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Three types of derivation lie algebras of isolated hypersurface singularities

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Abstract

In our previous work, we introduced three different ways to associate Lie algebras to isolated hypersurface singularities. In this paper, we analyze their relations in the case of weighted homogeneous singularities. Moreover, explicit formulas of the dimensions of three series of Lie algebras are given for fewnomial singularities. Several conjectures are proposed and verified partially.

Keywords Isolated hypersurface singularity · Lie algebra · Moduli algebra

Mathematics Subject Classification 14B05 · 32S05

1 Introduction

Many highly non-trivial physical questions such as the Coulomb branch spectrum and the Seiberg–Witten solution [11, 12, 29, 35, 36] can be easily found by studying the miniversal deformation of the singularity. The second and third authors classify three dimensional isolated weighted homogeneous rational complete intersection singularities, which define many new four dimensional N = 2 superconformal field theories [10]. In this article, we will study new invariants introduced in our previous works [19]-[28]. These new invariants are very useful in the classification theory of singularities. For any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^n, 0)$ where $V = V(f) = \{f = 0\}$, in the early 80s, the second author introduced the Lie algebra of derivations of moduli algebra $A(V) := \mathcal{O}_n/(f, J(f))$

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where $J(f) := (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$, i.e., $L(V) := \mathrm{Der}(A(V), A(V))$. It is known that $L(V)$ is a finite dimensional solvable Lie algebra [40, 42]. $L(V)$ is called the Yau algebra of V (its dimension $\lambda(V)$ is called Yau number) in [43] and [15] in order to distinguish from Lie algebras of other types appearing in singularity theory [30]. Yau and his collaborators have been systematically studying the Lie algebras of isolated hypersurface singularities since early eighties (see, e.g., [40]-[42], [3, 8, 32, 37, 44, 45]).

In the last thirty years, the Lie algebra of derivations have become a very important tool in singularity theory and Lie theory ([6–9, 13, 15, 37, 39]). In the theory of isolated singularities, one always wants to find invariants associated to isolated singularities. Hopefully with enough invariants found, one can distinguish between isolated singularities. However, not many invariants are known. In recent years, we have introduced many derivations Lie algebras which are new invariants of isolated singularities. Three different ways to associate Lie algebras to isolated hypersurface singularities were introduced in [9, 20, 21, 26, 33].

Firstly, a new series of derivation Lie algebras $L_k(V), 0 \leq k \leq n$ associated to the isolated hypersurface singularity $(V, 0)$ defined by the holomorphic function $f(x_1, \cdots, x_n)$ was introduced in [26]. Let $Hess(f)$ be the Hessian matrix (f_{ij}) of the second order partial derivatives of f and $h(f)$ be the determinant of the matrix $Hess(f)$. More generally, for each k satisfying $0 \leq k \leq n$, we denote by $h_k(f)$ the ideal in $\mathcal{O}_n$ generated by all $k \times k$-minors in the matrix $Hess(f)$. In particular, $h_0(f) = 0$, the ideal $h_n(f) = (h(f))$ is a principal ideal. For each k as above, the graded k-th Hessian algebra of the polynomial f is defined by

$$H_k(V) = \mathcal{O}_n/(f + J(f) + h_k(f)).$$

The dimension of $H_k(V)$ as a $\mathbb{C}$-vector space is denoted as $h_k(V)$.

It is known that the isomorphism class of the local k-th Hessian algebra $H_k(f)$ is contact invariant of f, i.e. depends only on the isomorphism class of the germ $(V, 0)$ ([14], Lemma 2.1). In [26], we investigated the new Lie algebra $L_k(V)$ which is the Lie algebra of derivations of k-th Hessian algebra $H_k(f)$. The dimension of $L_k(V)$, denoted by $\lambda_k(V)$, is a new numerical analytic invariant of an isolated hypersurface singularity.

In particular, when $k = 0$, those are exactly the previous Yau algebra and Yau number, i.e., $L_0(V) = L(V), \lambda_0(V) = \lambda(V)$. Thus, the $L_k(V)$ is a generalization of Yau algebra $L(V)$. Moreover, $L_n(V)$ has been investigated intensively and many interesting results were obtained. In [9], it was shown that $L_n(V)$ completely distinguish ADE singularities. Furthermore, the authors have proven Torelli-type theorems for some simple elliptic singularities. Therefore, this new Lie algebra $L_n(V)$ is a subtle invariant of isolated hypersurface singularities. It is a natural question whether we can distinguish singularities by only using part of the information of $L_n(V)$. In [22], we studied generalized Cartan matrices of the new Lie algebra $L_n(V)$ for simple hypersurface singularities and simple elliptic singularities. We introduced many other numerical invariants, namely, the dimension of the a maximal nilpotent subalgebras (i.e., nilradical of nilpotent Lie algebra) $g(V)$ of $L_n(V)$; dimension of maximal torus of $g(V)$, etc. We have proven that the generalized Cartan matrix of $L_n(V)$ can be used to characterize the ADE singularities except the pair of A_6 and D_5 singularities [22].

Secondly, let $(V, 0)$ be an isolated hypersurface singularity defined by a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The multiplicity $mult(f)$ of the singularity $(V, 0)$ is defined to be the order of the lowest nonvanishing term in the power series expansion of f at 0.

Definition 1.1 Let $(V, 0) = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \quad f(x_1, \ldots, x_n) = 0\}$ be an isolated hypersurface singularity with $mult(f) = m$. Let $J_k(f)$ be the ideal generated by all the k-th order partial derivative of f, i.e., $J_k(f) = < \frac{\partial^k f}{\partial z_{i_1} \ldots \partial z_{i_k}} \mid 1 \leq i_1, \ldots, i_k \leq n >$. For $1 \leq k \leq m$, we define the new k-th local algebra, $M_k(V) := \mathcal{O}_n/(f + J_1(f) + \cdots + J_k(f))$. In particular, $M_m = 0$. The dimension of $M_k(V)$ as a $\mathbb{C}$-vector space is denoted as $d_k(V)$. In particular $d_m(V) = 0$.

Recall that a polynomial $f \in \mathbb{C}[x_1, \cdots, x_n]$ is said to be the weighted homogeneous if there exist positive rational numbers $w_1, \cdots, w_n$ (weights of $x_1, \cdots, x_n$) and d such that, $\sum a_i w_i = d$ for each monomial $\prod x_i^{a_i}$ appearing in f with non-zero coefficient. The number d is called weighted homogeneous degree (w-degree) of f with respect to weights w_j. The weight type of f is denoted as $(w_1, \cdots, w_n; d)$. Without loss of generality, we can assume that w-deg$f = 1$. An isolated hypersurface singularity $(V, 0)$ is called weighted homogeneous if it is defined by a weighted homogeneous polynomial f.

Remark 1.1 If f defines a weighted homogeneous isolated singularity at the origin, then $f \in J_1(f) \subset J_2(f) \subset \cdots \subset J_k(f)$, thus $M_k(V) = \mathcal{O}_{n+1}/(f + J_1(f) + \cdots + J_k(f)) = \mathcal{O}_{n+1}/(J_k(f))$.

The isomorphism class of the k-th local algebra $M_k(V)$ is a contact invariant of $(V, 0)$, i.e. depends only on the isomorphism class of the germ $(V, 0)$. The dimension of $M_k(V)$ is denoted by $d_k(V)$ which is a numerical analytic invariant of an isolated hypersurface singularity.

Theorem 1.1 [33] *Suppose* $(V, 0) = \{(x_1, \cdots, x_n) \in \mathbb{C}^n \; f(x_1, \cdots, x_n) = 0\}$ *and* $(W, 0) = \{(x_1, \cdots, x_n) \in \mathbb{C}^n \; g(x_1, \cdots, x_n) = 0\}$ *are isolated hypersurface singularities. If* $(V, 0)$ *is biholomorphically equivalent to* $(W, 0)$*, then* $M_k(V)$ *is isomorphic to* $M_k(W)$ *as a* $\mathbb{C}$*-algebra for all* $1 \leq k \leq m$*, where* $m = mult(f) = mult(g)$.

Based on Theorem 1.1, it is natural for us to introduce the new series of k-th derivation Lie algebras $\mathcal{L}_k(V)$ (or $\mathcal{L}_k((V, 0))$) which are defined to be the Lie algebra of derivations of the k-th local algebra $M_k(V)$, i.e., $\mathcal{L}_k(V) = Der(M_k(V), M_k(V))$. Its dimension is denoted as $\delta_k(V)$ (or $\delta_k((V, 0))$). This number $\delta_k(V)$ is also a new numerical analytic invariant.

Finally, recall that the Mather-Yau theorem was slightly generalized.

Theorem 1.2 ([17], Theorem 2.26) *Let* $f, g \in \mathfrak{m} \subset \mathcal{O}_n$*. The following are equivalent:*
1) $(V(f), 0) \cong (V(g), 0)$*;*
2) *For all* $k \geq 0$*,* $\mathcal{O}_n/(f, \mathfrak{m}^k J(f)) \cong \mathcal{O}_n/(g, \mathfrak{m}^k J(g))$ *as* $\mathbb{C}$*-algebra;*
3) *There is some* $k \geq 0$ *such that* $\mathcal{O}_n/(f, \mathfrak{m}^k J(f)) \cong \mathcal{O}_n/(g, \mathfrak{m}^k J(g))$ *as* $\mathbb{C}$*-algebra, where* $J(f) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$.

In particular, if $k = 0$ and $k = 1$ above, then the claim of the equivalence of 1) and 3) is exactly the Mather–Yau theorem [31].

