62} & q_{63} & q_{64} & q_{65} & q_{66} & q_{67} & q_{68} \\ q_{71} & q_{72} & q_{73} & q_{74} & q_{75} & q_{76} & q_{77} & q_{78} \\ q_{81} & q_{82} & q_{83} & q_{84} & q_{85} & q_{86} & q_{87} & q_{88} \end{bmatrix}$$

has rank at most 2. Therefore the matrix Q has rank at most six, a contradiction to the fact that Q is nonsingular.

Case 3. $a_{10,10}=2$
Then (1.12), (1.13) and (1.14) imply

$$q_{51}=0=q_{55}=q_{56}=q_{57}$$
$$q_{61}=0=q_{65}=q_{66}=q_{67} \tag{1.16}$$
$$q_{71}=0=q_{75}=q_{76}=q_{77}.$$

$$\rho_0(a_{81}e_1+a_{82}e_2+\cdots+a_{8,10}e_{10})Q=Q\rho_t(e_8)$$

simplies

$$\begin{array}{llll} a_{8,10}q_{51}=0 & a_{8,10}q_{52}=0 & a_{8,10}q_{53}=q_{51} & a_{8,10}q_{54}=0 \\ a_{8,10}q_{61}=0 & a_{8,10}q_{62}=0 & a_{8,10}q_{63}=q_{61} & a_{8,10}q_{64}=0 \\ a_{8,10}q_{71}=0 & a_{8,10}q_{72}=0 & a_{8,10}q_{73}=q_{71} & a_{8,10}q_{74}=0 \\ a_{8,10}q_{55}=q_{52} & a_{8,10}q_{56}=0 & a_{8,10}q_{57}=tq_{53} & \\ a_{8,10}q_{65}=q_{62} & a_{8,10}q_{66}=0 & a_{8,10}q_{67}=tq_{63} & \\ a_{8,10}q_{75}=q_{72} & a_{8,10}q_{76}=0 & a_{8,10}q_{77}=tq_{73}. & \end{array}$$

If $a_{8,10}=0$, then

$$q_{52}=0=q_{53}$$
$$q_{62}=0=q_{63} \tag{1.17}$$
$$q_{72}=0=q_{73}.$$

(1.15), (1.16) and (1.17) imply the matrix Q is singular, a contradiction to our assumption.

If $a_{8,10}\neq 0$, then

$$q_{51}=0=q_{52}=q_{53}=q_{54}=q_{55}=q_{56}=q_{57}$$
$$q_{61}=0=q_{62}=q_{63}=q_{64}=q_{65}=q_{66}=q_{67}$$
$$q_{71}=0=q_{72}=q_{73}=q_{74}=q_{75}=q_{76}=q_{77}.$$

So the matrix Q is singular, which contradicts to our assumption.

Case 4. $a_{10,10}=1$
Then (1.12) (1,13) and (1.14) imply

$$q_{51}=0=q_{52}=q_{53}=q_{54}$$
$$q_{61}=0=q_{62}=q_{63}=q_{64} \tag{1.18}$$
$$q_{71}=0=q_{72}=q_{73}=q_{74}.$$

We compare the (21), (31), (32), (33), (34) and (41) entries of the matrix $\rho(a_{10,1}e_1+a_{10,2}e_2+\cdots+a_{10,10}e_{10})$ and the matrix $Q\rho_t(e_{10})$.

Using (1.18) and the fact that $a_{10,10}=1$, we conclude that

$$q_{21}=0=q_{31}=q_{32}=q_{33}=q_{34}=q_{41}. \tag{1.19}$$

By (1.15), (1.18) and (1.19), the matrix

$$\begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{21} & q_{22} & q_{23} & q_{24} \\ q_{31} & q_{32} & q_{33} & q_{34} \\ q_{41} & q_{42} & q_{43} & q_{44} \\ q_{51} & q_{52} & q_{53} & q_{54} \\ q_{61} & q_{62} & q_{63} & q_{64} \\ q_{71} & q_{72} & q_{73} & q_{74} \\ q_{81} & q_{82} & q_{83} & q_{84} \end{bmatrix}$$

has rank at most three. So the matrix Q cannot be non-singular, which is a contradiction. Q.E.D.

Let us consider a family of nonsingular elliptic curves in CP^2 defined by

$$x^3+y^3+z^3+txyz=0 \tag{1.20}$$

where $t^3+27\neq 0$. The complex structure of the elliptic curve depends on t. In fact, $j=-\dfrac{1}{27.4}\dfrac{t^6}{t^3+27}$. If we view (1.20) as an equation in affine 3-space, we have a family of simple elliptic singularities V_t. For each fixed t with $t^3+27\neq 0$, the moduli algebra

$$A(V_t)=\langle 1,\ x,\ y,\ z,\ xy,\ yz,\ zx,\ zyx\rangle$$

with multiplication rules:

$$x^2=-\frac{t}{3}yx, \quad y^2=-\frac{t}{3}zx, \quad z^2=-\frac{t}{3}xy$$
$$x^2y=xy^2=y^2z=yz^2=x^2z=xz^2=0$$

We shall assume $t\neq 0$ and $t^6/27-7t^3-216\neq 0$. Under these assumptions

$$L(V_t)=\Big\langle xy\frac{\partial}{\partial x}-\frac{t}{6}zx\frac{\partial}{\partial y},\ zx\frac{\partial}{\partial x}-\frac{t}{6}xy\frac{\partial}{\partial z},\ xyz\frac{\partial}{\partial x},$$
$$-\frac{t}{6}yz\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y},\ yz\frac{\partial}{\partial y}-\frac{t}{6}xy\frac{\partial}{\partial z},\ xyz\frac{\partial}{\partial y},$$
$$-\frac{t}{6}zx\frac{\partial}{\partial y}-yz\frac{\partial}{\partial z},\ -\frac{t}{6}yz\frac{\partial}{\partial x}+zx\frac{\partial}{\partial z},\ xyz\frac{\partial}{\partial z},$$
$$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\Big\rangle.$$

$L(V_t)$ is isomorphic to $L(\tilde{E}_6)$. The isomorphism is given by the following map.

$$\Psi: L(V_t)\longrightarrow L(\tilde{E}_6)$$
$$xy\frac{\partial}{\partial x}-\frac{t}{6}zx\frac{\partial}{\partial y}\longrightarrow e_1$$
$$zx\frac{\partial}{\partial x}-\frac{t}{6}xy\frac{\partial}{\partial z}\longrightarrow e_2$$
$$xyz\frac{\partial}{\partial x}\longrightarrow e_3$$
$$-\frac{t}{6}yz\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}\longrightarrow e_4$$
$$yz\frac{\partial}{\partial y}-\frac{t}{6}xy\frac{\partial}{\partial z}\longrightarrow e_5$$
$$xyz\frac{\partial}{\partial y}\longrightarrow e_6$$
$$-\frac{t}{6}zx\frac{\partial}{\partial y}+yz\frac{\partial}{\partial z}\longrightarrow e_7$$
$$-\frac{t}{6}yz\frac{\partial}{\partial x}+zx\frac{\partial}{\partial z}\longrightarrow e_8$$
$$xyz\frac{\partial}{\partial z}\longrightarrow e_9$$

$$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\longrightarrow e_{10}.$$

The representations in Proposition 1.1 is actually the natural representations of $L(V_{-6t})$ on $A(V_{-6t})$. In view of Proposition 1.1 we suspect the following is true.

Let $V=\{z \in C^{n+1}: f(z)=0\}$ be a hypersurface with an isolated singularity at origin. Then the natural rerpesentation of $L(V)$ on $A(V)$ determines the complex structure of the singularity $(V, 0)$.

Chapter 2. Kac-Moody Lie algebras and isolated hypersurface singularities

In this chapter we shall attach a Kac-Moody Lie algebra to every isolated hypersurface singularity. Let $(V, 0)$ be an isolated hypersurface singularity. Let $\mathfrak{g}(V)$ be the maximal ideal of $L(V)$ consisting of nilpotent elements. Following [16], we shall construct a generalized Cartan matrix $C(V)$ from $\mathfrak{g}(V)$, which is a new invariant of $(V, 0)$ (cf. [20]).

Definition 2.1. An $l\times l$ matrix with entries in Z, $C=(c_{ij})$ is a generalized Cartan matrix if

a) $c_{ii}=2 \quad \forall i=1, \cdots, l$

b) $c_{ij}\leq 0 \quad \forall i, j=1, \cdots, l, i\neq j$

c) $c_{ij}=0$ if and only if $c_{ji}=0 \quad \forall i, j=1, \cdots, l, i\neq j$.

To each generalized Cartan matrix $C(V)$, one can associate a Lie algebra $\mathrm{KM}(C)$ (called a Kac-Moody Lie algebra) defined by generators:

$$\{f_1, \cdots, f_l, h_1, \cdots, h_l, e_1, \cdots, e_l\}$$

and relations:

$$\begin{aligned}
&[h_i, e_j]=c_{ij}e_j, \quad [h_i, f_j]=-c_{ij}f_j \qquad (\forall i, j=1, \cdots, l)\\
&[h_i, h_j]=0, \qquad [e_i, f_i]=h_i, \qquad \forall i, j=1, \cdots, l\\
&[e_i, f_j]=0, \quad (\mathrm{ad}\, e_i)^{-c_{ij}+1}e_j=0=(\mathrm{ad} f_i)^{-c_{ij}+1}f_j \quad (\forall i\neq j)
\end{aligned}$$

Let $H=Ch_i+\cdots Ch_l$; denote $\mathscr{L}_+(C)$ (resp. $\mathscr{L}_-(C)$) the subalgebra of $\mathrm{KM}(C)$ generated by $\{e_1, \cdots, e_l\}$ (resp. $(f_1, \cdots, f_l\}$) One shows that:

$$\mathrm{KM}(C)=\mathscr{L}_+(C)\oplus H\oplus \mathscr{L}_-(C)$$

One can also define $\mathscr{L}_+(C)$ by generators: $\{e_1, \cdots, e_l\}$ and relations:

$$(\mathrm{ad}\, e_i)^{-c_{ij}+1}e_j=0 \qquad \forall i, j=1, \cdots, l, i\neq j.$$

We shall construct a generalized Cartan matrix from an isolated hypersurface singularity $(V, 0)$. Let $\mathfrak{g}(V)$ be the set of all nilpotent elements in $L(V)$. Then $\mathfrak{g}(V)$ is the maximal nilpotent Lie subalgebra of $L(V)$. Let Der $\mathfrak{g}(V)$ be its derivation algebra.

Definition 2.2. A torus on $\mathfrak{g}(V)$ is a commutative subalgebra of Der $\mathfrak{g}(V)$ whose elements are semi-simple endomorphisms. A maximal torus is a torus not contained in any other torus.

The dimension of maximal torus is called Mostow number. Mostow number is an invariant of isolated singularity $(V, 0)$.

Theorem 2.3 (Mostow 4.1 of [11]). *If T_1 and T_2 are maximal tori of $\mathfrak{g}(V)$, then there exists $\theta \in$ Aut $\mathfrak{g}(V)$ (automorphism group of $\mathfrak{g}(V)$ such that $\theta T_1 \theta^{-1} = T_2$.*

Let T be a maximal torus and consider the root space decomposition of $\mathfrak{g}(V)$ relatively to T:

$$\mathfrak{g}(V) = \sum_{\beta \in R(T)} \mathfrak{g}(V)^{\beta}$$

where $\mathfrak{g}(V)^{\beta} = \{x \in g(V) : tx = \beta(t)x, \ \forall t \in T\}$ and $R(T) = \{\beta \in T^* : \mathfrak{g}(V)^{\beta} \neq (0)\}$. We denote: $m = \dim T$

$$R^1(T) = \{\beta \in R(T) : \mathfrak{g}(V) \not\subseteq [\mathfrak{g}(V), \mathfrak{g}(V)]\}$$
$$l_{\beta} = \dim (\mathfrak{g}(V)^{\beta} / [\mathfrak{g}(V), \mathfrak{g}(V)] \cap \mathfrak{g}(V)^{\beta}) \qquad \forall \beta \in R^1(T)$$
$$d_{\beta} = \dim \mathfrak{g}(V)^{\beta} \qquad \beta \in R^1(T).$$

The map $\beta \to d_{\beta}$, $R^1(T) \to N^*$ gives the partition:

$$R^1(T) = R^1(T)_{p_1} \cup \cdots \cup R^1(T)_{p_q}$$

where $p_1 < \cdots < p_q$, $R^1(T)_{p_i} \neq \phi$ and $R^1(T)_p = \{\beta \in R^1(T) : d_{\beta} = p\}$.

Let $s_i = \# R^1(T)_{p_i}$ and $s = s_1 + \cdots + s_q$; we number the elements of $R^1(T) = \{\beta_1, \cdots, \beta_s\}$ in such a way that:

$$R^1(T)_{p_1} = \{\beta_1, \cdots, \beta_s\}, \ R^1(T)_{p_2} = \{\beta_{s_1+1}, \cdots, \beta_{s_1+s_2}\}, \cdots.$$

Let $d_i = d_{\beta_i}$, $l_i = l_{\beta_i}$ and $l = l_1 + \cdots + l_s$ (one checks that $l = \dim \mathfrak{g}(V) / [\mathfrak{g}(V), \mathfrak{g}(V)]$). Let $P_s^{s_1 \cdots s_q}$ be the group of permutations of $\{1, \cdots, s\}$ which leave $\{1, \cdots, s_1\}$, $\{s_1+1, \cdots, s_1+s_2\}$, $\cdots$ invariant.

Lemma 2.4. *The integers $m, q, p_1, \cdots, p_q, s_1, \cdots, s_q, d_1, \cdots, d_s, l_1, \cdots, l_s, l$ defined above are invariants of isolated hypersurface singularity $(V, 0)$.*

Proof. Let T' be another maximal torus; then there exists $\theta \in \mathrm{Aut}\,\mathfrak{g}(V)$ such that $\theta T\theta^{-1}=T'$ (by Theorem 2.3). For T', we use the previous notations with prime. We have $m=m'$. The map

$$\begin{aligned}\check{\theta}\colon T^* &\longrightarrow T'^* \\ \beta &\longrightarrow \check{\theta}\beta\end{aligned}$$

where $\check{\theta}\beta(\theta t\theta^{-1})=\beta(t)$ $\forall \beta \in T^*$, $\forall t \in T$ is a vector space isomorphism and one has obviously:

$$\theta\mathfrak{g}(V)^{\beta}=\mathfrak{g}(V)^{\check{\theta}\beta} \qquad \forall \beta \in R(T)$$

Therefore $d'_{\check{\theta}\beta}=d_\beta$ $\forall \beta \in R^1(T)$ which gives

$$q'=q, \quad p'_i=p_i, \quad s'_i=s_i, \quad 1\leq i\leq q, \quad s'=s.$$

Since $\theta[\mathfrak{g}(V), \mathfrak{g}(V)]=[\mathfrak{g}(V), \mathfrak{g}(V)]$, one has $l'_{\check{\theta}\beta}=l_\beta$ $\forall \beta \in R^1(T)$. Q.E.D.

The map θ induces a bijection between: $R(T)$ and $R(T')$, $R^1(T)$ and $R^1(T')$, $R^1(T)_{p_i}$ and $R^1(T')_{p_i}$ $1\leq i\leq q$; thus there exists $\tau \in P_s^{s_1\cdots s_q}$ such that

$$\check{\theta}\beta_a=\beta'_{\tau a} \qquad 1\leq a\leq s.$$

Therefore, if T, T' are two maximal torus on $\mathfrak{g}(V)$, then there exists $\theta \in \mathrm{Aut}\,\mathfrak{g}(V)$ and $\tau \in P_s^{s_1\cdots s_1}$ such that $\theta\mathfrak{g}(V)^{\beta_a}=\mathfrak{g}(V)^{\beta'_{\tau a}}$ $1\leq a\leq s$.

Let $f\colon \{1, \cdots, l\}\rightarrow\{1, \cdots, s\}$ be defined by

$$f(i)=\begin{cases} 1 & \text{if } 1\leq i\leq l_1 \\ 2 & \text{if } l_1<i\leq l_1+l_2 \\ \vdots & \\ s & \text{if } l_1+\cdots+l_{s-1}<i\leq l. \end{cases}$$

For $\sigma \in P_s^{s_1\cdots s_q}$, we lift σ to $\hat{\sigma} \in P_l$ (Permutation group of l elements) such that $f\circ\hat{\sigma}=\sigma\circ f$. Define an action of $P_s^{s_1\cdots s_q}$ on the set of $l\times l$ matrices by setting

$$\sigma(c_{ij})_{1\leq i,j\leq l}=(c_{\hat{\sigma}i\hat{\sigma}j})_{1\leq i,j\leq l}.$$

Theorem 2.5. *For* $i, j \in \{1, \cdots, l\}$, $i\neq j$; *let*

$$-c_{ij}(T)=\mathrm{Min}\,\{-n \in N\cup\{0\}\colon (\mathrm{ad}\,v)^{-n+1}w=0 \qquad \begin{aligned}&\forall v \in \mathfrak{g}(V)^{\beta_{f(i)}}\\ &\forall w \in \mathfrak{g}(V)^{\beta_{f(j)}}\end{aligned}$$

with $(\mathrm{ad}\,0)^\circ=0$ *and let* $c_{ii}(T)=2$ *for* $i=1, \cdots, l$. *Then*

(i) $C(T)=(c_{ij}(T))_{1\leq i,j\leq l}$ *is a Cartan Matrix*

(ii) *For any $\sigma \in P_s^{s_1\cdots s_q}$, the action of σ on $C(T)$ is independent of the lifting $\hat{\sigma}$ of σ. Furthermore the $P_s^{s_1\cdots s_1}$ orbit of $C(T)$ is an invariant of $(V, 0)$.*

Proof. (i) Since $\operatorname{ad} v$ is nilpotent, $c_{ij}(T)$ is a well-defined non-positive integer for $i \neq j$: if $[v, w]=0$, then $[w, v]=0$, therefore $c_{ij}(T)=0$ implies $c_{ji}(T)=0$. Since $c_{ii}(T)=2$ by definition, $C(T)$ is a Cartan matrix.

