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On T-maps and ideals of antiderivatives of hypersurface singularities

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Abstract. Mather–Yau's theorem leads to an extensive study about moduli algebras of isolated hypersurface singularities. In this paper, the Tjurina ideal is generalized as T-principal ideals of certain T-maps for Noetherian algebras. Moreover, we introduce the ideal of antiderivatives of a T-map, which creates many new invariants. Firstly, we compute two new invariants associated with ideals of antiderivatives for ADE singularities and conjecture a general pattern of polynomial growth of these invariants.

Secondly, the language of T-maps is applied to generalize the well-known theorem that the Milnor number of a semi quasi-homogeneous singularity is equal to that of its principal part. Finally, we use the *T-fullness* and *T-dependence* conditions to determine whether an ideal is a T-principal ideal and provide a constructive way of giving a generator of a T-principal ideal. As a result, the problem about reconstruction of a hypersurface singularitiy from its generalized moduli algebras is solved. It generalizes the results of Rodrigues in the cases of the 0th and 1st moduli algebra, which inspired our solution.

Keywords: isolated singularities, local rings, Kähler differential, semi quasi-homogeneous singularities, Tjurina ideals.

§ 1. Introduction

The motivation of the present research is Mather–Yau's theorem (see [17]). Let $\mathbb{C}\{x_1,\dots,x_n\}$ ($\mathbb{C}\{\boldsymbol{x}\}$ for short) be the ring of complex convergent power series of n variables at $(\mathbb{C}^n,0)$. For an isolated hypersurface singularity $(V,0)\subset(\mathbb{C}^n,0)$ defined by the analytic germ $f\colon(\mathbb{C}^n,0)\to\mathbb{C},0)$, one has the moduli algebra $A(V):=\mathcal{O}_n/(f,\partial f/\partial x_1,\dots,\partial f/\partial x_n)$, which is finite dimensional. The well-known Mather–Yau theorem is as follows. Let $(V_1,0)$ and $(V_2,0)$ be two isolated hypersurface singularities, $A(V_1)$ and $A(V_2)$ be their respective moduli algebras, then $(V_1,0)\cong(V_2,0)\iff A(V_1)\cong A(V_2)$. The biholomorphic classes of isolated hypersurface singularities correspond to isomorphism classes of commutative $\mathbb{C}$-algebras. The Mather–Yau theorem plays a very important role in the classification of isolated hypersurface singularities.

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In the classification theory of isolated singularities, one always wants to find invariants associated with the isolated singularities. Hopefully, with enough invariants found, one can distinguish between isolated singularities. Mather–Yau's theorem tells us that the moduli algebra $A(V)$ is a complete invariant of an isolated hypersurface singularity $(V,0)$. All the information about singularities can be taken from its moduli algebra. It is natural to ask if there are other $\mathbb{C}$-analytic algebras that play a similar role to the moduli algebra. In this paper, we call a local algebra which satisfies Mather–Yau theorem a *valid moduli algebra*. Since a valid moduli algebra is often a quotient ring of $\mathbb{C}\{\boldsymbol{x}\}$ modulo an ideal, we call a map $Q\colon \mathbb{C}\{\boldsymbol{x}\} \to \{\text{ideals of } \mathbb{C}\{\boldsymbol{x}\}\}$ a *moduli ideal map* if, for any $f \in \mathbb{C}\{\boldsymbol{x}\}$, $\mathbb{C}\{\boldsymbol{x}\}/Q(f)$ is a local algebra invariant of singularity $(V(f),0)$. For example, the kth Tjurina ideal map $Q = T_k\colon f \mapsto (f) + (\boldsymbol{x})^k J(f)$, $J(f) = (\partial f/\partial x_1, \dots, \partial f/\partial x_n)$ is a moduli ideal map. A map Q is called *valid* if each $\mathbb{C}\{\boldsymbol{x}\}/Q(f)$ is a valid moduli algebra if $(V(f),0)$ is an isolated hypersurface singularity. In past years, Yau, Zuo and their collaborators have introduced many new local algebras to singularities: higher Nash Blow-up local algebra (see [13]), kth local Hessian algebra (see [14]), kth moduli algebra (see [12]) and kth singular local moduli algebra (see [18]). These local algebras are new invariants of singularities. They play important roles in the classification theory of singularities. It is a natural question whether these new algebras are valid moduli algebras. The answer is yes for kth moduli algebra (see the *generalized Mather–Yau's theorem*, [9]). Moreover, it is known that the kth local Hessian algebra is also a valid moduli algebra for some k (see [5]).

For a hypersurface singularity $(V(f),0)$, its Tjurina ideal is defined by $T(f) := (f)+J(f)$, whose corresponding moduli algebra $\mathbb{C}\{\boldsymbol{x}\}/T(f)$ is also called a Tjurina algebra or a moduli algebra. In [19], Rodrigues proposed the problem of finding a necessary and sufficient condition that an ideal I of $\mathbb{C}\{\boldsymbol{x}\}$ is a Tjurina ideal. By introduction of the concepts of T-fullness and T-dependence, the problem was finally solved. If one can further find an $f \in \mathbb{C}\{\boldsymbol{x}\}$ such that $I = T(f)$, then the problem of reconstructing a hypersurface singularity from its moduli algebra is also solved, since an analytic algebra is given by $\mathbb{C}\{\boldsymbol{x}\}$ modulo an ideal. Motivated from his work, we propose a more general problem.

Question 1.1. Let $Q\colon \mathbb{C}\{\boldsymbol{x}\} \to \{\text{ideals of } \mathbb{C}\{\boldsymbol{x}\}\}$ be a valid moduli ideal map. For an ideal $I \lhd \mathbb{C}\{\boldsymbol{x}\}$, find a necessary and sufficient condition that $I = Q(g)$ for some $g \in \mathbb{C}\{\boldsymbol{x}\}$.

Many well-known valid moduli ideal maps are of the form $Q(f) = (Q_1(f), \dots, Q_m(f))$, where m is a fixed integer and all $Q_i\colon \mathbb{C}\{\boldsymbol{x}\} \to \mathbb{C}\{\boldsymbol{x}\}$ are $\mathbb{C}$-linear maps. For example, Tjurina ideal map is of this form. From this, the problem has an algebraic generalization stated as below.

Question 1.2. Let A be an algebra over a field F, $Q\colon A \to \{\text{ideals of } A\}$ is a map of the form $Q(f) = (Q_1(f), \dots, Q_m(f))$, where m is a fixed integer and $Q_i \in \operatorname{End}_F(A)$. Then for an ideal $I \lhd A$, how to find a necessary and sufficient condition that $I = Q(g)$ for some $g \in A$.

In our article, we solved Question 1.2 when Q is a T-map (see Definition 4.1) and A is a Noetherian F-algebra, where F is an infinite residue field. The introduction of T-map is of importance, since it includes many well-known moduli ideal maps: higher order Tjurina ideal map (the sum of higher order Jacobian ideals (see [6]), kth Tjurina ideal map (see [12]) and kth local Hessian ideal map (see [14]). Our solution is motivated from [19], with necessary adjustments. We introduce the ideal of antiderivatives, T-fullness and T-dependence with respect to (w.r.t., for short) T-maps (see § 4.5) and prove our main theorem.

Theorem A (Theorem 4.76 and Algorithm 4.82). *Let F be an infinite field and A be a Noetherian local F-algebra with maximal ideal $\mathfrak{m}$. Suppose $A/\mathfrak{m} \simeq F$. Let Q be a fixed T-map of F-algebra A. An ideal $I \triangleleft A$ is a T-principal ideal if and only if I is T-full and $\Delta(I)$ is T-dependent. Moreover, if I is a T-principal ideal, then a generator of I can be explicitly calculated.*

The notions "T-full" and "T-dependent" are conditions on Q. Besides, a T-principal ideal refers to an ideal of the form $Q(f)$, $f \in A$. For example, Tjurina ideals are those T-principal ideals in the $\mathbb{C}$-algebra $\mathbb{C}\{\boldsymbol{x}\}$ if $Q(f) = (f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ for all $f \in \mathbb{C}\{\boldsymbol{x}\}$. We point out that the theorem holds for an arbitrary infinite field, even with a positive characteristic. For example, $A = \mathbb{F}_p((t))[x_1, x_2, \ldots, x_n]$ with $F = \mathbb{F}_p((t))$ also satisfies the assumption. However, the correctness of the theorem when F is a finite field has not been verified, but we conjecture that it is also true.

Furthermore, we give a constructive method to recover a hypersurface singularity from its kth moduli ideal in Algorithm 4.82. This gives us an answer to well-known reconstruction problem in [23] given by the second author: how can one construct the singularity $(V, 0)$ explicitly from moduli algebra $A(V)$. The difficulty of this problem is reduced to the computation of the ideal of antiderivatives. In § 4.1, we provide approaches to finding ideals of antiderivatives w.r.t. higher order Tjurina ideal maps and kth Tjurina ideal maps.

Besides, we introduce various invariants associated with the ideal of antiderivatives (see § 4.1). In § 4.2, we introduce a series of invariants of singularities ρ_k, σ_k and T-threshold. Briefly, for $f \in \mathbb{C}\{\boldsymbol{x}\}$, which defines an isolated singularity at the origin, $\Delta(T_k(f))$ is defined to be the ideal of antiderivatives of kth Tjurina ideal $T_k(f)$ w.r.t. T_k. As a result, $\sigma_k := \dim_{\mathbb{C}} \Delta(T_k(f))/T_k(f)^2$ and $\rho_k := \dim_{\mathbb{C}} T_k(f)/\Delta(T_k(f))$ are two new invariants of singularities. We prove that ρ_k decreases to 0 as k tends to infinity and define the T-threshold of f to be the smallest number r such that $T_r(f) = \Delta(T_r(f))$.

We complete the computation of these invariants for ADE curve singularities. As a result, we have verified the following conjecture for ADE curve singularities.

Conjecture 4.35. Let $(X, 0) = (V(f), 0) \subset (\mathbb{C}^n, 0)$ be an isolated hypersurface singularity. Then T-threshold of f is the smallest integer N such that $\{\sigma_k\}_{k \geqslant N}$ is a polynomial in k of degree $n - 1$.

Theorem B. *Conjecture* 4.35 *holds for ADE curve singularities.*

We are able to find the leading term of the polynomial in Conjecture 4.35 by sandwiching σ_k between two polynomials of k.

Proposition 4.38. *Suppose* $(X, 0) = (V(f), 0) \subset (\mathbb{C}^n, 0)$ *is an isolated singularity. Then*

$$\sigma_k \sim \frac{2^{n-1} \operatorname{ord}(f)}{(n-1)!} k^{n-1}.$$

Here, $\operatorname{ord}(f)$ *denotes the minimal degree among all monomial terms appearing in* f*. For two sequences* $\{a_n\}, \{b_n\} \subset \mathbb{C}$*, we denote* $a_n \sim b_n$ *if* $a_n/b_n \to 1$ *as* $n \to \infty$*.*

Corollary 4.39. *If Conjecture* 4.35 *holds, then the leading term of this polynomial is* $(2^{n-1} \operatorname{ord}(f)/(n-1)!)k^{n-1}$.

In § 4.4, the language of T-maps is applied to the ring of formal power series. Despite contact equivalence, right equivalence is also an important relation in classification of singularities. Among all right invariants, the Milnor number is possibly the most widely known one. It is a well-known theorem that for a semi quasi-homogeneous (SQH for short) series $f \in K[[\boldsymbol{x}]] := K[[x_1, \ldots, x_n]]$, the Milnor number of f coincides with that of the principal part f_w of f (see [4]). In this paper, we generalize this theorem to the kth Milnor number $\mu_k(f)$, which is the dimension of the quotient ring of $K[[\boldsymbol{x}]]$ modulo the kth Jacobian ideal $J_k(f) = \mathfrak{m}^k J(f)$ (see [12]) and is also a right invariant. Using the machinery of regular sequences, we finally prove the following.

Theorem C (Theorem 4.63). *Suppose* $f \in K[[\boldsymbol{x}]]$ *is an SQH series w.r.t.* $w \in \mathbb{N}^n_{>0}$*, that is,* $\mu(f_w) < \infty$*. Then* $\mu_k(f_w) = \mu_k(f)$ *for* $k \leqslant \min_i\{\operatorname{ord}(\partial f/\partial x_i)\}$*.*

Moreover, we believe that the result is correct for all $k \geqslant 0$. Hence we propose the following conjecture.

Conjecture 4.65. Suppose $f \in K[[\boldsymbol{x}]]$ is an SQH series w.r.t. $w \in \mathbb{N}^n_{>0}$, that is, $\mu(f_w) < \infty$. Then $\mu_k(f_w) = \mu_k(f)$ for all $k \in \mathbb{N}$.

Apart from the three above theorems, we also give a geometric interpretation of ideals of antiderivatives w.r.t. Tjurina ideal map T_0. For an ideal $I \lhd \mathbb{C}\{\boldsymbol{x}\}$, the ideal of antiderivatives of I w.r.t. T_0 (namely, $\Delta(I)$) is closely related to the well-known second fundamental exact sequence for Kähler differential (see Theorem 2.9). We illustrate and prove this connection in § 4.3. Briefly, $\Delta(I)$ coincides with the kernel of the first homomorphism in the second fundamental exact sequence. We also call the ideal of antiderivatives defined above a locally defined ideal of antiderivatives. In fact, we can generalize the locally defined ideal of antiderivatives to a global version. Consider the global objects complex space $(X, \mathcal{O}_X)$ and coherent ideal sheaf $\mathcal{I}$ of $\mathcal{O}_X$. In § 4.3, we further define the (globally defined) ideal sheaf of antiderivatives $\Delta(\mathcal{I})$ for $\mathcal{I}$. Besides, we prove that if X is smooth, then, for each $p \in X$, the stalk $\Delta(\mathcal{I})_p$ is equal to the locally defined ideal $\Delta(\mathcal{I}_p) \lhd \mathcal{O}_{X,p} = \mathbb{C}\{\boldsymbol{x}\}$.

In the appendix, we give the code for computing ideals of antiderivatives for T_k and the invariants σ_k, ρ_k. We only provide the code for two variables and the code for three variables is similar.

Remark 1.3. After completing the project, we found the paper [20]. We would like to point out that our work overlaps merely a small part with this preprint. Our Theorem A and the main theorem of [20] are both related to reconstruction of a hypersurface singularity from its moduli algebra. We would like to emphasize that our Theorem A can be applied to T-maps on local algebras over infinite field F

with residue field F, which includes Tjurina ideal map and 1st Tjurina ideal map of $\mathbb{C}\{\boldsymbol{x}\}$. For example, it can be applied to all of the six maps in Examples 4.5. Moreover, we do not even require the characteristic of F to be zero.

§ 2. Preliminaries

2.1. Invariants of singularities. Let $(X,0) \subseteq (\mathbb{C}^n,0)$ be the common zero locus of some functions $f_1, f_2, \ldots, f_m$ which are analytic near 0. If $m=1$, $(X,0)$ is called a hypersurface singularity. The singular locus of $(X,0)$, denoted as $(\operatorname{Sing} X,0)$, is the zero locus of f_1 and its partial derivatives. The singular locus is often called the singularity of $(X,0)$. Sometimes, if not confusing, we call $(X,0)$ a singularity. A singularity is called isolated if $(\operatorname{Sing} X,0)$ is a single point. A morphism of two analytic space germs $(X,0) \subseteq (\mathbb{C}^n,0)$ and $(Y,0) \subseteq (\mathbb{C}^m,0)$ is a restriction of a holomorphic map germ $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^m,0)$ to $(X,0)$ such that $(f(X),0) \subseteq (Y,0)$. $(X,0)$ and $(Y,0)$ are called isomorphic if and only if there are two morphisms between them which are inverse to each other. It is equivalent to saying that $(X,0)$ and $(Y,0)$ are biholomorphic.

The classification of singularities is based on such isomorphisms. A natural idea of algebraic geometry is to consider the valid functions on spaces, that is, analytic space germs. The function on $(X,0)$ are those analytic germs. By *Hilbert–Rückert's theorem* (see [9]), the ring of holomorphic function of $(X,0)$ is $\mathbb{C}\{x_1,\ldots,x_n\}/I$, where I is the ideal of analytic germs vanishing at $(X,0)$. $\mathbb{C}\{x_1,\ldots,x_n\}$ is a Henselian, Noetherian UFD as corollaries of the *Weierstraß Preparation theorem* (see [9]). If not confusing, we abbreviate $\mathbb{C}\{x_1,\ldots,x_n\}$ as $\mathbb{C}\{\boldsymbol{x}\}$ and denote $\mathfrak{m}$ as its maximal ideal.

Two analytic germs f and g in $\mathbb{C}\{\boldsymbol{x}\}$ are called *right equivalent* if there exists a $\varphi \in \operatorname{Aut}(\mathbb{C}\{\boldsymbol{x}\})$ such that $\varphi(f)=g$, called *contact equivalent* if $\varphi(f)=ug$, where $\varphi \in \operatorname{Aut}(\mathbb{C}\{\boldsymbol{x}\})$ and $u \in \mathbb{C}\{\boldsymbol{x}\}^*$ is a unit. Note that the two types of equivalence induce an isomorphism of singularities since φ is always given by an isomorphism of analytic space germs. It is not difficult to verify that two analytic space germs are isomorphic if and only if their corresponding analytic algebras are isomorphic. Another question is whether such an isomorphism can be determined by simpler algebras. Mather and Yau (see [17]) proved that two isolated hypersurface singularities are isomorphic if and only if their moduli algebras are isomorphic. Mather Yau's theorem was slightly generalized in [9] as follows.

Theorem 2.1 (see [9], Theorem 2.26, and [10], Theorem 1). *Let $f, g \in \mathfrak{m} \subset \mathbb{C}\{\boldsymbol{x}\}$. Then the following conditions are equivalent*:

(1) *f is contact equivalent to g*;

(2) *for all $k \geqslant 1$, $\mathbb{C}\{\boldsymbol{x}\}/T_k(f) \simeq \mathbb{C}\{\boldsymbol{x}\}/T_k(g)$*;

(3) *there is $k \geqslant 1$ such that $\mathbb{C}\{\boldsymbol{x}\}/T_k(f) \simeq \mathbb{C}\{\boldsymbol{x}\}/T_k(g)$.*

Here, T_k is the kth Tjurina ideal $T_k(f) := (f) + \mathfrak{m}^k J(f)$. In particular, $T_0(f) = T(f)$.

Moreover, if f has an isolated singularity, then f is contact equivalent to g if and only if $T(f) \simeq T(g)$.

Now Mather–Yau's theorem leads to the massive study of moduli algebras, the generalization of which is the main objects studied in this paper.

For a hypersurface singularity, there are a series of invariants: Milnor number (see [9]), Tjurina number (see [9]), higher Jacobian algebra (the quotient for higher order Jacobian, [6]), spectrum number (see [22]), Igusa Zeta function (see [15]), and Bernstein–Sato polynomial (see [3]). Besides, moduli ideal maps often generate invariants, for examples, the Krull dimension or linear dimension over $\mathbb{C}$ of their quotient rings. The following are three kinds of moduli ideal maps.