Motivated from Theorem 1.2, in [20, 21], we introduced the new series of k-th Yau algebras $L^k(V)$ (or $L^k((V, 0))$) which are defined to be the Lie algebra of derivations of the moduli algebra $T^k(V) = \mathcal{O}_n/(f, \mathfrak{m}^k J(f))$, $k \geq 0$, where $\mathfrak{m}$ is the maximal ideal, i.e., $L^k(V) := \mathrm{Der}(T^k(V), T^k(V))$. Its dimension is denoted as $\lambda^k(V)$ (or $\lambda^k((V, 0))$). This series of integers $\lambda^k(V)$ are new numerical analytic invariants of singularities. It is natural to call it k-th Yau number. In particular, when $k = 0$, those are exactly the previous Yau algebra and Yau number, i.e., $L^0(V) = L(V)$, $\lambda^0(V) = \lambda(V)$. In [40], Yau observed that the Yau algebra for the one-parameter family of simple elliptic singularities $\tilde{E}_6$ is constant. It turns out that the 1-st Yau algebra $L^1(V)$ is also constant for the family of simple elliptic singularities $\tilde{E}_6$. However, Torelli-type theorem for $L^k(V)$ for all $k > 1$ do hold on $\tilde{E}_6$ ([18]). In general, the invariant $L^k(V)$, $k \geq 1$ are more subtle than the Yau algebra $L(V)$. In a word, we have reasons to believe that these three series of new Lie algebras and their numerical invariants will also play an important role in the study of singularities.

Two naturally interesting questions are: what are the relations between these invariants of an isolated hypersurface singularities $(V, 0)$ above? Whether one can give a sharp bound for these invariants? We proposed the following conjectures.

Conjecture 1.1 Recall that $d_k(V)$ is the dimension of the Artinian algebra $M_k(V)$. For each $k \geq 1$, assume that $d_k(\{x_1^{a_1} + \cdots + x_n^{a_n} = 0\}) = \ell_k(a_1, \cdots, a_n)$. Let $(V, 0) = \{(x_1, x_2, \cdots, x_n) \in \mathbb{C}^n : f(x_1, x_2, \cdots, x_n) = 0\}, (n \geq 2)$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \cdots, x_n)$ of weight type $(w_1, w_2, \cdots, w_n; 1)$. Then $d_k(V) \leq \ell_k(1/w_1, \cdots, 1/w_n)$.

Proposition 1.1 *Let $(V, 0)$ be an isolated hypersurface singularity defined by $f \in \mathcal{O}_n,\ n \geq 2,\ m = mult(f)$. And let $d_k(V), h_k(V), \tau^k(V)$ be the dimension of the Artinian algebras $M_k(V), H_k(V), T^k(V)$ respectively. For each $k \geq 0$, then $\cdots > \tau^{(k+1)}(V) > \tau^k(V) \cdots > \tau^0(V) = d_1(V) = h_0(V) = h_n(V) - 1 > h_{n-1}(V) > \cdots > h_1(V) = d_2(V) > d_3(V) > \cdots > d_m(V)$.*

Remark 1.2 The Proposition 1.1 follows easily from the definitions of $\tau^k(V)$, $d_k(V)$ and $h_k(V)$. However, the following Conjecture 1.2 is highly nontrivial.

Conjecture 1.2 With the above notations, let $(V, 0)$ be an isolated hypersurface singularity defined by $f \in \mathcal{O}_n,\ n \geq 2$. For each $k \geq 0$, then $\cdots > \lambda^{(k+1)}(V) > \lambda^k(V) \cdots > \lambda^0(V) = \delta_1(V) = \lambda_0(V) = \lambda_n(V) > \lambda_{n-1}(V) > \cdots > \lambda_1(V) = \delta_2(V) > \delta_3(V) > \cdots > \delta_{m-1}(V)$, where $m = mult(f)$.

The k-th Milnor number and k-th Tjurina number are defined as follows.

$$\mu^k := \dim \mathcal{O}_n/(\mathfrak{m}^k J(f)),\ \ \tau^k := \dim \mathcal{O}_n/(f, \mathfrak{m}^k J(f)).$$

It is obvious that $\mu^k \geq \tau^k$, thus $\frac{\mu^k}{\tau^k} \geq 1$. In particular, when $k = 0$, μ^0 and τ^0 are the classical Milnor number μ and Tjurina number τ respectively. Moreover, Saito showed [34] that $\frac{\mu}{\tau} = 1$, if and only if, that f defines a weighted homogeneous isolated singularity. However, even for weighted homogeneous isolated singularity, $\frac{\mu^k}{\tau^k} \neq 1, k \geq 2$. We give the following interesting example.

Example 1.1 Let $(V, 0)$ be an isolated hypersurface singularity defined by

$$f = x_1^m + x_2^m + x_3^m + x_1^{m+1}x_2^{m+1}x_3^{m+1}, m \geq 2.$$

Then

$$\begin{aligned}
\tau^0(V) &= \mu^0(V) = m^3 - 3m^2 + 3m - 1,\\
\tau^1(V) &= \mu^1(V) = m^3 - 3m^2 + 3m + 2,\\
\tau^2(V) &= \begin{cases} m^3 - 3m^2 + 3m + 10; & m \geq 3\\ 9; & m = 2,\end{cases}\\
\mu^2(V) &= \begin{cases} m^3 - 3m^2 + 3m + 12; & m \geq 3\\ 10; & m = 2.\end{cases}
\end{aligned}$$

It is interesting to note that $\mu^k = \tau^k$ for $k = 0, 1$, however, $\mu^2 \neq \tau^2$.

The following new result gives an upper bound of $\frac{\mu^k}{\tau^k}$.

Theorem 1.3 *Assume that* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *is a holomorphic function germ at the origin with only isolated singularity. Then, for each* $k \geq 0$*, we have*

$$\tau^k \leq \mu^k \leq n\tau^k + \binom{n+k-1}{k-1}.$$

Proof We only need to show that $\mu^k \leq n\tau^k + \binom{n+k-1}{k-1}$. We consider the following long exact sequence of $\mathbb{C}$-algebras:

$$0 \to Ker(f) \to M_f^k \xrightarrow{f} M_f^k \to T_f^k \to 0,$$

where $M_f^k = \mathcal{O}_n/\mathfrak{m}^k J(f)$, $T_f^k = \mathcal{O}_n/(f, \mathfrak{m}^k J(f))$, the middle map is multiplication by f, and Ker(f) is the kernel of this map.

Recall a well-known result given by Briançon and Skoda in [5] that $f^n \in J(f)$, so $(f^n)\mathfrak{m}^k = 0$ in M_f^k, i.e., $(f^{n-1})\mathfrak{m}^k \subset Ker(f)$. Thus we have the following finite decreasing filtration:

$$M_f^k \supset (f) \supset (f)\mathfrak{m}^k \supset (f^2)\mathfrak{m}^k \supset \cdots \supset (f^{n-1})\mathfrak{m}^k \supset (f^n)\mathfrak{m}^k = 0,$$

where (f^i) is the ideal in M_f^k generated by f^i.

Consider the following long exact sequence:

$$0 \to Ker(f) \cap (f^i)\mathfrak{m}^k \to (f^i)\mathfrak{m}^k \xrightarrow{f} (f^i)\mathfrak{m}^k \to (f^i)\mathfrak{m}^k/(f^{i+1})\mathfrak{m}^k \to 0,$$

where the middle map is multiplication by f. Then,

$$dim_{\mathbb{C}}\{(f^i)\mathfrak{m}^k/(f^{i+1})\mathfrak{m}^k\} = dim_{\mathbb{C}}\{Ker(f) \cap (f^i)\mathfrak{m}^k\} \leq dim_{\mathbb{C}} Ker(f) = \tau^k.$$

Therefore,

$$\mu^k = dim_{\mathbb{C}} M_f^k = dim_{\mathbb{C}} T_f^k + dim_{\mathbb{C}}\{(f)/(f)\mathfrak{m}^k\} + \sum_{i=1}^{n-1} dim_{\mathbb{C}}\{(f^i)\mathfrak{m}^k/(f^{i+1})\mathfrak{m}^k\}$$

$$\leq n\tau^k + dim_{\mathbb{C}}\{\mathcal{O}_n/\mathfrak{m}^k\} = n\tau^k + \binom{n+k-1}{k-1}.$$

□

Remark 1.3 Moreover, in Theorem 1.3, if $(V(f), 0)$ is a weighted homogeneous singularity, then

$$\tau^k \leq \mu^k \leq n\tau^k + \binom{n+k-2}{k-2}.$$

The proof is similar as the proof of Theorem 1.3, the only step which we need to improve is $dim_{\mathbb{C}}\{(f)/(f)\mathfrak{m}^k\} = dim_{\mathbb{C}}\{f \cdot \mathcal{O}_n/(f \cdot \mathfrak{m}^k, \mathfrak{m}^k J(f))\} \leq dim_{\mathbb{C}}\{\mathcal{O}_n/\mathfrak{m}^{k-1}\} = \binom{n+k-2}{k-2}$ which follows from the Euler equality for weighted homogeneous polynomial.

Theorem 1.3 tells us that, fixing n, k, the $\frac{\mu^k}{\tau^k}$ is sufficiently close to the n if τ^k is sufficiently large. It seems that the upper bound n can never be achieved. We don't have any example whose $\frac{\mu^k}{\tau^k}$ is sufficiently close to n. It is natural to ask what is the optimal upper bound of $\frac{\mu^k}{\tau^k}$. We propose the following two conjectures.