(ii) Let T' be another maximal torus on $\mathfrak{g}(V)$. There exist $\theta \in \operatorname{Aut} \mathfrak{g}(V)$ and $\tau \in P_s^{s_1\cdots s_q}$ such that $\theta\mathfrak{g}(V)^{\beta_a}=\mathfrak{g}(V)^{\beta'_{\tau a}}$ $1 \leq a \leq s$; if $v \in \mathfrak{g}(V)^{\beta_a}$ and $w \in \mathfrak{g}(V)^{\beta_b}$ and if $i, j \in \{1, \cdots, l\}$ are such that $f(i)=a, f(j)=b$, then $(\operatorname{ad} v)^{-c_{ij}(T)+1}w=0$; thus $(\operatorname{ad} \theta v)^{-c_{ij}(T)+1}\theta w=0$ with

$$\theta v \in \mathfrak{g}(V)^{\beta'_{\tau a}}=\mathfrak{g}(V)^{\beta'_{f(\hat{\tau} i)}} \quad \text{and} \quad \theta w \in \mathfrak{g}(V)^{\beta'_{\tau b}}=\mathfrak{g}(V)^{\beta'_{f(\hat{\tau} j)}};$$

therefore $-c_{\hat{\tau} i \hat{\tau} j}(T') \leq -c_{ij}(T)$, and by symmetry $c_{\hat{\tau} i \hat{\tau} j}(T')=c_{ij}(T)$ which proves that $\tau C(T')=C(T)$. Q.E.D.

Definition 2.6. We choose arbitrarily A in $P_s^{s_1\cdots s_q}$-orbit of $C(T)$ (which has most $s!/s_1!\cdots s_q!$ elements) and we say by an abuse of language: "$\mathfrak{g}(V)$ is of type C" or "C is the Cartan matrix of $\mathfrak{g}(V)$". We denote:

$$\mathscr{J}_V(C)=\{T\colon T \text{ is a maximal torus on } \mathfrak{g}(V),\ C(T)=C\}$$
$$P_s^{s_1\cdots s_q}(C)=\{\sigma \in P_s^{s_1\cdots s_q}\colon \sigma C=C\}$$

Lemma 2.7. *If $T, T' \in \mathscr{J}_V(C)$ then there exist $\theta \in \operatorname{Aut} \mathfrak{g}(V)$ and $\tau \in P_s^{s_1\cdots s_q}(C)$ such that:*

$$\theta \in \mathfrak{g}(V)^{\beta_a}=\mathfrak{g}(V)^{\beta'_{\tau a}} \qquad \forall a=1, \cdots, s$$

Proof. By Mostow's Theorem, there exists $\theta \in \operatorname{Aut} \mathfrak{g}(V)$ and $\tau \in P_s^{s_1\cdots s_q}$ such that $\theta\mathfrak{g}(V)^{\beta_a}=\mathfrak{g}(V)^{\beta'_{\tau a}}$; by the proof of Theorem 2.5 (ii), $\tau C(T') =C(T)$; therefore $\tau C=\tau C(T')=C(T)=C$. Q.E.D.

We denote by $\operatorname{msg}(\mathfrak{g}(V))$ the set of minimal systems of generators of $\mathfrak{g}(V)$; by [3, Sect. 4, p. 119]: $(x_1, x_2, \cdots) \in \operatorname{msg}(\mathfrak{g}(V))$ if and only if $(x_1+[\mathfrak{g}(V), \mathfrak{g}(V)], x_2+[\mathfrak{g}(V), \mathfrak{g}(V)], \cdots)$ is a basis of $\mathfrak{g}(V)/[\mathfrak{g}(V), \mathfrak{g}(V)]$. Therefore each element of $\operatorname{msg}(\mathfrak{g}(V))$ is an l-tuple $(x_1, \cdots, x_l)$ where $l= \dim \mathfrak{g}(V)/[\mathfrak{g}(V), \mathfrak{g}(V)]$.

Let $T \in \mathscr{J}_V(C)$ and denote:

$$\operatorname{msg}(T)=\operatorname{msg}(\mathfrak{g}(V)) \in ((\mathfrak{g}(V)^{\beta_1})^{l_1} \times \cdots \times (\mathfrak{g}(V)^{\beta_s})^{l_s})$$

For all $(x_1, \cdots, x_l) \in \operatorname{msg}(T)$ one has:

$$(\operatorname{ad} x_i)^{-c_{ij}+1}x_j=0 \qquad 1 \leq i \neq j \leq l.$$

We shall now apply the above theory to study Lie algebras of rational double points. We shall use the following convention:

$$\mathfrak{g}^1=[\mathfrak{g}, \mathfrak{g}], \cdots, \mathfrak{g}^{P+1}=[\mathfrak{g}, \mathfrak{g}^P].$$

Proposition 2.8. *Let $V=\{(x, y, z) \in C^3 : x^2+y^2-z^{k+1}=0\}$ be the A_k singularity, $k \geq 1$. Then*

$$A(V)=C\{z\}/(z_0^k)=\langle 1, z, z^2, \cdots, z^{k-1}\rangle \quad \textit{with multiplication rule } z^k=0$$

$$L(V)=\begin{cases} \left\langle z\dfrac{\partial}{\partial z}, z^2\dfrac{\partial}{\partial z}, \cdots, z^{k-1}\dfrac{\partial}{\partial z}\right\rangle & \textit{if } k\geq 2 \\ 0 & \textit{if } k=1 \end{cases}$$

$$\mathfrak{g}(V)=\begin{cases} \left\langle \underset{\substack{\| \\ x_1}}{z^2\dfrac{\partial}{\partial z}}, \underset{\substack{\| \\ x_2}}{z^3\dfrac{\partial}{\partial z}}, \cdots, \underset{\substack{\| \\ x_{k-2}}}{z^{k-1}\dfrac{\partial}{\partial z}}\right\rangle & \textit{if } k\geq 3 \\ \left\langle z\dfrac{\partial}{\partial z}\right\rangle & \textit{if } k=2 \\ 0 & \textit{if } k=1. \end{cases}$$

For A_4 singularity,

$$\mathfrak{g}(V)=\left\langle \underset{\substack{\| \\ x_1}}{z^2\frac{\partial}{\partial z}}, \underset{\substack{\| \\ x_2}}{z^3\frac{\partial}{\partial z}}\right\rangle$$

with multiplication rule $[x_1, x_2]=0$.
The type of A_4 singularity: $=\dim \mathfrak{g}(V)/[\mathfrak{g}(V), \mathfrak{g}(V)]=2$.
The nilpotency of A_4 singularity: $=\min\{p \in N\cup\{0\}; \mathfrak{g}(V)^{p+1}=0\}=0$.
Let t_1, t_2 be two derivations of $\mathfrak{g}(V)$ defined by the following rules:

$$\begin{array}{ll} t_1: \mathfrak{g}(V)\longrightarrow\mathfrak{g}(V) \qquad & t_2: \mathfrak{g}(V)\longrightarrow\mathfrak{g}(V) \\ \quad x_1\longrightarrow x_1 & \quad x_1\longrightarrow 0 \\ \quad x_2\longrightarrow 0 & \quad x_2\longrightarrow x_2 \end{array}$$

Then $T=Ct_1\oplus Ct_2$ is an uniqne maximal torus associated to $\mathfrak{g}(V)$.
Let β_i: $T\to C$ be a linear map with $\beta_i(t_j)=\delta_{ij}$ for $i, j=1, 2$.

$$\begin{aligned}\mathfrak{g}(V)&=Cx_1\oplus Cx_2\\ &=\mathfrak{g}^{\beta_1}\oplus\mathfrak{g}^{\beta_2}\end{aligned}$$

(x_1, x_2) is a T-minimal system of generators.

The generalized Cartan matrix associated to A_4 is

$$C(A_4)=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}.$$

For A_5 singularity

$$\mathfrak{g}(V)=\Big\langle \underset{\substack{\| \\ x_1}}{z^2\frac{\partial}{\partial z}},\ \underset{\substack{\| \\ x_2}}{z^3\frac{\partial}{\partial z}},\ \underset{\substack{\| \\ x_3}}{z^4\frac{\partial}{\partial z}} \Big\rangle$$

with multiplication rules:

$$\begin{aligned}[x_1, x_2]&=x_3\\ [x_1, x_3]&=0\\ [x_2, x_3]&=0.\end{aligned}$$

The type of A_5 singularity: $=\dim \mathfrak{g}(V)/[\mathfrak{g}(V), \mathfrak{g}(V)]=2$.
The nilpotency of A_5 singularity: $=\min\{p \in N\cup\{0\}: \mathfrak{g}(V)^{p+1}=0\}=1$.

Let t_1, t_2 be two derivations of $\mathfrak{g}(V)$ *defined by*

$$\begin{array}{ll} t_1;\ \mathfrak{g}\longrightarrow\mathfrak{g} & \qquad t_2:\ \mathfrak{g}\longrightarrow\mathfrak{g}\\ \quad x_1\longrightarrow x_1 & \qquad\quad x_1\longrightarrow 0\\ \quad x_2\longrightarrow 0 & \qquad\quad x_2\longrightarrow x_2\\ \quad x_3\longrightarrow x_3 & \qquad\quad x_3\longrightarrow x_3. \end{array}$$

Then $T=\boldsymbol{C}t_1\oplus\boldsymbol{C}t_2$ is a torus of $\mathfrak{g}(V)$. *Since* $\dim T=2=$ *the type of A_5, T is a maximal torus of* $\mathfrak{g}(V)$.

Let β_i: $T\rightarrow C$ be a linear map with $\beta_i(t_j)=\delta_{ij}$ for $i, j=1, 2$.

$$\mathfrak{g}(V)=\underset{\substack{\| \\ \boldsymbol{C}x_1}}{\mathfrak{g}^{\beta_1}}\oplus \underset{\substack{\| \\ \boldsymbol{C}x_2}}{\mathfrak{g}^{\beta_2}}\oplus \underset{\substack{\| \\ \boldsymbol{C}x_3}}{\mathfrak{g}^{\beta_1+\beta_2}}$$

(x_1, x_2) *is a T-minimal system of generators. The generalized Cartan matrix associated to A_5 is*

$$C(A_5)=\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}.$$

Far A_6 singularity

$$L(V)=\Big\langle \underset{\substack{\| \\ x_1}}{z^2\frac{\partial}{\partial z}},\ \underset{\substack{\| \\ x_2}}{z^3\frac{\partial}{\partial z}},\ \underset{\substack{\| \\ x_3}}{z^4\frac{\partial}{\partial z}},\ \underset{\substack{\| \\ x_4}}{z^5\frac{\partial}{\partial z}} \Big\rangle$$

with multiplication rules:

$$\begin{array}{lll} [x_1, x_2]=x_3 & [x_2, x_3]=0 & [x_3, x_4]=0 \\ [x_1, x_3]=2x_4 & [x_2, x_4]=0 & \\ [x_1, x_4]=0 & & \end{array}$$

The type of A_6 *singularity* $=\dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]=2$.
The nilpotency of A_6 *singularity* $=\min\{p \in \boldsymbol{N}\cup\{0\};\ \mathfrak{g}(V)^{p+1}=0\}=2$.

Let t_1, t_2 *be two derivations of* $\mathfrak{g}(V)$ *defined by*

$$\begin{array}{ll} t_1\colon \mathfrak{g}\longrightarrow\mathfrak{g} & t_2\colon \mathfrak{g}\longrightarrow\mathfrak{g} \\ \quad x_1\longrightarrow x_1 & \quad x_1\longrightarrow 0 \\ \quad x_2\longrightarrow 0 & \quad x_2\longrightarrow x_2 \\ \quad x_3\longrightarrow x_3 & \quad x_3\longrightarrow x_3 \\ \quad x_4\longrightarrow 2x_4 & \quad x_4\longrightarrow x_4. \end{array}$$

Then $T=\boldsymbol{C}t_1\oplus\boldsymbol{C}t_2$ *is a torus of* $\mathfrak{g}(V)$. *Since* $\dim T=2=$ *the type of* A_6, T *is a maximal torus of* $\mathfrak{g}(V)$. *Let* $\beta_i\colon T\to\boldsymbol{C}$ *be a linear map with* $\beta_i(t_j)=\delta_{ij}$ *for* $i, j=1, 2$.

$$\begin{array}{ccccccccc} \mathfrak{g}(V)= & \mathfrak{g}^{\beta_1} & \oplus & \mathfrak{g}^{\beta_2} & \oplus & \mathfrak{g}^{\beta_1+\beta_2} & \oplus & \mathfrak{g}^{2\beta_1+\beta_2} \\ & \| & & \| & & \| & & \| \\ & \boldsymbol{C}x_1 & & \boldsymbol{C}x_2 & & \boldsymbol{C}x_3 & & \boldsymbol{C}x_4 \end{array}$$

(x_1, x_2) *is a T-minimal system of generators. The generalized Cartan matrix associated to* A_6 *is*

$$C(A_6)=\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.$$

For A_k *singularity* $k\geq 7$,

$$\mathfrak{g}(V)=\Big\langle \underset{\substack{\|\\ x_1}}{z^2\frac{\partial}{\partial z}},\ \underset{\substack{\|\\ x_2}}{z^3\frac{\partial}{\partial z}},\ \cdots,\ \underset{\substack{\|\\ x_{k-2}}}{z^{k-1}\frac{\partial}{\partial z}} \Big\rangle$$

with multiplication rules:

$$\begin{array}{lll} [x_1, x_2]=x_3 & [x_2, x_3]=x_5 & \vdots \\ [x_1, x_3]=2x_4 & [x_2, x_4]=2x_6 & \vdots \\ \quad\vdots & \quad\vdots & \vdots \\ [x_1, x_{k-4}]=(k-5)x_{k-3} & [x_2, x_{k-4}]=(k-6)x_{k-2} & \vdots \\ [x_1, x_{k-3}]=(k-4)x_{k-2} & [x_2, x_{k-3}]=0 & \vdots \\ [x_1, x_{k-2}]=0 & [x_2, x_{k-2}]=0 & \vdots \end{array}$$

The type of A_k *singularity for* $k\geq 7$: $=\dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]=2$.

The nilpotency of A_k singularity for $k\geq 7$: $=\min\{p\in N\cup\{0\};\ g^{p+1}=0\}$
$=k-4.$

Let t be the derivation of $\mathfrak{g}(V)$ *defined by*

$$\begin{aligned} t\colon \mathfrak{g} &\longrightarrow \mathfrak{g} \\ x_1 &\longrightarrow x_1 \\ x_2 &\longrightarrow 2x_2 \\ \vdots\ & \qquad \vdots \\ x_{k-2} &\longrightarrow (k-2)x_{k-2}. \end{aligned}$$

We claim that $T=Ct$ is an unique maximal torus of $\mathfrak{g}(V)$. *Let β: $T\to \boldsymbol{C}$ be a linear map such that* $\beta(t)=1$.

$$\begin{array}{cccccc} \mathfrak{g}(V)= & \mathfrak{g}^{\beta} \oplus & \mathfrak{g}^{2\beta} \oplus & \cdots & \oplus & \mathfrak{g}^{(k-2)\beta} \\ & \| & \| & & & \| \\ & \boldsymbol{C}x_1 & \boldsymbol{C}x_2 & & & \boldsymbol{C}x_{k-2} \end{array}$$

(x_1, x_2) *is a T-minimal system of generators.*

Observe that $(\operatorname{ad} x_1)^{k-3}x_2=0$ *but* $(\operatorname{ad} x_1)^{k-4}x_2\neq 0$. *Therefore* $c_{12}=-(k-4)$.

In order to compute c_{21} we have two cases.

Case 1. *k is odd and $k=2l+5\geq 7$*

$$\operatorname{ad} x_2^{l+1}(x_1)=-(2.2-1)(2.3-1)\cdots(2l-1)x_{2l+3}$$
$$\operatorname{ad} x_2^{l+2}(x_1)=0$$
$$\therefore\quad c_{21}=-(l+1)=-\frac{k-3}{2}.$$

Case 2. *k is even and $k=2l+6\geq 7$*

$$\operatorname{ad} x_2^{l+1}(x_1)=-(2.2-1)(2.3-1)(2.4-1)\cdots(2l-1)x_{2l+3}$$
$$\operatorname{ad} x_2^{l+2}(x_1)=0$$
$$c_{21}=-(l+1)=-\frac{k-4}{2}.$$

The generalized Cartan matrix associated to A_k for $k\geq 7$ is

$$C(A_k)=\begin{cases} \begin{pmatrix} 2 & -(k-4) \\ -\dfrac{k-3}{2} & 2 \end{pmatrix} & \textit{if } k \textit{ is odd and } k\geq 7 \\ \begin{pmatrix} 2 & -(k-4) \\ -\dfrac{k-4}{2} & 2 \end{pmatrix} & \textit{if } k \textit{ is even and } k\geq 7. \end{cases}$$

We now provide the proof that the unique maximal torus is spanned by t defined as above.