Higher order Tjurina ideal. For $f \in \mathbb{C}\{\boldsymbol{x}\}$, $T(f) = (f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ is the Tjurina ideal of f. For an ideal $I \lhd \mathbb{C}\{\boldsymbol{x}\}$, we define the action of T over I as $T(I) = \sum_{f\in I} T(f)$. T^k is defined to be the compositions of T by k times, that is, $T^k(f) = T(T(\cdots T(f)))$ is the ideal generated by f and all its partial derivatives whose orders are not greater than k. It is well-known that, for any $\varphi \in \operatorname{Aut}(\mathbb{C}\{\boldsymbol{x}\})$, $\varphi(T(f)) = T(\varphi(f))$ and $T(uf) = T(f)$ for any unit u. By a simple induction, we have $\varphi(T^k(f)) = T^k(\varphi(f))$ for all $\varphi \in \operatorname{Aut}(\mathbb{C}\{\boldsymbol{x}\})$ and $T^k(uf) = T^k(f)$ for any unit u. Hence T^k is a moduli ideal map.

kth Tjurina ideal. In [12], $T_k(f) := (f) + \mathfrak{m}^k J(f)$ is called the kth Tjurina ideal, where $\mathfrak{m}$ is the maximal ideal of $\mathbb{C}\{\boldsymbol{x}\}$ and $J(f)$ is the Jacobian ideal of f. One can easily check T_k is a moduli ideal map by noticing two facts:

(1) $T_k(uf) = T_k(f)$ for any unit u;

(2) $\varphi(\mathfrak{m}) = \mathfrak{m}$ for any $\varphi \in \operatorname{Aut}(\mathbb{C}\{\boldsymbol{x}\})$.

kth local Hessian ideal. The kth local Hessian ideal was first introduced in [14]. Let $f \in \mathbb{C}\{\boldsymbol{x}\}$, $J(f)$ be its Jacobian ideal and $\operatorname{Hess}(f) = (\partial^2 f/(\partial x_i\, \partial x_j))_{ij}$ be its Hessian matrix. Let $h_k(f)$ denote the ideal generated by all $(k \times k)$-minors in $\operatorname{Hess}(f)$, then $I_k^H(f) := (f) + J(f) + h_k(f)$ is called the kth local Hessian ideal of f and $H_k(f) := \mathbb{C}\{\boldsymbol{x}\}/I_k^H(f)$ is called the kth Hessian algebra. As shown in [14], I_k^H is a moduli ideal map.

Let Q stand for anyone of the three above. It is a natural problem whether an ideal of $\mathbb{C}\{\boldsymbol{x}\}$ is of the form $Q(f), f \in \mathbb{C}\{\boldsymbol{x}\}$. For $Q = T_0$ and $Q = T_1$, the Tjurina ideal map, Rodrigues ([19] and [20]) gave two conditions and solve the problem. In this article, we will generalize his results, at least to ideal maps including the three above.

A simple observation is that all of the three ideals can be written as a sum of principal ideals associated with f, and, for all $a, g \in \mathbb{C}\{\boldsymbol{x}\}$, $Q(ag) \subseteq Q(g)$. This observation will important for our generalization in § 4.

For a hypersurface singularity $(V(f), 0)$,

$$\mu = \dim_{\mathbb{C}} \mathbb{C}\{\boldsymbol{x}\}/J(f) \quad \text{and} \quad \tau = \dim_{\mathbb{C}} \mathbb{C}\{\boldsymbol{x}\}/T(f)$$

are called the Milnor number and Tjurina number, respectively. They are two important invariants.

Lemma 2.2 (see [9], Lemma 2.3). *Let $U \subseteq \mathbb{C}^n$ be an open neighborhood of 0, and let $f\colon U \to \mathbb{C}$ be holomorphic. Then the following are equivalent*:

(a) 0 *is an isolated critical point of* f;

(b) $\mu(f, 0) < \infty$;

(c) 0 *is an isolated singularity of* $f^{-1}(f(0)) = V(f - f(0))$;

(d) $\tau(f - f(0), 0) < \infty$.

The lemma can be slightly generalized.

Lemma 2.3. *Let $f \in \mathbb{C}\{\boldsymbol{x}\}$ be a holomorphic function with $f(0) = 0$. Then the following are equivalent*:

(a) $\dim \mathbb{C}\{\boldsymbol{x}\}/\mathfrak{m}^k J(f) < \infty$ *for all* $k \geqslant 0$;

(b) $\dim \mathbb{C}\{\boldsymbol{x}\}/\mathfrak{m}^k J(f) < \infty$ *for some* $k \geqslant 0$;

(c) $(V(f), 0)$ *is an isolated singularity*;

(d) $\dim \mathbb{C}\{\boldsymbol{x}\}/(f) + \mathfrak{m}^k J(f) < \infty$ *for all* $k \geqslant 0$;

(e) $\dim \mathbb{C}\{\boldsymbol{x}\}/(f) + \mathfrak{m}^k J(f) < \infty$ *for some* $k \geqslant 0$;

(f) *there exists some* $r \geqslant 0$ *such that* $\mathfrak{m}^r \subseteq J(f)$.

Proof. Since $(f) + \mathfrak{m}^k J(f) \subseteq (f) + J(f)$ and $\mathfrak{m}^k J(f) \subseteq J(f)$, by Lemma 2.2, (a), (b), (d), and (e) all imply (c). Moreover, it is clear that (f) implies (a), (b), (d), and (e). So it suffices to prove that (c) implies (f). Suppose f defines an isolated singularity. By Lemma 2.2, $\dim_{\mathbb{C}} \mathbb{C}\{\boldsymbol{x}\}/J(f) < \infty$ and hence $\sqrt{J(f)} \supset \mathfrak{m}$. Since $\mathbb{C}\{\boldsymbol{x}\}$ is Noetherian, there exists some $r \geqslant 0$ such that $\mathfrak{m}^r \subseteq J(f)$. Lemma 2.3 is proved.

Remark 2.4. The proof of (c) $\Rightarrow$ (f) is also true for any ideal I other than $J(f)$, as long as $\dim \mathbb{C}\{\boldsymbol{x}\}/I < \infty$.

2.2. Commutative algebra. In this subsection, we survey some facts about commutative algebra and Kähler differential.

Theorem 2.5 (Artin–Rees, [2], Corollary 10.10). *Let A be a Noetherian ring, I be an ideal, and M be a finitely generated A-module. If M' is a submodule of M, then there exists a $k \geqslant 0$ such that $I^n M \cap M' = I^{n-k}(I^k M \cap M')$, for all $n \geqslant k$.*

The next is the basis theorem of finitely generated modules over a principal ideal domain (PID, for short).

Theorem 2.6 (Basis Theorem, [21], Theorem 9.12). *If R is a PID, then every finitely generated R-module is a direct sum of cyclic modules in which each cyclic summand is either primary or is isomorphic to R.*

If R is a discrete valuation ring (DVR for short), with a uniformizer ϖ, then every finitely generated R-module is a direct sum of a free module and some cyclic modules of the form $R/\varpi^k R$ for some k. If $M = R^a \oplus \left(\bigoplus_{i=1}^r R/\varpi^{k_i} R\right)$, $k_i \geqslant 1$, then $a+r$ is the minimal number of generators of M. We call $a+r$ the rank of M.

Lastly, we recall some notions about regular sequence. Let A be a local ring and M be a finitely generated A-module. $(f_1, \ldots, f_r) \in M^r$ is called a regular sequence if, for all $1 \leqslant i \leqslant r$, f_i is not a zero-divisor in $M/\sum_{j=1}^{i-1} A f_j$.

Proposition 2.7 (see [7], Corollary 17.2). *If R is a Noetherian local ring and $(x_1, \ldots, x_r)$ is a regular sequence in R, then any permutation of $(x_1, \ldots, x_r)$ is again a regular sequence.*

Theorem 2.8 (see [16], Theorem 31). *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay ring. Then*:

(i) *for every proper ideal I of A,*

$$\mathrm{ht} I + \dim A/I = \dim A;$$

(ii) *for every sequence* $a_1, \ldots, a_r$ *in* $\mathfrak{m}$, *the following are equivalent*:
(1) *the sequence* $a_1, \ldots, a_r$ *is* A*-regular*;
(2) $\mathrm{ht}(a_1, \ldots, a_r) = r$.

The following is the *second fundamental exact sequence for the Kähler differential.* We state it in a way more convenient for further uses.

Theorem 2.9 (second fundamental exact sequence for Kähler differential). *Let* $\pi\colon B \to C$ *be a surjection of* A*-algebras with kernel* I. *Then the following exact sequence holds*:

$$I \xrightarrow{d} \Omega_{B/A} \otimes C \to \Omega_{C/A} \to 0,$$

where $d(a) = da \otimes 1$ *for all* $a \in I$ *is a* B*-module homomorphism, and* $d\colon I \to \Omega_{B/A}$ *is the restriction of* $d\colon B \to \Omega_{B/A}$. *Furthermore,* I^2 *is contained in* $\ker d$.

We may refer to [11] and [9] for this theorem. Their statements are slightly different, where the first map of the corresponding sequence in these books is $I/I^2 \xrightarrow{d} \Omega_{B/A} \otimes C$, but in fact they are equivalent to ours.

For the complex space $(X, \mathcal{O}_X)$, we can also define the Kähler differential. If $X = D \subset \mathbb{C}^n$ is an open subset, Ω_X is the free module $\bigoplus_{i=1}^n \mathcal{O}_D \cdot dx_i$, and d is naturally defined. Locally, $(X, \mathcal{O}_X) = (V(\mathcal{I}), (\mathcal{O}_D/\mathcal{I})|_{V(\mathcal{I})})$ is a complex model space, $\Omega_X = \Omega_D/(\mathcal{I}\Omega_D + \mathcal{O}_D d\mathcal{I})$. The derivation is defined to be the pullback of the quotient map $d\colon \mathcal{O}_D/\mathcal{I} \to \Omega_D/(\mathcal{I}\Omega_D + \mathcal{O}_D d\mathcal{I})$ by the inclusion map $V(\mathcal{I}) \hookrightarrow D$.

§ 3. *T*-fullness and *T*-dependence for Tjurina ideal

3.1. *T*-fullness and *T*-dependence. In [19], Rodrigues first developed the conceptions of T-fullness and T-dependence. Those are two conditions characterizing whether an ideal of $\mathbb{C}\{\boldsymbol{x}\}$ is a Tjurina ideal. Let $I \lhd \mathbb{C}\{\boldsymbol{x}\}$ be an ideal and T be the Tjurina ideal map, that is, $T(f) = (f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$. The action of T can be naturally extended to the set of ideals: $T(I) := \sum_{f \in I} T(f)$. We call $\Delta(I) := \{f \in \mathbb{C}\{\boldsymbol{x}\} \mid T(f) \subseteq I\}$ the ideal of antiderivatives of I. Since $T(af) \subseteq T(f)$, $a, f \in \mathbb{C}\{\boldsymbol{x}\}$, and $T(f+g) \subseteq T(f) + T(g)$ for all $f, g \in \mathbb{C}\{\boldsymbol{x}\}$, $\Delta(I)$ is actually an ideal.

Definition 3.1. I is called T*-full* if $T(\Delta(I)) = I$.

For an ideal $J = (g_1, \ldots, g_m) \lhd \mathbb{C}\{\boldsymbol{x}\}$, let $S = \mathbb{C}\{\boldsymbol{x}\}[y_1, \ldots, y_m]$ be a polynomial ring over $\mathbb{C}\{\boldsymbol{x}\}$ and $\sigma := \sum_i g_i y_i$. $\mathcal{T}(\sigma) := (\sigma, \partial\sigma/\partial x_1, \ldots, \partial\sigma/\partial x_n)$ is the Tjurina ideal of σ and $T(J)S$ is a homogeneous ideal of S. The original definition of T-dependence is stated in the language of algebraic geometry. Here, for simplicity, we give an equivalent definition illustrated in commutative algebra.

Definition 3.2. J is called T*-dependent* if $(\mathcal{T}(\sigma)\colon T(J)S) \not\subset \mathfrak{m}S$.

A subtle thing is whether it is well-defined. In [19], Rodrigues proved that the definition is independent of the choice of generators of J and hence well-defined. The proof will also appear in § 4.5, which is slightly adjusted to fit in more general cases. Below is the main theorem of [19].

Theorem 3.3. *An ideal* I *is a Tjurina ideal if and only if* I *is* T*-full and* $\Delta(I)$ *is* T*-dependent.*

Roughly speaking, T-fullness guarantees that I can be generated by some analytic germs and their partial derivatives. It can be seen quite clearly in the monomial case.

3.2. An example: monomial ideal case. It is also an interesting problem when a monomial ideal of $\mathbb{C}\{\boldsymbol{x}\}$ is a Tjurina ideal. In this subsection, we give a characterization of a T-full monomial ideal and survey some recent results associated with the problem. The notation, which follows that in [19], is also surveyed in § 3.1. The following proposition shows the ideal of antiderivatives of a monomial ideal is also a monomial ideal.

Proposition 3.4. *If $I \lhd \mathbb{C}\{\boldsymbol{x}\}$ is a monomial ideal, then so is $\Delta(I)$. Moreover, $\Delta(I) = \bigcap_{i=1}^{n} Q_i$, where Q_k is the monomial ideal generated by $x_k \cdot I$ and $I \cap \mathbb{C}[x_1, \ldots, \widehat{x}_k, \ldots, x_n]$. Here, $\mathbb{C}[x_1, \ldots, \widehat{x}_k, \ldots, x_n]$ refers to the polynomial ring of $n-1$ variables apart from x_k.*

Remark 3.5. Throughout the article, we adopt the multi-index motation. That is, $\boldsymbol{x}^{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} = (\alpha^1, \ldots, \alpha^n) \in \mathbb{N}^n$, refers to the monomial $x_1^{\alpha^1} \cdots x_n^{\alpha^n}$ in $\mathbb{C}[x_1, \ldots, x_n]$. For $\boldsymbol{\alpha} \in \mathbb{N}^n$, $|\boldsymbol{\alpha}| := \sum_{i=1}^{n} \alpha^i$ is called the length of $\boldsymbol{\alpha}$. We call $\boldsymbol{\alpha}_1 \leqslant \boldsymbol{\alpha}_2$, if $\alpha_1^i \leqslant \alpha_2^i$, for all $1 \leqslant i \leqslant n$. If not confusing, we set $\boldsymbol{e}_j = (\delta_i^j)_{i=1}^n \in \mathbb{N}^n$ as the normal orthogonal vectors. For $\boldsymbol{\alpha} \in \mathbb{N}^n$, $\operatorname{Supp} \boldsymbol{\alpha} := \{i \mid \alpha_i \neq 0\}$ is called the support of $\boldsymbol{\alpha}$.

Proof of Proposition 3.4. It is obvious that $f \in \Delta(I)$ if and only if every monomial term of f is $\Delta(I)$, since I is a monomial ideal. So, it suffices to work on the second statement.

Let $P_k = \{f \in \mathbb{C}\{\boldsymbol{x}\} \mid f, \partial f/\partial x_k \in I\}$, then $\Delta(I) = \bigcap_{k=1}^{n} P_k$. We only need to show that $P_k = Q_k$. For a subset $W \subseteq \mathbb{C}\{\boldsymbol{x}\}$, we use $\partial(W)/\partial x_k$ to stand for $\{\partial w/\partial x_k \mid w \in W\}$.

On one hand, since I is a monomial ideal, we have $\partial(x_k I)/\partial x_k = I$. Moreover, $\partial(I \cap \mathbb{C}[x_1, \ldots, \widehat{x}_k, \ldots, x_n])/\partial x_k = 0$, and hence $Q_k \subseteq P_k$. On the other hand, for $\boldsymbol{x}^{\boldsymbol{\alpha}} \in P_k$, if x_k does not appear in $\boldsymbol{x}^{\boldsymbol{\alpha}}$, then $\boldsymbol{x}^{\boldsymbol{\alpha}} \in \mathbb{C}[x_1, \ldots, \widehat{x}_k, \ldots, x_n] \cap I$. Otherwise, we have $\boldsymbol{x}^{\boldsymbol{\alpha}} \in x_k I$, or $P_k \subseteq Q_k$. Proposition 3.4 is proved.

In the next theorem, we give a characterization of T-full monomial ideals.

Theorem 3.6. *Let $I \lhd \mathbb{C}\{\boldsymbol{x}\}$ be a monomial ideal, then I is T-full if and only if there exist $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m \in \mathbb{N}^n$ such that $I = (\{\boldsymbol{x}^{\boldsymbol{\alpha}_i - \boldsymbol{e}_j} \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n, \boldsymbol{\alpha}_i - \boldsymbol{e}_j \geqslant 0\})$.*

Proof. It is clear that the theorem is equivalent to saying that I is T-full if and only if, for any $\boldsymbol{x}^{\boldsymbol{\alpha}} \in I$, there exists an $1 \leqslant i \leqslant n$ such that $\boldsymbol{x}^{\boldsymbol{\alpha} + \boldsymbol{e}_i - \boldsymbol{e}_j} \in I$ for all $1 \leqslant j \leqslant n$ satisfying $\boldsymbol{\alpha} + \boldsymbol{e}_i - \boldsymbol{e}_j \geqslant 0$.

So, it suffices to prove the above statement. The argument for "only if" is easy. By the assumption, $\boldsymbol{x}^{\boldsymbol{\alpha} + \boldsymbol{e}_i} \in \Delta(I)$ and hence $\boldsymbol{x}^{\boldsymbol{\alpha}} \in T(\Delta(I))$. For "if", since $\Delta(I)$ is a monomial ideal, there exists an $\boldsymbol{x}^{\boldsymbol{\beta}}$ such that $\boldsymbol{x}^{\boldsymbol{\beta}} = \boldsymbol{x}^{\boldsymbol{\alpha}}$ or $\boldsymbol{x}^{\boldsymbol{\beta} - \boldsymbol{e}_i} = \boldsymbol{x}^{\boldsymbol{\alpha}}$ for some i. In both cases, we have $\boldsymbol{x}^{\boldsymbol{\alpha} + \boldsymbol{e}_i} \in \Delta(I)$ for some i, so $\boldsymbol{x}^{\boldsymbol{\alpha} + \boldsymbol{e}_i - \boldsymbol{e}_j} \in I$. This proves the theorem.

Remark 3.7. T-full monomial ideals can be easily distinguished through the Newton diagrams. For $n = 2$, they correspond to the diagrams whose corners towards left-down appear as twins different by $(1, -1)$ as Fig. 1 shows.

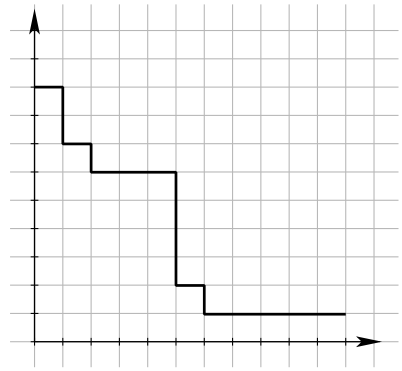

Figure 1

Combining Proposition 3.4 with Theorem 3.6, the ideal of antiderivatives of a T-full monomial ideal can be calculated as below.