Conjecture 1.3 Let $(V, 0)$ be an isolated hypersurface singularity defined by $f(x_1, x_2)$. Then for each $k \geq 0$,

$$\frac{\mu^k}{\tau^k} < \frac{4}{3}.$$

We believe that the $\frac{4}{3}$ in Conjecture 1.3 is optimal because of the following example: let $(V, 0)$ be an isolated hypersurface singularity defined by

$$f = x_1^{2m+1} + x_2^{2m+1} + x_1^{m+1}x_2^{m+1}, m \geq 1,$$

then we have

$$\begin{aligned}
&\mu^0(V) = 4m^2, \ \mu^1(V) = 4m^2 + 2, \\
&\tau^0(V) = 4m^2 - (m - 1^2), \ \tau^1(V) = 3m^2 + 2m + 1, \\
&\mu^2(V) = 4m^2 + 6, \\
&\tau^2(V) = \begin{cases} 3m^2 + 2m + 5; & m \geq 2 \\ 9; & m = 1. \end{cases}
\end{aligned}$$

Thus $\frac{\mu^1}{\tau^1}$ and $\frac{\mu^2}{\tau^2}$ are sufficiently close to $\frac{4}{3}$ when m is sufficiently large.

Conjecture 1.4 Let $(V, 0)$ be a isolated hypersurface singularity defined by $f(x_1, x_2, x_3)$. Then for each $k \geq 0$,

$$\frac{\mu^k}{\tau^k} < \frac{3}{2}.$$

We believe that the $\frac{3}{2}$ in Conjecture 1.4 is optimal. The reason is that, when $k = 0$, there are examples whose $\frac{\mu^0}{\tau^0}$ is sufficiently close to $\frac{3}{2}$ (see Example 3, [2]). When $k > 0$, we have the following two examples:

1) Let $(V, 0)$ be an isolated hypersurface singularity defined by

$$f = x_1^{3m+1} + x_2^{3m+1} + x_3^{3m+1} + x_1^{m+1}x_2^{m+1}x_3^{m+1}, m \geq 1,$$

then we have

$$\begin{aligned}
&\mu^0(V) = 27m^3, \\
&\tau^0(V) = \begin{cases} 19m^3 + 9m^2 + 6m - 11; & m \geq 2 \\ 26; & m = 1, \end{cases} \\
&\mu^1(V) = 27m^3 + 3, \\
&\tau^1(V) = \begin{cases} 19m^3 + 9m^2 + 6m - 8; & m \geq 2 \\ 29; & m = 1, \end{cases} \\
&\mu^2(V) = 27m^3 + 13, \\
&\tau^2(V) = \begin{cases} 19m^3 + 9m^2 + 6m + 2; & m \geq 2 \\ 38; & m = 1. \end{cases}
\end{aligned}$$

2) Let $(V, 0)$ be an isolated hypersurface singularity defined by

$$f = x_1^{3m-1} + x_2^{3m-1} + x_3^{3m-1} + x_1^m x_2^m x_3^m, m \geq 1,$$

then we have

$$\mu^0(V) = 27m^3 - 54m^2 + 36m - 8,$$
$$\tau^0(V) = \begin{cases} 19m^3 - 33m^2 + 24m - 12; & m \geq 2 \\ 1; & m = 1, \end{cases}$$
$$\mu^1(V) = 27m^3 - 54m^2 + 36m - 5$$
$$\tau^1(V) = \begin{cases} 19m^3 - 33m^2 + 24m - 9; & m \geq 2 \\ 4; & m = 1, \end{cases}$$
$$\mu^2(V) = \begin{cases} 27m^3 - 54m^2 + 36m + 5; & m \geq 2 \\ 10; & m = 1. \end{cases}$$
$$\tau^2(V) = \begin{cases} 19m^3 - 33m^2 + 24m + 1; & m \geq 2 \\ 9; & m = 1. \end{cases}$$

In the above two examples, $\frac{\mu^k}{\tau^k}$, $k = 0, 1, 2$ are sufficiently close to $\frac{27}{19} \approx 1.42 < \frac{3}{2}$ when m is sufficiently large. We currently don't have an example such that $\frac{\mu^k}{\tau^k}$, $k \geq 1$ sufficiently close to $\frac{3}{2}$. It seems very hard to find such examples.

Remark 1.4 When $k = 0$, the conjecture 1.3 and conjecture 1.4 are verified in some cases in [1, 38] and [2] respectively.

It is also interesting to give an optimal upper bound for the invariants $\delta_k(V)$.

Conjecture 1.5 Let $(V, 0) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n : f(x_1, x_2, \ldots, x_n) = 0\}$ $(n \geq 2)$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \ldots, x_n)$ of weight type $(w_1, w_2, \cdots, w_n; 1)$. Then the generalized k-th Yau number

$$\delta_k(V) \leq n \prod_{i=1}^{n} \left(\frac{1}{w_i} - k\right) - \sum_{i}^{n} \left(\frac{1}{w_1} - k\right)\left(\frac{1}{w_2} - k\right) \cdots \left(\widehat{\frac{1}{w_i} - k}\right) \cdots \left(\frac{1}{w_n} - k\right).$$

The main purpose of this paper is to verify the Conjectures 1.1–1.5 for binomial and trinomial singularities. We obtain the following results.

Theorem A *Let $(V, 0)$ be a weighted homogeneous fewnomial isolated singularity of type A which is defined by $f = x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$ $(a_i \geq k + 2,\ 1 \leq i \leq n)$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_n}; 1)$. Then generalized k-th Yau number*

$$\delta_k(V) = n \prod_{i=1}^{n} (a_i - k) - \sum_{i}^{n} (a_1 - k)(a_2 - k) \cdots (\widehat{a_i - k}) \cdots (a_n - k),$$

where $(\widehat{a_i - k})$ means that $a_i - k$ is omitted and $1 \leq k \leq m - 1$ and $m = mult(f)$.

Theorem B *Let $(V_f, 0) \subset (\mathbb{C}^n, 0)$ and $(V_g, 0) \subset (\mathbb{C}^m, 0)$ be defined by weighted homogeneous polynomials $f(x_1, x_2, \cdots, x_n) = 0$ of weight type $(w_1, w_2 \cdots, w_n; 1)$ and $g(y_1, y_2, \cdots, y_m) = 0$ of weight type $(w_{n+1}, w_{n+2}, \cdots, w_{n+m}; 1)$ respectively. Let $d_k(V_f)$ and $d_k(V_g)$ be the dimensions of moduli algebras $M_k(V_f)$ and $M_k(V_g)$ of $(V_f, 0)$ and $(V_g, 0)$ respectively. Then*

$$\delta_k(V_{f+g}) = d_k(V_f)\delta_k(V_g) + d_k(V_g)\delta_k(V_f). \tag{1.1}$$

and furthermore if both f and g satisfy the conjecture 1.5, then $f + g$ also satisfies the conjecture.

Theorem C *Let* $(V, 0)$ *be a binomial singularity defined by the weighted homogeneous polynomial* $f(x_1, x_2)$ *(see Corollary 2.1) with weight type* $(w_1, w_2; 1)$. *Then*

$$d_k(V) \leq \ell_k\left(\frac{1}{w_1}, \frac{1}{w_2}\right), \ k = 2, 3, 4.$$

Theorem D *Let* $(V, 0)$ *be a trinomial singularity defined by the weighted homogeneous polynomial* $f(x_1, x_2, x_3)$ *(see Proposition 2.2) with weight type* $(w_1, w_2, w_3; 1)$. *Then*

$$d_k(V) \leq \ell_k\left(\frac{1}{w_1}, \frac{1}{w_2}, \frac{1}{w_3}\right), \ k = 2, 3, 4.$$

Theorem E *Let* $(V, 0)$ *be a binomial singularity defined by the weighted homogeneous polynomial* $f(x_1, x_2)$ *(see Corollary 2.1). Then*

$$\frac{\mu^2}{\tau^2} < \frac{4}{3}.$$

Theorem F *Let* $(V, 0)$ *be a trinomial singularity defined by the weighted homogeneous polynomial* $f(x_1, x_2, x_3)$ *(see Proposition 2.2). Then*

$$\frac{\mu^2}{\tau^2} < \frac{3}{2}.$$

Remark 1.5 In fact, it is easy to see $\frac{\mu^0}{\tau^0} = \frac{\mu^1}{\tau^1} = 1$ for weighted homogeneous singularities.

Theorem G *Let* $(V, 0)$ *be a binomial singularity defined by the weighted homogeneous polynomial* $f(x_1, x_2)$ *(see Corollary 2.1). Then*

$\lambda^2(V) > \lambda^1(V) > \lambda^0(V) = \delta_1(V) = \lambda_0(V) = \lambda_2(V) > \lambda_1(V) = \delta_2(V) > \delta_3(V) > \delta_4(V)$.

Theorem H *Let* $(V, 0)$ *be a trinomial singularity defined by the weighted homogeneous polynomial* $f(x_1, x_2, x_3)$ *(see Proposition 2.2). Then*
$\lambda^2(V) > \lambda^1(V) > \lambda^0(V) = \delta_1(V) = \lambda_0(V) = \lambda_3(V) > \lambda_1(V) = \delta_2(V) > \delta_3(V) > \delta_4(V)$.

2 Generalities on derivation Lie algebras of isolated singularities

In this section, we shall briefly define the basic definitions and important results which are helpful to solve the problem. The following basic concepts and results will be used to compute the derivation Lie algebras of isolated hypersurface singularities. Let A, B be associative algebras over $\mathbb{C}$. The subalgebra of endomorphisms of A generated by the identity element and left and right multiplications by elements of A is called multiplication algebra $M(A)$ of A. The centroid $C(A)$ is defined as the set of endomorphisms of A which commute with all elements of $M(A)$. Obviously, $C(A)$ is an unital subalgebra of $\mathrm{End}(A)$. The following statement is a particular case of a general result from Proposition 1.2 of [4]. Let $S = A \otimes B$ be a tensor product of finite dimensional associative algebras with units. Then

$$\mathrm{Der}S \cong (\mathrm{Der}A) \otimes C(B) + C(A) \otimes (\mathrm{Der}B).$$

We will only use this result for commutative associative algebras with unit, in which case the centroid coincides with the algebra itself and one has the following result for commutative associative algebras A, B:

Theorem 2.1 ([4]) *For commutative associative algebras* A, B,

$$DerS \cong (DerA) \otimes B + A \otimes (DerB). \tag{2.1}$$

We shall use this formula in the sequel.