Let δ be a derivation of $\mathfrak{g}(V)$

$$\delta(x_i)=a_{i1}x_1+a_{i2}x_2+\cdots+a_{i,k-2}x_{k-2} \qquad \text{for } 1\leq i\leq k-2.$$

$$\begin{aligned}[x_1, x_{k-2}]=0&\Rightarrow[\delta(x_1), x_{k-2}]+[x_1, \delta(x_{k-2})]=0\\ &\Rightarrow a_{k-2,2}x_3+2a_{k-2,3}x_4+3a_{k-2,4}x_5+\cdots\\ &\qquad\qquad +(k-4)a_{k-2,k-3}x_{k-2}=0\\ &\Rightarrow a_{k-2,2}=0=a_{k-2,3}=\cdots=a_{k-2,k-3}=0,\\ [x_2, x_{k-2}]=0&\Rightarrow[\delta(x_2), x_{k-2}]+[x_2, \delta(x_{k-2})]=0\\ &\Rightarrow -a_{k-2,1}x_3+a_{k-2,3}x_5+2a_{k-2,4}x_6+\cdots\\ &\qquad\qquad +(k-6)a_{k-2,k-4}x_{k-2}=0\\ &\Rightarrow a_{k-2,1}=0.\end{aligned}$$

We assume that $a_{j+1,i}=0$ for $i\leq j$. We shall prove that $a_{ji}=0$ for $i\leq j-1$. We first consider $j\geq 3$. We may as well assume that $j\leq k-3$ by what we have proved above.

$$\begin{aligned}[x_1, x_j]&=(j-1)x_{j+1}\Rightarrow[\delta(x_1), x_j]+[x_1, \delta(x_j)]=(j-1)\delta(x_{j+1})\\ &\Rightarrow[a_{11}(j-1)x_{j+1}+a_{12}(j-2)x_{j+2}+\cdots]\\ &\quad +[a_{j2}x_3+2a_{j3}x_4+\cdots+(j-2)a_{j,j-1}x_j+(j-1)a_{j,j}x_{j+1}+\cdots]\\ &=(j-1)[a_{j+1,1}x_1+a_{j+1,2}x_2+\cdots+a_{j+1,k-2}x_{k-2}] \qquad (2.1)\\ &\Rightarrow a_{j,2}=(j-1)a_{j+1,3}=0\\ &\quad a_{j,3}=\frac{j-1}{2}a_{j+1,4}=0\\ &\quad\vdots\\ &\quad a_{j,j-1}=\frac{j-1}{j-2}a_{j+1,j}=0\end{aligned}$$

$$\begin{aligned}[x_2, x_j]&=\begin{cases}(j-2)x_{j+2} & \text{if } j+2\leq k-2\Rightarrow[\delta(x_2), x_j]+[x_2, \delta(x_j)]\\ 0 & \text{if } j+2\geq k-1\end{cases}\\ &=\begin{cases}(j-2)\delta(x_{j+2}) & \text{if } j+2\leq k-2\\ 0 & \text{if } j+2\geq k-1\end{cases} \qquad (2.2)\\ &\Rightarrow[a_{21}(j-1)x_{j+1}+a_{22}(j-2)x_{j+2}+\cdots]\\ &\qquad\qquad +[-a_{j1}x_3+a_{j3}x_5+2a_{j4}x_6+\cdots]\\ &=\begin{cases}(j-2)[a_{j+2,1}x_1+a_{j+2,2}x_2+\cdots+a_{j+2,k-2}x_{k-2}] & \text{if } j+2\leq k-2\\ 0 & \text{if } j+2\geq k-1\end{cases}\end{aligned}$$

(2.3) $\quad \Rightarrow a_{j,1} = -(j-2)a_{j+2,3} = 0.$

(2.1) and (2.2) imply that $a_{ji}=0$ for $j\geq 3$ and $i<j$.

Set $j=3$ in (2.2). By comparing the coefficient of x_4, we get

$$2a_{21} = a_{54}$$
$$\Rightarrow a_{21} = 0$$

Hence δ is represented by upper triangular matrix

$$A = \begin{pmatrix} a_{11} & a_{12}\cdots & \cdots a_{1,k-2} \\ & a_{22}\cdots & \cdots a_{2,k-2} \\ 0 & \ddots & \vdots \\ & & a_{k-2,k-2} \end{pmatrix}.$$

δ is semi-simple $\Leftrightarrow AA^* = A^*A$

$\Rightarrow$ the length of i^{th} row of A = the length of j^{th} column of A.

$\Rightarrow a_{ij}=0$ for $i\neq j$.

$$A = \begin{pmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & \ddots \\ & & & a_{k-2,k-2} \end{pmatrix}$$

(2.4) $\quad [x_1, x_2] = x_3 \quad \Rightarrow a_{11} + a_{22} = a_{33}$

(2.5) $\quad [x_1, x_3] = 2x_4 \quad \Rightarrow 2a_{11} + 2a_{33} = 2a_{44}$

(2.6) $\quad [x_1, x_4] = 3x_5 \quad \Rightarrow 3a_{11} + 3a_{44} = 3a_{55}$

(2.7) $\quad$ (2.4), (2.5) and (2.6) $\Rightarrow 3a_{11} + a_{22} = a_{55}$

(2.8) $\quad [x_2, x_3] = x_5 \quad \Rightarrow a_{22} + a_{33} = a_{55}$

Put (2.4) and (2.7) in (2.8),

$$a_{22} + a_{11} + a_{22} = 3a_{11} + a_{22}$$
$$\Rightarrow a_{22} = 2a_{11}.$$

We assume that $a_{j-1,j-1} = (j-1)a_{11}$. We shall prove that $a_{j,j} = ja_{11}$.

$$\begin{aligned}[x_1, x_{j-1}] = (j-2)x_j &\Rightarrow [a_{11}x_1, x_{j-1}] + [x_1, a_{j-1,j-1}x_{j-1}] = (j-2)a_{jj}x_j \\ &\Rightarrow (j-2)a_{11} + (j-2)a_{j-1,j-1} = (j-2)a_{jj} \\ &\Rightarrow a_{jj} = ja_{11} + a_{j-1,j-1} \\ &\qquad = ja_{11}.\end{aligned}$$

We have proved that any derivation of $\mathfrak{g}(V)$ must be a constant multiple of t. To prove that t is really a derivation, we first observe that the multiplication rule of $\mathfrak{g}(V)$ is described by the following formula

$$[x_i, x_j]=\begin{cases}(j-i) \quad x_{j+i} & \text{for } i+j\leq k-2\\ 0 & \text{for } i+j\geq k-1.\end{cases}$$

Assume $j+i\leq k-2$. Then

$$t[x_i, x_j]=t[(j-i)x_{j+i}]=(j-i)t(x_{j+i})=(j-i)(j+i)x_{j+i}$$
$$\begin{aligned}[t(x_i), x_j]+[x_i, t(x_j)]&=[ix_i, x_j]+[x_i, jx_j]=(i+j)[x_i, x_j]\\ &=(j-i)(j+i)x_{i+j}.\end{aligned}$$

Assume $j+i\geq k-1$. Then

$$t[x_i, x_j]=t(0)=0$$
$$[t(x_i), x_j]+[x_i, t(x_j)]=[ix_i, x_j]+[x_i, jx_j]=0.$$

In both cases, $t[x_i, x_j]=[t(x_i), x_j]+[x_i, t(x_j)]$. Hence t is a derivation.

Proposition 2.9. *Let $V=\{(x, y, z)\in C^3: z^{k-1}+zy^2+x^2=0\}$ be the D_k singularity, $k\geq 4$. Then*

$$A(V)=\langle 1, z, y, z^2, z^3, \cdots, z^{k-2}\rangle \quad \textit{with multiplication rule}$$
$$zy=0$$
$$y^2=-(k-1)z^{k-2}$$
$$z^{k-1}=0.$$

$$L(V)=\left\langle z\frac{\partial}{\partial z}+\frac{k-2}{2}y\frac{\partial}{\partial y}, y\frac{\partial}{\partial z}+(k-1)z^{k-3}\frac{\partial}{\partial y}, z^{k-2}\frac{\partial}{\partial y}, z^2\frac{\partial}{\partial z}, \cdots, z^{k-2}\frac{\partial}{\partial z}\right\rangle$$

$$\mathfrak{g}(V)=\begin{cases}\Big\langle \underbrace{y\frac{\partial}{\partial z}+(k-1)z^{k-3}\frac{\partial}{\partial y}}_{x_1}, \underbrace{z^{k-2}\frac{\partial}{\partial y}}_{x_2}, \underbrace{z^2\frac{\partial}{\partial z}}_{x_3}, \cdots, \underbrace{z^{k-2}\frac{\partial}{\partial z}}_{x_{k-1}}\Big\rangle & \textit{for } k\geq 5\\ \Big\langle \underbrace{z^2\frac{\partial}{\partial y}}_{x_1}, \underbrace{z^2\frac{\partial}{\partial z}}_{x_2}\Big\rangle & \textit{for } k=4.\end{cases}$$

For D_4 singularity, $\mathfrak{g}(V)=\langle x_1, x_2\rangle$ with multiplication rule $[x_1, x_2]=0$. The type of D_4 singularity: $=\dim \mathfrak{g}(V)/[\mathfrak{g}(V), \mathfrak{g}(V)]=2$. The nilpotency *of D_4 singularity*: $=\min\{p \in N \cup \{0\}: \mathfrak{g}(V)^{p+1}=0\}=0$. *Let t_1, t_2 be two derivations of $\mathfrak{g}(V)$ defined by the following rules.*

$$\begin{array}{lll} t_1: \mathfrak{g}(V) \longrightarrow \mathfrak{g}(V) & \qquad & t_2: \mathfrak{g}(V) \longrightarrow \mathfrak{g}(V) \\ \quad x_1 \longrightarrow x_1 & & \quad x_1 \longrightarrow 0 \\ \quad x_2 \longrightarrow 0 & & \quad x_2 \longrightarrow x_2 \end{array}$$

Then $T=Ct_1 \oplus Ct_2$ is an unique maximal torus associated to $\mathfrak{g}(V)$. Let β_i: $T \to C$ be a linear map with $\beta_i(t_j)=\delta_{ij}$ for $i, j=1, 2$.

$$\begin{aligned} \mathfrak{g}(V) &= Cx_1 \oplus Cx_2 \\ &= \mathfrak{g}^{\beta_1} \oplus \mathfrak{g}^{\beta_2} \end{aligned}$$

(x_1, x_2) is a T-minimal system of generators. The generalized Cartan matrix associated to D_4 is

$$C(D_4)=\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

For D_5 singularity,

$$\mathfrak{g}(V)=\Big\langle \underset{\substack{\| \\ x_1}}{y\frac{\partial}{\partial z}+4z^2\frac{\partial}{\partial y}},\ \underset{\substack{\| \\ x_2}}{z^3\frac{\partial}{\partial y}},\ \underset{\substack{\| \\ x_3}}{z^2\frac{\partial}{\partial z}},\ \underset{\substack{\| \\ x_4}}{z^3\frac{\partial}{\partial z}} \Big\rangle$$

with multiplication rules:

$$\begin{array}{lll} [x_1, x_2]=-x_4 & [x_2, x_3]=0 & [x_3, x_4]=0 \\ [x_1, x_3]=-8x_2 & [x_2, x_4]=0 & \\ [x_1, x_4]=0 & & \end{array}$$

The type of D_5 singularity: $=\dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]=2$.
The nilpotency of D_5 singularity: $=\min\{p \in N \cup \{0\}: \mathfrak{g}^{p+1}=0\}=1$.

Let t_1, t_2 be two derivations of $\mathfrak{g}(V)$ defined by

$$\begin{array}{lll} t_1: \mathfrak{g} \longrightarrow \mathfrak{g} & \qquad & t_2: \mathfrak{g} \longrightarrow \mathfrak{g} \\ \quad x_1 \longrightarrow x_1 & & \quad x_1 \longrightarrow 0 \\ \quad x_2 \longrightarrow 0 & & \quad x_2 \longrightarrow x_2 \\ \quad x_3 \longrightarrow -x_3 & & \quad x_3 \longrightarrow x_3 \\ \quad x_4 \longrightarrow x_4 & & \quad x_4 \longrightarrow x_4. \end{array}$$

Then $T=Ct_1 \oplus Ct_2$ is an unique maximal torus of $\mathfrak{g}(V)$. The root space $R(T)$ is $\langle \beta_1, \beta_2 \rangle$ where β_i: $T \to C$ is a linear map with $\beta_i(t_j)=\delta_{ij}$ for $i, j=$ 1, 2.

$$\begin{array}{l} \mathfrak{g}(V)=\mathfrak{g}^{\beta_1}\oplus\mathfrak{g}^{\beta_2}\oplus\mathfrak{g}^{-\beta_1+\beta_2}\oplus\mathfrak{g}^{\beta_1+\beta_2} \\ \qquad\quad \| \qquad\; \| \qquad\;\; \| \qquad\qquad \| \\ \qquad\; Cx_1 \;\; Cx_2 \;\; Cx_3 \qquad\; Cx_4 \end{array}$$

(x_1, x_3) is a T-minimal system of generators. The generalized Cartan matrix associated to D_5 is

$$C(D_5)=\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.$$

For D_6 singularity,

$$\mathfrak{g}(V)=\Big\langle \underbrace{y\frac{\partial}{\partial z}+5z^3\frac{\partial}{\partial y}}_{x_1},\ \underbrace{z^4\frac{\partial}{\partial y}}_{x_2},\ \underbrace{z^2\frac{\partial}{\partial z}}_{x_3},\ \underbrace{z^3\frac{\partial}{\partial z}}_{x_4},\ \underbrace{z^4\frac{\partial}{\partial z}}_{x_5}\Big\rangle$$

with multiplication rules:

$$\begin{array}{llll} [x_1, x_2]=-x_5 & [x_2, x_3]=0 & [x_3, x_4]=x_5 & [x_4, x_5]=0 \\ [x_1, x_3]=-15x_2 & [x_2, x_4]=0 & [x_3, x_5]=0 & \\ [x_1, x_4]=0 & [x_2, x_5]=0 & & \\ [x_1, x_5]=0 & & & \end{array}$$

The type of D_6 singularity: $=\dim \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]=3$.
The nilpotency of D_6 singularity: $=\min\{p\in N\cup\{0\}: \mathfrak{g}^{p+1}=0\}=2$.

Let t_1, t_2 be two derivations of $\mathfrak{g}(V)$ defined by

$$\begin{array}{ll} t_1\colon \mathfrak{g}\longrightarrow\mathfrak{g} & t_2\colon \mathfrak{g}\longrightarrow\mathfrak{g} \\ \quad x_1\longrightarrow x_1 & \quad x_1\longrightarrow 0 \\ \quad x_2\longrightarrow 0 & \quad x_2\longrightarrow x_2 \\ \quad x_3\longrightarrow -x_3 & \quad x_3\longrightarrow x_3 \\ \quad x_4\longrightarrow 2x_4 & \quad x_4\longrightarrow 0 \\ \quad x_5\longrightarrow x_5 & \quad x_5\longrightarrow x_5. \end{array}$$

Then $T=Ct_1\oplus Ct_2$ is an unique maximal torus of $\mathfrak{g}(V)$.

Let β_i: $T\rightarrow C$ be a linear map with $\beta_i(t_j)=\delta_{ij}$ for $i, j=1, 2$.

$$\begin{array}{l} \mathfrak{g}(V)=\mathfrak{g}^{\beta_1}\oplus\mathfrak{g}^{\beta_2}\oplus\mathfrak{g}^{-\beta_1+\beta_2}\oplus\mathfrak{g}^{2\beta_1}\oplus\mathfrak{g}^{\beta_1+\beta_2} \\ \qquad\quad \| \qquad\; \| \qquad\;\; \| \qquad\qquad \| \qquad\; \| \\ \qquad\; Cx_1 \;\; Cx_2 \;\; Cx_3 \qquad\; Cx_4 \;\; Cx_5 \end{array}$$

(x_1, x_3, x_4) is a T-minimal system of generators. The generalized Cartan matrix associated to D_6 is

$$C(D_6)=\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

For D_7 singularity,

$$\mathfrak{g}(V)=\left\langle \underset{x_1}{\underset{\|}{y\frac{\partial}{\partial z}+6z^4\frac{\partial}{\partial y}}},\ \underset{x_2}{\underset{\|}{z^5\frac{\partial}{\partial y}}},\ \underset{x_3}{\underset{\|}{z^2\frac{\partial}{\partial z}}},\ \underset{x_4}{\underset{\|}{z^3\frac{\partial}{\partial z}}},\ \underset{x_5}{\underset{\|}{z^4\frac{\partial}{\partial z}}},\ \underset{x_6}{\underset{\|}{z^5\frac{\partial}{\partial z}}} \right\rangle$$

with multiplication rules:

$$\begin{array}{lllll}
[x_1, x_2]=-x_6 & [x_2, x_3]=0 & [x_3, x_5]=x_5 & [x_4, x_5]=0 & [x_5, x_6]=0 \\
[x_1, x_3]=-24x_2 & [x_2, x_4]=0 & [x_3, x_5]=2x_6 & [x_4, x_6]=0 & \\
[x_1, x_4]=0 & [x_2, x_5]=0 & [x_3, x_6]=0 & & \\
[x_1, x_5]=0 & [x_2, x_6]=0 & & & \\
[x_1, x_6]=0 & & & &
\end{array}$$

The type of D_7 singularity: $=\dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]=3$.
The nilpotency of D_7 singularity: $=\min\{p \in N\cup\{0\}\colon \mathfrak{g}^{p+1}=0\}=2$.
Let t_1, t_2 be two derivations of $\mathfrak{g}(V)$ defined by

$$\begin{array}{ll}
t_1\colon \mathfrak{g}\longrightarrow\mathfrak{g} & t_2\colon \mathfrak{g}\longrightarrow\mathfrak{g} \\
\quad x_1\longrightarrow x_1 & \quad x_1\longrightarrow 0 \\
\quad x_2\longrightarrow 0 & \quad x_2\longrightarrow x_2 \\
\quad x_3\longrightarrow -x_3 & \quad x_3\longrightarrow x_3 \\
\quad x_4\longrightarrow 3x_4 & \quad x_4\longrightarrow -x_4 \\
\quad x_5\longrightarrow 2x_5 & \quad x_5\longrightarrow 0 \\
\quad x_6\longrightarrow x_6 & \quad x_5\longrightarrow x_6.
\end{array}$$