Corollary 3.8. *Let $I \lhd \mathbb{C}\{\boldsymbol{x}\}$ be a T-full monomial ideal. By Theorem* 3.6, *we may assume that $I = (\{\boldsymbol{x}^{\boldsymbol{\alpha}_i - \boldsymbol{e}_j} \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n, \boldsymbol{\alpha}_i - \boldsymbol{e}_j \geqslant 0\})$. Then $\Delta(I) = \bigcap_{i=1}^{n} Q_k$, where $Q_k = x_k I + (\{\boldsymbol{x}^{\boldsymbol{\alpha} - \boldsymbol{e}_k}, \boldsymbol{\alpha} \in A\}) + (\{\boldsymbol{x}^{\boldsymbol{\beta} - \boldsymbol{e}_l}, \boldsymbol{\beta} \in B, l \neq k\})$ and $A = \{\boldsymbol{\alpha} \in \mathbb{N}^n \mid \boldsymbol{e}_k \cdot \boldsymbol{\alpha} = 1\}$, $B = \{\boldsymbol{\beta} \in \mathbb{N}^n \mid \boldsymbol{e}_k \cdot \boldsymbol{\beta} = 0\}$.*

Very recently, Epure and Schulze [8] answered the question when a Tjurina ideal is a monomial ideal by introducing Jacobian semigroup ideals and applying the machinery of matroids. Their main theorem is as follows.

Theorem 3.9 (see [8]). *Let $0 \neq f \in \mathbb{C}\{\boldsymbol{x}\}$ and $I := T(f)$ be its Tjurina ideal. Then I is a monomial ideal if and only if f is right equivalent to a Thom–Sebastiani polynomial. Here, a Thom–Sebastiani polynomial refers to a polynomial of the form $\sum_{i=1}^{m} \boldsymbol{x}^{\boldsymbol{\alpha}^i}$, where $\boldsymbol{\alpha}^i \in \mathbb{N}^n$ and* $\operatorname{Supp} \boldsymbol{\alpha}^i$ *are disjoint subsets of $\{1, \dots, n\}$.*

At last, we give two examples of monomial ideals. The first one is T-full but not T-dependent and the second one is T-dependent but not T-full.

Example 3.10. $n = 3$, $I = (xy^2z^3, x^2yz^3, x^2y^2z^2, y^7, xy^6, zy^6)$.

By Theorem 3.6, I is T-full. But one may compute that the $\mathbb{C}$-dimension of $I/(x,y,z)I$ is 6. By Nakayama's lemma, the minimal number of generators of I is 6 and hence I is not a Tjurina ideal. By Theorem 3.3, I not T-dependent.

Example 3.11. $n = 2$, $I = (xy)$.

We have $\Delta(I) = (x^2y^2)$, $T(\Delta(I)) = (x^2y, xy^2)$, $\sigma = x^2y^2\alpha$, $\mathcal{T}(\sigma) = (xy^2\alpha, x^2y\alpha)$. Since $\alpha T(\Delta(I)) = \mathcal{T}(\sigma)$ and $\alpha \notin \mathfrak{m}[\alpha]$, $\Delta(I)$ is T-dependent. There is a single corner in the Newton diagram of I and hence I is not T-full.

§ 4. *T*-map

In this section, we will introduce the conception of T-map and some of its applications. In § 4.1, we introduce the notions of T-map, T-principal ideal and ideal of antiderivatives. In § 4.2, we introduce two new invariants $\sigma_k = \dim_{\mathbb{C}} \Delta(I)/I^2$ and

$\rho_k = \dim_{\mathbb{C}} I/\Delta(I)$ associated with kth Tjurina ideal I and its ideal of antiderivatives $\Delta(I)$ w.r.t. kth Tjurina ideal map. We show that there exists a polynomial $P \in \mathbb{Z}[x]$ such that $\sigma_k = P(k)$ for all k sufficiently large. In § 4.3, we give a geometric interpretation of ideals of antiderivatives w.r.t. Tjurina ideal map. In § 4.4, we first survey the well-known theorem that the Milnor number of a semi quasi-homogeneous series $f \in K[[\boldsymbol{x}]]$ coincides with the Milnor number of its principal part. Then we generalize the theorem to μ_k, whenever $k \leqslant \min_i\{\mathrm{ord}(\partial f/\partial x_i)\}$. In §§ 4.5 and 4.6, we generalize the main theorem of [19] to some types of Noetherian local algebras so that many kinds of moduli ideal maps in § 2.1 can be included. Furthermore, we give an approach to finding a generator for a T-principal ideal.

4.1. *T*-map and ideal of antiderivatives. From now on, R is a ring and A is a Noetherian R-algebra. We will define abstract "Tjurina ideals" for A.

Definition 4.1. The set of all ideals of A is denoted as $\mathfrak{I}$. A map $Q\colon A \to \mathfrak{I}$ is called a *quasi-T-map* if there is an integer m and R-linear maps $Q_1, \dots, Q_m\colon A \to A$ such that $Q(f) = (Q_1(f), \dots, Q_m(f))$ for all $f \in A$. A quasi-T-map Q is called a *T-map* if it has the following property:

$$Q(af) \subseteq Q(f), \quad \text{for all} \;\; a, f \in A. \tag{$*$}$$

Remark 4.2. The following properties of a T-map Q can be easily derived from the definition.

(i) $Q(f+g) \subseteq Q(f) + Q(g)$;
(ii) if $(f) = (g)$, then $Q(f) = Q(g)$.

Remark 4.3. We note that R-linear maps $Q_1, \dots, Q_m$ are also parts of the definition.

T-maps appear rather frequently in singularity theory. Here, we mention some typical examples. (3), (4) and (5) in Example 4.5 are those moduli ideal maps mentioned in § 2.1.

Example 4.4 (quasi-T-maps).

(1) $A = \mathbb{C}\{\boldsymbol{x}\}$, $R = \mathbb{C}$ and $Q(f) = J_k(f) = (\boldsymbol{x})^k J(f)$ is the kth Jacobian ideal.
(2) $A = \mathbb{C}\{\boldsymbol{x}\}$, $R = \mathbb{C}$ and $Q(f)$ is the Nash blow-up ideal of f in [13].

Example 4.5 (T-maps).

(1) A is an arbitrary R-algebra and $Q(f) = (f)$;
(2) A is an arbitrary R-algebra and $Q(f) = (f, \partial_1(f), \dots, \partial_k(f))$, where $\partial_i \in \mathrm{Der}_R(A)$ are R-derivations;
(3) $A = \mathbb{C}\{\boldsymbol{x}\}$, $R = \mathbb{C}$, $Q = T^k$;
(4) $A = \mathbb{C}\{\boldsymbol{x}\}$, $R = \mathbb{C}$, $Q = T_k$;
(5) $A = \mathbb{C}\{\boldsymbol{x}\}$, $R = \mathbb{C}$, $Q = I_k^H$;
(6) $A = \overline{\mathbb{F}_p}[[\boldsymbol{x}]]$, $R = \overline{\mathbb{F}_p}$, $Q(f) = (f, \partial^p f/\partial x_1^p, \dots, \partial^p f/\partial x_n^p)$.

Fix a T-map Q, we call an ideal $I \lhd A$ a *T-principal ideal* if there exists an $f \in A$ such that $I = Q(f)$. Such an f is called a *generator* of I w.r.t. Q. It is an interesting problem when an ideal is a T-principal ideal. Before solving this problem, we will develop some basic notions.

Definition 4.6. With the same notation as in Definition 4.1, for an ideal $I = (g_1, \dots, g_n) \lhd A$, we have $Q(I) := (\{Q(f) \mid f \in I\}) = Q(g_1) + \cdots + Q(g_n)$. For another T-map $Q' = (Q'_1, \dots, Q'_r)$, we define the composition of Q' and Q as $(Q'Q)(f) = Q'(Q(f)) = (\{Q'_i(Q_j(f)),\ 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant m\})$. It is also a T-map.

One can check that the composition of T-maps satisfies the associative rule and $U(f) = (f)$ is the unit of this operation. We write this property as below.

Proposition 4.7. *With the above notation. $M_T := \{T\text{-maps of } A\}$ with the composition as the multiplication is a semigroup with unit element $U(f) = (f)$.*

From now on, we will always assume that Q is the fixed T-map of A. When stating properties of T-maps, we will omit the notion "with respect to Q". Following the step of [19], it is natural to introduce the ideal of antiderivatives.

Definition 4.8. Let $I \lhd A$ be an ideal. Then the ideal of antiderivatives $\Delta(I)$ is defined to be the set of all the elements whose images under Q are contained in I, that is,

$$\Delta(I) := \{f \in A \mid Q(f) \subseteq I\}.$$

Remark 4.9. By property $(*)$, one can easily check $\Delta(I)$ is an ideal.

Proposition 4.10. *With the as above notation, let Q' be another T-map. To avoid confusion, we denote Δ_Q, $\Delta_{Q'}$, and $\Delta_{Q'Q}$ as the ideals of antiderivatives w.r.t. Q, Q', and $Q'Q$, respectively. Then $\Delta_{Q'Q}(I) = \Delta_Q(\Delta_{Q'}(I))$ for any ideal $I \lhd A$.*

Proof. Suppose $f \in \Delta_{Q'Q}(I)$, then $(Q'Q)(f) \subseteq I$. Hence, for all $g \in Q(f)$, $Q'(g) \subseteq I$. Therefore, $Q(f) \subseteq \Delta_{Q'}(I)$, that is, $f \in \Delta_Q(\Delta_{Q'}(I))$. Conversely, since $f \in \Delta_Q(\Delta_{Q'}(I))$, we have $Q(f) \subseteq \Delta_{Q'}(I)$ and hence $Q'(Q(f)) \subset I$. It is equivalent to saying that $f \in \Delta_{Q'Q}(I)$. Proposition 4.10 is proved.

For convenience of applying the language of T-maps to singularity theory, we may give some definitions as counterparts of right and contact equivalence.

Definition 4.11. For $f, g \in A$, we call them *right* (respectively, contact) *equivalent* if there exists a $\varphi \in \mathrm{Aut}_R(A)$ such that $\varphi(f) = g$ (respectively, $\varphi(f) = ug$, for some unit $u \in A^*$). Clearly, the definition coincides with the original one with $R = \mathbb{C}$ and $A = \mathbb{C}\{\boldsymbol{x}\}$.

Definition 4.12. A T-map Q is called *stable under contact equivalence* if, for any $\varphi \in \mathrm{Aut}_R(A)$, Q is compatible with φ, that is, $Q(\varphi(f)) = \varphi(Q(f))$ for all $f \in A$.

Proposition 4.13. *If Q is stable under contact equivalence and $I \lhd A$ is an ideal, then $\varphi(\Delta(I)) = \Delta(\varphi(I))$ for all $\varphi \in \mathrm{Aut}_R(A)$.*

Proof. On one hand, for any $f \in \varphi(\Delta(I))$, there exists a $g \in \Delta(I)$ such that $f = \varphi(g)$. Since $Q(f) = Q(\varphi(g)) = \varphi(Q(g)) \subseteq \varphi(I)$, we have $\varphi(\Delta(I)) \subseteq \Delta(\varphi(I))$. On the other hand, for any $f \in \Delta(\varphi(I))$, we have $Q(f) \subseteq \varphi(I)$. Hence $Q(\varphi^{-1}(f)) \subseteq I$, then $\varphi^{-1}(f) \in \Delta(I)$, or $f \in \varphi(\Delta(I))$. Proposition 4.13 is proved.

Remark 4.14. The three kinds of T-maps in § 2.1 are all stable under contact equivalence. A simple corollary of Proposition 4.13 is that T-maps stable under contact equivalence induce moduli invariants.

Corollary 4.15. *Suppose Q is stable under contact equivalence. For all $\varphi \in \mathrm{Aut}_R(A)$ and an ideal $J \subseteq \Delta(I)$, the homomorphism $\Delta(I)/J \xrightarrow{\overline{\varphi}} \Delta(\varphi(I))/\varphi(J)$ induced by φ is an isomorphism.*

For $A = \mathbb{C}\{\boldsymbol{x}\}, R = \mathbb{C}$ and $Q = T$ the Tjurina ideal map, [19] gave an algorithm to compute $\Delta(I)$. We give a brief description of it as below. The algorithm also holds for $Q = T_k$.

Algorithm 4.16. Let $Q : g \mapsto (Q_1(g), ..., Q_m(g))$ be the T-map and I be an ideal of A. Suppose $f_1, ..., f_r$ is a set of generators of the ideal I.
Step 1. Compute $M_k = \left\{(\underline{a}) \in A^m \;\middle|\; \sum_i a_i Q_k(f_i) \in I\right\}$.
Step 2. Let $I_k = \left\{\sum_i a_i f_i \;\middle|\; (\underline{a}) \in M_k\right\}$. Compute $\bigcap_k I_k = \Delta(I)$.

Now let $Q = T$ be the Tjurina ideal map. By a simple induction we have the ideal of antiderivatives w.r.t. T^k is Δ^k, composing the Δ w.r.t. T by k times. This gives a method to compute the ideal of antiderivatives for higher order Tjurina ideal map.

In the next subsection, we will apply this algorithm to compute a series of new invariants associated with kth Tjurina ideal and its ideal of antiderivatives for ADE singularities.

4.2. Invariants associated with $\boldsymbol{T_k}$ and its ideal of antiderivatives. Suppose $(X, 0) = (V(f), 0)$ is an isolated hypersurface singularity and $I = T_k(f) \lhd A := \mathbb{C}\{\boldsymbol{x}\}$, then A/I is of finite dimension over $\mathbb{C}$ by Lemma 2.3. Since A is Noetherian, I/I^2 is a finitely generated A/I-module and hence has finite dimension over $\mathbb{C}$. For $Q = T_k$, since $I^2 \subseteq \Delta(I)$ and $\Delta(I)/I^2 \subseteq I/I^2$, we have $\Delta(I)/I^2$ is also of finite dimension. By Corollary 4.15, $\dim_{\mathbb{C}}(\Delta(I)/I^2)$ is a contact invariant. The same properties hold for $I/\Delta(I)$ as well. Hence, for each k, we obtain two new invariants. For $I = T_k(f)$ and $Q = T_k$, we denote σ_k as $\dim_{\mathbb{C}} \Delta(I)/I^2$ and ρ_k as $\dim_{\mathbb{C}} I/\Delta(I)$. Next, we will prove the stationary property of ρ_k as k tends to infinity and calculate σ_k, ρ_k and T-threshold (defined later) for ADE curve singularities (for classification, see [1]). The code for computing $\Delta(I)$, $\dim \Delta(I)/I^2$ and $\dim I/\Delta(I)$ is Code 5.1 in § 5.

Proposition 4.17. *Suppose $(X, 0) = (V(f), 0)$ is an isolated singularity, then $\{\rho_k\}_{k \geqslant 0}$ is a decreasing sequence. Moreover, there exists an N such that $\rho_k = 0$ for all $k \geqslant N$. We call the minimum of such N the T-threshold of f, denoted as $Tt(f)$.*

Proof. We first prove $\{\rho_k\}$ is decreasing. To avoid confusion, let Δ_k and Δ_{k+1} be the ideals of antiderivatives of $(I, Q) = (T_k(f), T_k)$ and $(T_{k+1}(f), T_{k+1})$, respectively. Set $I_k = T_k(f)$ and $I_{k+1} = T_{k+1}(f)$. Since $I_{k+1} \subseteq I_k$, $i\colon I_{k+1}/(I_{k+1} \cap \Delta_k) \hookrightarrow I_k/\Delta_k$ is an inclusion. It suffices to show that $I_{k+1} \cap \Delta_k \subseteq \Delta_{k+1}$. For any $g \in I_{k+1} \cap \Delta_k$, we have $g \in I_{k+1} \subseteq I_k$ and $\mathfrak{m}^k J(g) \subseteq (f) + \mathfrak{m}^k J(f)$, where $\mathfrak{m} = (\boldsymbol{x})$ is the maximal ideal of $\mathbb{C}\{\boldsymbol{x}\}$. Then $\mathfrak{m}^{k+1} J(g) \subseteq \mathfrak{m}(f) + \mathfrak{m}^{k+1} J(f) \subseteq I_{k+1}$, hence $g \in \Delta_{k+1}$.

Now we prove $\rho_k = 0$ for k sufficiently large. Let $Q = T_k$, $I = T_k(f)$ and $\Delta(I)$ be the ideal of antiderivatives of I w.r.t. Q. Since $(X, 0)$ is an isolated singularity, there is an integer l such that $\mathfrak{m}^l \subseteq J(f)$ and hence $(f) + \mathfrak{m}^{k+l} \subseteq I$. Clearly, $f \in \Delta(I)$, so it suffices to show for any k large enough, $\boldsymbol{\alpha} \in \mathbb{N}^n, |\boldsymbol{\alpha}| = k$, and

$u \in J(f)$, then $\mathfrak{m}^k J(\boldsymbol{x}^{\boldsymbol{\alpha}} u) \subseteq \mathfrak{m}^{k+l}$. Notice that $J(\boldsymbol{x}^{\boldsymbol{\alpha}} u) \subseteq \mathfrak{m}^{k-1} J(f) + \mathfrak{m}^k$, then $\mathfrak{m}^k J(\boldsymbol{x}^{\boldsymbol{\alpha}} u) \subseteq \mathfrak{m}^{2k-1}$. Thus $\Delta(I) = I$ for all $k \geqslant l+1$. Proposition 4.17 is proved.

Lemma 4.18. *For $(R, A, Q) = (\mathbb{C}, \mathbb{C}\{\boldsymbol{x}\}, T_k)$, if I is a monomial ideal, then $\Delta(I)$ is also monomial given by*

$$\Delta(I) = \bigcap_{\boldsymbol{i},j} Q_{(\boldsymbol{i},j)}, \qquad \boldsymbol{i} \in \mathbb{N}^n, \quad |\boldsymbol{i}|t = k, \quad 1 \leqslant j \leqslant n.$$

Here,

$$\begin{aligned} Q_{(\boldsymbol{i},j)} &= A_{(\boldsymbol{i},j)} + B_{(\boldsymbol{i},j)}, \\ A_{(\boldsymbol{i},j)} &= (\{x_j \cdot \boldsymbol{x}^{\boldsymbol{\alpha}} \mid \boldsymbol{x}^{\boldsymbol{\alpha}} \cdot \boldsymbol{x}^{\boldsymbol{i}} \in I\}), \\ B_{(\boldsymbol{i},j)} &= (\{\boldsymbol{x}^{\boldsymbol{\alpha}} \in I \mid \alpha^j = 0\}). \end{aligned}$$

Proof. The argument is the same as in Proposition 3.4. Let

$$P_{(\boldsymbol{i},j)} = \left\{ f \in \mathbb{C}\{\boldsymbol{x}\} \;\middle|\; f,\, \boldsymbol{x}^{\boldsymbol{i}} \cdot \frac{\partial f}{\partial x_j} \in I \right\}.$$

Hence $I = \bigcap_{\boldsymbol{i},j} P_{(\boldsymbol{i},j)}$. It suffices to show $Q_{(\boldsymbol{i},j)} = P_{(\boldsymbol{i},j)}$. On one hand, for a generator $x_j \cdot \boldsymbol{x}^{\boldsymbol{\alpha}}$ of $A_{(\boldsymbol{i},j)}$, one finds $\boldsymbol{x}^{\boldsymbol{i}} \partial(x_j \cdot \boldsymbol{x}^{\boldsymbol{\alpha}})/\partial x_j = (\alpha^j + 1)\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{i}} \in I$. Besides, $\partial(\boldsymbol{x}^{\boldsymbol{\alpha}})/\partial x_j = 0$ for $\alpha^j = 0$, and hence we have $Q_{(\boldsymbol{i},j)} \subseteq P_{(\boldsymbol{i},j)}$. On the other hand, since I is a monomial ideal and the operator $\boldsymbol{x}^{\boldsymbol{i}} \cdot \partial/\partial x_j$ sends momomials to monomials, $P_{(\boldsymbol{i},j)}$ is also monomial. For a monomial $\boldsymbol{x}^{\boldsymbol{\alpha}} \in P_{(\boldsymbol{i},j)}$, if x_j is not a factor, then $\boldsymbol{x}^{\boldsymbol{\alpha}} \in B_{(\boldsymbol{i},j)}$. Otherwise, $\boldsymbol{x}^{\boldsymbol{\alpha}} \in A_{(\boldsymbol{i},j)}$. Therefore, $P_{(\boldsymbol{i},j)} = Q_{(\boldsymbol{i},j)}$. Lemma 4.18 is proved.