Definition 2.1 Let J be an ideal in an analytic algebra S. Then $\mathrm{Der}_J S \subseteq \mathrm{Der}_{\mathbb{C}} S$ is Lie subalgebra of all $\sigma \in \mathrm{Der}_{\mathbb{C}} S$ for which $\sigma(J) \subset J$.

We shall use the following well-known result to compute the derivations.

Theorem 2.2 ([45]) *Let* J *be an ideal in* $R = \mathbb{C}\{x_1, \cdots, x_n\}$. *Then there is a natural isomorphism of Lie algebras*

$$(Der_J R)/(J \cdot Der_{\mathbb{C}} R) \cong Der_{\mathbb{C}}(R/J).$$

Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$. Thus for such an algebra A one can consider the Lie algebra of its derivations $Der(A, A)$ with the bracket defined by the commutator of linear endomorphisms.

Definition 2.2 Let $(V, 0)$ be an isolated hypersurface singularity. The new series of k-th derivation Lie algebras $\mathcal{L}_k(V)$ (or $\mathcal{L}_k((V, 0))$) which are defined to be the Lie algebra of derivations of the k-th local algebra $M_k(V)$, i.e.,$\mathcal{L}_k(V) = Der(M_k(V), M_k(V))$. Its dimension is denoted as $\delta_k(V)$ (or $\delta_k((V, 0))$). This number $\delta_k(V)$ is also a new numerical analytic invariant.

Definition 2.3 A polynomial $f \in \mathbb{C}[x_1, x_2, \cdots, x_n]$ is called quasi-homogeneous (or weighted homogeneous) if there exist positive rational numbers $w_1, \ldots, w_n$ (called weights of indeterminates x_j) and d such that, for each monomial $\prod x_j^{k_j}$ appearing in f with non-zero coefficient, one has $\sum w_j k_j = d$. The number d is called the quasi-homogeneous degree (w-degree) of f with respect to weights w_j and is denoted $\deg f$. The collection $(w; d) = (w_1, \cdots, w_n; d)$ is called the quasi-homogeneity type (qh-type) of f.

Definition 2.4 An isolated hypersurface singularity in $\mathbb{C}^n$ is fewnomial if it can be defined by a n-nomial in n variables and it is a weighted homogeneous fewnomial isolated singularity if it can be defined by a weighted homogeneous fewnomial. 3-nomial isolated hypersurface singularity is also called trinomial singularity.

Proposition 2.1 *Let* f *be a weighted homogeneous fewnomial isolated singularity with* $mult(f) \geq 3$. *Then* f *analytically equivalent to a linear combination of the following three series:*

Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$, $n \geq 1$,
Type B. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$, $n \geq 2$,
Type C. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1$, $n \geq 2$.

Proposition 2.1 has an immediate corollary.

Corollary 2.1 *Each binomial isolated singularity is analytically equivalent to one from the three series: A)* $x_1^{a_1} + x_2^{a_2}$, *B)* $x_1^{a_1}x_2 + x_2^{a_2}$, *C)* $x_1^{a_1}x_2 + x_2^{a_2}x_1$.

Wolfgang and Atsushi [16] give the following classification of weighted homogeneous fewnomial singularities in case of the three variables.

Proposition 2.2 *Let $f(x_1, x_2, x_3)$ be a weighted homogeneous fewnomial isolated singularity with mult$(f) \geq 3$. Then f is analytically equivalent to the following five types:*

Type 1. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$,
Type 2. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$,
Type 3. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$,
Type 4. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}x_1$,
Type 5. $x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$.

3 Proof of theorems

To prove the main theorems we need following propositions. The detailed proofs can be found in our previous papers.

Proposition 3.1 ([19, 21, 26–28, 45]) *Let $(V, 0)$ be a weighted homogeneous fewnomial isolated singularity of type A which is defined by $f = x_1^{a_1} + x_2^{a_2}$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_n}; 1)$. Then*

$$\tau(V) = (a_1 - 1)(a_2 - 1); a_1 \geq 2,\ a_2 \geq 2.$$
$$\tau^1(V) = a_1a_2 - (a_1 + a_2) + 3; a_1 \geq 2,\ a_2 \geq 2.$$
$$\tau^2(V) = \begin{cases} a_1a_2 - (a_1 + a_2) + 6; & a_1 \geq 3, a_2 \geq 3 \\ a_2 + 3; & a_1 = 2, a_2 \geq 2. \end{cases}$$
$$\mu^2(V) = \begin{cases} a_1a_2 - (a_1 + a_2) + 7; & a_1 \geq 3, a_2 \geq 3 \\ a_2 + 4; & a_1 = 2, a_2 \geq 2. \end{cases}$$
$$d_2(V) = (a_1 - 2)(a_2 - 2); a_1 \geq 3, a_2 \geq 3.$$
$$d_3(V) = (a_1 - 3)(a_2 - 3); a_1 \geq 5, a_2 \geq 5.$$
$$d_4(V) = (a_1 - 4)(a_2 - 4); a_1 \geq 6, a_2 \geq 6.$$
$$\delta_2(V) = 2a_1a_2 - 5(a_1 + a_2) + 12; a_1 \geq 3, a_2 \geq 3.$$
$$\delta_3(V) = 2a_1a_2 - 7(a_1 + a_2) + 24; a_1 \geq 5, a_2 \geq 5.$$
$$\delta_4(V) = 2a_1a_2 - 9(a_1 + a_2) + 40; a_1 \geq 6, a_2 \geq 6.$$
$$\lambda(V) = a_1a_2 - 3(a_1 + a_2) + 4; a_1 \geq 3, a_2 \geq 3.$$
$$\lambda^1(V) = \begin{cases} 2a_1a_2 - 3(a_1 + a_2) + 10; & a_1 \geq 3, a_2 \geq 3 \\ a_1 + 2; & a_1 \geq 2, a_2 = 2. \end{cases}$$
$$\lambda^2(V) = \begin{cases} 2a_1a_2 - 3(a_1 + a_2) + 17; & a_1 \geq 5, a_2 \geq 5 \\ 3a_2 + 5; & a_1 = 3, a_2 \geq 4 \\ 13; & a_1 = 3, a_2 = 3 \\ 5a_2 + 4; & a_1 = 4, a_2 \geq 5 \\ 23; & a_1 = 4, a_2 = 4 \\ a_2 + 5; & a_1 = 2, a_2 \geq 3 \\ 6; & a_1 = 2, a_2 = 2 \\ 1; & a_1 = 1, a_2 \geq 1. \end{cases}$$

Remark 3.1 Note that $\mu = \tau$, $\mu^1 = \tau^1$ in Proposition 3.1–3.8, thus τ, μ^1 are not listed.

Proposition 3.2 ([19, 21, 26–28, 45]) *Let $(V, 0)$ be a binomial isolated singularity of type B which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ with weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. Then*

$$\tau(V) = a_2(a_1 - 1) + 1; a_1 \geq 1,\ a_2 \geq 2.$$

$$\tau^1(V) = a_2(a_1-1)+3;\ a_1 \geq 1,\ \ a_2 \geq 2.$$

$$\tau^2(V) = \begin{cases} a_1a_2 - a_2 + 6 & a_1 \geq 2, a_2 \geq 3 \\ 5; & a_1 = 1, a_2 \geq 2 \\ 7; & a_1 = 2, a_2 = 2. \end{cases}$$

$$\mu^2(V) = \begin{cases} a_1a_2 - a_2 + 7 & a_1 \geq 2, a_2 \geq 3 \\ 6; & a_1 = 1, a_2 \geq 2 \\ 8; & a_1 = 2, a_2 = 2. \end{cases}$$

$$d_2(V) = a_1a_2 - 2(a_1+a_2)+5;\ a_1 \geq 2, a_2 \geq 3.$$

$$d_3(V) = a_1a_2 - 3(a_1+a_2)+10;\ a_1 \geq 4, a_2 \geq 5.$$

$$d_4(V) = a_1a_2 - 4(a_1+a_2)+17;\ a_1 \geq 5, a_2 \geq 6.$$

$$\delta_2(V) = \begin{cases} 2a_1a_2 - 5(a_1+a_2)+15; & a_1 \geq 4, a_2 \geq 3 \\ a_2 - 2; & a_1 = 3, a_2 \geq 3 \\ 0; & a_1 \geq 3, a_2 = 2. \end{cases}$$

$$\delta_3(V) = 2a_1a_2 - 7(a_1+a_2)+27;\ a_1 \geq 4, a_2 \geq 5.$$

$$\delta_4(V) = 2a_1a_2 - 9(a_1+a_2)+43;\ a_1 \geq 5, a_2 \geq 6.$$

$$\lambda(V) = a_1a_2 - 2a_1 - 3a_2 + 5;\ a_1 \geq 2, a_2 \geq 3.$$

$$\lambda^1(V) = \begin{cases} 2a_1a_2 - 2a_1 - 3a_2 + 11; & a_1 \geq 2, a_2 \geq 3 \\ 2a_1 + 2; & a_1 \geq 2, a_2 = 2 \\ 4; & a_1 = 1, a_2 \geq 2. \end{cases}$$

$$\lambda^2(V) = \begin{cases} 2a_1a_2 - 2a_1 - 3a_2 + 20; & a_1 \geq 5, a_2 \geq 5 \\ 5a_2 + 12; & a_1 = 4, a_2 \geq 5 \\ 31; & a_1 = 4, a_2 = 4 \\ 4a_1 + 7; & a_1 \geq 3, a_2 = 3 \\ 2a_1 + 5; & a_1 \geq 2, a_2 = 2 \\ a_2 + 11; & a_1 = 2, a_2 \geq 4 \\ 13; & a_1 = 2, a_2 = 3 \\ 6; & a_1 = 1, a_2 \geq 2. \end{cases}$$