Then $T=Ct_1\oplus Ct_2$ is an unique maximal torus on $\mathfrak{g}(V)$. Let $\beta_i\colon T\to C$ be a linear map with $\beta_i(t_j)=\delta_{ij}$ for $i, j=1, 2$.

$$\mathfrak{g}(V)=\underset{Cx_1}{\underset{\|}{\mathfrak{g}^{\beta_1}}}\oplus\underset{Cx_2}{\underset{\|}{\mathfrak{g}^{\beta_2}}}\oplus\underset{Cx_3}{\underset{\|}{\mathfrak{g}^{\beta_2-\beta_1}}}\oplus\underset{Cx_4}{\underset{\|}{\mathfrak{g}^{3\beta_1-\beta_2}}}\oplus\underset{Cx_5}{\underset{\|}{\mathfrak{g}^{2\beta_1}}}\oplus\underset{Cx_6}{\underset{\|}{\mathfrak{g}^{\beta_1+\beta_2}}}$$

(x_1, x_3, x_4) is a T-minimal system of generators. The generalized Cartan matrix associated to D_7 is

$$C(D_7)=\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

For D_k singularity, $k \geq 8$.

$$\mathfrak{g}(V) = \Big\langle \underbrace{y\frac{\partial}{\partial z} + (k-1)z^{k-3}\frac{\partial}{\partial y}}_{x_1},\ \underbrace{z^{k-2}\frac{\partial}{\partial y}}_{x_2},\ \underbrace{z^2\frac{\partial}{\partial z}}_{x_3},\ \underbrace{z^3\frac{\partial}{\partial z}}_{x_4},\ \cdots,\ \underbrace{z^{k-2}\frac{\partial}{\partial z}}_{x_{k-1}} \Big\rangle$$

with multiplication rules:

$$\begin{array}{lll}
[x_1, x_2] = -x_{k-1} & [x_2, x_3] = 0 & [x_3, x_4] = x_5 \\
[x_1, x_3] = -(k-3)(k-1)x_2 & [x_2, x_4] = 0 & [x_3, x_5] = 2x_6 \\
[x_1, x_4] = 0 & \vdots & [x_3, x_6] = 3x_7 \\
\vdots & [x_2, x_{k-1}] = 0 & \vdots \\
[x_1, x_{k-1}] = 0 & & [x_3, x_{k-1}] = (k-6)x_{k-2} \\
 & & [x_3, x_{k-2}] = (k-5)x_{k-1} \\
 & & [x_3, x_{k-1}] = 0
\end{array}$$

$$\begin{array}{ll}
[x_4, x_5] = x_7 & \vdots \\
[x_4, x_6] = 2x_8 & \vdots \\
[x_4, x_7] = 3x_9 & \vdots \\
\vdots & \vdots \\
[x_4, x_{k-3}] = (k-7)x_{k-1} & \vdots \\
[x_4, x_{k-2}] = 0 & \vdots \\
[x_4, x_{k-1}] = 0 & \vdots
\end{array}$$

The type of D_k singularity for $k \geq 8$: $= \dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = 3$.
The nilpotency of D_k singularity for $k \geq 8$: $= \min\{p \in N \cup \{0\};\ \mathfrak{g}^{p+1} = 0\} = k-5$.

Let t be the derivation of $\mathfrak{g}(V)$ defined by

$$\begin{aligned}
t:\ & \mathfrak{g} \longrightarrow \mathfrak{g} \\
& x_1 \longrightarrow x_1 \\
& x_2 \longrightarrow \frac{k-2}{k-4}x_2 \\
& x_3 \longrightarrow \frac{2.1}{k-4}x_3 \\
& x_4 \longrightarrow \frac{2.2}{k-4}x_4 \\
& x_5 \longrightarrow \frac{2.3}{k-4}x_5
\end{aligned}$$

$$\vdots \qquad\qquad \vdots$$
$$x_{k-1} \longrightarrow \frac{2(k-3)}{k-4} x_{k-1}.$$

We claim that $T = Ct$ *is an unique maximal torus of* $\mathfrak{g}(V)$. *Let* $\beta\colon T \to C$ *be a linear map such that* $\beta(t) = 1$. *Then*

$$\begin{array}{cccccccc} \mathfrak{g}(V) = \mathfrak{g}^{((k-2)/(k-4))\beta} & \oplus & \mathfrak{g}^{(2/(k-4))\beta} & \oplus & \mathfrak{g}^{((2\cdot 2)/(k-4))\beta} & \oplus \cdots \oplus & \mathfrak{g}^{(2(k-3)/(k-4))\beta}. \\ \| & & \| & & \| & & \| \\ Cx_1 & & Cx_2 & & Cx_3 & & Cx_{k-1} \end{array}$$

(x_1, x_3, x_4) *is a T-minimal system of generators. The generalized Cartan matrix associated to* D_k *for* $k \geq 8$ *is*

$$C(D_k) = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -(k-5) \\ 0 & -(k-6) & 2 \end{pmatrix}.$$

We now provide the proof that the unique maximal torus is spanned by t defined as sbove.

Let δ be a derivation of $\mathfrak{g}(V)$

$$\delta(x_i) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{i,k-1}x_{k-1} \qquad \text{for } 1 \leq i \leq k-1.$$

We shall prove by induction that $a_{ij} = 0$ for $i \geq 5$ and $i > j$.

1st *Step.* $i = k-1$

$$\begin{aligned} [x_3, x_{k-1}] = 0 &\Rightarrow [\delta(x_3), x_{k-1}] + [x_3, \delta(x_{k-1})] = 0 \\ &\Rightarrow (k-3)(k-1)a_{k-1,1}x_2 + a_{k-1,4}x_5 + 2a_{k-1,5}x_6 \\ &\qquad\qquad + \cdots + (k-5)a_{k-1,k-2}x_{k-1} = 0 \\ &\Rightarrow a_{k-1,1} = 0 = a_{k-1,4} = a_{k-1,5} = \cdots = a_{k-1,k-2} \end{aligned}$$

$$\begin{aligned} [x_1, x_{k-1}] = 0 &\Rightarrow [\delta(x_1), x_{k-1}] + [x_1, \delta(x_{k-1})] = 0 \\ &\Rightarrow -a_{k-1,2}x_{k-1} - (k-3)(k-1)a_{k-1,3}x_2 = 0 \\ &\Rightarrow a_{k-1,2} = 0 = a_{k-1,3} \end{aligned}$$

2nd *Step.* Assuming that it is true for i_0, we shall prove that it is also true for $i_0 - 1$. Notice that we may assume that $6 \leq i_0 \leq k-2$

$$[x_3, x_{i_0-1}] = (i_0-4)x_{i_0} \Rightarrow [\delta(x_3), x_{i_0-1}] + [x_3, \delta(x_{i_0-1})] = (i_0-4)\delta(x_{i_0}).$$

For $i_0 \geq 5$. we have

$$[(i_0-4)a_{33}x_{i_0}+(i_0-5)a_{34}x_{i_0+1}+\cdots]+[(k-3)(k-1)a_{i_0-1,1}x_2$$
$$+a_{i_0-1,4}x_5+2a_{i_0-1,5}x_6+\cdots+(k-5)a_{i_0-1,k-2}x_{k-1}]$$
$$=(i_0-4)a_{i_01}x_1+(i_0-4)a_{i_02}x_2+\cdots+(i_0-4)a_{i_0,k-1}x_{k-1}$$

$$(2.9)\qquad \begin{aligned}(k-1)(k-3)a_{i_0-2,1}&=(i_0-4)a_{i_0,2}\\ a_{i_0-1,4}&=(i_0-4)a_{i_0,5}\\ 2a_{i_0-1,5}&=(i_0-4)a_{i_0,6}\\ &\vdots\\ (i_0-5)a_{i_0-1,i_0-2}&=(i_0-4)a_{i_0,i_0-1}\end{aligned}$$

$$[x_1, x_{i_0-1}]=0\Rightarrow[\delta(x_1), x_{i_0-1}]+[x_1, \delta(x_{i_0-1})]=0.$$

For $i_0\geq 5$

$$(2.10)\qquad \Rightarrow[(i_0-4)a_{13}x_{i_0}+(i_0-5)a_{14}x_{i_0+1}+\cdots+(2i_0-k-3)a_{1,k+2-i_0}x_{k-1}]$$
$$+[-a_{i_0-1,2}x_{k-1}-(k-3)(k-1)a_{i_0-1,3}x_2]=0$$
$$\Rightarrow a_{i_0-1,3}=0$$

$$(2.11)\qquad a_{i_0-1,2}=-(2i_0-k-3)a_{1,k+2-i_0}$$

$$[x_1, x_3]=-(k-3)(k-1)x_2$$
$$\Rightarrow[-(k-3)(k-1)a_{11}x_2-a_{14}x_5-2a_{15}x_6-3a_{16}x_7-\cdots$$
$$-(k-6)a_{1,k-3}x_{k-2}-(k-5)a_{1,k-2}x_{k-1}]$$
$$+[-a_{32}x_{k-1}-(k-3)(k-1)a_{33}x_2]$$
$$=(k-1)\cdot[-(k-3)a_{21}x_1-(k-3)a_{22}x_2-\cdots-(k-3)a_{2,k-1}x_{k-1}]$$
$$\Rightarrow(k-3)(k-1)a_{21}x_1+(k-1)[(k-3)a_{22}-(k-3)a_{11}-(k-3)a_{33}]x_2$$
$$+(k-1)(k-3)a_{23}x_3+(k-1)(k-3)a_{24}x_4$$
$$+[(k-1)(k-3)a_{25}-a_{14}]x_5+[(k-1)(k-3)a_{26}-2a_{15}]x_6+\cdots$$
$$+[(k-1)(k-3)a_{2,k-2}-(k-6)a_{1,k-3}]x_{k-2}+[(k-1)(k-3)a_{2,k-1}$$
$$-(k-5)a_{1,k-1}-a_{32}]x_{k-1}=0$$

$$(2.12)\qquad \begin{aligned}&a_{23}=0\\ &a_{14}=(k-1)(k-3)a_{25}\\ \Rightarrow\ &a_{15}=(k-1)(k-3)a_{26}\\ &\vdots\\ &a_{1,k-3}=(k-1)(k-3)a_{2,k-2}\end{aligned}$$

$$(2.13)\qquad a_{32}=(k-5)a_{1,k-2}-(k-1)(k-3)a_{2,k-1}$$

$$[x_1, x_4]=0 \Rightarrow [\delta(x_1), x_4]+[x_1, \delta(x_4)]=0$$

(2.14) $$\Rightarrow [a_{13}x_5-a_{15}x_7+\cdots]+[-a_{42}x_{k-1}-(k-3)(k-1)a_{43}x_2]=0$$
$$\Rightarrow a_{13}=0=a_{15}=a_{16}=\cdots=a_{1,k-4}$$

Here we have used the fact that $k-1\geq 7>5$.

(2.15) $$a_{42}=-(k-7)a_{1,k-3}$$

(2.16) $$a_{43}=0$$

$$[x_2, x_3]=0 \Rightarrow [\delta(x_2), x_3]+[x_2, \delta(x_3)]=0$$
$$\Rightarrow -(k-3)(k-1)a_{21}x_2-a_{24}x_5-2a_{25}x_6-3a_{26}x_7-\cdots$$
$$-(k-5)a_{2,k-2}x_{k-1}+a_{31}x_{k-1}=0$$

(2.17) $$\Rightarrow a_{21}=0=a_{24}=a_{25}=a_{26}=\cdots=a_{2,k-3}$$

(2.18) $$a_{31}=(k-5)a_{2,k-2}.$$

Using (2.12), (2.14) and (2.17), we deduce that

(2.19) $$a_{1,k+2-i_0}=0 \qquad \text{for } 6\leq i_0\leq k-1$$

(2.9), (2.10), (2.11) and (2.19) and the induction hypothesis give that $a_{i_0-1,j}=0$ for $6\leq i_0\leq k-2$ and $i_0-1>j$. This finishes our claim.

Put $i_0=5$ in (2.9), we have

(2.20) $$(k-3)a_{4,1}=a_{5,2}=0$$
$$\Rightarrow a_{4,1}=0.$$

Hence δ is represented by the following matrix

$$A=\begin{pmatrix}
a_{11} & a_{12} & 0 & 0 & 0 & 0 & a_{1,k-3} & a_{1,k-2} & a_{1,k-1} \\
0 & a_{22} & 0 & 0 & 0 & 0 & 0 & a_{2,k-2} & a_{2,k-1} \\
\begin{matrix}(k-5)\\ a_{2,k-2}\end{matrix} & \begin{matrix}(k-5)a_{1,k-2}\\ -(k-3)a_{1,k-1}\end{matrix} & a_{33} & a_{34} & a_{35} & a_{3,k-4} & a_{3,k-3} & a_{3,k-2} & a_{3,k-1} \\
0 & -(k-7)a_{1,k-3} & 0 & a_{44} & a_{45} & a_{4,k-4} & a_{4,k-3} & a_{4,k-2} & a_{4,k-1} \\
0 & 0 & 0 & 0 & a_{55} & a_{5,k-4} & a_{5,k-3} & a_{5,k-2} & a_{5,k-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{k-1,k-1}
\end{pmatrix}$$

δ is semi simple $\Leftrightarrow AA^*=A^*A$

$\Rightarrow$ the length of i^{th} row of $A=$ the length of j^{th} column of A

$\Rightarrow a_{ij}=0 \quad \text{for } i\neq j$

$$A = \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{k-1,k-1} \end{bmatrix}$$

(2.21, 1) $[x_1, x_2] = -x_{k-1} \Rightarrow a_{11} + a_{22} = a_{k-1,k-1}$

(2.21, 2) $[x_1, x_3] = -(k-3)(k-1)x_2 \Rightarrow a_{22} - a_{11} = a_{33}$

(2.21, 3) $[x_3, x_4] = x_5 \Rightarrow a_{33} + a_{44} = a_{55}$

(2.21, 4) $[x_3, x_5] = 2x_6 \Rightarrow a_{33} + a_{55} = a_{66}$

(2.21, 5) $[x_3, x_6] = 3x_7 \Rightarrow a_{33} + a_{66} = a_{77}$

$\vdots$ $\vdots$

(2.21, k–4) $[x_3, x_{k-3}] = (k-6)x_{k-2} \Rightarrow a_{33} + a_{k-3,k-3} = a_{k-2,k-2}$

(2.21, k–3) $[x_3, x_{k-2}] = (k-5)x_{k-1} \Rightarrow a_{33} + a_{k-2,k-2} = a_{k-1,k-1}$

(2.21, k–2) $[x_4,, x_5] = x_7 \Rightarrow a_{44} + a_{55} = a_{77}$

(2.21, k–1) $[x_4, x_6] = 2x_8 \Rightarrow a_{44} + a_{66} = a_{88}$

(2.22, 1) (2.21, 3)+(2.21, 4) $\Rightarrow a_{66} = a_{44} + 2a_{33}$

(2.22, 2) $\Rightarrow a_{77} = a_{44} + 3a_{33}$

(2.21, $k-2$), (2.21, 3) and (2.22, 2) $\Rightarrow a_{44} = 2a_{33}$

(2.23)
$$\begin{aligned} a_{55} &= 3a_{33} \\ a_{66} &= 4a_{33} \\ &\vdots \\ a_{k-1,k-1} &= (k-3)a_{33} \end{aligned}$$

(2.23), (2.21, 2) and (2.21, 1) $\Rightarrow a_{22} = \dfrac{k-2}{k-4} a_{11}$

$$\begin{aligned} a_{33} &= a_{22} - a_{11} = \frac{2}{k-4} a_{11} \\ a_{44} &= 2a_{33} = \frac{2.2}{k-4} a_{11} \\ a_{55} &= 3a_{33} = \frac{2.3}{k-4} a_{11} \\ &\vdots \\ a_{k-1,k-1} &= (k-3)a_{k-3,k-3} = \frac{2(k-3)}{k-4} a_{11}. \end{aligned}$$

We have proved that any derivation of $\mathfrak{g}(V)$ must be a constant multiple of t. To prove that t is really a derivation, we first observe that the multiplication rule of $\mathfrak{g}(V)$ is described by the following formulas:

$$\begin{aligned} &[x_1, x_2] = -x_{k-1} \\ &[x_1, x_3] = -(k-3)x_3 \\ (2.24) \qquad &[x_1, x_4] = 0 \\ &\quad\vdots \\ &[x_1, x_{k-1}] = 0 \end{aligned}$$

$$\begin{aligned} &[x_2, x_3] = 0 \\ (2.25) \qquad &[x_2, x_4] = 0 \\ &\quad\vdots \\ &[x_2, x_{k-1}] = 0 \end{aligned}$$

$$[x_i, x_j] = \begin{cases} (j-i)x_{j+i-2} & \text{if } j+i-2 \le k-1, \quad 3 \le i, j \le k-1 \\ 0 & \text{if } j+i-2 \ge k, \quad 3 \le i, j \le k-1 \end{cases}$$

Then it is an easy matter to check $t[x_i, x_j] = [t(x_i), x_j] + [x_i, t(x_j)]$ for all $1 \le i, j \le k-1$.