We will compute ρ_k, σ_k and T-threshold for ADE curve singularities A_m, D_m, E_6, E_7, E_8. To avoid repetition, we compute those invariants for D_m and only provide results for other types. Besides, we will compute the ideal of antiderivatives for ADE surface singularities.

Proposition 4.19. *For $D_m = V(x^{m-1} + xy^2)$, $m \geqslant 4$,*

$$\begin{aligned} &\Delta(T_k(x^{m-1} + xy^2)) \\ &= \begin{cases} (x^{m-1} + xy^2, 3(m-1)x^{m-2}y + y^3, x^m, y^4, xy^3, x^2y^2, x^{m-1}), & k = 0, \\ (x^{m-1}, x^{m-2}y, xy^2, y^3), & k = 1, \\ (x^{m-1} + xy^2) + (x^{m+k-2}, x^k y^2, \ldots, xy^{k+1}, y^{k+2}, x^{m-2}y), & 2 \leqslant k \leqslant m-3, \\ (x^{m-1} + xy^2) + (x^{m+k-2}, x^k y^2, \ldots, xy^{k+1}, y^{k+2}, x^{k+1}y), & k \geqslant m-2, \end{cases} \end{aligned}$$

$$\rho_k = \begin{cases} m, & k = 0, \\ m - 3 - k, & 1 \leqslant k \leqslant m-4, \\ 0, & k \geqslant m-3, \end{cases} \qquad \sigma_k = \begin{cases} m, & k = 0, \\ m + 7k + 4, & 1 \leqslant k \leqslant m-4, \\ 2m + 6k + 1, & k \geqslant m-3, \end{cases}$$

and $Tt(x^{m-1} + xy^2) = m - 3$.

Remark 4.20. For an ideal $I \lhd \mathbb{C}\{\boldsymbol{x}\}$, we sometimes split I into a sum of finite dimensional $\mathbb{C}$-vector spaces and a monomial ideal as $\mathbb{C}$-vector spaces. This will simplify the computation of $\dim_{\mathbb{C}} \mathbb{C}\{\boldsymbol{x}\}/I$. For example, in the proof, we use $I_1 = (x^m, y^3, x^2y) + \operatorname{span}_{\mathbb{C}}\{x^{m-1} + xy^2\}$, which means the sum of monomial ideal (x^m, y^3, x^2y) and $\mathbb{C}$-vector space $\operatorname{span}_{\mathbb{C}}\{x^{m-1} + xy^2\}$.

Proof of Proposition 4.19. We first compute σ_0 by definition:

$$I = T(x^{m-1} + xy^2) = ((m-1)x^{m-2} + y^2, xy) = ((m-1)x^{m-2} + y^2, x^{m-1}, y^3, xy),$$
$$I^2 = ((m-1)^2x^{2m-4} + y^4, (m-1)x^{m-1}y + xy^3, x^{2m-3}, y^5, xy^4, x^my, x^2y^2).$$

Following Algorithm 4.16, we compute $\Delta(I)$ as follows:

$$M_1 = \{(a,b) \in A^2 \mid a(m-2)(m-1)x^{m-2} + by \in I\},$$
$$M_2 = \{(a,b) \in A^2 \mid 2ay + bx \in I\}.$$

Hence

$$I_1 = (x^m, y^3, x^2y) + \operatorname{span}_{\mathbb{C}}\{x^{m-1} + xy^2\},$$
$$I_2 = (x^{m-1}, y^4, xy^2) + \operatorname{span}_{\mathbb{C}}\{3(m-1)x^{m-2}y + y^3\}.$$

So, we have $\Delta(I) = I_1 \cap I_2 = \operatorname{span}_{\mathbb{C}}\{x^{m-1} + xy^2\} + \operatorname{span}_{\mathbb{C}}\{3(m-1)x^{m-2}y + y^3\} + (y^4, xy^3, x^2y^2, x^{m-1})$. It can be checked that $x^m, \dots, x^{2m-4}, xy^3, x^{m-1} + xy^2, x^{m-2}y + y^3$ is a basis of $\Delta(I)/I^2$. Therefore, $\sigma_0 = m$.

Since $I = \operatorname{span}_{\mathbb{C}}\{(m-1)x^{m-2} + y^2\} + (x^{m-1}, y^3, xy)$, we have $\rho_0 = m$.

For σ_1, one can compute $I = T_1(x^{m-1} + xy^2) = (x^{m-1}, x^2y, xy^2, y^3)$, which is a monomial ideal. By Lemma 4.18, $\Delta(I)$ is a monomial ideal as well. One can compute $P_{(\boldsymbol{i},j)}$ in Lemma 4.18:

$$P_{x\,(\partial/\partial x)} = I, \qquad P_{x\,(\partial/\partial y)} = (x^{m-1}, x^{m-2}y, xy^2, y^3),$$
$$P_{y\,(\partial/\partial y)} = I, \qquad P_{y\,(\partial/\partial x)} = I.$$

Hence

$$\Delta(I) = (x^{m-1}, x^{m-2}y, xy^2, y^3),$$
$$I^2 = (x^{2m-2}, x^{m+1}y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6).$$

Since

$$x^{m-1}, \dots, x^{2m-3}, x^{m-2}y, x^{m-1}y, x^my, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3, y^4, xy^4, y^5$$

is a basis of $\Delta(I)/I^2$, we have $\sigma_1 = m + 11$. Since $x^2y, \dots, x^{m-3}y$ is a basis of $I/\Delta(I)$, we have $\rho_1 = m - 4$.

Next, we compute σ_k, $k \geqslant 2$. A simple observation shows that

$$I = T_k(x^{m-1} + xy^2) = (x^{m-1} + xy^2) + (x^{m+k-2}, x^{k+1}y, x^ky^2, \dots, xy^{k+1}, y^{k+2}).$$

Let

$$U_i = \left\{ u \in I \;\middle|\; x^iy^{k-i}\frac{\partial u}{\partial x} \in I \right\} \quad \text{and} \quad V_i = \left\{ v \in I \;\middle|\; x^iy^{k-i}\frac{\partial v}{\partial y} \in I \right\}.$$

Suppose $a((m-1)x^{m-2}+y^2)+bx^{m+k-2}+c_1x^{k+1}y+\cdots+c_{k+2}y^{k+2}\in U_i$. Then

$$x^iy^{k-i}[a((m-1)x^{m-2}+y^2)+(m+k-2)bx^{m+k-3} \\ +(k+1)c_1x^ky+\cdots+c_{k+1}y^{k+1}]\in I.$$

Hence $U_i=I$, for all $0\leqslant i\leqslant k$. Applying the same argument to V_i, for

$$a((m-1)x^{m-2}+y^2)+bx^{m+k-2}+c_1x^{k+1}y+\cdots+c_{k+2}y^{k+2}\in V_i,$$

we have

$$x^iy^{k-i}[2axy+c_1x^{k+1}+\cdots+c_{k+2}(k+2)y^{k+1}]\in I.$$

Hence $V_i=I$, for all $0\leqslant i\leqslant k-1$. As for V_k, the only restriction is $c_1x^{k+1}\in I$. Thus $c_1\in(y,x^{\max\{m-k-3,0\}})$ and

$$\Delta(I)=(x^{m-1}+xy^2)+(x^{m+k-2},x^ky^2,\ldots,xy^{k+1},y^{k+2})+(x^{\max\{m-2,k+1\}}y).$$

Hence $Tt(x^{m-1}+xy^2)=m-3$. Let $J:=(x^{m+k-2},x^{k+1}y,\ldots,xy^{k+1},y^{k+2})$. Consequently, $I^2=((x^{m-1}+xy^2)^2)+(x^{m-1}+xy^2)J+J^2$.

Case 1: $m-2\geqslant k+1$.

$$\begin{aligned}\Delta(I)&=(x^{m-1}+xy^2)+(x^{m-2}y)+(x^{m+k-2},x^ky^2,\ldots,xy^{k+1},y^{k+2})\\ &=(x^{m-1}+xy^2)+(x^{m-2}y,xy^3)+(x^{m+k-2},x^ky^2,y^{k+2})\\ &=\sum_{i=0}^{k-2}\operatorname{span}_{\mathbb{C}}\{x^i(x^{m-1}+xy^2)\}+(x^{m+k-2},x^{m-2}y,x^ky^2,xy^3,y^{k+2})\\ &:=\sum_{i=0}^{k-2}\operatorname{span}_{\mathbb{C}}\{x^i(x^{m-1}+xy^2)\}+L.\end{aligned}$$

Moreover, $\{x^i(x^{m-1}+xy^2)\}_{0\leqslant i\leqslant k-2}$ is linearly independent in $\mathbb{C}\{\boldsymbol{x}\}/L$, that is, $\Delta(I)=\left(\bigoplus_{i=0}^{k-2}\operatorname{span}_{\mathbb{C}}\{x^i(x^{m-1}+xy^2)\}\right)\oplus L$ is a direct sum of $\mathbb{C}$-linear spaces.

Since $J^2=(x^{2m+2k-4},x^{m+2k-1}y,x^{2k+2}y^2,x^{2k+1}y^3,\ldots,y^{2k+4})$, we can write I^2 as below:

$$\begin{aligned}&I^2=(A)+(B)+K,\\ &A=2x^my^2+x^2y^4-x^my^3,\\ &B=x^{m+k}y+x^{k+2}y^3,\\ &K=(x^{2m+k-3},x^{m+2k-1}y,x^{2k+2}y^2,x^{2k+1}y^3,x^{k+1}y^4,x^ky^5,x^2y^6,xy^{k+4},y^{2k+4}).\end{aligned}$$

Moreover, $A,\ldots,x^{k-2}A,yA,\ldots,x^{k-3}yA,B,\ldots,x^{k-2}B$ is a basis of I^2/K. Since $K\subseteq L$, $\sigma_k=\dim(L/K)+k-1-(k-1)-(k-2)-(k-1)=m+7k+4$.

A similar argument shows that $\rho_k=m-3-k$.

Case 2: $k\geqslant m-2$.

$$\Delta(I)=(x^{m-1}+xy^2)+(x^{m+k-2},x^{k+1}y,x^ky^2,\ldots,xy^{k+1},y^{k+2})=I.$$

One can obtain the following decomposition:

$$\begin{aligned}
I &= (f) + J,\\
I^2 &= (C) + (B) + (A_1, A_2, \dots, A_{k+2}) + K,\\
f &= x^{m-1} + xy^2, \quad C = f^2, \quad B = x^{m+k-2} f,\\
A_i &= x^{k+m+1-i} y^i + x^{k+3-i} y^{2+i},\\
J &= (x^{m+k-2}, x^{k+1}y, \dots, xy^{k+1}, y^{k+2}),\\
K &= (x^{m+2k}, x^{m+2k-1}y, x^{2k+2}y^2, x^{2k+1}y^3, x^{2k-m+4}y^4, \dots, x^{k-m+4}y^{k+4},\\
&\quad x^{k-m+3}y^{k+m+1}, \dots, y^{2k+4}).
\end{aligned}$$

Moreover, the following are $\mathbb{C}$-bases of I/J and I^2/K, respectively:

$$f,\ \dots,\ x^{k-2}f,\ yf,\ \dots,\ x^{k-3}y,\ \dots,\ y^{k-2}f$$

and

$$\begin{gathered}
A_1,\ \dots,\ x^{k-2}A_1,\ A_2,\ \dots,\ x^{k-m+2}A_2,\ \dots, A_{k+1},\ \dots,\ x^{k-m+2}A_{k+1},\\
A_{k+2},\ \dots,\ x^{k-m+2}A_{k+2},\ \dots,\ A_{k+2},\ \dots,\ x^{k-m+2}y^{m-4}A_{k+2},\\
y^{m-3}A_{k+2},\ \dots,\ x^{k-m+1}y^{m-3}A_{k+2},\\
y^{m-2}A_{k+2},\ \dots,\ x^{k-m}y^{m-2}A_{k+2},\ \dots,\ y^{k-2}A_{k+2},\\
B,\ \dots,\ x^{k-m+2}B,\\
C,\ \dots,\ x^{k-2}C,\ yC,\ \dots,\ x^{k-3}yC,\ \dots,\ y^{k-2}C.
\end{gathered}$$

Since $K \subseteq J$, one can calculate σ_k by $\sigma_k = \dim I/J + \dim J/K - \dim I^2/K = 2m + 6k + 1$. Proposition 4.19 is proved.

Applying the same argument, one can obtain the results for A_m, E_6, E_8.

Proposition 4.21. *For $A_m = V(x^{m+1} + y^2)$, $m \geqslant 2$,*

$$\begin{aligned}
&\Delta(T_k(x^{m+1} + y^2))\\
&\quad = \begin{cases}
(x^{m+1}, x^m y, y^2), & k = 0, 1,\\
(x^{m+1} + y^2) + (x^{m+k}, x^m y, x^{k-1}y^2, \dots, y^{k+1}), & 2 \leqslant k \leqslant m-1,\\
(x^{m+1} + y^2) + (x^k y, x^{k-1}y^2, \dots, y^{k+1}), & k \geqslant m,
\end{cases}
\end{aligned}$$

$$\rho_k = \begin{cases}
m+1, & k = 0,\\
m-k, & 1 \leqslant k \leqslant m-1,\\
0, & k \geqslant m,
\end{cases} \qquad
\sigma_k = \begin{cases}
m-1, & k = 0, 1,\\
m+6, & k = 1,\\
m+5k+1, & 2 \leqslant k \leqslant m-1,\\
2m+4k+1, & k \geqslant m,
\end{cases}$$

and $Tt(x^{m+1} + y^2) = m$.

Proposition 4.22. *For* $E_6 = V(x^3 + y^4)$,

$$\Delta(T_k(x^3+y^4)) = \begin{cases} (x^3, x^2y^3, y^4), & k=0, \\ (x^3, x^2y, xy^3, y^4), & k=1, \\ (x^3+y^4) + (x^{2+k}, \dots, x^2y^k, xy^{k+2}), & k \geqslant 2, \end{cases}$$

$$\rho_k = \begin{cases} 5, & k=0, \\ 1, & k=1, \\ 0, & k \geqslant 2, \end{cases} \qquad \sigma_k = \begin{cases} 7, & k=0, \\ 18, & k=1, \\ 6k+13, & k \geqslant 2, \end{cases}$$

and $Tt(x^3+y^4) = 2$.

Proposition 4.23. *For* $E_7 = V(x^3 + xy^3)$,

$$\begin{aligned} &\Delta(T_k(x^3+xy^3)) \\ &= \begin{cases} (x^3+xy^3, 15x^2y^2+2y^5, xy^5, y^6), & k=0, \\ (3x^2y+y^3, x^4, x^3y, x^2y^2, xy^4, y^5), & k=1, \\ (x^3+xy^3) + (3x^2y^k + y^{k+3}, x^{k+2}, \dots, x^3y^{k-1}, x^2y^{k+1}, xy^{k+2}, y^{k+4}), & k \geqslant 2, \end{cases} \end{aligned}$$

$$\rho_k = \begin{cases} 6, & k=0, \\ 2, & k=1, \\ 0, & k \geqslant 2, \end{cases} \qquad \sigma_k = \begin{cases} 8, & k=0, \\ 19, & k=1, \\ 6k+15, & k \geqslant 2, \end{cases}$$

and $Tt(x^3+xy^3) = 2$.

Proposition 4.24. *For* $E_8 = V(x^3 + y^5)$,

$$\Delta(T_k(x^3+y^5)) = \begin{cases} (x^3, x^2y^4, y^5), & k=0, \\ (x^3, x^2y^3, xy^4, y^5), & k=1, \\ (x^3+y^5) + (x^{k+2}, \dots, x^2y^k, xy^{k+3}, y^{k+4}), & k \geqslant 2, \end{cases}$$

$$\rho_k = \begin{cases} 6, & k=0, \\ 2, & k=1, \\ 1, & k=2, \\ 0, & k \geqslant 3, \end{cases} \qquad \sigma_k = \begin{cases} 10, & k=0, \\ 21, & k=1, \\ 28, & k=2, \\ 17+6k, & k \geqslant 3, \end{cases}$$

and $Tt(x^3+y^5) = 3$.

Next, we provide a lemma, which relates the ideal of antiderivatives of ADE surface singularities to ADE curve singularities.

Lemma 4.25. *Suppose* $f \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \dots, x_n\}$ *is an analytic germ with an isolated singularity at the origin. Let* u *be a new variable and* $\widetilde{f} = u^2 + f \in \mathbb{C}\{u, \boldsymbol{x}\}$ *be another analytic germ with an isolated singularity. The notation is given in the remark below.*

For $k = 0$,

$$\Delta(T_0(\widetilde{f})) = (u^2) + (u \cdot T_0^x(f)) + (\Delta^x(T_0^x(f))),$$

and for $k \geqslant 1$,

$$\Delta(T_k(\widetilde{f})) = (\widetilde{f}) + (\Psi_k) + (\Lambda_k \cdot u) + (\mathfrak{m}_x^{k-1} u^2) + \cdots + (\mathfrak{m}_x u^k) + (u^{k+1}),$$

where $(S), S \subset \mathbb{C}\{u, \boldsymbol{x}\}$ *is the ideal generated by* S. Λ_k *and* Ψ_k *are ideals in* $\mathbb{C}\{\boldsymbol{x}\}$ *given by* $\Lambda_k = (\mathfrak{m}_x^k \cdot J(f) + \mathfrak{m}_x^{k-1} \cdot f : \mathfrak{m}_x^k) \cap \mathfrak{m}_x^k$ *and* $\Psi_k = \Delta^x(T_k^x(f)) \cap \mathfrak{m}_x^k J(f)$.