Proposition 3.3 ([19, 21, 26–28, 45]) *Let* $(V, 0)$ *be a binomial isolated singularity of type C which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ *with weight type* $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$. *Then*

$$\tau(V) = \begin{cases} a_1a_2; & a_1 \geq 2, a_2 \geq 2 \\ 1; & a_1 = 1, a_2 \geq 1. \end{cases}$$

$$\tau^1(V) = \begin{cases} a_1a_2 + 2; & a_1 \geq 2, a_2 \geq 2 \\ 3; & a_1 = 1, a_2 \geq 1. \end{cases}$$

$$\tau^2(V) = \begin{cases} a_1a_2 + 5 & a_1 \geq 2, a_2 \geq 2 \\ 5; & a_1 = 1, a_2 \geq 1. \end{cases}$$

$$\mu^2(V) = \begin{cases} a_1a_2 + 6 & a_1 \geq 2, a_2 \geq 2 \\ 6; & a_1 = 1, a_2 \geq 1. \end{cases}$$

$$d_2(V) = a_1a_2 - 2(a_1+a_2)+7;\ a_1 \geq 2, a_2 \geq 2.$$

$$d_3(V) = a_1a_2 - 3(a_1+a_2)+11;\ a_1 \geq 4, a_2 \geq 4.$$

$$d_4(V) = a_1a_2 - 4(a_1+a_2)+18;\ a_1 \geq 5, a_2 \geq 5.$$

$$\delta_2(V) = \begin{cases} 2a_1a_2 - 5(a_1+a_2) + 19; & a_1 \geq 5, a_2 \geq 5 \\ a_2 + 1; & a_1 = 3, a_2 \geq 3 \\ 3a_2 - 2; & a_1 = 4, a_2 \geq 5 \\ 9; & a_1 = 4, a_2 = 4 \\ 0; & a_1 = 2, a_2 \geq 2. \end{cases}$$

$$\delta_3(V) = \begin{cases} 2a_1a_2 - 7(a_1+a_2) + 30; & a_1 \geq 5, a_2 \geq 5 \\ a_2; & a_1 = 4, a_2 \geq 4. \end{cases}$$

$$\delta_4(V) = \begin{cases} 2a_1a_2 - 9(a_1+a_2) + 46; & a_1 \geq 6, a_2 \geq 6 \\ a_2 - 1; & a_1 = 5, a_2 \geq 5. \end{cases}$$

$$\lambda(V) = \begin{cases} a_1a_2 - 2a_1 - 2a_2 + 6 & a_1 \geq 3, a_2 \geq 3 \\ 2a_2; & a_1 = 2, a_2 \geq 2. \end{cases}$$

$$\lambda^1(V) = \begin{cases} 2a_1a_2 - 2a_1 - 2a_2 + 12; & a_1 \geq 3, a_2 \geq 3 \\ 2a_1 + 6; & a_1 \geq 2, a_2 = 2 \\ 4; & a_1 \geq 1, a_2 = 1 \\ 4; & a_1 = 1, a_2 \geq 2. \end{cases}$$

$$\lambda^2(V) = \begin{cases} 2a_1a_2 - 2(a_1+a_2) + 21; & a_1 \geq 4, a_2 \geq 4 \\ 4a_2 + 13; & a_1 = 3, a_2 \geq 4 \\ 2a_2 + 10; & a_1 = 2, a_2 \geq 3 \\ 23; & a_1 = 3, a_2 = 3 \\ 13; & a_1 = 2, a_2 = 2 \\ 6; & a_1 = 1, a_2 \geq 1. \end{cases}$$

Proposition 3.4 ([19–21, 23, 26–28]) *Let* $(V, 0)$ *be a fewnomial surface isolated singularity of type 1 which is defined by* $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$ *with weight type* $(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}; 1)$. *Then*

$$\tau(V) = (a_1-1)(a_2-1)(a_3-1);\ a_1 \geq 3, a_2 \geq 3, a_3 \geq 3.$$

$$\tau^1(V) = a_1a_2a_3 - a_1a_2 - a_1a_3 - a_2a_3 + a_1 + a_2 + a_3 + 2;\ a_1 \geq 2, a_2 \geq 2, a_3 \geq 2.$$

$$\tau^2(V) = \begin{cases} a_1a_2a_3 - (a_1a_2 + a_1a_3 + a_2a_3) + (a_1+a_2+a_3) + 10; & a_1 \geq 3, a_2 \geq 3, a_3 \geq 3 \\ a_2a_3 - a_2 - a_3 + 10; & a_1 = 2, a_2 \geq 3, a_3 \geq 3 \\ a_3 + 7; & a_1 = 2, a_2 = 2, a_3 \geq 3 \\ 9; & a_1 = 2, a_2 = 2, a_3 = 2. \end{cases}$$

$$\mu^2(V) = \begin{cases} a_1a_2a_3 - (a_1a_2 + a_1a_3 + a_2a_3) + (a_1+a_2+a_3) + 12; & a_1 \geq 3, a_2 \geq 3, a_3 \geq 3 \\ a_2a_3 - a_2 - a_3 + 12; & a_1 = 2, a_2 \geq 3, a_3 \geq 3 \\ a_3 + 9; & a_1 = 2, a_2 = 2, a_3 \geq 3 \\ 10; & a_1 = 2, a_2 = 2, a_3 = 2. \end{cases}$$

$$d_2(V) = (a_1-2)(a_2-2)(a_3-2);\ a_1 \geq 3, a_2 \geq 3, a_3 \geq 3.$$

$$d_3(V) = (a_1-3)(a_2-3)(a_3-3);\ a_1 \geq 5, a_2 \geq 5, a_3 \geq 5.$$

$$d_4(V) = (a_1-4)(a_2-4)(a_3-4);\ a_1 \geq 6, a_2 \geq 6, a_3 \geq 6.$$

$$\delta_2(V) = 3a_1a_2a_3 + 16(a_1+a_2+a_3) - 7(a_1a_2 + a_1a_3 + a_2a_3) - 36;$$
$$a_1 \geq 3, a_2 \geq 3, a_3 \geq 3.$$

$$\delta_3(V) = 3a_1a_2a_3 + 33(a_1+a_2+a_3) - 10(a_1a_2 + a_1a_3 + a_2a_3) - 108;$$
$$a_1 \geq 5, a_2 \geq 5, a_3 \geq 5.$$

$$\delta_4(V) = 3a_1a_2a_3 + 56(a_1+a_2+a_3) - 13(a_1a_2 + a_1a_3 + a_2a_3) - 240;$$
$$a_1 \geq 6, a_2 \geq 6, a_3 \geq 6.$$

$$\lambda(V) = 3(a_1-1)(a_2-1)(a_3-1) - (a_1-1)(a_2-1)$$

$$-(a_1-1)(a_3-1)-(a_2-1)(a_3-1);\ a_i\geq 3.$$

$$\lambda^1(V)=3a_1a_2a_3+5(a_1+a_2+a_3)-4(a_1a_2+a_1a_3+a_2a_3)+6;$$
$$a_1\geq 3, a_2\geq 3, a_3\geq 3.$$
$$\lambda^2(V)=3a_1a_2a_3+5(a_1+a_2+a_3)-4(a_1a_2+a_1a_3+a_2a_3)+34;$$
$$a_1\geq 3, a_2\geq 3, a_3\geq 3.$$

Proposition 3.5 ([19–21, 23, 26–28]) *Let* $(V,0)$ *be a fewnomial isolated singularity of type 2 which is defined by* $f=x_1^{a_1}x_2+x_2^{a_2}x_3+x_3^{a_3}$ *with weight type* $(\frac{1-a_3+a_2a_3}{a_1a_2a_3}, \frac{a_3-1}{a_2a_3}, \frac{1}{a_3}; 1)$. *Then*