Proposition 2.10. *Let $V = \{(x, y, z) \in C^3 : z^4 + y^3 + x^2 = 0\}$ be the E_6 singularity. Then*

$A(V) = \langle 1, z, z^2, y, yz, z^2y \rangle$ *with multiplication rule* $z^3 = 0$, $y^2 = 0$

$$L(V) = \left\langle z\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}, yz\frac{\partial}{\partial z}, yz^2\frac{\partial}{\partial z}, y\frac{\partial}{\partial y}, yz\frac{\partial}{\partial y}, yz^2\frac{\partial}{\partial y} \right\rangle$$

$$\mathfrak{g}(V) = \left\langle \underset{x_1}{\underset{\|}{z^2\frac{\partial}{\partial z}}}, \underset{x_2}{\underset{\|}{yz\frac{\partial}{\partial z}}}, \underset{x_3}{\underset{\|}{yz\frac{\partial}{\partial y}}}, \underset{x_4}{\underset{\|}{yz^2\frac{\partial}{\partial z}}}, \underset{x_5}{\underset{\|}{yz^2\frac{\partial}{\partial y}}} \right\rangle$$

with multiplication rules:

$$\begin{array}{llll} [x_1, x_5] = -x_4 & [x_2, x_3] = -x_4 & [x_3, x_4] = 0 & [x_4, x_5] = 0 \\ [x_1, x_3] = x_5 & [x_2, x_4] = 0 & [x_3, x_5] = 0 & \\ [x_1, x_4] = 0 & [x_2, x_5] = 0 & & \\ [x_1, x_5] = 0. & & & \end{array}$$

The type of E_6 singularity: $= \dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = 3$.
The nilpotency of E_6 singularity: $= \min\{p \in N \cup \{0\} : \mathfrak{g}^{p+1} = 0\} = 1$.

Let t_1, t_2, t_3 *be three derivations of* $\mathfrak{g}(V)$ *defined by*

$$\begin{array}{lll} t_1\colon \mathfrak{g}\longrightarrow\mathfrak{g} & t_2\colon \mathfrak{g}\longrightarrow\mathfrak{g} & t_3\colon \mathfrak{g}\longrightarrow\mathfrak{g} \\ x_1\longrightarrow x_1 & x_1\longrightarrow 0 & x_1\longrightarrow x_3 \\ x_2\longrightarrow 0 & x_2\longrightarrow x_2 & x_2\longrightarrow 0 \\ x_3\longrightarrow x_3 & x_3\longrightarrow 0 & x_3\longrightarrow x_1 \\ x_4\longrightarrow x_4 & x_4\longrightarrow x_4 & x_4\longrightarrow -x_4 \\ x_5\longrightarrow 2x_5 & x_5\longrightarrow 0 & x_5\longrightarrow 0. \end{array}$$

Then $T=\boldsymbol{C}t_1\oplus \boldsymbol{C}t_2\oplus \boldsymbol{C}t_3$ *is a torus of* $\mathfrak{g}(V)$. *Since* $\dim T=3=$ *the type of* E_6, T *is a maximal torus of* $\mathfrak{g}(V)$. *Let* $\beta_i\colon T\to\boldsymbol{C}$ *be a linear map with* $\beta_i(t_j)=\delta_{ij}$ *for* $i,j=1,2,3$.

$$\begin{array}{ccccccccc} \mathfrak{g}(V)=\mathfrak{g}^{\beta_2} & \oplus & \mathfrak{g}^{2\beta_1} & \oplus & \mathfrak{g}^{\beta_1+\beta_3} & \oplus & \mathfrak{g}^{\beta_1-\beta_3} & \oplus & \mathfrak{g}^{\beta_1+\beta_2-\beta_3} \\ \| & & \| & & \| & & \| & & \| \\ \boldsymbol{C}x_2 & & \boldsymbol{C}x_5 & & \boldsymbol{C}(x_1+x_3) & & \boldsymbol{C}(x_1-x_3) & & \boldsymbol{C}x_4 \end{array}$$

(x_1+x_3, x_1-x_3, x_2) *is a T-minimal system of generators. The generalized Cartan matrix associated to* E_6 *singularity is*

$$C(E_6)=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Proposition 2.11. *Let* $V=\{(x,y,z)\in \boldsymbol{C}^3\colon z^3y+y^3+x^2=0\}$ *be the* E_7 *singularity. Then*

$A(V)=\langle 1, z, y, z^2, yz, z^3, z^4\rangle$ *with multiplication rule*

$$yz^2=0,\quad y^2=-\frac{1}{3}z^3,\quad z^5=0,\quad y^3=0$$

$$L(V)=\Big\langle 3y\frac{\partial}{\partial z}+2z^2\frac{\partial}{\partial y},\ 2z\frac{\partial}{\partial z}+3y\frac{\partial}{\partial y},\ 2z^2\frac{\partial}{\partial z}+3yz\frac{\partial}{\partial y},\ yz\frac{\partial}{\partial z},$$
$$z^3\frac{\partial}{\partial z},\ z^4\frac{\partial}{\partial z},\ z^3\frac{\partial}{\partial y},\ z^4\frac{\partial}{\partial y}\Big\rangle$$

$$\mathfrak{g}(V)=\Big\langle \underbrace{3y\frac{\partial}{\partial z}+2z^2\frac{\partial}{\partial y}}_{x_1},\ \underbrace{2z^2\frac{\partial}{\partial z}+3yz\frac{\partial}{\partial y}}_{x_2},\ \underbrace{yz\frac{\partial}{\partial z}}_{x_3},\ \underbrace{z^3\frac{\partial}{\partial z}}_{x_4},\ \underbrace{z^4\frac{\partial}{\partial z}}_{x_5},$$
$$\underbrace{z^3\frac{\partial}{\partial y}}_{x_6},\ \underbrace{z^4\frac{\partial}{\partial y}}_{x_7}\Big\rangle$$

with multiplication rules:

$$\begin{array}{lll}
[x_1, x_2]=3x_3-5x_6 & [x_2, x_3]=x_7 & [x_3, x_4]=0 \\
[x_1, x_3]=x_4 & [x_2, x_4]=2x_5 & [x_3, x_5]=0 \\
[x_1, x_4]=-4x_7 & [x_2, x_5]=0 & [x_3, x_6]=-x_5 \\
[x_1, x_5]=0 & [x_2, x_6]=3x_7 & [x_3, x_7]=0 \\
[x_1, x_6]=-3x_4 & [x_2, x_7]=0 & \\
[x_1, x_7]=-3x_5 & & \\
[x_4, x_5]=0 & [x_5, x_6]=0 & [x_6, x_7]=0 \\
[x_4, x_6]=0 & [x_5, x_7]=0 & \\
[x_4, x_7]=0. & &
\end{array}$$

The type of E_7 singularity: $=\dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]=3$.
The nilpotency of E_7 singularity: $=\min\{p \in N\cup\{0\}: \mathfrak{g}^{p+1}=0\}=4$.
Let t be a derivation of $\mathfrak{g}(V)$ defined by

$$\begin{aligned}
t: \mathfrak{g} &\longrightarrow \mathfrak{g} \\
x_1 &\longrightarrow x_1 \\
x_2 &\longrightarrow 2x_2 \\
x_3 &\longrightarrow 3x_3 \\
x_4 &\longrightarrow 4x_4 \\
x_5 &\longrightarrow 6x_5 \\
x_6 &\longrightarrow 3x_6 \\
x_7 &\longrightarrow 5x_7
\end{aligned}$$

$T=Ct$ is the unique maximal torus on $\mathfrak{g}$. Let β: $T\to C$ be a linear map such that $\beta(t)=1$. Then

$$\begin{array}{ccccccccccc}
\mathfrak{g}(V)= \mathfrak{g}^{\beta} & \oplus & \mathfrak{g}^{2\beta} & \oplus & \mathfrak{g}^{3\beta} & \oplus & \mathfrak{g}^{4\beta} & \oplus & \mathfrak{g}^{5\beta} & \oplus & \mathfrak{g}^{6\beta}. \\
\| & & \| & & \| & & \| & & \| & & \| \\
Cx_1 & & Cx_2 & & Cx_3\oplus Cx_6 & & Cx_4 & & Cx_7 & & Cx_5
\end{array}$$

(x_1, x_2, x_3) is a T-minimal system of generators. The generalized Cartan matrix attached to E_7 singularity is

$$C(E_7)=\begin{pmatrix} 2 & -4 & -3 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

We shall now prove that there is only one semi-simple derivation of

$\mathfrak{g}(V)$ up to multiplicative constant. Let δ: $\mathfrak{g}\to\mathfrak{g}$ be a derivation

$$\delta(x_i)=a_{i1}x_1+a_{i2}x_2+\cdots+a_{i7}x_7 \qquad 1\leq i\leq 7$$

$$\begin{aligned}
[x_1, x_2]=3x_3-5x_6 &\Rightarrow [\delta(x_1), x_2]+[x_1, \delta(x_2)]=3\delta(x_3)-5\delta(x_6)\\
&\Rightarrow [3a_{11}x_3-5a_{11}x_6-a_{13}x_7-2_{14}x_5-3a_{16}x_7]\\
&\quad +[3a_{22}x_3-5a_{22}x_6+a_{23}x_4-4a_{24}x_7-3a_{26}x_4-3a_{27}x_5]\\
&\quad =(3a_{31}-5a_{61})x_1+(3a_{32}-5a_{62})x_2+(3a_{33}-5a_{63})x_3+(3a_{34}-5a_{64})x_4\\
&\quad +(3a_{35}-5a_{65})x_5+(3a_{36}-5a_{66})x_6+(3a_{37}-5a_{67})x_7
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow 3a_{31}-5a_{61}=0\\
&\quad 3a_{32}-5a_{62}=0\\
&\quad 3a_{33}-5a_{63}-3a_{11}-3a_{22}=0\\
\text{(2.26)}\qquad &\quad 3a_{24}-5a_{64}-a_{23}+3a_{26}=0\\
&\quad 3a_{35}-5a_{65}+2a_{14}+3a_{27}=0\\
&\quad 3a_{36}-5a_{66}+5a_{11}+5a_{22}=0\\
&\quad 3a_{37}-5a_{67}+a_{13}+3a_{16}+4a_{24}=0
\end{aligned}$$

$$\begin{aligned}
[x_1, x_3]=x_4 &\Rightarrow [\delta(x_1), x_3]+[x_1, \delta(x_3)]=\delta(x_4)\\
&\Rightarrow [a_{11}x_4+q_{12}x_7+a_{16}x_5]+[3a_{32}x_3-5a_{32}x_6+a_{33}x_4-4a_{34}x_7\\
&\quad -3a_{36}x_4-3a_{37}x_5]=a_{41}x_1+a_{42}x_2+\cdots+a_{47}x_7
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow a_{41}=0\\
&\quad a_{42}=0\\
&\quad a_{43}-3a_{32}=0\\
\text{(2.27)}\qquad &\quad a_{44}-a_{33}-a_{11}+3a_{36}=0\\
&\quad a_{45}-a_{16}+3a_{37}=0\\
&\quad a_{46}+5a_{32}=0\\
&\quad a_{47}-a_{12}+4a_{34}=0
\end{aligned}$$

$$\begin{aligned}
[x_1, x_4]=-4x &\Rightarrow [\delta(x_1), x_4]+[x_1, \delta(x_4)]=-4\delta(x_7)\\
&\Rightarrow [-4a_{11}x_7+2a_{12}x_5]+[3a_{42}x_3-5a_{42}x_6+a_{43}x_4-4a_{44}x_7\\
&\quad -3a_{46}x_4-3a_{47}x_5]=-4a_{71}x_1-4a_{72}x_2-\cdots-4a_{77}x_7\\
&\Rightarrow a_{71}=0\\
&\quad a_{72}=0\\
&\quad 4a_{73}+3a_{42}=0
\end{aligned}$$

$$(2.28)\qquad \begin{aligned} &4a_{74}+a_{43}-3a_{46}=0\\ &4a_{75}+2a_{12}-3a_{47}=0\\ &4a_{76}-5a_{42}=0\\ &4a_{77}-4a_{11}-4a_{44}=0 \end{aligned}$$

$$[x_1, x_5]=0\Rightarrow[\delta(x_1), x_5]+[x_1, \delta(x_5)]=0$$
$$\Rightarrow 3a_{52}x_3-5a_{52}x_6+a_{53}x_4-4a_{54}x_7-3a_{56}x_4-3a_{57}x_5=0$$

$$(2.29)\qquad \begin{aligned} \Rightarrow &a_{52}=0\\ &a_{53}-3a_{56}=0\\ &a_{57}=0\\ &a_{54}=0 \end{aligned}$$

$$[x_1, x_6]=-3x_4\Rightarrow[\delta(x_1), x_6]+[x_1, \delta(x_6)]=-3\delta(x_4)$$
$$\Rightarrow[-3a_{11}x_4+3a_{12}x_7-a_{13}x_5]+[3a_{62}x_3-5a_{62}x_6+a_{63}x_4-4a_{64}x_7$$
$$-3a_{66}x_4-3a_{67}x_5]=-3a_{41}x_1-3a_{42}x_2-\cdots-3a_{47}x_7$$

$$(2.30)\qquad \begin{aligned} \Rightarrow &a_{43}+a_{62}=0\\ &3a_{44}-3a_{11}+a_{63}-3a_{66}=0\\ &3a_{45}-a_{13}-3a_{67}=0\\ &3a_{46}-5a_{62}=0\\ &3a_{47}+3a_{12}-4a_{64}=0 \end{aligned}$$

$$[x_1, x_7]=-3x_5\Rightarrow[\delta(x_1), x_7]+[x_1, \delta(x_7)]=-3\delta(x_5)$$
$$\Rightarrow -3a_{11}x_5+(3a_{72}x_3-5a_{72}x_6+a_{73}x_4-4a_{74}x_7-3a_{76}x_4$$
$$-3a_{77}x_5 \;=-3a_{51}x_1-3a_{52}x_2-\cdots-3a_{57}x_7$$

$$(2.31)\qquad \begin{aligned} \Rightarrow &a_{51}=0\\ &a_{53}+a_{72}=0\\ &3a_{54}+a_{73}-3a_{76}=0\\ &a_{55}-a_{11}-a_{77}=0\\ &3a_{56}-5a_{72}=0\\ &3a_{57}-4a_{74}=0 \end{aligned}$$

$$[x_2, x_3]=x_7\Rightarrow[\delta(x_2), x_3]+[x_2, \delta(x_3)]=\delta(x_7)$$
$$\Rightarrow[a_{21}x_4+a_{22}x_7+a_{26}x_5]+[-3a_{31}x_3+5a_{31}x_6+a_{33}x_7+2a_{34}x_5$$
$$+3a_{36}x_7]=a_{71}x_1+a_{72}x_2+\cdots+a_{77}x_7$$

$$\Rightarrow a_{73}+3a_{31}=0$$

(2.32)
$$a_{74}-a_{21}=0$$
$$a_{75}-a_{26}-2a_{34}=0$$
$$a_{76}-5a_{31}=0$$
$$a_{77}-a_{22}-a_{33}-3a_{36}=0$$

$$[x_2, x_4]=2x_5 \Rightarrow [\delta(x_2), x_4]+[x_2, \delta(x_4)]=2\delta(x_5)$$
$$\Rightarrow [-4a_{21}x_7+2a_{22}x_5]+[-3a_{41}x_3+5a_{41}x_6+a_{43}x_7+2a_{44}x_5$$
$$+3a_{46}x_7]=2a_{51}x_1+2a_{52}x_2+\cdots+2a_{57}x_7$$
$$\Rightarrow 2a_{53}+3a_{41}=0$$
$$a_{54}=0$$

(2.33)
$$a_{55}-a_{22}-a_{44}=0$$
$$2a_{56}-5a_{41}=0$$
$$2a_{57}+4a_{21}-3a_{46}-a_{43}=0$$

$$[x_2, x_5]=0 \Rightarrow [\delta(x_2), x_5]+[x_2, \delta(x_5)]=0$$

(2.34)
$$\Rightarrow -3a_{51}x_3+5a_{51}x_6+a_{53}x_7+2a_{54}x_5+3a_{56}x_7=0$$
$$\Rightarrow a_{53}+3a_{56}=0$$

$$[x_2, x_6]=3x_7 \Rightarrow [\delta(x_2), x_6]+[x_2, \delta(x_6)]=3\delta(x_7)$$
$$\Rightarrow [-3a_{21}x_4+3a_{22}x_7-a_{23}x_5]+[-3a_{61}x_3+5a_{61}x_6+a_{63}x_7$$
$$+2a_{64}x_5+3a_{66}x_7]=3a_{71}x_1+3a_{72}x_2+\cdots+3a_{77}x_7$$
$$\Rightarrow a_{73}+a_{61}=0$$
$$a_{74}+a_{21}=0$$

(2.34)
$$3a_{75}+a_{23}-2a_{64}=0$$
$$3a_{76}-5a_{61}=0$$
$$3a_{77}-3a_{22}-a_{63}-3a_{66}=0$$

$$[x_2, x_7]=0 \Rightarrow [\delta(x_2), x_7]+[x_2, \delta(x_7)]=0$$
$$\Rightarrow -3a_{21}x_5+[-3a_{71}x_3+5a_{71}x_6+a_{73}x_7+2a_{74}x_5+3a_{76}x_7]=0$$