Moreover, if f *is quasi-homogeneous, then*

$$\Delta(T_k(\widetilde{f})) = (\widetilde{f}) + (\Psi_k) + (\Lambda_k \cdot u).$$

Remark 4.26. Let k be an positive integer. To avoid confusion, let T_k^x denote the kth Tjurina ideal map in $\mathbb{C}\{\boldsymbol{x}\}$ and T_k is the Tjurina ideal map in $\mathbb{C}\{u, \boldsymbol{x}\}$. Moreover, $\mathfrak{m}_x$ refers to the maximal ideal of $\mathbb{C}\{\boldsymbol{x}\}$ and $J(f)$ the Jacobian ideal of f in $\mathbb{C}\{\boldsymbol{x}\}$. Besides, for an ideal I in $\mathbb{C}\{\boldsymbol{x}\}$, $\Delta^x(I) \subseteq \mathbb{C}\{\boldsymbol{x}\}$ is the ideal of antiderivatives w.r.t. T_k^x. For ideal J in $\mathbb{C}\{u, \boldsymbol{x}\}$, $\Delta(J)$ is the ideal of antiderivatives w.r.t. T_k.

In all, the notation attached with an ^{x} or $_x$ is the one associated with $\mathbb{C}\{\boldsymbol{x}\}$, while others are associated with $\mathbb{C}\{u, \boldsymbol{x}\}$.

Proof of Lemma 4.25. We will also follow Algorithm 4.16.

For $k = 0$, we have $T_0(\widetilde{f}) = (u, f, J(f))$. Suppose $au + \sum_{i=1}^n b_i(\partial f/\partial x_i) \in \Delta(T_0(\widetilde{f}))$, where $a, b_i \in \mathbb{C}\{u, \boldsymbol{x}\}$. Then, by taking $\partial/\partial u$ and $\partial/\partial x_j$, we have $a \in T_0^x(f) + (u)$ and $\sum_{i=1}^n b_i(\partial^2 f/(\partial x_i\, \partial x_j)) \in T_0(\widetilde{f})$. Since $u \in T_0(\widetilde{f})$, we may assume $b_i \in \mathbb{C}\{\boldsymbol{x}\}$. Under this assumption, we obtain $\sum_{i=1}^n b_i(\partial^2 f/(\partial x_i\, \partial x_j)) \in T_0^x(f)$ and hence the assertion is proven.

When $k \geqslant 1$, we have $T_k(\widetilde{f}) = (\widetilde{f}) + (\mathfrak{m}_x, u)^k(u, J(f))$. Suppose $\sum a_{i\boldsymbol{\alpha}} u^{i+1} \boldsymbol{x}^{\boldsymbol{\alpha}} + \sum b_{jl\boldsymbol{\beta}} u^j \boldsymbol{x}^{\boldsymbol{\beta}}(\partial f/\partial x_l) \in \Delta(T_k(\widetilde{f}))$, where $i, j \geqslant 0$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n$ with $i + |\boldsymbol{\alpha}| = j + |\boldsymbol{\beta}| = k$. For $s \geqslant 0$, $\boldsymbol{\gamma} \in \mathbb{N}^n$, $s + |\boldsymbol{\gamma}| = k$, apply $u^s \boldsymbol{x}^{\boldsymbol{\gamma}}(\partial/\partial u)$. If $s \geqslant 1$, we obtain no restriction to all a and b. If $s = 0$, we get one restriction

$$\mathfrak{m}_x^k \left(\sum_{|\boldsymbol{\alpha}|=k} a_{0\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \right) \subseteq T_k(\widetilde{f}).$$

Since $u\boldsymbol{x}^{\boldsymbol{\alpha}} \in T_k(\widetilde{f})$ if $|\boldsymbol{\alpha}| = k$, we may assume $a_{0\boldsymbol{\alpha}} \in \mathbb{C}\{\boldsymbol{x}\}$. Under these circumstances, by considering the degree of u, we have $\sum_{|\boldsymbol{\alpha}|} a_{0\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \in \Lambda_k$.

For general s, $\boldsymbol{\gamma}$ as before, $u^s \boldsymbol{x}^{\boldsymbol{\gamma}}(\partial/\partial x_q)$ provides no restriction to $a_{i\boldsymbol{\alpha}}$, since $u^{s+i+1}\boldsymbol{x}^{\boldsymbol{\beta}+\boldsymbol{\gamma}-\boldsymbol{e}_q}$ is always in $T_k(\widetilde{f})$. Hence we can focus only on $\sum b_{jl\boldsymbol{\beta}} u^j \boldsymbol{x}^{\boldsymbol{\beta}}(\partial f/\partial x_l)$. Also, if $s \geqslant 1$, then

$$u^{j+s}\boldsymbol{x}^{\boldsymbol{\beta}+\boldsymbol{\gamma}-\boldsymbol{e}_q} \frac{\partial f}{\partial x_l} \in T_k(\widetilde{f}) \quad \text{and} \quad u^{j+s}\boldsymbol{x}^{\boldsymbol{\beta}+\boldsymbol{\gamma}} \frac{\partial^2 f}{\partial x_l\, \partial x_q} \in T_k(\widetilde{f}).$$

So, these $u^s \boldsymbol{x}^{\boldsymbol{\gamma}}(\partial/\partial x_q)$ give no restriction. For $s = 0$, we have

$$\sum b_{jl\boldsymbol{\beta}} \left[u^j \boldsymbol{x}^{\boldsymbol{\beta}+\boldsymbol{\gamma}-\boldsymbol{e}_q} \frac{\partial f}{\partial x_l} + u^j \boldsymbol{x}^{\boldsymbol{\beta}+\boldsymbol{\gamma}} \frac{\partial^2 f}{\partial x_l\, \partial x_q} \right] \in T_k(\widetilde{f}).$$

If $j \geqslant 1$, we also have

$$b_{jl\boldsymbol{\beta}} \left(u^j \boldsymbol{x}^{\boldsymbol{\beta}+\boldsymbol{\gamma}-\boldsymbol{e}_q} \frac{\partial f}{\partial x_l} + u^j \boldsymbol{x}^{\boldsymbol{\beta}+\boldsymbol{\gamma}} \frac{\partial^2 f}{\partial x_l\, \partial x_q} \right) \in T_k(\widetilde{f}).$$

So, it suffices to consider the condition

$$W=\sum_{l}\sum_{|\boldsymbol{\beta}|=k} b_{0l\boldsymbol{\beta}}\left(\boldsymbol{x}^{\boldsymbol{\beta}+\boldsymbol{\gamma}-\boldsymbol{e}_q}\,\frac{\partial f}{\partial x_l}+\boldsymbol{x}^{\boldsymbol{\beta}+\boldsymbol{\gamma}}\,\frac{\partial^2 f}{\partial x_l\,\partial x_q}\right)\in T_k(\widetilde{f}).$$

We may assume as before that $b_{0l\boldsymbol{\beta}}\in\mathbb{C}\{\boldsymbol{x}\}$. Considering again the degree of u, we have $W\in(\widetilde{f})+(\mathfrak{m}_x,u)^k(u,J(f))$ if and only if

$$W\in(\mathfrak{m}_x,u)^{k-1}(\widetilde{f})+(\mathfrak{m}_x,u)^k(u,J(f))=(\mathfrak{m}_x,u)^{k-1}(f)+(\mathfrak{m}_x,u)^k(u,J(f)).$$

Therefore,

$$\sum b_{0l\boldsymbol{\beta}}\boldsymbol{x}^{\boldsymbol{\beta}}\,\frac{\partial f}{\partial x_l}\in\Delta^x(T_k^x(\widetilde{f})).$$

Concluding all the restrictions, we verify the first assertion.

For the second assertion, let $I=(\widetilde{f})+(\Psi_k)+(\Lambda_k\cdot u)$. Since $f\in\mathfrak{m}_xJ(f)$, we have $\mathfrak{m}_x^{k-1}f\subseteq\mathfrak{m}_x^kJ(f)$. Next, $\Delta^x(T_k^x(f))$ is an ideal and $f\in\Delta^x(T_k^x(f))$, and so $\mathfrak{m}_x^{k-1}f\subset\Psi_k$. For all $\boldsymbol{x}^{\boldsymbol{\alpha}}\in\mathfrak{m}_x^{k-1}$, $\boldsymbol{x}^{\boldsymbol{\alpha}}u^2=\boldsymbol{x}^{\boldsymbol{\alpha}}\widetilde{f}-\boldsymbol{x}^{\boldsymbol{\alpha}}f\in I$. This implies $\mathfrak{m}_x^{k-1}u^2\subset I$. To show that $\mathfrak{m}_x^{k-2}u^3\subset I$, it suffices to verify that $\mathfrak{m}_x^{k-2}u\cdot f\in(\Lambda_k\cdot u)$, but this is clear by definition.

Next, we induct on r to show that $\mathfrak{m}_x^{k-r}u^{r+1}\subset I$. The case for $k=1,2$ proceeds as above. Suppose it is true for all $r\leqslant k-1$. Since $(\mathfrak{m}^{k-r+1}u^{r-1})\subset I$ by the induction hypothesis and $\mathfrak{m}_x^{k-r}u^{r-1}f\subset\mathfrak{m}_x^{k-r+1}u^{r-1}$, for all $\boldsymbol{x}^{\boldsymbol{\alpha}}u^{r+1}\in\mathfrak{m}_x^{k-r}u^{r+1}$, we have $\boldsymbol{x}^{\boldsymbol{\alpha}}u^{r+1}=\boldsymbol{x}^{\boldsymbol{\alpha}}u^{r-1}\widetilde{f}-\boldsymbol{x}^{\boldsymbol{\alpha}}u^{r-1}f\in I$. Lemma 4.25 is proved.

We have the following easy corollary to this lemma.

Corollary 4.27. *Under the notation of Lemma* 4.25, *let $\rho_k(f)$ and $Tt(f)$ be invariants of $f\in\mathbb{C}\{\boldsymbol{x}\}$ and $\rho_k(\widetilde{f})$, and $Tt(\widetilde{f})$. be those of $\widetilde{f}\in\mathbb{C}\{u,\boldsymbol{x}\}$. Then*

$$\rho_0(\widetilde{f})=\tau_0(f)+\rho_0(f)\quad\text{and}\quad\sigma_0(\widetilde{f})=\sigma_0(f),$$

where $\tau_0(f)$ is the Tjurina number of f. Furthermore, if f is quasi-homogeneous, then

$$Tt(\widetilde{f})=\max\{Tt(f),\mathrm{mm}(f)\},$$

where $\mathrm{mm}(f)$ is the smallest integer r such that $\mathfrak{m}_x^{2r}\subseteq\mathfrak{m}_x^rJ(f)$.

Proof. The first assertion follows from the isomorphism

$$T_0(f)/\Delta(T_0(f))\simeq T_0^x(f)/\Delta^x(T_0^x(f))\oplus(\mathbb{C}\{\boldsymbol{x}\}/T_0^x(f))u$$

between vector spaces. The second assertion is a consequence of $T_0(\widetilde{f})^2=(u^2)+(T_0^x(f)\cdot u)+(T_0^x(f))^2$, which implies $\Delta(T_0(\widetilde{f}))/(T_0(\widetilde{f}))^2\simeq\Delta^x(T_0^x(f))/(T_0^x(f))^2$.

As for the third assertion, since $(\Psi_k\cdot u)\subseteq(\Lambda_k\cdot u)$ and $(\Lambda_k\cdot u^2)\subseteq(\mathfrak{m}_x^{k-1}u^2)$, we have $\Delta(T_k(\widetilde{f}))\cap\mathbb{C}\{\boldsymbol{x}\}=(\Psi_k)+(\mathfrak{m}_x^{k-1}\cdot f)$. Because $\mathfrak{m}_x^{k-1}\cdot f\subseteq\mathfrak{m}^kJ(f)$ and $f\in\Delta^x(T_k^x(f))$, we have $\Delta(T_k(\widetilde{f}))\cap\mathbb{C}\{\boldsymbol{x}\}=\Psi_k$. Moreover, $\Delta(T_k(\widetilde{f}))\cap(\mathbb{C}\{\boldsymbol{x}\}\cdot u)=\Lambda_k\cdot u$. Hence $\Delta(T_k(\widetilde{f}))=T_k(\widetilde{f})$ if and only if $\Psi_k=\mathfrak{m}_x^kJ(f)$ and $\Lambda_k=\mathfrak{m}_x^k$. The smallest number k satisfying the respective conditions are $Tt(f)$ and $\mathrm{mm}(f)$, respectively. Corollary 4.27 is proved.

Next, we compute the ideal of antiderivatives for ADE surface singularities. We only give Λ_k and Ψ_k so that the reader can recover $\Delta(T_k(\widetilde{f}))$ by Lemma 4.25. We point out that we have deliberately written Δ in the form $(f)+(\Delta\cap\mathfrak{m}^kJ(f))$ in the previous computation for curve singularities.

Proposition 4.28. *For $D_m = V(x^{m-1}+xy^2+u^2), m\geqslant 4$,*

$$\Lambda_k = (xy, x^{m-2}, y^2)\cap\mathfrak{m}_x^k,$$
$$\Psi_k = \begin{cases}(x^{m+k-2}, x^ky^2,\dots,xy^{k+1},y^{k+2},x^{m-2}y), & 1\leqslant k\leqslant m-3,\\ (x^{m+k-2}, x^ky^2,\dots,xy^{k+1},y^{k+2},x^{k+1}y), & k\geqslant m-2.\end{cases}$$

Proposition 4.29. *For $A_m = V(x^{m+1}+y^2+u^2), m\geqslant 2$,*

$$\Lambda_k = (x^m, y)\cap\mathfrak{m}_x^k,$$
$$\Psi_k = \begin{cases}(x^{m+k}, x^my, x^{k-1}y^2,\dots,y^{k+1}), & 2\leqslant k\leqslant m-1,\\ (x^ky, x^{k-1}y^2,\dots,y^{k+1}), & k\geqslant m.\end{cases}$$

Proposition 4.30. *For $E_6 = V(x^3+y^4+u^2)$,*

$$\Lambda_k = (x^2, xy^2, y^3)\cap\mathfrak{m}_x^k,\qquad \Psi_k = (x^{2+k},\dots,x^2y^k,xy^{k+2}).$$

Proposition 4.31. *For $E_7 = V(x^3+xy^3+u^2)$,*

$$\Lambda_k = \begin{cases}(3x^2+y^3, xy^2), & k=1,\\ (x^2,y^3,xy^2)\cap\mathfrak{m}_x^k, & k\geqslant 2,\end{cases}$$
$$\Psi_k = (3x^2y^k+y^{k+3}, x^{k+2},\dots,x^3y^{k-1},x^2y^{k+1},xy^{k+2},y^{k+4}).$$

Proposition 4.32. *For $E_8 = V(x^3+y^5+u^2)$,*

$$\Lambda_k = (x^2, xy^3, y^4)\cap\mathfrak{m}_x^k,$$
$$\Psi_k = \begin{cases}(x^3, x^2y^3, xy^4, y^5), & k=1,\\ (x^3+y^5)+(x^{k+2},\dots,x^2y^k,xy^{k+3},y^{k+4}), & k\geqslant 2.\end{cases}$$

Below are the invariants for D_6, E_6, E_7 when $0\leqslant k\leqslant 12$.

Example 4.33. We distinguish invariants of f and $\widetilde{f}$ by adding a tilde over those of $\widetilde{f}$.

(1) $f = x^5+xy^2$, $\widetilde{f} = f+u^2$.

k	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ_k	6	2	1	0	0	0	0	0	0	0	0	0	0
$\widetilde{\rho}_k$	12	6	3	1	0	0	0	0	0	0	0	0	0
σ_k	6	17	24	31	37	43	49	55	61	67	73	79	85
$\widetilde{\sigma}_k$	6	28	55	89	130	178	234	298	370	450	538	634	738

One can find $\widetilde{\sigma}_k = 4k^2+12k+18$, $4\leqslant k\leqslant 12$, and $\sigma_k = 6k+13$, $k\geqslant 3$.

(2) $f = x^3 + y^4$, $\widetilde{f} = f + u^2$.

k	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ_k	5	1	0	0	0	0	0	0	0	0	0	0	0
$\widetilde{\rho}_k$	11	5	2	0	0	0	0	0	0	0	0	0	0
σ_k	7	18	25	31	37	43	49	55	61	67	73	79	85
$\widetilde{\sigma}_k$	7	29	56	90	130	178	234	298	370	450	538	634	738

One can find $\widetilde{\sigma}_k = 4k^2 + 12k + 18$, $3 \leqslant k \leqslant 12$, and $\sigma_k = 6k + 13$, $k \geqslant 2$.

(3) $f = x^3 + xy^3$, $\widetilde{f} = f + u^2$.

k	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ_k	6	2	0	0	0	0	0	0	0	0	0	0	0
$\widetilde{\rho}_k$	13	7	3	1	0	0	0	0	0	0	0	0	0
σ_k	8	19	27	33	39	45	51	57	63	69	75	81	87
$\widetilde{\sigma}_k$	8	30	58	92	133	181	237	301	373	453	541	637	741

One can find $\widetilde{\sigma}_k = 4k^2 + 12k + 21$, $4 \leqslant k \leqslant 12$, and $\sigma_k = 6k + 15$, $k \geqslant 2$.

Remark 4.34. As in [1], germs $f \in \mathbb{C}\{\boldsymbol{x}\}$ and $\widetilde{f} = f + u^2$ are called stable equivalent. Moduli algebra itself can not tell the difference between stable equivalent singularities if not given the dimension of ambient space. However, by Proposition 4.38, we can read the dimension of the singularities from invariants $\{\sigma_k\}_{k\geqslant 0}$ and hence stable equivalent singularities of different dimensions can be separated apart.

So far, some interesting things have happened:
(a) there is a polynomial $P \in \mathbb{Z}[x]$ such that $\{\sigma_k\}_{k\geqslant Tt(f)} = \{P(k)\}_{k\geqslant Tt(f)}$;
(b) Tt is the smallest integer N such that $\{\sigma_k\}_{k\geqslant N}$ fits a polynomial of k. We state the findings in the following conjecture.

Conjecture 4.35. Let $(X, 0) = (V(f), 0) \subset (\mathbb{C}^n, 0)$ be an isolated hypersurface singularity. Then $Tt(f)$ is the smallest integer N such that $\{\sigma_k\}_{k\geqslant N}$ is a polynomial of k of degree $n - 1$.

Remark 4.36. Our calculation shows that the conjecture holds for ADE curve singularities. Below, we give some examples other than ADE singularities that support our conjecture.

Example 4.37. (1) $f = x^6 + xy^7$.

k	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ_k	57	13	7	2	0	0	0	0	0	0	0	0	0
σ_k	17	68	85	103	117	129	141	153	165	177	189	201	213

One can find $\sigma_k = 12k + 69$, $3 \leqslant k \leqslant 12$.

(2) $f = x^{10}y + xy^{15}$.

k	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ_k	44	40	34	26	18	10	6	2	0	0	0	0	0
σ_k	256	267	284	307	334	365	391	417	441	463	485	507	529

One can find $\sigma_k = 22k + 265$, $8 \leqslant k \leqslant 12$.