$$\tau(V)=a_1a_2a_3-1+a_3-a_2a_3;\ a_1\geq 1, a_2\geq 1, a_3\geq 2.$$
$$\tau^1(V)=a_1a_2a_3-a_2a_3+a_3+2;\ a_1\geq 1, a_2\geq 1, a_3\geq 2.$$
$$\tau^2(V)=\begin{cases} a_1a_2a_3-a_2a_3+a_3+10; & a_1\geq 2, a_2\geq 2, a_3\geq 3\\ 2a_1a_2-2a_2+10; & a_1\geq 2, a_2\geq 2, a_3=2\\ a_3+7; & a_1=1, a_2\geq 1, a_3\geq 2\\ a_1a_3+7; & a_1\geq 2, a_2=1, a_3\geq 2.\end{cases}$$
$$\mu^2(V)=\begin{cases} a_1a_2a_3-a_2a_3+a_3+12; & a_1\geq 2, a_2\geq 2, a_3\geq 3\\ 2a_1a_2-2a_2+11; & a_1\geq 2, a_2\geq 2, a_3=2\\ a_3+8; & a_1=1, a_2\geq 1, a_3\geq 2\\ a_1a_3+8; & a_1\geq 2, a_2=1, a_3\geq 2.\end{cases}$$
$$d_2(V)=a_1a_2a_3-2(a_1a_2+a_1a_3+a_2a_3)+5(a_1+a_3)+4a_2-12;$$
$$a_1\geq 2, a_2\geq 2, a_3\geq 3.$$
$$d_3(V)=a_1a_2a_3-3(a_1a_2+a_1a_3+a_2a_3)+10(a_1+a_3)+9a_2-33;$$
$$a_1\geq 4, a_2\geq 4, a_3\geq 5.$$
$$d_4(V)=a_1a_2a_3-4(a_1a_2+a_1a_3+a_2a_3)+17(a_1+a_3)+16a_2-72;$$
$$a_1\geq 5, a_2\geq 5, a_3\geq 6.$$
$$\delta_2(V)=\begin{cases} 3a_1a_2a_3-7(a_1a_2+a_1a_3+a_2a_3)+20(a_1+a_3)+16a_2-55; & a_1\geq 4, a_2\geq 4, a_3\geq 4\\ 2a_1a_3-a_1-3a_3-1; & a_1\geq 3, a_2=3, a_3\geq 4\\ 2a_2a_3-5a_2-a_3+5; & a_1=3, a_2\geq 4, a_3\geq 4\\ 2a_1a_2-3a_1-5a_2+10; & a_1\geq 3, a_2\geq 3, a_3=3\\ a_3-3; & a_1=2, a_2\geq 2, a_3\geq 3\\ a_1-3; & a_1\geq 3, a_2=2, a_3\geq 3.\end{cases}$$
$$\delta_3(V)=\begin{cases} 3a_1a_2a_3-10(a_1a_2+a_1a_3+a_2a_3)+37(a_1+a_3) & \\ +33a_2-135; & a_1\geq 4, a_2\geq 5, a_3\geq 5\\ 2a_1a_3-3a_1-5a_2+5; & a_1\geq 4, a_2=4, a_3\geq 5.\end{cases}$$
$$\delta_4(V)=\begin{cases} 3a_1a_2a_3-13(a_1a_2+a_1a_3+a_2a_3)+60(a_1+a_3) & \\ +56a_2-275; & a_1\geq 5, a_2\geq 6, a_3\geq 6\\ 2a_1a_3-5a_1-7a_3+15; & a_1\geq 5, a_2=5, a_3\geq 6.\end{cases}$$
$$\lambda(V)=\begin{cases} 3a_1a_2a_3-2a_1a_3-4a_2a_3+6a_3+2a_1-2a_1a_2+2a_2-7; & a_1\geq 2, a_2\geq 3, a_3\geq 3\\ 4a_1a_3-3a_3-2a_1-1; & a_1\geq 2, a_2=2, a_3\geq 3.\end{cases}$$
$$\lambda^1(V)=\begin{cases} 5a_3+7; & a_1=2, a_2=2, a_3\geq 3\\ 4a_1a_3-2a_1-3a_3+11; & a_1\geq 3, a_2=2, a_3\geq 3\\ 3a_1a_2a_3-2a_1a_2-2a_1a_3-4a_2a_3+2a_1+2a_2+6a_3+5; & Otherwise.\end{cases}$$
$$\lambda^2(V)=\begin{cases} 3a_1a_2a_3+2a_1+2a_2+6a_3-4a_2a_3-2a_1a_2-2a_1a_3+37; & a_1\geq 3, a_2\geq 3, a_3\geq 3\\ 4a_2+42; & a_1=2, a_2\geq 3, a_3=3\\ 2a_2a_3-2a_2+2a_3+38; & a_1=2, a_2\geq 3, a_3\geq 4\\ 4a_1a_3-3a_3-2a_1+42; & a_1\geq 3, a_2=2, a_3\geq 4\\ 5a_3+35; & a_1=2, a_2=2, a_3\geq 4\\ 46; & a_1=2, a_2=2, a_3=3\\ 10a_1+32; & a_1\geq 3, a_2=2, a_3=3.\end{cases}$$

Proposition 3.6 ([19–21, 23, 26–28]) *Let* $(V, 0)$ *be a fewnomial isolated singularity of type 3 which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$ *with weight type* $(\frac{1-a_3+a_2a_3}{1+a_1a_2a_3}, \frac{1-a_1+a_1a_3}{1+a_1a_2a_3}, \frac{1-a_2+a_1a_2}{1+a_1a_2a_3}; 1)$. *Then*

$$
\begin{aligned}
\tau(V) &= a_1a_2a_3; a_1 \geq 1, a_2 \geq 1, a_3 \geq 1.\\
\tau^1(V) &= a_1a_2a_3 + 3; a_1 \geq 1, a_2 \geq 1, a_3 \geq 1.\\
\tau^2(V) &= \begin{cases} a_1a_2a_3 + 11; & a_1 \geq 2, a_2 \geq 2, a_3 \geq 2\\ a_2a_3 + 8; & a_1 = 1, a_2 \geq 1, a_3 \geq 1.\end{cases}\\
\mu^2(V) &= \begin{cases} a_1a_2a_3 + 13; & a_1 \geq 2, a_2 \geq 2, a_3 \geq 2\\ a_2a_3 + 9; & a_1 = 1, a_2 \geq 1, a_3 \geq 1.\end{cases}\\
d_2(V) &= a_1a_2a_3 - 2(a_1a_2 + a_1a_3 + a_2a_3) + 5(a_1 + a_2 + a_3) - 14;\\
&\qquad a_1 \geq 2, a_2 \geq 2, a_3 \geq 2.\\
d_3(V) &= a_1a_2a_3 - 3(a_1a_2 + a_1a_3 + a_2a_3) + 10(a_1 + a_2 + a_3) - 36;\\
&\qquad a_1 \geq 4, a_2 \geq 4, a_3 \geq 4.\\
d_4(V) &= a_1a_2a_3 - 4(a_1a_2 + a_1a_3 + a_2a_3) + 17(a_1 + a_2 + a_3) - 76;\\
&\qquad a_1 \geq 5, a_2 \geq 5, a_3 \geq 5.\\
\delta_2(V) &= \begin{cases} 3a_1a_2a_3 + 20(a_1 + a_2 + a_3) - 7(a_1a_2 + a_1a_3 + a_2a_3) - 63; & a_1 \geq 3, a_2 \geq 3, a_3 \geq 3\\ a_3 - 2; & a_1 = 2, a_2 \geq 2, a_3 \geq 2.\end{cases}\\
\delta_3(V) &= \begin{cases} 3a_1a_2a_3 + 37(a_1 + a_2 + a_3) - 10(a_1a_2 + a_1a_3 + a_2a_3) & \\ -147; & a_1 \geq 5, a_2 \geq 5, a_3 \geq 5\\ 2a_2a_3 - 5a_2 - 3a_3 + 9; & a_1 = 4, a_2 \geq 5, a_3 \geq 4\\ 2a_1a_3 - 3a_1 - 5a_3 + 9; & a_1 \geq 4, a_2 = 4, a_3 \geq 4\\ 2a_1a_2 - 5a_1 - 3a_2 + 9; & a_1 \geq 5, a_2 \geq 5, a_3 = 4.\end{cases}\\
\delta_4(V) &= \begin{cases} 3a_1a_2a_3 + 60(a_1 + a_2 + a_3) - 13(a_1a_2 + a_1a_3 + a_2a_3) & \\ -291; & a_1 \geq 6, a_2 \geq 6, a_3 \geq 6\\ 2a_2a_3 - 7a_2 - 5a_3 + 19; & a_1 = 5, a_2 \geq 6, a_3 \geq 5\\ 2a_1a_3 - 5a_1 - 7a_3 + 19; & a_1 \geq 5, a_2 = 5, a_3 \geq 5\\ 2a_1a_2 - 7a_1 - 5a_2 + 19; & a_1 \geq 6, a_2 \geq 6, a_3 = 5.\end{cases}\\
\lambda(V) &= \begin{cases} 12; & a_1 = 2, a_2 = 2, a_3 = 2\\ 3a_1a_2a_3 - 2(a_1a_2 + a_1a_3 + a_2a_3) + 2(a_1 + a_2 + a_3) - 1; & Otherwise.\end{cases}\\
\lambda^1(V) &= \begin{cases} 24; & a_1 = 2, a_2 = 2, a_3 = 2\\ 3a_1a_2a_3 + 2(a_1 + a_2 + a_3) - 2(a_1a_2 + a_1a_3 + a_2a_3) + 11; & Otherwise.\end{cases}\\
\lambda^2(V) &= \begin{cases} 3a_1a_2a_3 + 2(a_1 + a_2 + a_3) - 2(a_1a_2 + a_1a_3 + a_2a_3) + 43; & a_1 \geq 3, a_2 \geq 3, a_3 \geq 3\\ 4a_2a_3 - 2(a_2 + a_3) + 46; & a_1 = 2, a_2 \geq 3, a_3 \geq 3.\end{cases}
\end{aligned}
$$