(2.35)
$$\Rightarrow -2a_{74}-3a_{21}=0$$
$$a_{73}+3a_{76}=0$$

$$[x_3, x_4]=0 = \rangle [\delta(x_3), x_4]+[x_3, \delta(x_4)]=0$$
$$[-4a_{31}x_7+2a_{32}x_5]+[-a_{41}x_4-a_{42}x_7-a_{46}x_5]=0$$

$$\begin{aligned}
(2.35)\qquad & \Rightarrow 2a_{32}-a_{46}=0\\
& \quad 4a_{31}+a_{42}=0
\end{aligned}$$

$$\begin{aligned}
[x_3, x_5]=0 & \Rightarrow [\delta(x_3), x_5]+[x_3, \delta(x_5)]=0\\
(2.36)\qquad & \Rightarrow -a_{51}x_4-a_{52}x_7-a_{56}x_5=0\\
& \Rightarrow a_{56}=0
\end{aligned}$$

$$\begin{aligned}
[x_3, x_6]=-x_5 & \Rightarrow [\delta(x_3), x_6]+[x_3, \delta(x_6)]=-\delta(x_5)\\
& \Rightarrow [-3a_{31}x_4+3a_{32}x_7-a_{33}x_5]+[-a_{61}x_4-a_{62}x_7-a_{66}x_5]\\
& \quad =-a_{51}x_1-a_{52}x_2-\cdots-a_{57}x_7\\
& \Rightarrow a_{53}=0\\
(2.36)\qquad & \quad a_{54}-3a_{31}-a_{61}=0\\
& \quad a_{55}-a_{33}-a_{66}=0\\
& \quad a_{57}+3a_{32}-a_{62}=0
\end{aligned}$$

$$\begin{aligned}
[x_3, x_7]=0 & \Rightarrow [\delta(x_3), x_7]+[x_3, \delta(x_7)]=0\\
(2.37)\qquad & \Rightarrow -3a_{31}x_5+[-a_{71}x_4-a_{72}x_7-a_{76}x_5]=0\\
& \Rightarrow 3a_{31}+a_{76}=0
\end{aligned}$$

$$\begin{aligned}
[x_4, x_5]=0 & \Rightarrow [\delta(x_4), x_5]+[x_4, \delta(x_5)]=0\\
& \Rightarrow 4a_{51}x_7-2a_{52}x_5=0
\end{aligned}$$

$$\begin{aligned}
[x_4, x_6]=0 & \Rightarrow [\delta(x_4), x_6]+[x_4, \delta(x_6)]=0\\
(2.48)\qquad & \Rightarrow [-3a_{41}x_4+3a_{42}x_7-a_{43}x_5]+[4a_{61}x_7-2a_{62}x_5]=0\\
& \Rightarrow a_{43}+2a_{62}=0\\
& \quad a_{61}=0
\end{aligned}$$

$$\begin{aligned}
[x_4, x_7]=0 & \Rightarrow [\delta(x_4), x_7]+[x_4, \delta(x_7)]=0\\
& \Rightarrow -3a_{41}x_5-[4a_{71}x_7-2a_{72}x_5]=0
\end{aligned}$$

$$\begin{aligned}
[x_5, x_6]=0 & \Rightarrow [\delta(x_5), x_6]+[x_5, \delta(x_6)]=0\\
& \Rightarrow -3a_{51}x_4+3a_{52}x_7-a_{53}x_5=0
\end{aligned}$$

$$\begin{aligned}
[x_5, x_7]=0 & \Rightarrow [\delta(x_5), x_7]+[x_5, \delta(x_7)]=0\\
& \Rightarrow -3a_{51}x_5=0
\end{aligned}$$

$$\begin{aligned}
[x_6, x_7]=0 & \Rightarrow [\delta(x_6), x_7]+[x_6, \delta(x_7)]=0\\
& \Rightarrow -3a_{61}x_5+[3a_{71}x_4-3a_{72}x_7+a_{73}x_5]=0
\end{aligned}$$

(2.26), (2.27), $\cdots$, (2.38) imply

$$a_{21}=0$$

$$a_{31}=0=a_{32}=a_{36}$$

$$a_{41}=0=a_{42}=a_{43}=a_{46}$$

$$a_{51}=0=a_{52}=a_{53}=a_{54}=a_{56}=a_{57}$$

$$a_{61}=0=a_{62}=a_{63}$$

$$a_{71}=0=a_{72}=a_{73}=a_{74}=a_{76}$$

$$a_{22}=2a_{11}$$

$$a_{24}=\frac{1}{12}(27a_{12}+4a_{23})$$

$$a_{26}=\frac{1}{3}(12a_{12}-5a_{23})$$

$$a_{33}=3a_{11}$$

$$a_{34}=\frac{1}{12}(-9a_{12}+4a_{23})$$

$$a_{37'}=\frac{1}{27}\left(3a_{16}-4a_{13}-\frac{54}{4}a_{12}-2a_{23}\right)$$

$$a_{44}=4a_{11}$$

$$a_{45}=\frac{1}{18}(12a_{16}+8a_{13}+27a_{12}+4a_{23})$$

$$a_{47}=\frac{4}{3}(3a_{12}-a_{23})$$

$$a_{55}=6a_{11}$$

$$a_{64}=\frac{1}{4}(15a_{12}-4a_{23})$$

$$a_{65}=\frac{1}{5}(3a_{35}+2a_{14}+3a_{27})$$

$$a_{66}=3a_{11}$$

$$a_{67}=\frac{1}{18}(12a_{16}+2a_{13}+27a_{12}+4a_{23})$$

$$a_{75}=\frac{1}{2}(5a_{12}-2a_{23})$$

$$a_{77}=5a_{11}$$

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ 0 & 2a_{11} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ 0 & 0 & 3a_{11} & a_{34} & a_{35} & 0 & a_{37} \\ 0 & 0 & 0 & 4a_{11} & a_{45} & 0 & a_{47} \\ 0 & 0 & 0 & 0 & 6a_{11} & 0 & a_{57} \\ 0 & 0 & 0 & \frac{1}{4}(15a_{12}-4a_{23}) & a_{65} & 3a_{11} & a_{67} \\ 0 & 0 & 0 & 0 & a_{75} & 0 & 5a_{11} \end{bmatrix}$$

δ is semi simple $\Leftrightarrow AA^*=A^*A$

$\Rightarrow$ the length of i^{th} row of $A=$the length of j^{th} column of A.

$\Rightarrow a_{ij}=0$ for $i\neq j$.

$$A=\begin{bmatrix} a_{11} & & & & & & \\ & 2a_{11} & & & & 0 & \\ & & 3a_{11} & & & & \\ & & & 4a_{11} & & & \\ & & & & 6a_{11} & & \\ & 0 & & & & 3a_{11} & \\ & & & & & & 5a_{11} \end{bmatrix}.$$

We have proved that any derivation of $\mathfrak{g}(V)$ must be a constant multiple of t. It is a trivial matter to prove that t is really a derivation.

Proposition 2.12. *Let $V=\{(x, y, z)\in \boldsymbol{C}^3: z^5+y^3+x^2=0\}$ be the E_8 singularity. Then*

$A(V)=\langle 1, z, z^2, z^3, y, yz, yz^2, yz^3\rangle$ with multiplication rules $z^4=0$, $y^2=0$

$$L(V)=\left\langle z\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}, z^3\frac{\partial}{\partial z}, yz\frac{\partial}{\partial z}, yz^2\frac{\partial}{\partial z}, yz^3\frac{\partial}{\partial z}, y\frac{\partial}{\partial y}, yz\frac{\partial}{\partial y}, yz^2\frac{\partial}{\partial y}, yz^3\frac{\partial}{\partial y}\right\rangle$$

$$\mathfrak{g}(V)=\Big\langle \underset{x_1}{\underset{\|}{z^2\frac{\partial}{\partial z}}}, \underset{x_2}{\underset{\|}{z^3\frac{\partial}{\partial z}}}, \underset{x_3}{\underset{\|}{yz\frac{\partial}{\partial z}}}, \underset{x_4}{\underset{\|}{yz^2\frac{\partial}{\partial z}}}, \underset{x_5}{\underset{\|}{yz^3\frac{\partial}{\partial z}}}, \underset{x_6}{\underset{\|}{yz\frac{\partial}{\partial y}}}, \underset{x_7}{\underset{\|}{yz^2\frac{\partial}{\partial y}}}, \underset{x_8}{\underset{\|}{yz^3\frac{\partial}{\partial y}}}\Big\rangle$$

with multiplication rules:

$[x_1, x_2]=0$	$[x_2, x_3]=-2x_5$	$[x_3, x_4]=0$
$[x_1, x_3]=-x_4$	$[x_2, x_4]=0$	$[x_3, x_5]=0$
$[x_1, x_4]=0$	$[x_2, x_5]=0$	$[x_3, x_6]=-x_4$
$[x_1, x_5]=0$	$[x_2, x_6]=x_8$	$[x_3, x_7]=-x_5$

$$
\begin{array}{llll}
[x_1, x_6]=x_7 & [x_2, x_7]=0 & & [x_3, x_8]=0 \\
[x_1, x_7]=2x_8 & [x_2, x_8]=0 & & \\
[x_1, x_8]=0 & & & \\
[x_4, x_5]=0 & [x_5, x_6]=0 & [x_6, x_7]=0 & [x_7, x_8]=0 \\
[x_4, x_6]=-x_5 & [x_5, x_7]=0 & [x_6, x_8]=0 & \\
[x_4, x_7]=0 & [x_5, x_8]=0 & & \\
[x_4, x_8]=0. & & &
\end{array}
$$

The type of E_8 singularity: $=\dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]=4$.
The nilpotency of E_8 singularity: $=\min\{p \in N \cup \{0\} : \mathfrak{g}^{p+1}=0\}=2$.

Let t_1, t_2 be two derivations of $\mathfrak{g}(V)$ defined by

$$
\begin{array}{ll}
t_1: \mathfrak{g} \longrightarrow \mathfrak{g} & t_2: \mathfrak{g} \longrightarrow \mathfrak{g} \\
x_1 \longrightarrow x_1 & x_1 \longrightarrow 0 \\
x_2 \longrightarrow 2x_2 & x_2 \longrightarrow 0 \\
x_3 \longrightarrow 0 & x_3 \longrightarrow x_3 \\
x_4 \longrightarrow x_4 & x_4 \longrightarrow x_4 \\
x_5 \longrightarrow 2x_5 & x_5 \longrightarrow x_5 \\
x_6 \longrightarrow x_6 & x_6 \longrightarrow 0 \\
x_7 \longrightarrow 2x_7 & x_7 \longrightarrow 0 \\
x_8 \longrightarrow 3x_8 & x_8 \longrightarrow 0.
\end{array}
$$

Then $T=\boldsymbol{C}t_1 \oplus \boldsymbol{C}t_2$ is an unique maximal torus on $\mathfrak{g}(V)$. Let $\beta_i: T \to \boldsymbol{C}$ be a linear map such that $\beta_i(t_j)=\partial_{ij}$ for $1 \le i, j \le 2$.

$$
\begin{array}{ccccccccccc}
\mathfrak{g}(V)= \mathfrak{g}^{\beta_1} & \oplus & \mathfrak{g}^{2\beta_1} & \oplus & \mathfrak{g}^{\beta_2} & \oplus & \mathfrak{g}^{\beta_1+\beta_2} & \oplus & \mathfrak{g}^{2\beta_1+\beta_2} & \oplus & \mathfrak{g}^{3\beta_1} \\
\| & & \| & & \| & & \| & & \| & & \| \\
\boldsymbol{C}x_1 \oplus \boldsymbol{C}x_6 & & \boldsymbol{C}x_2 \oplus \boldsymbol{C}x_7 & & \boldsymbol{C}x_3 & & \boldsymbol{C}x_4 & & \boldsymbol{C}x_5 & & \boldsymbol{C}x_8
\end{array}
$$

(x_3, x_2, x_1, x_6) *is a T-minimal system of generators.*

$$
\begin{aligned}
R^1(T) &= \{\beta_2, 2\beta_1, \beta_1\} \\
&= R^1(T)_1 \cup R^1(T)_2 \\
\textit{where } & R^1(T_1)=\{\beta_2\},\ R^1(T_2)=\{2\beta_1, \beta_1\}
\end{aligned}
$$

We number the number of elements of $R^1(T)$ in such a way

$$R^1(T)=\{\beta^1, \beta^2, \beta^3\}$$

$$\textit{where } \beta^1=\beta_2,\ \beta^2=2\beta_1,\ \beta^3=\beta_1$$

$$\mathfrak{g}^{\beta^1}=\boldsymbol{C}x_3 \quad \mathfrak{g}^{\beta^2}=\boldsymbol{C}x_2 \oplus \boldsymbol{C}x_7 \mathfrak{g}^{\beta^3}=\boldsymbol{C}x_1 \oplus \boldsymbol{C}x_6$$

The generalized Cartan matrix associated to E_8 singularity is

$$C(E_8)=\begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -2 & -1 & 2 & -2 \\ -2 & -1 & -2 & 2 \end{bmatrix}.$$

We shall now show that $\mathfrak{g}(V)$ has an unique maximal torus spanned by t_1 and t_2 defined as above. Let δ be a derivation of $\mathfrak{g}(V)$.

$$\delta(x_i)=a_{ii}x_1+a_{i2}x_2+\cdots+a_{i8}x_8 \qquad \text{for } 1\leq i\leq 8$$

$$\begin{aligned}
[x_1, x_2]=0 &\Rightarrow [\delta(x_1), x_2]+[x_1, \delta(x_2)]=0 \\
&\Rightarrow [2a_{13}x_5-a_{16}x_8]+[-a_{23}x_4+a_{26}x_7+2a_{27}x_8]=0 \\
&\Rightarrow a_{23}=0
\end{aligned}$$

$$\begin{aligned}
(2.39)\qquad & a_{13}=0 \\
& a_{26}=0 \\
& 2a_{27}-a_{16}=0
\end{aligned}$$

$$\begin{aligned}
[x_1, x_3]=-x_4 &\Rightarrow [\delta(x_1), x_3]+[x_1, \delta(x_3)]=-\delta(x_4) \\
&\Rightarrow [-a_{11}x_4-2a_{12}x_5+a_{16}x_4+a_{17}x_5]+[-a_{33}x_4+a_{36}x_7+2a_{37}x_8] \\
&\quad =-a_{41}x_1-a_{42}x_2-\cdots-a_{48}x_8 \\
&\Rightarrow a_{41}=0
\end{aligned}$$

$$\begin{aligned}
& a_{42}=0 \\
& a_{43}=0 \\
(2.40)\qquad & a_{44}-a_{11}-a_{33}+a_{16}=0 \\
& a_{45}-2a_{12}+a_{17}=0 \\
& a_{46}=0 \\
& a_{47}+a_{36}=0 \\
& a_{48}+2a_{37}=0
\end{aligned}$$

$$\begin{aligned}
[x_1, x_4]=0 &\Rightarrow [\delta(x_1), x_4]+[x_1, \delta(x_4)]=0 \\
&\Rightarrow a_{16}x_5+[-a_{43}x_4+a_{46}x_7+2a_{47}x_8]=0 \\
(2.41)\qquad &\Rightarrow a_{16}=0 \\
& a_{47}=0
\end{aligned}$$

$$\begin{aligned}
[x_1, x_5]=0 &\Rightarrow [\delta(x_1), x_5]+[x_1, \delta(x_5)]=0 \\
&\Rightarrow -a_{53}x_4+a_{56}x_7+2a_{57}x_8=0
\end{aligned}$$

$$(2.42) \qquad \begin{aligned} &\Rightarrow a_{53}=0 \\ &a_{56}=0 \\ &a_{57}=0 \end{aligned}$$

$$[x_1, x_6]=x_7 \Rightarrow [\delta(x_1), x_6]+[x_1, \delta(x_6)]=\delta(x_7)$$

$$\Rightarrow [a_{11}x_7+a_{12}x_8-a_{13}x_3-a_{14}x_5]+[-a_{63}x_4+a_{66}x_7+2a_{67}x_8]$$

$$=a_{71}x_1+a_{72}x_2+\cdots+a_{78}x_8$$

$$(2.43) \qquad \begin{aligned} &\Rightarrow a_{71}=0 \\ &a_{72}=0 \\ &a_{73}=0 \\ &a_{74}+a_{63}=0 \\ &a_{75}+a_{14}=0 \\ &a_{76}=0 \\ &a_{77}-a_{11}-a_{66}=0 \\ &a_{78}-a_{12}-2a_{61}=0 \end{aligned}$$

$$[x_1, x_7]=2x_8 \Rightarrow [\delta(x_1), x_7]+[x_1, \delta(x_7)]=2(x_8)$$

$$\Rightarrow [2a_{11}x_8-a_{13}x_5]+[-a_{73}x_4+a_{76}x_7+2a_{77}x_8]$$

$$=2a_{81}x_1+2a_{82}x_2+\cdots+2a_{88}x_8$$

$$(2.44) \qquad \begin{aligned} &\Rightarrow a_{81}=0 \\ &a_{82}=0 \\ &a_{83}=0 \\ &2a_{84}+a_{73}=0 \\ &2a_{85}+a_{13}=0 \\ &a_{86}=0 \\ &a_{87}=0 \\ &a_{88}-a_{11}-a_{77}=0 \end{aligned}$$

$$[x_1, x_8]=0 \Rightarrow [\delta(x_1), x_8]+[x_1, \delta(x_8)]=0$$

$$(2.45) \qquad \begin{aligned} &\Rightarrow -a_{83}x_4+a_{86}x_7+2a_{87}x_8=0 \\ &a_{87}=0 \end{aligned}$$

$$[x_2, x_3]=-2x_5 \Rightarrow [\delta(x_2), x_3]+[x_2, \delta(x_3)]=-2\delta(x_5)$$

$$\Rightarrow [-a_{21}x_4-2a_{22}x_5+a_{26}x_4+a_{27}x_5]+[-2a_{33}x_5+a_{36}x_8]$$

$$=-2a_{51}x_1-2a_{52}x_2-2a_{53}x_3\cdots-2a_{58}x_8$$

$$(2.46) \qquad \Rightarrow a_{51}=0$$

$$a_{52}=0$$
$$2a_{54}-a_{21}+a_{26}=0$$
$$2a_{55}-2a_{22}-2a_{33}+a_{27}=0$$
$$2a_{58}+a_{36}=0$$

$$[x_2, x_4]=0 \Rightarrow [\delta(x_2), x_4]+[x_2, \delta(x_4)]=0$$
$$\Rightarrow a_{26}x_5+[-2a_{43}x_5+a_{46}x_8]=0$$

$$[x_2, x_5]=0 \Rightarrow [\delta(x_2), x_5]+[x_2, \delta(x_5)]=0$$
$$\Rightarrow -2a_{53}x_5+a_{56}x_8=0$$

$$[x_2, x_6]=x_8 \Rightarrow [\delta(x_2), x_6]+[x_2, \delta(x_6)]=\delta(x_8)$$
$$\Rightarrow [a_{21}x_7+a_{22}x_8-a_{23}x_4-a_{24}x_5]+[-2a_{63}x_5+a_{66}x_8]$$
$$=a_{81}x_1+a_{82}x_2+\cdots+a_{88}x_8$$