(3) $f = x(x + y)(x + 2y)(x + 3y)(x + 4y)(x + 5y)$.

k	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ_k	14	10	4	0	0	0	0	0	0	0	0	0	0
σ_k	36	47	64	81	93	105	117	129	141	153	165	177	189

One can find $\sigma_k = 12k + 45$, $3 \leqslant k \leqslant 12$.

(4) $f = x^3 + y^5 + z^6$.

k	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ_k	38	32	20	8	0	0	0	0	0	0	0	0	0
σ_k	82	109	172	252	339	430	533	648	775	914	1065	1228	1403

One can find $\sigma_k = 6k^2 + 37k + 95$, $4 \leqslant k \leqslant 12$.

(5) $f = x^2y + y^2z + z^2y$.

k	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ_k	12	6	0	0	0	0	0	0	0	0	0	0	0
σ_k	12	39	85	136	199	274	361	460	571	694	829	976	1135

One can find $\sigma_k = 6k^2 + 21k + 19$, $2 \leqslant k \leqslant 12$.

Even though the correctness of the conjecture is not verified, we can prove the following estimate.

Proposition 4.38. *Suppose* $(X, 0) = (V(f), 0) \subset (\mathbb{C}^n, 0)$ *is an isolated singularity, then*

$$\sigma_k \sim \frac{2^{n-1}\operatorname{ord}(f)}{(n-1)!} k^{n-1}.$$

Here, $\operatorname{ord}(f)$ *denotes the minimal degree among all monomial terms appearing in* f*. Besides, for two sequences* $\{a_n\}, \{b_n\} \subset \mathbb{C}$, $a_n \sim b_n$ *means* $a_n/b_n \to 1$, *when* $n \to \infty$.

Proof. For $t \in \mathbb{N}$, let $l(t) = \binom{n+t}{t}$ be the cardinality of the set

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n \mid x_1 + x_2 + \cdots + x_n \leqslant t\}.$$

By Lemma 2.3, there exists an integer w such that $\mathfrak{m}^w \subseteq J(f)$. For a non-negative integer t, we set $L_t = \dim \mathbb{C}\{\boldsymbol{x}\}/((f^2)+(f)\mathfrak{m}^t+\mathfrak{m}^{2t})$ and $R_t = \dim \mathbb{C}\{\boldsymbol{x}\}/((f)+\mathfrak{m}^t)$. Since $\sigma_k = \dim \mathbb{C}\{\boldsymbol{x}\}/T_k(f)^2 - \dim \mathbb{C}\{\boldsymbol{x}\}/T_k(f)$ for large k (Lemma 4.17), we have the estimate

$$L_k - R_{k+w} \leqslant \sigma_k \leqslant L_{k+w} - R_k. \tag{$\sim$}$$

In fact, L_t and R_t can be explicitly calculated for $t > \operatorname{ord}(f)$.

$$\begin{aligned} R_t &= l(t-1) - l(t - \operatorname{ord}(f) - 1) \sim \frac{\operatorname{ord}(f)}{(n-1)!} t^{n-1}, \\ L_t &= l(2t-1) - \big(l(2t - 1 - \operatorname{ord}(f)) - l(t-1)\big) - l(t - 1 - \operatorname{ord}(f)) \\ &\sim \frac{(2^{n-1}+1)\operatorname{ord}(f)}{(n-1)!} t^{n-1}. \end{aligned}$$

Applying the calculation to ($\sim$), we are done. Proposition 4.38 is proved.

With this proposition, we have a direct corollary.

Corollary 4.39. *If Conjecture* 4.35 *holds, the leading term of this polynomial is* $(2^{n-1}\operatorname{ord}(f)/(n-1)!)k^{n-1}$.

Remark 4.40. In Proposition 4.38, we have shown that

$$\dim_{\mathbb{C}} I/I^2 = R_t - L_t = l(2t-1) - l(2t-1-\operatorname{ord}(f)),$$

where $I = (f) + \mathfrak{m}^t \subseteq \mathbb{C}\{\boldsymbol{x}\}$ is an ideal. If $f = \sum_{i=1}^n x_i^r$ is a homogeneous Brieskorn singularity, then $\mathfrak{m}^k J(f) = \mathfrak{m}^{k+r}$ for large k. Consequently, for $k \gg 0$,

$$\sigma_k = l(2(k+r)-1) - l(2(k+r)-1-\operatorname{ord}(f)),$$

which is a polynomial in k. This also verifies Conjecture 4.35.

4.3. A geometric interpretation for the ideal of antiderivatives. In this subsection, we are going to give a geometric interpretation of the ideal of antiderivatives w.r.t. Tjurina ideal map. Hence all Δs in this subsection refer to the ideals of antiderivatives w.r.t. Tjurina ideal map. The motivation of the following construction comes from the well-known exact sequence for Kähler differential (Theorem 2.9).

Lemma 4.41. *Let $I \subseteq \mathfrak{m} \subset \mathbb{C}\{\boldsymbol{x}\}$ be an ideal. Then the following exact sequence holds*:

$$0 \to \Delta(I) \to I \xrightarrow{d} \Omega_{\mathbb{C}\{\boldsymbol{x}\}} \otimes \mathbb{C}\{\boldsymbol{x}\}/I \to \Omega_{\mathbb{C}\{\boldsymbol{x}\}/I} \to 0.$$

Proof. It suffices to check that $\Delta(I) = \ker d$. Since $\Omega_{\mathbb{C}\{\boldsymbol{x}\}} = \bigoplus_{i=1}^n \mathbb{C}\{\boldsymbol{x}\}\, dx_i$, $\Omega_{\mathbb{C}\{\boldsymbol{x}\}} \otimes \mathbb{C}\{\boldsymbol{x}\}/I \simeq \bigoplus_{i=1}^n (\mathbb{C}\{\boldsymbol{x}\}/I)\, dx_i$. Therefore, $f \in \ker d$ if and only if $\partial f/\partial x_i \in I$ for all $1 \leqslant i \leqslant n$. By definition, $\ker d = \Delta(I) = \{f \in I \mid J(f) \subseteq I\}$. This proves the lemma.

Now we suppose that $(X, \mathcal{O}_X)$ is a complex space and Z is the complex subspace given by coherent ideal sheaf $\mathcal{I}$. We have the natural morphism $\alpha\colon \mathcal{I} \to \Omega_X \otimes \mathcal{O}_X/\mathcal{I}$ given by $f \mapsto df \otimes 1$. It gives a global exact sequence for X/Z.

Theorem 4.42. *With the above notation, we have the exact sequence*

$$\mathcal{I} \xrightarrow{\alpha} \Omega_X \otimes \mathcal{O}_X/\mathcal{I} \to i_* \mathcal{O}_Z \to 0, \tag{E}$$

where $i\colon Z \hookrightarrow X$ is the natural closed embedding.

Proof. For $p \notin Z$, $(\mathcal{O}_X/\mathcal{I})_p = 0$ and $(i_*\mathcal{O}_Z)_p = 0$. For all $p \in Z$, taking stalks of (E), the sequences coincide with the (algebraic) sequences in Theorem 2.9. So, we are done.

Definition 4.43. For a coherent ideal sheaf $\mathcal{I}$ of $\mathcal{O}_X$, the ideal sheaf of antiderivatives is defined by the kernel of α in the above exact sequence.

Remark 4.44. Since i is a closed embedding and hence finite, by Theorem 1.67 in [9], $i_*\mathcal{O}_Z$ is coherent. Since $\mathcal{I}$, $\Omega_X \otimes \mathcal{O}_X/\mathcal{I}$ and $i_*\mathcal{O}_Z$ are all coherent, then so is $\Delta(\mathcal{I})$.

The following theorem shows that, for each $p \in X$, the stalk $\Delta(\mathcal{I})_p$ coincides with $\Delta(\mathcal{I}_p) \triangleleft \mathcal{O}_{X,p}$ in the local sense. Hence our global definition gives Δ a geometric interpretation.

Theorem 4.45. *Let $X = D \subset \mathbb{C}^n$ be an open subset and $\mathcal{I}$ be a coherent ideal sheaf of $\mathcal{O}_X$. Then, for each $p \in X$, $\Delta(\mathcal{I})_p$ is the ideal of antiderivatives of the ideal $\mathcal{I}_p$ in the local ring $\mathcal{O}_{X,p} = \mathbb{C}\{\boldsymbol{x} - p\}$.*

Proof. Without loss of generality, we set $p = 0$. Under the above assumptions, we have $\mathcal{O}_{X,p} = \mathbb{C}\{\boldsymbol{x}\}$. Applying Theorem 4.42 and taking stalk at p, we see that it is the exact sequence in Lemma 4.41. The theorem is proved.

4.4. kth Milnor number of semi quasi-homogeneous singularity. In [4], the notion of semi quasi-homogeneity (SQH for short) was provided. The authors proved that the Milnor number of an SQH series is equal to the Milnor number of its principal part. We will apply their method to prove an inequality associated with some types of quasi-T-maps. Besides, we prove the equality between the kth Milnor number of an SQH series and the Milnor number of its principal part.

In this section, $K[[x_1, \dots, x_n]]$ always refers to the ring of formal power series over a field K. For brevity, we write $K[[x_1, \dots, x_n]]$ as $K[[\boldsymbol{x}]]$. Let $\mathfrak{m}$ be the maximal ideal of $K[[\boldsymbol{x}]]$. We will focus on some quasi-T-maps on the K-algebra $K[[\boldsymbol{x}]]$, where K is an arbitrary field. We first define the notions *continuous* and *efficient* for a linear endomorphism of $K[[\boldsymbol{x}]]$.

Definition 4.46. A linear map $P \in \operatorname{End}_K(K[[\boldsymbol{x}]])$ is called *continuous* if there exists an integer d such that $\operatorname{ord}(P(f)) \geqslant \operatorname{ord}(f) - d$ for all $f \in K[[\boldsymbol{x}]]$.

Remark 4.47. There is a natural topology on $K[[\boldsymbol{x}]]$, that is, the $\mathfrak{m}$-adic topology. The open basis near 0 is given by the filtration $\mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots$. A sequence $\{f_i\}_{i=1}^{\infty} \subset K[[\boldsymbol{x}]]$ is called a Cauchy sequence if, for any integer $k > 0$, there exists $N_k > 0$ such that $f_i - f_{i+1} \in \mathfrak{m}^k$ for all $i \geqslant N_k$. It is not hard to check each Cauchy sequence in $K[[\boldsymbol{x}]]$ converges to a unique series. A continuous endomorphism is automatically a continuous map from $K[[\boldsymbol{x}]]$ to itself when considering the $\mathfrak{m}$-adic topology.

Lemma 4.48. *Let $P \in \operatorname{End}_K(K[[\boldsymbol{x}]])$ be continuous. Then $P(f) = \sum_{\boldsymbol{v}} a_{\boldsymbol{v}} P(\boldsymbol{x}^{\boldsymbol{v}})$ for all $f = \sum_{\boldsymbol{v}} a_{\boldsymbol{v}} \boldsymbol{x}^{\boldsymbol{v}} \in K[[\boldsymbol{x}]]$.*

Proof. Let $C_k \colon K[[\boldsymbol{x}]] \to K[[\boldsymbol{x}]]/\mathfrak{m}^{k+1} \hookrightarrow K[[\boldsymbol{x}]]$ be the canonical truncation. Namely, it maps $\sum_{\boldsymbol{v}} a_{\boldsymbol{v}} \boldsymbol{x}^{\boldsymbol{v}}$ to $\sum_{|\boldsymbol{v}| \leqslant k} a_{\boldsymbol{v}} \boldsymbol{x}^{\boldsymbol{v}}$. For all $k \in \mathbb{N}$, we have $f = C_k(f) + (f - C_k(f))$, where $\operatorname{ord}(f - C_k(f)) \geqslant k+1$. Since $P(C_k(f)) = \sum_{|\boldsymbol{v}| \leqslant k} a_{\boldsymbol{v}} P(\boldsymbol{x}^{\boldsymbol{v}})$ and $\operatorname{ord}(P(f - C_k(f))) \geqslant k + 1 - d$, $P(C_k(f))$ is a Cauchy sequence tending to $P(f)$ in $\mathfrak{m}$-adic topology. Therefore, we have $\sum_{|\boldsymbol{v}| \leqslant k} a_{\boldsymbol{v}} P(\boldsymbol{x}^{\boldsymbol{v}}) \to P(f)$, that is, $P(f) = \sum_{\boldsymbol{v}} a_{\boldsymbol{v}} P(\boldsymbol{x}^{\boldsymbol{v}})$, proving the lemma.

Remark 4.49. The lemma is not trivial since it works for infinite sums.

Definition 4.50. Given $f \in K[[\boldsymbol{x}]]$ and a quasi-T-map Q, we define

$$\mu_Q(f) := \dim_K K[[\boldsymbol{x}]]/Q(f).$$

If the dimension is infinite, we simply write $\mu_Q(f) = \infty$.

As in [4], for $w = (w^1, \dots, w^n) \in \mathbb{N}^n_{>0}$ and $f = \sum_{\boldsymbol{v}} a_{\boldsymbol{v}} \boldsymbol{x}^{\boldsymbol{v}} \in K[[\boldsymbol{x}]]$, the principal part of f w.r.t. w is defined to be $f_w = \sum_{\boldsymbol{v} \cdot w \text{ minimal}} a_{\boldsymbol{v}} \boldsymbol{x}^{\boldsymbol{v}}$. f is called semi quasi-homogeneous (SQH for short) w.r.t. a continuous quasi-T-map Q and w ((Q, w) in short) if $\mu_Q(f_w)$ is finite.

The support of $f = \sum_{\boldsymbol{v}} a_{\boldsymbol{v}} \boldsymbol{x}^{\boldsymbol{v}} \in K[[\boldsymbol{x}]]$ is defined by $\operatorname{Supp}(f) := \{\boldsymbol{v} \in \mathbb{N}^n \mid a_{\boldsymbol{v}} \neq 0\}$.

A quasi-T-map $Q\colon K[[\boldsymbol{x}]] \to \{\text{Ideals of } K[[\boldsymbol{x}]]\}$ can be naturally extended to $K[[\boldsymbol{x},t]] \to \{\text{Ideals of } K[[\boldsymbol{x},t]]\}$ in a natural way. That is, assuming $Q = (Q_1,\dots, Q_m)$, for $f = \sum_{i=0}^{\infty} f_i \cdot t^i \in K[[\boldsymbol{x},t]]$, we have $Q_j(f) := \sum_{i=0}^{\infty} Q_j(f_i)\cdot t^i$ and $Q(f) := (Q_1(f),\dots,Q_m(f))$. Let $d = \min_{\boldsymbol{v}\in \mathrm{Supp}(f)} \boldsymbol{v}\cdot w$ and $\widehat{f} = t^{-d} f(t^{w^1}x_1,\dots,t^{w^n}x_n) = f_w + tg$, where $g \in K[[\boldsymbol{x},t]]$. So, $Q_i(\widehat{f}) = Q_i(f_w) + tQ_i(g)$ for each i.

Definition 4.51. Let $w \in \mathbb{N}^n_{>0}$ and $\varphi_w\colon K[[\boldsymbol{x},t]] \to K[[\boldsymbol{x},t]]$ be such that $x_i \mapsto t^{w^i}x_i$, $t \mapsto t$. A linear map $P \in \mathrm{End}_K(K[[\boldsymbol{x}]])$ is called *efficient w.r.t.* w if P is continuous and there is an integer e such that $\varphi_w(P(\boldsymbol{x}^{\boldsymbol{v}})) = t^e P(\varphi_w(\boldsymbol{x}^{\boldsymbol{v}}))$ for each monomial $\boldsymbol{x}^{\boldsymbol{v}}$.

Example 4.52. Consider $K[[x,y]]$ and $w = (1,1)$. Then $P = x^2y^3\,\partial_x$ is efficient with $e = 4$, while $P = \partial_x + \partial_x\,\partial_y$ is continuous yet not efficient.

Definition 4.53. A quasi-T-map $Q = (Q_1,\dots,Q_m)$ (that is, $Q(f) = (Q_1(f),\dots, Q_m(f))$ for all $f \in K[[\boldsymbol{x}]]$) is called continuous (efficient, respectively) if all Q_i are continuous (efficient, respectively).

Proposition 4.54. *With the above notation, let Q be an efficient quasi-T-map and $w \in \mathbb{N}_{>0}$. Suppose $f = \sum a_{\boldsymbol{v}}\boldsymbol{x}^{\boldsymbol{v}}$ is SQH w.r.t. (Q,w), and $K[[\boldsymbol{x},t]]/Q(\widehat{f})K[[\boldsymbol{x},t]]$ is finitely generated as a $K[[t]]$-module, then $\mu_Q(f_w) \geqslant \mu_Q(f)$. The equality holds if and only if $K[[\boldsymbol{x},t]]/Q(\widehat{f})K[[\boldsymbol{x},t]]$ is torsion-free as a $K[[t]]$-module.*

Proof. Let L be the fraction field of $K[[t]]$ and $\varphi_w\colon x_i \to t^{w^i}x_i$, $t \mapsto t$, be an automorphism of $L[[\boldsymbol{x}]]$. By Lemma 4.48 and the definition of efficiency, we have $\varphi_w(Q(f))L[[\boldsymbol{x}]] = Q(\varphi_w(f))L[[\boldsymbol{x}]] = Q(\widehat{f})L[[\boldsymbol{x}]]$, and the isomorphisms

$$\begin{aligned} K[[\boldsymbol{x},t]]/Q(\widehat{f})K[[\boldsymbol{x},t]] \otimes_{K[[t]]} L &\simeq L[[\boldsymbol{x}]]/Q(\widehat{f})L[[\boldsymbol{x}]] \\ &\simeq L[[\boldsymbol{x}]]/\varphi(Q(f))L[[\boldsymbol{x}]] \simeq L[[\boldsymbol{x}]]/Q(f)L[[\boldsymbol{x}]]. \end{aligned}$$

The first isomorphism is due to Lemma 4.56 below. Am appeal to Lemma 4.57 below shows that $\dim_L L[[\boldsymbol{x}]]/Q(f)L[[\boldsymbol{x}]] = \mu_Q(f)$.

Since $K[[t]]$ is a discrete valuation ring, it follows that the L-dimension of $K[[\boldsymbol{x},t]]/Q(\widehat{f})K[[\boldsymbol{x},t]] \otimes_{K[[t]]} L$ is the free rank of $K[[\boldsymbol{x},t]]/Q(\widehat{f})K[[\boldsymbol{x},t]]$ by Theorem 2.6. Since $K[[\boldsymbol{x},t]]/Q(\widehat{f})K[[\boldsymbol{x},t]] \otimes_{K[[t]]} K \simeq K[[\boldsymbol{x}]]/Q(f_w)$ and $\mu(f_w)$ is the rank of $K[[\boldsymbol{x},t]]/Q(\widehat{f})K[[\boldsymbol{x},t]]$, we have $\mu_Q(f_w) \geqslant \mu_Q(f)$. The condition for equality is obvious.