Proposition 3.7 ([19–21, 23, 26–28]) *Let* $(V, 0)$ *be a fewnomial surface isolated singularity of type 4 which is defined by* $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}x_2$ *with weight type* $(\frac{1}{a_1}, \frac{1}{a_2}, \frac{a_2-1}{a_2a_3}; 1)$. *Then*

$$
\begin{aligned}
\tau(V) &= a_1a_2a_3 - a_1a_2 - a_2a_3 + a_1 + a_2 - 1; a_1 \geq 2, a_2 \geq 2, a_3 \geq 1.\\
\tau^1(V) &= a_1a_2a_3 - a_1a_2 - a_2a_3 + a_1 + a_2 + 2; a_1 \geq 2, a_2 \geq 2, a_3 \geq 1.\\
\tau^2(V) &= \begin{cases} a_1a_2a_3 - a_1a_2 - a_2a_3 + a_1 + a_2 + 10; & a_1 \geq 3, a_2 \geq 3, a_3 \geq 2\\ 2a_1a_3 - a_1 - 2a_3 + 10; & a_1 \geq 3, a_2 = 2, a_3 \geq 2\\ a_1 + 7; & a_1 \geq 2, a_2 \geq 2, a_3 = 1\\ a_2a_3 - a_2 + 10; & a_1 = 2, a_2 \geq 3, a_3 \geq 2\\ 2a_3 + 7; & a_1 = 2, a_2 = 2, a_3 \geq 2.\end{cases}
\end{aligned}
$$

$$\mu^2(V) = \begin{cases} a_1a_2a_3 - a_1a_2 - a_2a_3 + a_1 + a_2 + 12; & a_1 \geq 3, a_2 \geq 3, a_3 \geq 3 \\ a_1a_2 + a_1 - a_2 + 11; & a_1 \geq 3, a_2 \geq 3, a_3 = 2 \\ 2a_1a_3 - a_1 - 2a_3 + 12; & a_1 \geq 3, a_2 = 2, a_3 \geq 2 \\ a_1 + 8; & a_1 \geq 2, a_2 \geq 2, a_3 = 1 \\ a_2a_3 - a_2 + 11; & a_1 = 2, a_2 \geq 3, a_3 \geq 2 \\ 2a_3 + 9; & a_1 = 2, a_2 = 2, a_3 \geq 2. \end{cases}$$

$$d_2(V) = a_1a_2a_3 - 2(a_1a_2 + a_1a_3 + a_2a_3) + 4(a_1 + a_3) + 5a_2 - 10;$$
$$a_1 \geq 3, a_2 \geq 3, a_3 \geq 2.$$

$$d_3(V) = a_1a_2a_3 - 3(a_1a_2 + a_1a_3 + a_2a_3) + 9(a_2 + a_3) + 10a_1 - 30;$$
$$a_1 \geq 5, a_2 \geq 5, a_3 \geq 4.$$

$$d_4(V) = a_1a_2a_3 - 4(a_1a_2 + a_1a_3 + a_2a_3) + 16(a_2 + a_3) + 17a_1 - 68;$$
$$a_1 \geq 6, a_2 \geq 6, a_3 \geq 5.$$

$$\delta_2(V) = \begin{cases} 3a_1a_2a_3 + 16(a_1 + a_3) + 20a_2 - 7(a_1a_2 + a_1a_3 + a_2a_3) - 45; & a_1 \geq 4, a_2 \geq 3, a_3 \geq 3 \\ 2a_2a_3 - 3a_2 - 5a_3 + 7; & a_1 = 3, a_2 \geq 3, a_3 \geq 2 \\ a_2 - 3; & a_1 \geq 4, a_2 \geq 3, a_3 = 2. \end{cases}$$

$$\delta_3(V) = 3a_1a_2a_3 + 37a_1 + 33(a_2 + a_3) - 10(a_1a_2 + a_1a_3 + a_2a_3) - 121;$$
$$a_1 \geq 5, a_2 \geq 5, a_3 \geq 4.$$

$$\delta_4(V) = 3a_1a_2a_3 + 60a_1 + 56(a_2 + a_3) - 13(a_1a_2 + a_1a_3 + a_2a_3) - 257;$$
$$a_1 \geq 6, a_2 \geq 6, a_3 \geq 5.$$

$$\lambda(V) = 3a_1a_2a_3 - 4a_1a_2 - 4a_2a_3 - 2a_1a_3 + 6a_1 + 5a_2 + 2a_3 - 7; a_1 \geq 3, a_2 \geq 3, a_3 \geq 2.$$

$$\lambda^1(V) = 3a_1a_2a_3 - 4a_1a_2 - 4a_2a_3 - 2a_1a_3 + 6a_1 + 5a_2 + 2a_3 + 5; a_1 \geq 3, a_2 \geq 3, a_3 \geq 2.$$

$$\lambda^2(V) = \begin{cases} 7a_1a_3 - 6a_1 - 10a_3 + 48; & a_1 \geq 3, a_2 = 3, a_3 \geq 3 \\ 5a_2a_3 - 7a_2 - 4a_3 + 53; & a_1 = 3, a_2 \geq 4, a_3 \geq 3 \\ 3a_1a_2a_3 + 6a_1 + 5a_2 + 2a_3 - 4a_2a_3 - 4a_1a_2 - 2a_1a_3 + 35; & a_1 \geq 4, a_2 \geq 4, a_3 \geq 3 \\ 11a_1a_2 - 3a_1 - 15a_2 + 41; & a_1 \geq 4, a_2 \geq 4, a_3 = 5 \\ 46; & a_1 = 3, a_2 = 3, a_3 = 2 \\ 2a_1a_2 + 2a_1 - 3a_2 + 36; & a_1 \geq 4, a_2 \geq 4, a_3 = 2 \\ 3a_2 + 40; & a_1 = 3, a_2 \geq 4, a_3 = 2 \\ 8a_1 + 26; & a_1 \geq 4, a_2 = 3, a_3 = 2. \end{cases}$$

Proposition 3.8 ([19–21, 23, 26–28]) *Let* $(V, 0)$ *be a fewnomial surface isolated singularity of type 5 which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$ *with weight type* $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}, \frac{1}{a_3}; 1)$. *Then*

$$\tau(V) = a_1a_2a_3 - a_1a_2; a_1 \geq 1, a_2 \geq 1, a_3 \geq 2.$$

$$\tau^1(V) = a_1a_2a_3 - a_1a_2 + 3; a_1 \geq 1, a_2 \geq 1, a_3 \geq 2.$$

$$\tau^2(V) = \begin{cases} a_1a_2a_3 - a_1a_2 + 11; & a_1 \geq 2, a_2 \geq 2, a_3 \geq 3 \\ a_1a_2 + 9; & a_1 \geq 2, a_2 \geq 2, a_3 = 2 \\ a_3 + 7; & a_1 = 1, a_2 \geq 1, a_3 \geq 2 \\ a_3 + 7; & a_1 \geq 2, a_2 = 1, a_3 \geq 2. \end{cases}$$

$$\mu^2(V) = \begin{cases} a_1a_2a_3 - a_1a_2 + 13; & a_1 \geq 2, a_2 \geq 2, a_3 \geq 3 \\ a_1a_2 + 10; & a_1 \geq 2, a_2 \geq 2, a_3 = 2 \\ a_3 + 8; & a_1 = 1, a_2 \geq 1, a_3 \geq 2 \\ a_3 + 8; & a_1 \geq 2, a_2 = 1, a_3 \geq 2. \end{cases}$$

$$d_2(V) = a_1a_2a_3 - 2(a_1a_2 + a_1a_3 + a_2a_3) + 4(a_1 + a_2) + 7a_3 - 14;$$
$$a_1 \geq 2, a_2 \geq 2, a_3 \geq 3.$$

$$d_3(V) = a_1a_2a_3 - 3(a_1a_2 + a_1a_3 + a_2a_3) + 9(a_1 + a_2) + 11a_3 - 33;$$
$$a_1 \geq 4, a_2 \geq 4, a_3 \geq 5.$$

$$d_4(V) = a_1a_2a_3 - 4(a_1a_2 + a_1a_3 + a_2a_3) + 16(a_1 + a_2) + 18a_3 - 72;$$

$a_1 \geq 5, a_2 \geq 5, a_3 \geq 6.$

$$\delta_2(V) = \begin{cases} 3a_1a_2a_3 + 16(a_1+a_2) + 26a_3 - 7(a_1a_2 + a_1a_3 + a_2a_3) - 59; & a_1 \geq 5, a_2 \geq 5, a_3 \geq 3 \\ a_3 - 3; & a_1 = 2, a_2 \geq 2, a_3 \geq 3 \\ 2a_2a_3 - 5a_2 + 2a_3 - 5; & a_1 = 3, a_2 \geq 3, a_3 \geq 3 \\ 6a_2a_3 - 14a_2 - 8a_3 + 17; & a_1 = 4, a_2 \geq 4, a_3 \geq 3. \end{cases}$$

$$\delta_3(V) = \begin{cases} 3a_1a_2a_3 + 33(a_1+a_2) + 41a_3 - 10(a_1a_2 + a_1a_3 + a_2a_3) & \\ -134; & a_1 \geq 5, a_2 \geq 5, a_3 \geq 5 \\ 2a_2a_3 - 7a_2 - a_3 + 4; & a_1 = 4, a_2 \geq 4, a_3 \geq 5. \end{cases}$$

$$\delta_4(V) = \begin{cases} 3a_1a_2a_3 + 56(a_1+a_2) + 64a_3 - 13(a_1a_2 + a_1a_3 + a_2a_3) & \\ -274; & a_1 \geq 6, a_2 \geq 6, a_3 \geq 6 \\ 2a_2a_3 - 9a_2 - 4a_3 + 20; & a_1 = 5, a_2 \geq 5, a_3 \geq 6. \end{cases}$$

$$\lambda(V) = \begin{cases} 3a_1a_2a_3 - 4a_1a_2 - 2(a_2a_3 + a_1a_3) + 2(a_1+a_2) + 6a_3 - 6; & a_1 \geq 3, a_2 \geq 3, a_3 \geq 3 \\ 4a_2a_3 - 6a_2; & a_1 = 2, a_2 \geq 2, a_3 \geq 3. \end{cases}$$

$$\lambda^1(V) = \begin{cases} 4a_2a_3 - 6a_2 + 12; & a_1 = 2, a_2 \geq 2, a_3 \geq 2 \\ 3a_1a_2a_3 - 4a_1a_2 - 2a_2a_3 - 2a_1a_3 + 2a_1 + 2a_2 + 6a_3 + 6; & Otherwise. \end{cases}$$

$$\lambda^2(V) = \begin{cases} 3a_1a_2a_3 - 4a_1a_2 - 2a_2a_3 - 2a_1a_3 + 2a_1 + 2a_2 + 6a_3 + 36; & a_1 \geq 4, a_2 \geq 4, a_3 \geq 3 \\ 7a_2a_3 - 10a_2 + 40; & a_1 = 3, a_2 \geq 3, a_3 \geq 3 \\ 4a_2a_3 - 6a_2 + 39; & a_1 = 2, a_2 \geq 4, a_3 \geq 4 \\ 8a_3 + 26; & a_1 = 2, a_2 = 2, a_3 \geq 4 \\ 46; & a_1 = 2, a_2 = 2, a_3 = 3 \\ 12a_3 + 21; & a_1 = 2, a_2 = 3, a_3 \geq 4 \\ 55; & a_1 = 2, a_2 = 3, a_3 = 3. \end{cases}$$