(2.47)
$$a_{84}+a_{23}=0$$
$$a_{85}+a_{24}+2a_{63}=0$$
$$a_{21}=0$$
$$a_{88}-a_{22}-a_{66}=0$$

$$[x_2, x_7]=0 \Rightarrow [\delta(x_2), x_7]+[x_2, \delta(x_7)]=0$$
$$\Rightarrow [2a_{21}x_8-a_{23}x_5]+[-2a_{73}x_5+a_{76}x_8]=0$$

(2.48)
$$\Rightarrow a_{21}=0$$
$$a_{23}+2a_{73}=0$$

$$[x_2, x_8]=0 \Rightarrow [\delta(x_2), x_8]+[x_2, \delta(x_8)]=0$$
$$\Rightarrow -2a_{83}x_5+a_{86}x_8=0$$

$$[x_3, x_4]=0 \Rightarrow [\delta(x_3), x_4]+[x_3, \delta(x_4)]=0$$

(2.49)
$$\Rightarrow a_{36}x_5+[a_{41}x_4+2a_{42}x_5-a_{46}x_4-a_{47}x_5]=0$$
$$a_{36}+2a_{42}-a_{47}=0$$

$$[x_3, x_5]=0 \Rightarrow [\delta(x_3), x_5]+[x_3, \delta(x_5)]=0$$
$$\Rightarrow a_{51}x_4+2a_{52}x_5-a_{56}x_4-a_{57}x_5=0$$

$$[x_3, x_6]=-x_4 \Rightarrow \delta[(x_3), x_6]+[x_3, \delta(x_6)]=-\delta(x_4)$$
$$\Rightarrow [a_{31}x_7+a_{32}x_8-a_{33}x_4-a_{34}x_5]+[a_{61}x_4+2a_{62}x_5-a_{66}x_4-a_{67}x_5]$$
$$=-a_{41}x_1-a_{42}x_2-\cdots-a_{48}x_8$$

(2.50)
$$a_{44}-a_{33}+a_{61}-a_{66}=0$$
$$a_{45}-a_{34}+2a_{62}-a_{67}=0$$

$$a_{47}+a_{31}=0$$
$$a_{48}+a_{32}=0$$

$$[x_3, x_7]=-x_5 \Rightarrow [\delta(x_3), x_7]+[x_3, \delta(x_7)]=-\delta(x_5)$$
$$\Rightarrow [2a_{31}x_8-a_{33}x_5]+[a_{71}x_4+2a_{72}x_5-a_{76}x_4-a_{77}x_5]$$
$$=-a_{51}x_1-a_{52}x_2-\cdots-a_{58}x_8$$
$$\Rightarrow a_{54}=0 \tag{2.51}$$
$$a_{55}-a_{33}+2a_{75}-a_{77}=0$$
$$a_{58}+2a_{31}=0$$

$$[x_3, x_8]=0 \Rightarrow [\delta(x_3), x_8]+[x_3, \delta(x_8)]=0$$
$$\Rightarrow a_{81}x_4+2a_{82}x_5-a_{86}x_4-a_{87}x_5=0$$

$$[x_4, x_5]=0 \Rightarrow [\delta(x_4), x_5]+[x_4, \delta(x_5)]=0$$
$$-a_{56}x_5=0$$

$$[x_4, x_6]=-x_5 \Rightarrow [\delta(x_4), x_6]+[x_4, \delta(x_6)]=-\delta(x_6)$$
$$\Rightarrow [a_{41}x_7+a_{42}x_8-a_{43}x_4-a_{44}x_5]-a_{66}x_5$$
$$=-a_{51}x_1-a_{52}x_2-\cdots-a_{58}x_8 \tag{2.52}$$
$$\Rightarrow a_{55}-a_{44}-a_{66}=0$$
$$a_{58}=0$$

$$[x_4, x_7]=0 \Rightarrow [\delta(x_4), x_7]+[x_4, \delta(x_7)]=0$$
$$\Rightarrow [2a_{41}x_8-a_{43}x_5]-a_{76}x_5=0$$

$$[x_4, x_8]=0 \Rightarrow [\delta(x_4), x_8]+[x_4, \delta(x_8)]=0$$
$$\Rightarrow -a_{86}x_5=0$$

$$[x_5, x_6]=0 \Rightarrow [\delta(x_5), x_6]+[x_5, \delta(x_6)]=0$$
$$\Rightarrow a_{51}x_7+a_{52}x_8-a_{53}x_4-a_{54}x_5=0$$

$$[x_5, x_7]=0 \Rightarrow [\delta(x_5), x_7]+[x_5, \delta(x_7)]=0$$
$$\Rightarrow 2a_{51}x_8-a_{53}x_5=0$$

$$[x_5, x_8]=0 \Rightarrow [\delta(x_5), x_8]+[x_5, \delta(x_8)]=0$$

$$[x_6, x_7]=0 \Rightarrow [\delta(x_6), x_7]+[x_6, \delta(x_7)]=0$$
$$\Rightarrow [2a_{61}x_8-a_{63}x_5]+[-a_{71}x_7-a_{72}x_8+a_{73}x_4+a_{74}x_5]=0$$
$$a_{74}-a_{63}=0 \tag{2.53}$$
$$2a_{61}=0$$

$$[x_6, x_8]=0 \Rightarrow [\delta(x_6), x_8]+[x_6, \delta(x_8)]=0$$

$$-a_{81}x_7-a_{82}x_8+a_{83}x_4+a_{84}x_5=0 \tag{2.54}$$

$$a_{84}=0$$

$$[x_7, x_8]=0 \Rightarrow [\delta(x_7), x_8]+[x_7, \delta(x_8)]=0$$
$$\Rightarrow -2a_{81}x_8+a_{83}x_5=0$$

(2.39), (2.40), $\cdots$, (2.54) imply

$$\begin{aligned}
&a_{13}=a_{16}=0\\
&a_{21}=a_{23}=a_{24}=a_{26}=a_{27}=0\\
&a_{31}=a_{36}=0\\
&a_{41}=a_{42}=a_{43}=a_{46}=a_{47}=0\\
&a_{51}=a_{52}=a_{53}=a_{54}=a_{56}=a_{57}=a_{58}=0\\
&a_{61}=a_{63}=0\\
&a_{71}=a_{72}=a_{73}=a_{74}=a_{76}=0\\
&a_{81}=a_{82}=a_{83}=a_{84}=a_{85}=a_{86}=a_{87}=0
\end{aligned}$$

$$\begin{aligned}
&a_{22}=2a_{11} && a_{66}=a_{11}\\
&a_{37}=\tfrac{1}{2}a_{32} && a_{67}=2a_{12}-a_{17}-a_{34}+2a_{62}\\
&a_{44}=a_{11}+a_{33} && a_{75}=-a_{14}\\
&a_{45}=2a_{12}-a_{17} && a_{77}=2a_{11}\\
&a_{48}=-a_{32} && a_{78}=5a_{12}-2a_{17}-2a_{34}+4a_{62}\\
&a_{55}=a_{33}+2a_{11} && a_{88}=3a_{11}.
\end{aligned}$$

Then δ has the following matrix representation

$$A=\begin{pmatrix}
a_{11} & a_{12} & 0 & a_{14} & a_{15} & 0 & a_{17} & a_{18}\\
0 & 2a_{11} & 0 & 0 & a_{25} & 0 & 0 & a_{28}\\
0 & a_{32} & a_{33} & a_{34} & a_{35} & 0 & \frac{1}{2}a_{32} & a_{38}\\
0 & 0 & 0 & a_{11}+a_{33} & 2a_{12}-a_{17} & 0 & 0 & -a_{32}\\
0 & 0 & 0 & 0 & 2a_{11}+a_{33} & 0 & 0 & 0\\
0 & a_{62} & 0 & a_{64} & a_{65} & a_{11} & \begin{matrix}2a_{12}-a_{17}\\-a_{34}+2a_{62}\end{matrix} & a_{68}\\
0 & 0 & 0 & 0 & -a_{14} & 0 & 2a_{11} & \begin{matrix}5a_{12}-2a_{17}\\-2a_{34}+4a_{62}\end{matrix}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3a_{11}
\end{pmatrix}$$

δ is semi simple $\Leftrightarrow AA^*=A^*A$

$\Rightarrow$ the length of i^{th} row of $A=$ the length of j^{th} column of A.

$a_{ij}=0$ for $i\neq j$.

Hence

$$A=\begin{bmatrix} a_{11} &&&&&&& \\ & 2a_{11} &&&&& 0 & \\ && a_{33} &&&&& \\ &&& a_{11}+a_{33} &&&& \\ &&&& 2a_{11}+a_{33} &&& \\ & 0 &&&& a_{11} && \\ &&&&&& 2a_{11} & \\ &&&&&&& 3a_{11} \end{bmatrix}.$$

We have proved that any derivation of $\mathfrak{g}(V)$ must be a linear combination of t_1 and t_2. It is an easy matter to check that t_1 and t_2 are really a derivation of $\mathfrak{g}(V)$.

Chapter 3. Using a computer to calculate the Lie algebra of derivations and Lie algebra cohomology

3.1. Overview

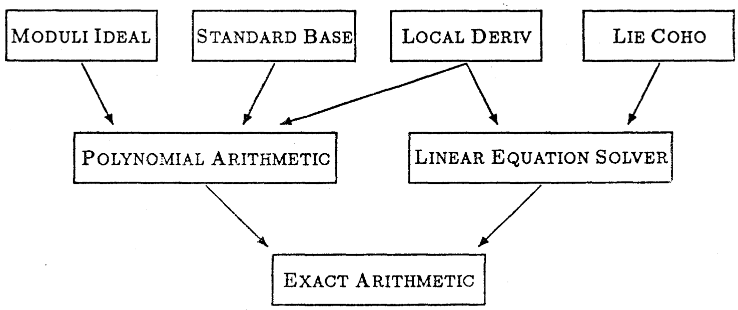
A computer can be used to calculate many singularity invariants. In this paper we will describe how the Lie algebra of derivations of an isolated hypersurface's moduli algebra can be computed. We will also show how to calculate its Lie algebra cohomology.

The computation methods we will discuss have been implemented by M. Benson in the C programming language. His programs presently run on several versions of UNIX[1] but should be portable to other operating systems. They are designed for making calculations in local rings which are finite dimensional quotients of power series rings.

The implementation approach is modular. Each program works more or less independently of the others, but in most cases the standard output of one can be used as standard input for another. Because of this, the modules are easily glued together using UNIX pipelines. Output documenting the computation is sent to the standard error and a disk file.

The calculations we are interested in are performed by four distinct program modules MODULI IDEAL, STANDARD BASE, LOCAL DERIV, and LIE COHO pipelined in this order. These program modules rely on a large library of support routines. The module dependency is depicted below.

1) UNIX is a trademark of AT & T Bell Laboratories.



Organization of Program Modules

We will begin by indicating how some of the library routines have been implemented.

3.2. Representation of polynomials

Polynomials are stored as linked lists of terms, where each term is represented by a vector specifying the exponents of the top level variables and by a coefficient. This coefficient can be either a pointer to polynomial in parameter level variables or else an exact fraction.

The methods of implementing the algebraic operations with polynomials are standard. Operations include reading and writing polynomials, addition, subtraction, negation, multiplication, exponentiation, truncation of high degree terms, and reduction modulo a list of polynomials. The last two operations facilitate computations in quotient rings.

3.3. Representation of derivations

Any derivation D of a local ring which is a quotient of $\boldsymbol{C}[\![x_1, x_2, \cdots, x_n]\!]$ can be written in the form

$$D=f_1\frac{\partial}{\partial x_1}+f_2\frac{\partial}{\partial x_2}+\cdots+f_n\frac{\partial}{\partial x_n} \tag{3.1}$$

where $f_i=Dx_i$ for $i=1, 2, \cdots, n$. This can be seen by first using induction on the degree to check that the left and right hand sides of (3.1) agree when you apply them to any monomial. Then (3.1) must hold true in all cases by linearity.

In our implementation, a derivation is stored as an array of the polynomial coefficients $f_1, f_2, \cdots, f_n$. Operations like applying derivations to polynomials or calculating Lie brackets were easy to implement.

3.4. Solving systems of linear equations

Large systems of linear equations with fractional coefficients appear in both LOCAL DERIV and LIE COHO. Examples that a mathematician would normally choose turn out to be very sparse.

A system of linear equations is represented by a linked list of sparse coefficient vectors representing the equations. Thus each equation in the system is represented by a linked list of its nonzero coefficients. To facilitate adding new columns to the system, a pointer to the last node in each row is stored.

The uses of the systems of linear equations in the programs differ. In one case the basis of the kernel of a linear system of equations is required, in another only the rank is needed. In a third case a set of particular solutions to a multiple system of equations of the form

$$\begin{aligned} a_{11}z_1+a_{12}z_2+\cdots+a_{1m}z_m &= b_{11},\ b_{12},\ \cdots,\ b_{1k}\\ a_{21}z_1+a_{22}z_2+\cdots+a_{2m}z_m &= b_{21},\ b_{22},\ \cdots,\ b_{2k}\\ &\cdots\\ a_{p1}z_1+a_{p2}z_2+\cdots+a_{pm}z_m &= b_{p1},\ b_{p2},\ \cdots,\ b_{pk} \end{aligned} \tag{3.2}$$

is required. In the first two situations the rows can be generated sequentially so Gaussian elimination can be performed as the rows are produced. This appears to be a big space saver because the systems are extremely over-determined. In the last case the columns are produced in sequence, meaning that the system must be stored and then solved at the end.

Both situations can be handled adequately with the approach used here. Gaussian elimination is performed sequentially by adding a row to a matrix already in echelon form and eliminating. If the row reduces to zero, the row is dropped. When the row vectors are generated sequentially this gives a space efficient implementation. And, it works satisfactorily for the other case as well.

Once the matrix is in echelon form it can be put in reduced echelon form by Gauss-Jordan elimination if particular solutions or a basis of the kernel are required. All of these procedures have been implemented. They return matrices whose sparse rows are the desired vectors.

3.5. The user interface

Moduli Ideal is a small program used to read the defining equation, calculate the partial derivatives of this polynomial, and then write information about the moduli ideal to the standard output.

3.6. Finding the standard base

Standard Base reads the generators of an ideal I in $C[\![x_1, x_2, \cdots, x_n]\!]$, and attempts to find a standard base for this ideal. The calculation will terminate whenever the quotient ring is finite dimensional. The standard base is a new set of generators for which testing membership in the ideal I is easy. It is implicit in many of papers of H. Hironaka, and it has been widely used in the study of polynomial ideals. A nice presentation of the concepts for polynomial ideals is given by D. Bayer in [1]. He and M. Stillman have used a computer to calculate standard bases of polynomial ideals.

STANDARD BASE uses the ideal base it finds to construct a basis of monomials for the quotient algebra $A = C[\![x_1, x_2, \cdots, x_n]\!]/I$ considered as a complex vector space. This is easy to do because the initial forms of the standard base elements lie on a "staircase" which delineate the elements in the monomial basis from the rest.

Here is an example which illustrates the use of the standard basis. In this case the ideal is the moduli ideal for the singularity defined by

$$f(x_1, x_2, x_3) = x_3^2 - (x_1^2 + x_2^2)(x_1^2 + x_2^3) \tag{3.3}$$

The generators of the moduli ideal are

$$\begin{aligned} &2x_3 \\ &-2x_1^2x_2 - 3x_1^2x_2^2 - 5x_2^4 \\ &-4x_1^3 - 2x_1x_2^2 - 2x_1x_2^3 \\ &x_3^2 - (x_1^2 + x_2^2)(x_1^2 + x_2^3) \end{aligned} \tag{3.4}$$

and a standard base is given by

$$\begin{aligned} &x_3 \\ &x_1^4 \\ &x_1x_2^2 + 2x_1^3 \\ &2x_1^2x_2 + 5x_2^4 \\ &x_2^5. \end{aligned} \tag{3.5}$$

We can depict the portion of the "staircase" lying in the xy-plane by using asterisks to indicate the monomials which are initial forms of elements of the ideal. The monomials which lie outside the asterisks are elements of the monomial basis.

$$\begin{array}{ccccc} * & * & * & * & * \\ x_2^4 & * & * & * & * \\ x_2^3 & * & * & * & * \\ x_2^2 & * & * & * & * \\ x_2 & x_1x_2 & * & * & * \\ 1 & x_1 & x_1^2 & x_1^3 & * \end{array} \tag{3.6}$$

Given the standard base and the basis of monomials, it is easy to reduce any power series to a linear combination of basis elements by subtracting off multiples of members of the standard basis. Reduction of a polynomial by a list of polynomials is one of the basic routines in the library which performs arithmetic of multivariate polynomials.