Remark 4.55. (1) It is clear that $Q\colon f \mapsto (f, \partial f/\partial x_1,\dots,\partial f/\partial x_n)$, the Tjurina ideal map, satisfies all the conditions. And $<$ holds for $f \notin (\partial f/\partial x_1,\dots,\partial f/\partial x_n)$.

(2) The "finitely generated" condition is necessary. Let $K = \mathbb{C}$, $f = x^2 + xy^3 + y^4 \in \mathbb{C}[[x,y]]$, and $w = (1/2, 1/4)$, then $f_w = x^2 + y^3$. Consider the quasi-T-maps Q_1 and Q_2 defined by

$$\begin{aligned} Q_1\colon \sum_{i,j} a_{ij}x^iy^j &\mapsto \Bigg(\sum_{i,j\geqslant 5} a_{ij}x^iy^j\Bigg) + a_{13}(xy^3 - x^2) + a_{20}x^2, \\ Q_2\colon \sum_{i,j} a_{ij}x^iy^j &\mapsto \Bigg(\sum_{i,j\geqslant 5} a_{ij}x^iy^j\Bigg) + a_{04}y^4, \end{aligned}$$

$$Q := (Q_1, Q_2).$$

We have $Q(f_w) = (x^2, y^4)$ and $Q(f) = (xy^3, y^3)$, and $K[[x, y, t]]/Q(\widehat{f})K[[x, y, t]]$ is not finitely generated. Next, $\mu(f_w) = 3 < \mu(f) = \infty$.

(3) We will soon later see that the kth Jacobian ideal map, $J_k \colon g \mapsto \mathfrak{m}^k \cdot (\partial g/\partial x_1, \ldots, \partial g/\partial x_n)$, satisfies all conditions.

The following two lemmas may be well-known for experts. However, we did not find suitable references. Hence we give complete proofs below.

Lemma 4.56. *Let $I \subseteq K[[\boldsymbol{x}, t]]$ be an ideal of $K[[\boldsymbol{x}, t]]$ such that $K[[\boldsymbol{x}, t]]/I$ is a finitely generated $K[[t]]$-module. $L := K((t))$ is the field of Laurent series over K. Then $K[[\boldsymbol{x}, t]]/I \otimes_{K[[t]]} L \simeq L[[\boldsymbol{x}]]/IL[[\boldsymbol{x}]]$ as L-algebras.*

Proof. Let $A = K[[\boldsymbol{x}, t]]$, then $A/I \otimes_{K[[t]]} L = A_t/IA_t$ is the localization of A/I. We claim that $\mathfrak{m}^r \subseteq IA_t$ for some $r > 0$. If not, there exists an x_i such that $x_i^k \notin IA_t$ for all k. Since A/I is a finitely generated $K[[t]]$-module, A_t/IA_t is a finite L-linear space. Hence $x_i, x_i^2, \ldots, x_i^p$ is linearly dependent for some p, which implies $x_i^p \in IA_t$, a contradiction.

Consequently, we have an isomorphism $A_t/IA_t = A_t/(IA_t + \mathfrak{m}^r A_t) \simeq B/I'B$, where $B = A_t/\mathfrak{m}^r A_t = L[[\boldsymbol{x}]]/\mathfrak{m}^r L[[\boldsymbol{x}]] = L[\boldsymbol{x}]/((\boldsymbol{x})L[[\boldsymbol{x}]])^r$ and I' is the image of I in B. The same argument holds for $L[[\boldsymbol{x}]]/IL[[\boldsymbol{x}]]$, for we only need to notice $\mathfrak{m}^r \subseteq IL[[\boldsymbol{x}]]$. Lemma 4.56 is proved.

Lemma 4.57. *Let $I = (f_1, \ldots, f_m) \subseteq K[[\boldsymbol{x}]]$ be an ideal, and let $L := K((t))$ be the field of Laurent series over K. Then $\dim_L L[[\boldsymbol{x}]]/IL[[\boldsymbol{x}]] < \infty$ if and only if $\dim_K K[[\boldsymbol{x}]]/I < \infty$. Moreover, if the finiteness holds, then those dimensions coincide.*

Proof. Suppose that $K[[\boldsymbol{x}]]/I$ is finite dimensional. Then $\mathfrak{m}^r \subseteq I$ for some r. Since $IL[[\boldsymbol{x}]] \supseteq \mathfrak{m}^r L[[\boldsymbol{x}]]$, $\dim_L L[[\boldsymbol{x}]]/IL[[\boldsymbol{x}]]$ is finite. Conversely, let $L[[\boldsymbol{x}]]/IL[[\boldsymbol{x}]]$ be finite dimensional. Then $\mathfrak{m}^r \subseteq IL[[\boldsymbol{x}]]$ for some r. Hence $\boldsymbol{x}^{\boldsymbol{\alpha}} = \sum_j f_j g_j, g_j \in L[[\boldsymbol{x}]]$, for all $\boldsymbol{\alpha} \in \mathbb{N}^n$, $|\boldsymbol{\alpha}| = r$. Considering the degree-zero part of all g_j w.r.t. t, we have $\boldsymbol{x}^{\boldsymbol{\alpha}} \in I$ for all i. Therefore, $I \supseteq \mathfrak{m}^r$.

If both finiteness holds, it suffices to show that a finite set of monomials $\{\boldsymbol{x}^{\boldsymbol{\alpha}_i}\}_{i \in I}$ is linearly dependent in $K[[\boldsymbol{x}]]/I$ if and only if in $L[[\boldsymbol{x}]]/IL[[\boldsymbol{x}]]$. The "only if" is trivial. As for "if", suppose $\sum h_i \boldsymbol{x}^{\boldsymbol{\alpha}_i} = \sum_j f_j l_j$ $(+)$, $h_i \in L$, and $l_j \in L[[\boldsymbol{x}]]$. We may assume that the degree-zero part of h_1 w.r.t. t is not 0. By considering the degree-zero part of $(+)$ w.r.t. t, we are done. Lemma 4.57 is proved.

Let $J_k(f) := \mathfrak{m}^k J(f)$ be the kth Jacobian ideal. The dimension of its quotient algebra is called the kth Milnor number $\mu_k(f) := \dim_K K[[\boldsymbol{x}]]/J_k(f)$. One can prove that $Q = J_k$ is efficient w.r.t. any weight $w \in \mathbb{N}^n_{>0}$. To see this, it suffices to check that $g \mapsto \boldsymbol{x}^{\boldsymbol{\alpha}} \cdot (\partial g/\partial x_i)$ is efficient for all i and $|\boldsymbol{\alpha}| = k$. It is not hard to check $\varphi_w(\boldsymbol{x}^{\boldsymbol{\alpha}}) \cdot (\partial \varphi_w(\boldsymbol{x}^{\boldsymbol{\beta}})/\partial x_i) = t^{w^i} \cdot \varphi_w(\boldsymbol{x}^{\boldsymbol{\alpha}} \cdot (\partial \boldsymbol{x}^{\boldsymbol{\beta}}/\partial x_i))$ for all $\beta \in \mathbb{N}^n_{>0}$.

Suppose $w \in \mathbb{N}^n_{>0}$ is a weight. In [4], the authors proved when $Q = J = J_0$ and $\mu(f_w) < \infty$, that $K[[\boldsymbol{x}]]/Q(\widehat{f})$ is a free $K[[t]]$-module of rank $\mu(f_w)$ and hence torsion free and finitely generated. We will base on this fact and prove $\mu_k(f) = \mu_k(f_w)$, whenever $k \leqslant \min_i\{\mathrm{ord}(\partial f/\partial x_i)\}$ and $\mu(f_w) < \infty$.

Lemma 4.58. *Suppose $I = (g_1, \ldots, g_m) \subseteq K[[\boldsymbol{x}, t]]$ is an ideal such that $K[[\boldsymbol{x}, t]]/I$ is a finitely generated $K[[t]]$-module. Then $K[[\boldsymbol{x}, t]]/\mathfrak{m}I$ is also finitely generated.*

Proof. We may emphasize $\mathfrak{m}$ is the maximal ideal of $K[[\boldsymbol{x}]]$. Let $e_1, \ldots, e_r \in K[[\boldsymbol{x}, t]]$ be such that their image in the quotient ring $K[[\boldsymbol{x}, t]]/I$ is a set of generators. Since $\sum g_i \cdot K[[\boldsymbol{x}, t]] + \mathfrak{m}I = I$, we have $e_1, \ldots, e_r$ together with $g_1, \ldots, g_m$ generates $K[[\boldsymbol{x}, t]]/\mathfrak{m}I$, proving the lemma.

As a corollary, if $\mu(f_w) < \infty$. then $K[[\boldsymbol{x}, t]]/\mathfrak{m}^k J(\widehat{f})$ is finitely generated as a $K[[t]]$-module.

We have so far proved the "finitely generated" condition in Proposition 4.54. Hence we have a simple corollary as below.

Corollary 4.59. *Suppose f is SQH w.r.t. $Q = J_k$ and $w \in \mathbb{N}^n_{>0}$ is a weight such that $\mu_k(f_w) < \infty$ (equivalently, $\mu(f_w) < \infty$). Then $\mu_k(f) \leqslant \mu_k(f_w)$.*

Remark 4.60. The same argument also holds for $Q = T_k$ (that is, $Q(f) = (f) + \mathfrak{m}^k J(f)$ for all $f \in K[[\boldsymbol{x}]]$), since $T_k(\widehat{f}) \supseteq J_k(\widehat{f})$. Thus we have $\tau_k(f) \leqslant \tau_k(f_w)$ for all $k \in \mathbb{N}$ if $\tau(f_w) < \infty$, where τ_k is the kth Tjurina number. But the equality does not generally hold.

Let us next show that the equality holds for μ_k when $k \leqslant \min_i\{\mathrm{ord}(\partial f/\partial x_i)\}$. By Proposition 4.54, it is equivalent that $K[[\boldsymbol{x}, t]]/Q(\widehat{f})K[[\boldsymbol{x}, t]]$ is torsion-free. Before giving a proof, we need to do some preparation for regular sequence.

Let A be a ring. We define $A\langle\langle t\rangle\rangle := \prod_{i\in\mathbb{Z}} A$ whose elements are written as $\sum_{i\in\mathbb{Z}} a_i t^i$, $a_i \in A$. It is an $A[t]$-module but not an $A[[t]]$-module. For $A = K[[\boldsymbol{x}]]$ and $L := K((t))$, $L[[\boldsymbol{x}]]$ is contained in $A\langle\langle t\rangle\rangle$ in the set-theoretical sense. The following lemma tells us how elements of $L[[\boldsymbol{x}]]$ look like.

Lemma 4.61.

$$L[[\boldsymbol{x}]] = \left\{ \sum_{i\in\mathbb{Z}} a_i(\boldsymbol{x})t^i \;\middle|\; a_i \in K[[\boldsymbol{x}]] \text{ such that } a_i \to 0,\ i \to -\infty \right\} \subseteq K[[\boldsymbol{x}]]\langle\langle t\rangle\rangle.$$

Here, the convergence is in $\mathfrak{m}$-adic topological sense.

The proof is to simply swap the order of summation.

Lemma 4.62. *Suppose $(f_1, \ldots, f_r)$ is a regular sequence in $K[[\boldsymbol{x}]]$. Then it is also regular in $L[[\boldsymbol{x}]]$, where $L = K((t))$.*

Proof. Without a loss of generality, it suffices to prove f_r is a non-zero divisor in the quotient ring $K[[\boldsymbol{x}]]/(f_1, \ldots, f_{r-1})L[[\boldsymbol{x}]]$. Suppose $af_r \in (f_1, \ldots, f_{r-1})L[[\boldsymbol{x}]]$, $a = \sum a_i(\boldsymbol{x})t^i \in L[[x]] \subset K[[\boldsymbol{x}]]\langle\langle t\rangle\rangle$. Considering the grading w.r.t. t, we have $a_i \in I := (f_1, \ldots, f_{r-1}) \subseteq K[[\boldsymbol{x}]]$. Therefore, $a_i = \sum_{j=1}^{r-1} a_i^j f_j$, $a_i^j \in K[[\boldsymbol{x}]]$. We need to select suitable a_i^j such that $a_i^j \to 0$, $i \to -\infty$. Suppose $a_i \in \mathfrak{m}^{n_i}$, $n_i \to \infty$, as $i \to -\infty$. By Artin–Rees's theorem (Theorem 2.5), there exists an $N > 0$ such that, for all $n \geqslant N$ and $k \geqslant 0$, $\mathfrak{m}^{k+n} \cap I = \mathfrak{m}^k(\mathfrak{m}^n \cap I)$. We may assume $n_i > N$ for all $i \leqslant 0$, after which we can select $a_i^j \in \mathfrak{m}^{n_i - N}$, hence tending to 0. Lemma 4.62 is proved.

Theorem 4.63. *Suppose $f \in K[[\boldsymbol{x}]]$ is SQH w.r.t. $Q = J_k$ and $w \in \mathbb{N}^n_{>0}$, that is, $\mu(f_w) < \infty$. Then $\mu_k(f_w) = \mu_k(f)$ for $k \leqslant \min_i\{\mathrm{ord}(\partial f/\partial x_i)\}$.*

Proof. By Lemma 4.58 and Proposition 4.54, it will be sufficient to show that $K[[\boldsymbol{x},t]]/J_k(\widehat{f})$ is torsion free. We prove it by induction. The case $k=0$ was verified in [4]. Suppose that $K[[\boldsymbol{x},t]]/J_k(\widehat{f})$ is torsion-free.

The notation is as in Proposition 4.54. Since $\mu(f)<\infty$, Theorem 2.8 shows that $(\partial f/\partial x_1,\ldots,\partial f/\partial x_n)$ is a regular sequence in $K[[\boldsymbol{x}]]$. By Lemma 4.62, we have $(\partial f/\partial x_1,\ldots,\partial f/\partial x_n)$ is also regular in $L[[\boldsymbol{x}]]$. Next, $\varphi\colon L[[\boldsymbol{x}]]\to L[[\boldsymbol{x}]]$, with $x_i\mapsto t^{w^i}x_i$, $t\mapsto t$, is an automorphism of $L[[\boldsymbol{x}]]$. Since $\partial\widehat{f}/\partial x_i=t^{-w^i}\varphi(\partial f/\partial x_i)$, $(\partial\widehat{f}/\partial x_1,\ldots,\partial\widehat{f}/\partial x_n)$ is also a regular sequence.

Suppose $t\cdot a(\boldsymbol{x})\in J_{k+1}(\widehat{f})\subseteq J_k(\widehat{f})$. On the one hand, by the induction hypothesis $a(\boldsymbol{x})\in J_k$. We may assume $a(\boldsymbol{x})=\sum_i\sum_{|\boldsymbol{\alpha}|=k}a_{i\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}}\,(\partial\widehat{f}/\partial x_i)$, $a_{i\boldsymbol{\alpha}}\in K[[t]]$. On the other hand, since $ta(\boldsymbol{x})\in J_{k+1}(\widehat{f})=\mathfrak{m}^{k+1}\cdot(\partial\widehat{f}/\partial x_1,\ldots,\partial\widehat{f}/\partial x_n)$, one can write $ta(\boldsymbol{x})=\sum_i b_i\,(\partial\widehat{f}/\partial x_i)$, where $b_i\in L[[\boldsymbol{x}]]$ and $\operatorname{ord}(b_i)\geqslant k+1$. Let $c_i:=\sum_{|\boldsymbol{\alpha}|=k}a_{i\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}}-b_i$, then $\sum_i c_i\,(\partial\widehat{f}/\partial x_i)=0\in L[[\boldsymbol{x}]]$.

Since $(\partial\widehat{f}/\partial x_1,\ldots,\partial\widehat{f}/\partial x_n)$ is regular in $L[[\boldsymbol{x}]]$, we have $c_n\in(\partial\widehat{f}/\partial x_1,\ldots,\partial\widehat{f}/\partial x_{n-1})$. However, $k<\min_i\{\operatorname{ord}(\partial f/\partial x_i)\}$ implies $\sum_{|\boldsymbol{\alpha}|=k}a_{n\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}}=0$. By Proposition 2.7, regularity is independent of permutation. So, we have $a=0$. Theorem 4.63 is proved.

However, it seems that $\mu_k(f_w)=\mu_k(f)$ as well for $k>\min_i\{\operatorname{ord}(\partial f/\partial x_i)\}$. Here are some examples.

Example 4.64. The following are computed by SINGULAR.

(1) $f=x^3+y^3+z^3+\lambda x^2y^2z^2$, $\lambda\in\mathbb{C}$, $w=(1,1,1)$.

$\mu_k\setminus k$	0	1	2	3	4	5	6	7	8	9	10
$x^3+y^3+z^3+\lambda x^2y^2z^2$	8	11	20	35	56	84	120	165	220	286	364
$x^3+y^3+z^3$	8	11	20	35	56	84	120	165	220	286	364

(2) $f=x^3+y^4+z^5+\lambda x^3y^4z^5$, $\lambda\in\mathbb{C}$, $w=(20,15,12)$.

$\mu_k\setminus k$	0	1	2	3	4	5	6	7	8	9	10
$x^3+y^4+z^5+\lambda x^3y^4z^5$	24	27	36	52	75	105	143	190	247	315	395
$x^3+y^4+z^5$	24	27	36	52	75	105	143	190	247	315	395

(3) $f=x^2y+y^2z+z^2x+\lambda(x^4+y^4+z^4)$, $\lambda\in\mathbb{C}$, $w=(1,1,1)$.

$\mu_k\setminus k$	0	1	2	3	4	5	6	7	8	9	10
$x^2y+y^2z+z^2x+\lambda(x^4+y^4+z^4)$	8	14	24	39	60	88	124	169	224	290	368
$x^2y+y^2z+z^2x$	8	14	24	39	60	88	124	169	224	290	368

Hence we make a conjecture.

Conjecture 4.65. Suppose $f\in K[[\boldsymbol{x}]]$ is SQH w.r.t. $Q=J_k$ and $w\in\mathbb{N}^n_{>0}$. Equivalently, $\mu(f_w)<\infty$. Then for all $k\in\mathbb{N}$, $\mu_k(f_w)=\mu_k(f)$.

4.5. *T*-fullness and *T*-dependence: a new definition. In this and the next subsections, we will determine whether an ideal of a local Noetherian algebra over an infinite field is a T-principal ideal. We further assume A is a local ring and $\mathfrak{m}$ is its maximal ideal. Recall that Q is a fixed T-map and we omit the notion of "w.r.t. Q" when stating properties about T-maps.

Definition 4.66. An ideal I of A is called T-full if and only if $Q(\Delta(I))=I$.

With this definition, the following proposition is straightforward.

Proposition 4.67. *Suppose $I \lhd A$ is a T-principal ideal, then I is T-full.*

However, as a typical example, a T-full ideal of $(R, A, Q) = (\mathbb{C}, \mathbb{C}\{\boldsymbol{x}\}, T_0)$ is not necessarily a Tjurina ideal, thus we need an additional condition. On some scale, T-full implies surjectivity, showing an ideal is possibly generated by the image Q. The following condition essentially tests whether it can be generated by one element.