Proof of Theorem A. Since

$$\begin{aligned} M_k(V) &= \mathcal{O}_n \big/ J_k(f) = \mathcal{O}_n \big/ (x_1^{a_1-k}, \cdots, x_n^{a_n-k}) \\ &\cong \mathbb{C}\{x_1\} \big/ (x_1^{a_1-k}) \otimes \mathbb{C}\{x_2\} \big/ (x_2^{a_2-k}) \otimes \cdots \otimes \mathbb{C}\{x_n\} \big/ (x_n^{a_n-k}), \end{aligned}$$

so we have

$$\begin{aligned} \delta_k(V) =& \dim\Big(Der\, M_k(V) \Big) \\ =& \sum_{i=1}^{n} \dim\Big(\mathbb{C}\{x_1\} \big/ (x_1^{a_1-k}) \Big) \cdots \dim\Big(\mathbb{C}\{x_{i-1}\} \big/ (x_{i-1}^{a_{i-1}-k}) \Big) \cdot \dim\Big(Der(\mathbb{C}\{x_i\} \big/ (x_i^{a_i-k})) \Big) \\ & \cdot \dim\Big(\mathbb{C}\{x_{i+1}\} \big/ (x_{i+1}^{a_{i+1}-k}) \Big) \cdots \dim\Big(\mathbb{C}\{x_n\} \big/ (x_n^{a_n-k}) \Big). \end{aligned}$$

Since $\mathrm{Der}\Big(\mathbb{C}\{x_i\} \big/ (x_i^{a_i-k}) \Big)$ is spanned by $x_i^j \partial x_i, \; 1 \leq j \leq a_i - k - 1,$

so, $\dim\Big(Der(\mathbb{C}\{x_i\} \big/ (x_i^{a_i-k})) \Big) = a_i - k - 1$. Notice that $\dim\Big(\{x_i\} \big/ (x_i^{a_i-k}) \Big) = a_i - k$, $1 \leq i \leq n$.

Therefore

$$\delta_k(V) = \sum_{i}^{n} (a_1 - k) \cdots (a_{i-1} - k)(a_i - k - 1)(a_{i+1} - k) \cdots (a_n - k)$$

$$= n\prod_{i=1}^{n}(a_i-k) - \sum_{i}^{n}(a_1-k)(a_2-k)\cdots\widehat{(a_i-k)}\cdots(a_n-k).$$

□

Proof of Theorem B.

$$\begin{aligned}
\mathcal{L}_k(V_{f+g}) &= \mathrm{Der}\Big(\mathcal{O}_{n+m}\big/J_k(f+g)\Big)\\
&= \mathrm{Der}\Big(\mathcal{O}_{n+m}\big/(J_k(f)+J_k(g))\Big)\\
&= \mathrm{Der}\Big((\mathcal{O}_n\big/J_k(f))\otimes(\mathcal{O}_m\big/J_k(g))\Big)\\
&= \mathrm{Der}\Big(\mathcal{O}_n\big/J_k(f)\Big)\otimes\mathcal{O}_m\big/J_k(g)+\mathcal{O}_n\big/J_k(f)\otimes\mathrm{Der}\Big(\mathcal{O}_m\big/J_k(g)\Big)\\
&= \mathcal{L}_k(V_f)\otimes M_k(g)+\mathcal{L}_k(V_g)\otimes M_k(f)\\
&\Rightarrow \delta_k(V_{f+g}) = d_k(V_f)\delta_k(V_g)+d_k(V_g)\delta_k(V_f).
\end{aligned}$$

The first equality above comes from the fact that f, g are weighted homogeneous while the fourth equality follows from the Theorem 2.1.

Since both f and g saitisfy the conjecture 1.5, we have

$$\delta_k(V_f) \le n\prod_{i=1}^{n}\left(\frac{1}{w_i}-k\right)-\sum_{i}^{n}\left(\frac{1}{w_1}-k\right)\left(\frac{1}{w_2}-k\right)\cdots\widehat{\left(\frac{1}{w_i}-k\right)}\cdots\left(\frac{1}{w_n}-k\right), \tag{3.1}$$

where $\widehat{(\frac{1}{w_i}-k)}$ denotes the omission of $\frac{1}{w_i}-k$.

$$\delta_k(V_g) \le m\prod_{i=1}^{m}\left(\frac{1}{w_{n+i}}-k\right)-\sum_{i}^{m}\left(\frac{1}{w_{n+1}}-k\right)\left(\frac{1}{w_{n+2}}-k\right)\cdots\widehat{\left(\frac{1}{w_{n+i}}-k\right)}\cdots\left(\frac{1}{w_{n+m}}-k\right). \tag{3.2}$$

We have

$$d_k(V_f) = \prod_{i=1}^{n}\left(\frac{1}{w_i}-k\right), \tag{3.3}$$

$$d_k(V_g) = \prod_{i=1}^{m}\left(\frac{1}{w_{n+i}}-k\right). \tag{3.4}$$

From above we can see that $M_k(V_{f+g}) = M_k(V_f)\otimes M_k(V_g)$, thus $d_k(V_{f+g}) = d_k(V_f)d_k(V_g)$. Combining this fact with the (3.1), (3.2), (3.3), and (3.4), we have

$$\begin{aligned}
\delta_k(V_{f+g}) &= d_k(V_f)\delta_k(V_g)+d_k(V_g)\delta_k(V_f)\\
&\le (n+m)d_k(f+g)-\sum_{i}^{n+m}\left(\frac{1}{w_1}-k\right)\left(\frac{1}{w_2}-k\right)\cdots\widehat{\left(\frac{1}{w_i}-k\right)}\cdots\left(\frac{1}{w_{n+m}}-k\right)\\
&= (n+m)\prod_{i=1}^{n+m}\left(\frac{1}{w_i}-k\right)-\sum_{i}^{n+m}\left(\frac{1}{w_1}-k\right)\left(\frac{1}{w_2}-k\right)\cdots\widehat{\left(\frac{1}{w_i}-k\right)}\cdots\left(\frac{1}{w_n}-k\right),
\end{aligned}$$

which shows that $f+g$ satisfies the Conjecture 1.5. □

Proof of Theorem C. Let $f \in \mathbb{C}\{x_1, x_2\}$ be a weighted homogeneous fewnomial isolated singularity. Then f can be divided into the following three types same as in Corollary 2.1:

Type A. $x_1^{a_1} + x_2^{a_2}$,
Type B. $x_1^{a_1}x_2 + x_2^{a_2}$,
Type C. $x_1^{a_1}x_2 + x_2^{a_2}x_1$.

It follows from Propositions 3.1–3.3 that the inequalities $d_k(V) \le \ell_k(\frac{1}{w_1}, \frac{1}{w_2})$, $k = 2, 3, 4$, hold true. □

Proof of Theorem D. Let $f \in \mathbb{C}\{x_1, x_2, x_3\}$ be a weighted homogeneous fewnomial isolated surface singularity. Then f can be divided into the five types same as in Proposition 2.2. It follows from Propositions 3.4–3.8 that the inequalities $d_k(V) \le \ell_k(\frac{1}{w_1}, \frac{1}{w_2}, \frac{1}{w_3})$, $k = 2, 3, 4$, hold true. □

Proof of Theorem E. Let $f \in \mathbb{C}\{x_1, x_2\}$ be a weighted homogeneous fewnomial isolated singularity. Then f can be divided into the three types same as in Corollary 2.1. It follows from Propositions 3.1–3.3 that the inequality $\frac{\mu^2}{\tau^2} < \frac{4}{3}$, holds true. □

Proof of Theorem F. Let $f \in \mathbb{C}\{x_1, x_2, x_3\}$ be a weighted homogeneous fewnomial isolated surface singularity. Then f can be divided into the five types same as in Proposition 2.2. It follows from Propositions 3.4–3.8 that the inequality $\frac{\mu^2}{\tau^2} < \frac{3}{2}$, holds true. □

Proof of Theorem G. Let $f \in \mathbb{C}\{x_1, x_2\}$ be a weighted homogeneous fewnomial isolated singularity. Then f can be divided into the three types same as in Corollary 2.1. It follows from Propositions 3.1–3.3 that the inequality

$$\lambda^2(V) > \lambda^1(V) > \lambda^0(V) = \delta_1(V) = \lambda_0(V) = \lambda_2(V) > \lambda_1(V) = \delta_2(V) > \delta_3(V) > \delta_4(V),$$

holds true. □

Proof of Theorem H. Let $f \in \mathbb{C}\{x_1, x_2, x_3\}$ be a weighted homogeneous fewnomial isolated surface singularity. Then f can be divided into the five types same as in Proposition 2.2. It follows from Propositions 3.4–3.8 that the inequality

$$\lambda^2(V) > \lambda^1(V) > \lambda^0(V) = \delta_1(V) = \lambda_0(V) = \lambda_3(V) > \lambda_1(V) = \delta_2(V) > \delta_3(V) > \delta_4(V),$$

holds true. □

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Declarations

Conflicts of Interest There is no conflict of interest among authors.

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