The algorithms used to construct the standard base of a finite dimensional power series ring are important for many calculations. A complete description of these algorithms will be given elsewhere.

3.7. Computing the derivations

LOCAL DERIV reads the standard base for an ideal I in $C[\![x_1, x_2, \cdots, x_n]\!]$ and the monomial basis of the quotient algebra $A = C[\![x_1, x_2, \cdots, x_n]\!]/I$. It uses this information to compute the Lie algebra, Der A, of derivations of A.

Any element of Der A can be written in the form (3.1). Since the coefficients $f_1, f_2, \cdots, f_n$ are defined only up congruence modulo I, we can choose representatives which are linear combinations of the monomial basis. For the example of the previous section, any derivation of the moduli algebra must be of this form

$$\begin{aligned}&(a_1+a_2x_1+a_3x_2+a_4x_1^2+a_5x_1x_2+a_6x_2^2+a_7x_1^3+a_8x_2^3+a_9x_2^4)\frac{\partial}{\partial x_1}\\ &+(b_1+b_2x_1+b_3x_2+b_4x_1^2+b_5x_1x_2+b_6x_2^2+b_7x_1^3+b_8x_2^3+b_9x_2^4)\frac{\partial}{\partial x_2}\\ &+(c_1+c_2x_1+c_3x_2+c_4x_1^2+c_5x_1x_2+c_6x_2^2+c_7x_1^3+c_8x_2^3+c_9x_2^4)\frac{\partial}{\partial x_3}.\end{aligned} \tag{3.7}$$

LOCAL DERIV actually generates a symbolic expression like (3.7). Not every expression of this form is really a derivation of A. Derivations must send each generator of I back into I. To determine the possible derivations of A we must apply the form (3.7) to each of the generators of I, reduce this formal expression modulo I, and then set it equal to zero.

For example, if we apply (3.7) to the standard base given in (3.5) we get the equations

$$\begin{aligned}&c_1+c_2x_1+c_3x_2+c_4x_1^2+c_5x_1x_2+c_6x_2^2+c_7x_1^3+c_8x_2^3+c_9x_2^4=0\\ &4a_1x_1^3=0\\ &6a_1x_1^2+2b_1x_1x_2+a_1x_2^2+(4a_2-4b_3)x_1^3+a_3x_2^3+(-15a_3-5b_2+a_6)x_2^4=0\\ &2b_1x_1^2+4a_1x_1x_2+(-8a_3+2b_2)x_1^3+20b_1x_2^3+(-10a_2+15b_3)x_2^4=0\\ &5b_1x_2^4=0.\end{aligned} \tag{3 8}$$

The coefficients of the basis elements must vanish independently of each other. Each of these coefficients is a linear form in the parameter variables. We end up with a homogeneous system of linear equations determining which values of the prameters in (3.7) give derivations.

These equations are solved to get a basis for the kernel, and therefore also a basis for the Lie algebra of derivations. Here is the basis of derivations computed from the system of equations arising from (3.8).

$$
\begin{array}{lll}
D_1 = x_2^4 \dfrac{\partial}{\partial x_1} & D_2 = x_1^3 \dfrac{\partial}{\partial x_1} & D_3 = x_2^3 \dfrac{\partial}{\partial x_1} \\
D_4 = x_1^2 \dfrac{\partial}{\partial x_1} & D_5 = x_1 x_2 \dfrac{\partial}{\partial x_1} & D_6 = x_2^4 \dfrac{\partial}{\partial x_2} \\
D_7 = x_1^3 \dfrac{\partial}{\partial x_2} & D_8 = x_2^3 \dfrac{\partial}{\partial x_2} & D_9 = x_1^2 \dfrac{\partial}{\partial x_2} \\
D_{10} = x_1 x_2 \dfrac{\partial}{\partial x_2} & D_{11} = x_2^2 \dfrac{\partial}{\partial x_2} &
\end{array}
\tag{3.9}
$$

The final step is compute the structure constants for the Lie algebra Der A. The Lie brackets $[D_i, D_j]$, for $i<j$ are computed as derivations and then expressed back in terms of the basis elements. In principle, expressing each of the Lie brackets in terms of the basis means solving a system of linear equations, although typically many of the brackets are zero and therefore can be eliminated.

Here is what LOCAL DERIV found when it calculated the Lie brackets of each pair of elements of the basis given in (3.8).

$$
\begin{array}{lll}
[D_3, D_5] = D_1 & [D_3, D_{10}] = D_6 & [D_3, D_{11}] = -D_1 \\
[D_4, D_5] = \frac{5}{2} D_1 & [D_4, D_9] = 2D_7 & [D_4, D_{10}] = \frac{5}{2} D_6 \\
[D_5, D_9] = -D_2 - 5D_6 & [D_5, D_{10}] = \frac{5}{2} D_1 - 2D_7 & [D_5, D_{11}] = 2D_2 \\
[D_8, D_{11}] = -D_6 & [D_9, D_{10}] = D_7 & [D_{10}, D_{11}] = -2D_7
\end{array}
\tag{3.10}
$$

All of the other brackets vanish.

It is important from the standpoint of computing to notice that fractional coefficients often appear in the Lie brackets even though the original basis had integer or even monic coefficients. This occurs when we reduce modulo I. This phenomenon forces us to work with exact fractions instead of trying to stay with exact integers for efficiency.

Here is summary of the algorithm used by LOCAL DERIV to find the Lie algebra of derivations of a finite dimensional power series quotient ring.

Algorithm 3.1 (Local Deriv).

1. Read data specifying power series ideal from the standard input.
2. Generate $n \times \dim A$ parameter variables.
3. Construct the general form, D, of a derivation as in (3.7) using the standard basis monomials and the parameter variables constructed in the last step.
4. Initialize KERNEL SYSTEM to be an empty list of rows.
5. For each element e_i of the standard basis.
   (a) Compute De_i and reduce moduli I. The result will be a linear combination of basis monomials with coefficients being linear forms in the parameter variables.
   (b) For each term in the list representing De_i.
      i. Convert the coefficient of that term to an $n \times \dim A$ dimensional sparse coefficient vector v.
      ii. Add v as a row of KERNEL SYSTEM. Bring the matrix back to echelon form by elimination. Drop v if it reduces to zero.
6. Bring KERNEL SYSTEM to reduced echelon form.
7. Compute a basis for the kernel of KERNEL SYSTEM.
8. Construct a derivation D_i for each element of the kernel basis.
9. Initialize BRACKET SYSTEM to be a list of $n \times \dim A$ empty rows.
10. For each D_i in the basis of derivations
    (a) Add a new column to BRACKET SYSTEM representing D_i.
11. Initialize the set NON ZERO to be empty.
12. For each D_i in the basis of derivations
    (a) For each D_j in the basis of derivations, $j > i$
       i. Compute the Lie bracket $[D_i, D_j]$
       ii. If this new derivation is not zero, then
          A. Insert (i, j) into NON ZERO
          B. Add a new column to BRACKET SYSTEM representing $[D_i, D_j]$.
13. Bring BRACKET SYSTEM to echelon form by Gaussian elimination, and then to reduced echelon form by Gauss-Jordan elimination.
14. Construct a list SOLUTIONS of particular solutions to the multi-system of equations given by BRACKET SYSTEM. Each row in SOLUTIONS expresses one of the nonzero Lie brackets in terms of the D_i.
15. Output the derivations and the nonzero brackets to the standard error and file. The contents of NON ZERO indicate which brackets are in the list of SOLUTIONS.
16. Output dimension and structure constants to the standard output.

3.8. Calculating Lie algebra cohomology

LIE COHO calculates the Betti numbers of the cohomology groups $H^i(L, W)$ where L is a Lie algebra and W is an L-module. Before discussing the algorithm that LIE COHO uses, we will review the definition of Lie algebra cohomology.

The set of k-cochains $C^k(L, W)$ of L with coefficients in W is defined to be the vector space of all linear maps from the k-fold tensor product $L^{(k)} = \bigotimes_{i=1}^k L$ into W. Let $n = \dim L$, then $C^k(L, W) = 0$ for $k > n$, and both $C^0(L, W)$ and $C^n(L, W)$ can be identified with the set of constant maps into W. $C(L, W)$ is defined to be the direct sum $\bigoplus_{k \geq 0} C^k(L, W)$.

The coboundary operator δ: $C(L, W) \to C(L, W)$ is a homogeneous linear operator of degree 1. It is defined on each of the homogeneous pieces of $C(L, W)$ as follows. If $f \in C^k(L, W)$, then $\delta_k f \in C^{k+1}(L, W)$ where

$$
\begin{aligned}
\delta_k f(x_0, x_1, \cdots, x_k) = &\sum_{i=0}^{k} (-1)^i x_i \cdot f(x_0 \otimes \cdots \otimes \hat{x}_i \otimes \cdots \otimes x_k) \\
&+ \sum_{i<j} (-1)^{i+j} f([x_i, x_j] \otimes x_0 \cdots \hat{x}_i \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes x_k).
\end{aligned} \tag{3.11}
$$

In the first summation, the dot is used to represent the action of $x_i \in L$ on $f(x_0 \otimes \cdots \otimes \hat{x}_i \otimes \cdots \otimes x_k) \in W$.

The pair $(C(L, W), \delta)$ forms a cochain complex. The Lie algebra cohomology of (L, W) is defined to be the homology of this complex.

Computing Lie algebra cohomology turns out to be just a gigantic linear algebra problem. As we know, $H^i(L, W)$ measures the exactness of the sequence

$$
C^0(L, W) \xrightarrow{\delta_0} C^1(L, W) \xrightarrow{\delta_1} C^2(L, W) \xrightarrow{\delta_2} C^3(L, W) \xrightarrow{\delta_3} \cdots \tag{3.12}
$$

that is

$$
\dim H^k(L, W) = \dim(\operatorname{Ker} \delta_k) - \dim(\operatorname{Im} \delta_{k-1}). \tag{3.13}
$$

In this equation the second term on the right hand side is assumed to be zero for $k=0$.

We can bring (3.13) into an even simpler form for computing the Betti numbers. Using the *Rank + Nullity Theorem* we get

$$
\dim H^k(L, W) = \dim C^k(L, W) - (\operatorname{rank} \delta_k + \operatorname{rank} \delta_{k-1}) \tag{3.14}
$$

where again if $k=0$, the rightmost term of the right hand side of (3.14) is understood to be zero. This shows that it is enough to compute the ranks of the linear maps $\delta_0, \delta_1, \cdots, \delta_{n-1}$.

We are going to introduce coordinate systems in the $C^k(L, W)$ in order to compute these ranks. Let $v_1, v_2, \cdots, v_n$ be a basis for L and $w_1, w_2, \cdots, w_m$ for W. In terms of these bases, the Lie algebra structure and the action of L on W can be expressed as follows.

$$[v_i, v_j] = \sum_{l=1}^{n} \Gamma^l_{ij} v_l \tag{3.15}$$

$$v_i \cdot w_j = \sum_{l=1}^{m} P^l_{ij} \cdot w_l \tag{3.16}$$

We will call any k-element subset $I = \{i_1, i_2, \cdots, i_k\}$ of $\{1, 2, \cdots, n\}$ a *k-index*. For any k-index I we define $v_I = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$ where $i_1 < i_2 < \cdots < i_k$. The set B_k of all such v_I forms the product basis of $L^{(k)}$.

We can now define a basis B_k^* of $C^k(L, W)$. B_k^* consists of all $v^*_{I,p}$ for I a k-index and $p = 1, 2, \cdots, m$, where $v^*_{I,p}$ is defined on the elements of B_k by

$$v^*_{I,p}(v_J) = \begin{cases} w_p & J = I \\ 0 & J \neq I \end{cases} \tag{3.17}$$

and extended to $L^{(k)}$ by linearity. We see from this construction that $\dim C^k(L, W) = m \times \binom{n}{k}$.

We will introduce some more notation. The sign of the permutation of $I \cup \{p\}$ that you get by listing p first, followed by the elements of $I \backslash \{p\}$ will denoted as follows

$$\operatorname{sign}(p, I) = (-1)^{\operatorname{card}\{i \in I \mid i < p\}}. \tag{3.18}$$

We can now compute $\delta_k v^*_{I,p}$ for each k-index I and $p = 1, 2, \cdots, m$. The first step is to evaluate it on each element of the basis B_k of $L^{(k)}$.

Lemma 3.2. *Suppose I is a k-index, J is a $k+1$-index, and $p = 1, 2, \cdots, m$. Then*

1. *If I and J share less than $k-1$ indices, then*

$$\delta_k v^*_{I,p}(v_J) = 0. \tag{3.19}$$

2. *If I and J share exactly $k-1$ indices, where $J \backslash I = \{r, s\}$, $r < s$ and $I \backslash J = \{t\}$, then*

$$\delta_k v^*_{I,p}(v_J) = -\operatorname{sign}(r, J) \cdot \operatorname{sign}(s, J) \cdot \operatorname{sign}(t, I) \cdot \Gamma^t_{rs} w_p. \tag{3.20}$$

3. *If I and J share exactly k indices, where $J \backslash I = \{r\}$, then*

$$\delta_k v^*_{I,p}(v_J) = \operatorname{sign}(r, I)(v_r \cdot w_p + \sum_{q \in I} \Gamma^q_{qr} w_p). \tag{3.21}$$

This lemma gives us enough information to express each $\delta_k v^*_{I,p}$ in terms of the basis B^*_{k+1} of $C^{k+1}(L, W)$.

Corollary 3.3. *Suppose I is a k-index and $p=1, 2, \cdots, m$. Then we have*

$$\begin{aligned}\delta_k v_{I,p}^* = & -\sum_{\substack{r,s\in I'\\ r<s}} \sum_{t\in I} \operatorname{sign}(r, J)\cdot \operatorname{sign}(s, J)\cdot \operatorname{sign}(t, I)\cdot \Gamma_{rs}^t v_{I(r,s;t),p}^* \\ & + \sum_{r\in I'} \operatorname{sign}(r, I)\left(\sum_{q=1}^m P_{rp}^q v_{I(r),q}^* + \sum_{q\in I} \Gamma_{qr}^q v_{I(r),p}^*\right)\end{aligned} \tag{3.22}$$

where $I'=\{1, 2, \cdots, n\}\backslash I$, $I(r)=I\cup\{r\}$, and $I(r, s; t)=I\cup\{r, s\}\backslash\{t\}$.

Lie Coho uses (3.22) to compute the ranks of the δ_k. The image $\delta_k v_{I,p}^*$ of each basis element in B_k^* is written in terms of the basis B_{k+1}^*. The matrix formed by using the coefficients of these linear combinations as row vectors has the same rank as δ_k. We can deduce from (3.22) that this matrix is very sparse. Each row has at most

$$\frac{k(n-k)(n-k-1)}{2}+m(n-k)$$

nonzero coefficients, far less than $m\times\binom{n}{k+1}$, which is the dimension of $C^{k+1}(L, W)$.

Lie Coho uses a bit vector to store a k-index. When m and n are not too large, then each pair (I, p) can be represented in a compact format by one computer word. The exact encoding method is not important for stating the algorithms which follow. We will use the notation index (I, p) to represent the integer corresponding to (I, p).

Here is an outline of the algorithm used by Lie Coho to compute the Betti numbers of Lie algebra cohomology.

Algorithm 3.4 (Lie Coho).

1. Read dimension n and structure constants Γ_{ij}^l of Lie algebra from standard input. Store as sparse matrix.
2. If present, read dimension m and constants P_{ij}^l of the representation. Otherwise assume $L=W$ and take the adjoint representation.
3. For $k=0$ to $n-1$
   (a) Call Compute Rank to find rank of δ_k.
4. For $k=0$ to n
   (a) Output k-th Betti number using the formula

$$h^k=m\binom{n}{k}-\operatorname{rank}\delta_k-\operatorname{rank}\delta_{k-1}.$$

Algorithm 3.5 (Compute Rank).

1. Set Basis Images to be an empty matrix.

2. For each k-index set $I \subset \{1, 2, \cdots, n\}$
   (a) For $p=1$ to m
      i. Call COMPUTE IMAGE to find $E_k v^*_{I,p}$. Returns sparse row vector giving this quantity as a linear combination of the B^*_{k+1}.
      ii. Insert row into BASIS IMAGES bringing matrix back into echelon form. Drop row if it reduces to all zeros.
3. Return the number of rows of BASIS IMAGES.

Algorithm 3.6 (Compute Image).
1. Initialize a sparse row to be empty.
2. Let J be the first $k+1$-set in lexicographical order which meets I in at least $k-1$ members.
3. While J exists do
   (a) If J meets I in exactly $k-1$ members with $J=I \cup \{r, s\} \backslash \{t\}$, $r<s$, then
      i. Compute $-\operatorname{sign}(r, J) \cdot \operatorname{sign}(s, J) \cdot \operatorname{sign}(t, I) \cdot \Gamma^t_{rs}$.
      ii. Place result in position index (J, p) of the sparse row.
   (b) Otherwise, J meets I in exactly k members with $J=I \cup \{r\}$
      i. Copy the product of the sparse row representing $v_r \cdot w_p$ with $\operatorname{sign}(r, I)$ into position index (J, J), index $(J, 2)$, $\cdots$, index (J, m) of the sparse row.
      ii. Add the quantity $\operatorname{sign}(r, I) \cdot \sum_{q \in I} \Gamma^q_{qr}$ to the entry in position index (J, p) of the sparse row.
   (c) Set J to be the next $k+1$-set in lexicographical order which meets I in at least $k-1$ members.
4. Return sparse row constructed.

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