Definition 4.68. Suppose $J = (g_1, \ldots, g_r) \lhd A$ is an ideal and consider the graded ring $S = A[y_1, \ldots, y_r]$. Let $\sigma = \sum g_i y_i$ and $\mathcal{Q}(\sigma) := (Q_1(\sigma), \ldots, Q_m(\sigma))$, where each Q_i acts on the coefficient ring A and acts as identity on $y_1, \ldots, y_r$. We call J *T-dependent* if $(\mathcal{Q}(\sigma) \colon Q(J)S) \not\subseteq \mathfrak{m}S$. Equivalently, there is a $P \in \mathbb{P}^{r-1}_A$ such that $\mathfrak{m}S \subseteq P$ and $(Q(J)S)_{(P)} = (\mathcal{Q}(\sigma))_{(P)}$.

Clearly, there is some trouble with "well-defined": whether the condition is independent of the choice of $g_1, \ldots, g_r$.

Proposition 4.69. *The definition of T-dependence is independent of the choice of generators of J.*

Before proving the proposition, we shall translate the definition into the language of algebraic geometry. The main notation is taken from [11]. First, we identify $\mathcal{Q}(\sigma)$ with its homogeneous sheafification, which is an ideal sheaf of $\mathbb{P}^{r-1}_A$. Second, let $\pi \colon \mathbb{P}^{r-1}_A \to \operatorname{Spec} A$ be the canonical projection, then $\pi^*(Q(J))$ is equal to $(IS)^{\sim}$, another ideal sheaf. It is clear that $\mathcal{Q}(\sigma) \hookrightarrow \pi^*(Q(J))$. Set $\mathcal{F} = \pi^*(Q(J))/\mathcal{Q}(\sigma)$, which is a coherent $\mathcal{O}_{\mathbb{P}^{r-1}_A}$-module and thus $\operatorname{Supp} \mathcal{F}$ is a closed subset of $\mathbb{P}^{r-1}_A$. Since $(J : I)^{\sim} = (J^{\sim} : I^{\sim})$ for finitely generated graded ideals I, J, we have $P \notin \operatorname{Supp} \mathcal{F}$ if and only if $(Q(J)S)_{(P)} = (\mathcal{Q}(\sigma))_{(P)}$. Therefore, $(\mathcal{Q}(\sigma) \colon Q(J)S) \not\subseteq \mathfrak{m}S$ in Definition 4.68 can be restated as $\mathfrak{m}S \notin \operatorname{Supp} \mathcal{F}$. Since $\mathfrak{m}S$ is the minimal element of $\pi^{-1}(\mathfrak{m})$ under the order "containing", it is also equivalent to saying that $\pi^{-1}(\mathfrak{m}) \not\subseteq \operatorname{Supp} \mathcal{F}$. The method of following proof is based on that of Lemma 3.8 in [19].

Proof of Proposition 4.69. Suppose $h_1, \ldots, h_l$ is another set of generators of J. For it, we correspondingly define σ', S', π', and $\mathcal{F}'$, where $z_1, \ldots, z_l$ are variables of S'. We may assume $J \neq 0$. It suffices to show that $\pi^{-1}(\mathfrak{m}) \not\subseteq \operatorname{Supp} \mathcal{F}$ implies $\pi'^{-1}(\mathfrak{m}) \not\subseteq \operatorname{Supp} \mathcal{F}'$.

By definition, $g_i = \sum_j r_{ij} h_j$ for some $r_{ij} \in A$. Suppose that all $r_{ij} \in \mathfrak{m}$. Then $J \subset \mathfrak{m}J$. By Nakayama's lemma, $J = 0$, a contradiction. Therefore, at least one $r_{ij} \notin \mathfrak{m}$. We construct $\Phi \colon S' \to S$ by $z_j \mapsto \sum_i r_{ij} y_i$. It is a homomorphism of graded A-algebras. Hence it induces $\varphi \colon U \to \mathbb{P}^{l-1}_A$ for $\operatorname{Spec} A$-schemes by (see Chap. II, Exercise 2.14 in [11]), where U is the open subscheme given by $U = \{\mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \not\supset \Phi(S'_+)\}$. One can find that $\pi^{-1}(\mathfrak{m}) \cap U \neq \varnothing$, since there is an $r_{ij} \notin \mathfrak{m}$ and then $\mathfrak{m}S \not\supset \Phi(S'_+)$. Consider the following commutative diagram:

$$\begin{tikzcd}
U \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & \mathbb{P}^{l-1}_A \arrow[dl, "\pi'"] \\
& \operatorname{Spec} A. &
\end{tikzcd}$$

By the construction of φ, we have $\sigma|_U = \varphi^*\sigma'$ since $\Phi(\sigma') = \sigma$. Then $\pi^*(Q(J)) = \varphi^*\pi'^*(Q(J))$ and $\mathcal{Q}(\sigma)|_U = \varphi^*(\mathcal{Q}(\sigma'))$. Therefore, $\mathcal{F}|_U = \varphi^*\mathcal{F}'$.

As above, $\mathfrak{m}S \in U \setminus \operatorname{Supp}\mathcal{F} = \varphi^{-1}(\mathbb{P}^{l-1}_A \setminus \operatorname{Supp}\mathcal{F}')$. Hence $\varphi(\mathfrak{m}S) \notin \operatorname{Supp}\mathcal{F}'$. Observing that $\mathfrak{m}S' \subseteq \Phi^{-1}(\mathfrak{m}S)$, we have $\varphi(\mathfrak{m}S) \in \pi'^{-1}(\mathfrak{m})$. proving Proposition 4.69.

In what follows, we will consider only the case where R is an infinite field, even though the definition of T-full and T-dependence is valid for local Noetherian algebras over arbitrary rings.

4.6. Determination of a T-principal ideal and construction of a generator. In this subsection, we are going to generalize the main theorem of [19] up to the level of commutative algebra. From now on, F is an *infinite field* and A is a *Noetherian local* F-algebra with maximal ideal $\mathfrak{m}$. Let Q be a fixed T-map of A.

Definition 4.70. For $\lambda \in \mathbb{P}^{r-1}_F$, define p_λ as the prime ideal $(\{\lambda_i y_j - \lambda_j y_i \mid 1 \leqslant i, j \leqslant r\}) \lhd F[y_1, \ldots, y_r]$. Note that the definition is reasonable, say it does not depend on the choice of representative element of λ. Here and below, $\mathbb{P}^{r-1}_F$ is always understood in the set-theoretical sense, while $\mathbb{P}^{r-1}_A$ is in the scheme-theoretical sense.

Lemma 4.71. *For $F[y_1, \ldots, y_r]$, p_λ as above, $f \in p_\lambda$ if and only if $f(\lambda) = 0$. (This lemma also applies to finite F.)*

Proof. The necessity is trivial, we only prove the sufficiency. Since p_λ is a homogeneous ideal, it suffices to prove the following result.

Given $\lambda \in F^r$, $\mathfrak{m}_\lambda := (z_1 - \lambda_1, \ldots, z_r - \lambda_r) \lhd F[z_1, \ldots, z_r]$, a polynomial f lies in $\mathfrak{m}_\lambda$ if and only if $f(\lambda) = 0$.

If $r = 1$, it is trivial. Suppose that the required result holds for $r - 1$. Since $f(z_1, \ldots, z_r) - f(\lambda_1, z_2, \ldots, z_r)$ is a multiple of $z_1 - \lambda_1$ and since $f(\lambda_1, z_2, \ldots, z_r) \in (z_2 - \lambda_2, \ldots, z_r - \lambda_r)$ by the induction hypothesis, we are done. The lemma is proved.

Lemma 4.72. *For $R = F[y_1, \ldots, y_r]$, p_λ as above, $\bigcap_{\lambda \in \mathbb{P}^{r-1}_F} p_\lambda = 0$.*

Proof. We first notice that $I = \bigcap_{\lambda \in \mathbb{P}^{r-1}_F} p_\lambda$ is a homogeneous ideal, so we only need to consider the homogeneous polynomials. Suppose $f \in I$ is a homogeneous polynomial. By Lemma 4.71, $f \in p_\lambda$ implies $f(\lambda_1, \ldots, \lambda_r) = 0$. Since λ runs through the whole $\mathbb{P}^{r-1}_F$, f must be 0, proving the lemma.

Lemma 4.73. *Suppose $A/\mathfrak{m} \simeq F$. For an ideal $I \lhd A$, if $IS \not\subset \mathfrak{m}S$, then $I \not\subset p_\lambda S + \mathfrak{m}S$, for some $\lambda \in \mathbb{P}^{r-1}_F$*

Proof. Let $\pi\colon A[y_1, \ldots, y_r] \twoheadrightarrow F[y_1, \ldots, y_r]$ be the projection of coefficients. Suppose $I \subset p_\lambda S + \mathfrak{m}S$ for all $\lambda \in \mathbb{P}^{r-1}_F$ Then $I \subset \bigcap_{\lambda \in \mathbb{P}^{r-1}_F}(\mathfrak{m}S + p_\lambda S) := J$. Since J contains the kernel of π, that is, $\mathfrak{m}S$, $J = \pi^{-1}(\pi(J))$. However, we have $\pi(J) = \bigcap \pi(\mathfrak{m}S + p_\lambda S) = \bigcap p_\lambda = 0$ by Lemma 4.72, a contradiction, which proves the lemma.

Remark 4.74. The assumption that F is infinite is necessary. For if F is finite, then $\bigcap p_\lambda$ is a finite intersection of finitely generated ideals. We may take the product of all the generators throughout all the components, then it is in the intersection.

Lemma 4.75. *Let B be a local ring and $\mathfrak{n}$ be its maximal ideal. For ideals $I, J \lhd B$, let P be a prime ideal of $C = B[y_1, \ldots, y_r]$ containing $\mathfrak{n}C$. If $IC_{(P)} = JC_{(P)}$, then $I = J$.*

Proof. Symmetrically, it suffices to show $I \subseteq J$. For any $h \in I$, $h = g(G_1/G_2)$, for some $g \in J$ and some homogeneous polynomials G_1, G_2 with the same degree and $G_2 \notin P$. Hence there exists a $G_3 \in C \setminus P$ homogeneous such that $G_3(hG_2 - gG_1) = 0$ $(*)$. Noting that G_2 and G_3 both have some coefficients in $B \setminus \mathfrak{n}$, and hence there is a term in G_2G_3 whose coefficient is in B^*. Considering the coefficient of this term in $(*)$, we have $h \in (g)$. Therefore, $I \subseteq J$, proving the lemma.

Theorem 4.76. *Suppose $A/\mathfrak{m} \simeq F$. An ideal $I \lhd A$ is a T-principal ideal if and only if I is T-full and $\Delta(I)$ is T-dependent.*

Proof. We use the same notation as in Definition 4.68. Let us first prove the necessity.

Suppose $I = Q(f)$ is a T-principal ideal. As in Proposition 4.67, I is automatically T-full. Suppose $\Delta(I) = (g_1, \ldots, g_r)$ and without a loss of generality, we set $g_1 = f$. Hence $f = g_1 + 0 \cdot g_2 + \cdots + 0 \cdot g_r$. So, we may assume $f = \sum \lambda_i g_i$, where $(\lambda_1, \ldots, \lambda_r) \in F^r \setminus \{0\}$. Consider $P = \mathfrak{m}S + p_\lambda S$ and assume $P \in D_+(y_l)$. Hence $\lambda_l \neq 0$. We have the lower estimate

$$\begin{aligned} IS_{(P)} = Q(f)S_{(P)} &\subseteq Q\left(\sum_i \left(\frac{\lambda_i}{\lambda_l} - \frac{y_i}{y_l}\right) g_i\right) S_{(P)} + Q\left(\frac{\sigma}{y_l}\right) S_{(P)} \\ &\subseteq IP_{(P)} + (\mathcal{Q}(\sigma))_{(P)} \subseteq IS_{(P)}. \end{aligned}$$

Uisng Nakayama's lemma, and considering the $S_{(P)}$-module $IS_{(P)}$ and its submodule $(\mathcal{Q}(\sigma))_{(P)}$, we have $(IS)_{(P)} = (\mathcal{Q}(\sigma))_{(P)}$.

Next, let us prove the sufficiency. Let I be T-full and $\Delta(I)$ is T-dependent. Then $(\mathcal{Q}(\sigma) : IS) \not\subset \mathfrak{m}S$. By Lemma 4.73, it is not even contained in some $P = \mathfrak{m}S + p_\lambda S \in \mathbb{P}^{r-1}_A$, say $(\mathcal{Q}(\sigma))_{(P)} = (IS)_{(P)}$. We may assume $P \in D_+(y_l)$ and hence $\lambda_l \neq 0$. We have the lower estimate

$$\begin{aligned} IS_{(P)} = (\mathcal{Q}(\sigma))_{(P)} &= Q\left(\frac{\sigma}{y_l}\right) S_{(P)} \\ &\subseteq Q\left(\sum_i \left(\frac{\lambda_i}{\lambda_l} - \frac{y_i}{y_l}\right) g_i\right) S_{(P)} + Q\left(\sum_i \left(\frac{\lambda_i}{\lambda_l} g_i\right)\right) S_{(P)} \\ &\subseteq IP_{(P)} + Q\left(\sum_i \lambda_i g_i\right) S_{(P)} \subseteq IS_{(P)}. \end{aligned}$$

Therefore, $IP_{(P)} + Q\big(\sum_i \lambda_i g_i\big) S_{(P)} = IS_{(P)}$. By Nakayama's lemma, considering $S_{(P)}$-modules $IS_{(P)}$ and $Q(\sum_i \lambda_i g_i) S_{(P)}$, we have $Q\big(\sum_i \lambda_i g_i\big) S_{(P)} = IS_{(P)}$. By Lemma 4.75, we have $I = Q(\sum_i \lambda_i g_i)$. Theorem 4.76 is proved.

Remark 4.77. Essentially, the proof shows that $I = Q\big(\sum_i \lambda_i g_i\big)$ if and only if $(\mathcal{Q}(\sigma) : IS) \not\subset \mathfrak{m}S + p_\lambda S$. And the sufficiency part shows that if $\Delta(I)$ is T-dependent and I is T-full, then such p_λ exists.

Remark 4.78. The theorem can be applied to all of the examples in Example 4.5.

Although the theorem provides a criterion for local Noetherian F-algebra with infinite residue field F, the behaviour for finite F is note quite clear. We make the following conjecture.

Conjecture 4.79. Let F be an arbitrary field, let A be a local Noetherian F-algebra with maximal ideal $\mathfrak{m}$, and let Q be a fixed T-map. Suppose $A/\mathfrak{m} \simeq F$ Then, for an ideal $I \triangleleft A$, I is T-principal if and only if I is T-full and $\Delta(I)$ is T-dependent.

Rodrigues [19] proved the following result.

Corollary 4.80 (see [19], Corollary 3.13). *Suppose $0 \neq I \triangleleft \mathbb{C}\{\boldsymbol{x}\}$ is a Tjurina ideal and $\Delta(I) = (g_1, \dots, g_r)$. Then $I = T\big(\sum_k \lambda_k g_k\big)$ for $[\lambda_1, \dots, \lambda_r]$ in a non-empty open set of $\mathbb{P}^{r-1}_{\mathbb{C}}$.*

The following lemma gives a detailed description of such an open set.

Lemma 4.81. *Under the same notation as in Theorem 4.76, suppose $0 \neq I = Q(f) \triangleleft A$ is a T-principal ideal. Let $J = (\mathcal{Q}(\sigma) : IS)$, where $\Delta(I) = (g_1, \dots, g_r)$, $S = A[y_1, \dots, y_r]$, and $\sigma = \sum g_i y_i$. $\pi\colon A[y_1, \dots, y_r] \twoheadrightarrow F[y_1, \dots, y_r]$ is the projection of coefficients. Then the set $U = \{\lambda \in \mathbb{P}^{r-1}_F \mid I = Q\big(\sum_k \lambda_k g_k\big)\}$ coincides with the open set $Z(\pi(J))^c$, where $Z(\pi(J))$ refers to the common zero locus of polynomials in $\pi(J)$.*

Proof. Proceeding as in the proof of Theorem 4.76, we have $U = \{\lambda \mid J \not\subset \mathfrak{m}S + p_\lambda S\} = \{\lambda \mid \pi(J) \not\subset p_\lambda\}$. By Lemma 4.71, $U = \{\lambda \mid \lambda \notin Z(\pi(J))\} = Z(\pi(J))^c$, which proves the lemma.

The above lemma also provides an algorithm to compute a generator for a T-principal ideal I. For $Q = T_k$ or T^k, we can compute the ideal of antiderivatives explicitly. We write the algorithm as following.

Algorithm 4.82. The notation is as in Lemma 4.81. By the following steps, one can check that an ideal is T-principal with respect to Q and obtain a generator if the ideal is T-principal.
Step 1. Compute a set of generators $g_1, \dots, g_r$ of $\Delta(I)$.
Step 2. Check if $T(\Delta(I)) = I$.
Step 3. If $T(\Delta(I)) \neq I$, return false; otherwise, compute the colon ideal $J = (\mathcal{Q}(\sigma) : IS)$.
Step 4. If $J \subseteq \mathfrak{m}S$, return false; otherwise, find a $\lambda \in F^r$ such that $\lambda \in Z(\pi(J))^c$, then $\sum_i \lambda_i g_i$ is a generator.

§ 5. Appendix: codes

Code 5.1. Computing $\Delta(I)$, ρ_k and σ_k. (SINGULAR)

```
LIB ''hnoether.lib'';
ring r = 0, (x,y),ds;
int k = 20;
poly f = x^4*y+x*y^5;
def J = jacob(f);
ideal m = x,y;
```

```
ideal Tt = f,m^k*J;
ideal Tk = std(Tt);
int u = size(Tk);
matrix B[1][u] = Tk;
matrix C1[k+1][u];
matrix C2[k+1][u];
matrix temp[1][u];
int i;
int j;
for( i = 1 ; i <= u ; i ++){
    temp = jacob(B[1,i]);
        for( j = 0 ; j <= k ; j ++){
       C1[j+1,i] = x^j*y^(k-j)*temp[1,1];
       C2[j+1,i] = x^j*y^(k-j)*temp[1,2];
    }
}
matrix ttp[1][u];
def res = modulo(ttp,ttp);
for(j = 1 ; j <= k+1 ; j ++){
        for(i = 1 ; i <= u ; i ++){
       ttp[1,i] = C1[j,i];
    }
    def rec = modulo(ttp,B);
    def recc = intersect(res, rec);
    res = recc;
}

for(j = 1 ; j <= k+1 ; j ++){
        for(i = 1 ; i <= u ; i ++){
       ttp[1,i] = C2[j,i];
    }
    def rec = modulo(ttp,B);
    def recc = intersect(res, rec);
    res = recc;
}
matrix Res = res;
int uu = size(res);
matrix D[1][uu];
for(i = 1 ; i <= uu ; i ++){
        for(j = 1 ; j <= u ; j ++){
       D[1,i] = D[1,i]+Res[j,i]*B[1,j];
    }
}
ideal Delta = std(D);
ideal I = std(Tk^2);
```

```
ideal D1 = groebner(Delta);
int sigma = vdim(I)-vdim(D1);
sigma;
int rho = -vdim(Tk)+vdim(D1);
rho;
```

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