ON THE NAKAI CONJECTURE FOR SOME SINGULARITIES

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Abstract. The well-known Nakai Conjecture concerns a very natural question: For an algebraic variety, how does the differential operators of its coordinate ring imply the smoothness of it? It has been shown that all higher derivations of a smooth complex variety can be generated by the first order derivations, and Nakai proposed the converse question: if the algebra of differential operators is generated by the first order derivations, is the variety smooth? In this paper, we verify the Nakai Conjecture for weighted homogeneous fewnomial isolated singularities and hypersurface cusp singularities, this extends the existing works.

Keywords. higher derivations, isolated hypersurface singularity, weighted homogeneous, Nakai Conjecture.

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1. Introduction

In this paper we always assume that k is a field of characteristic zero. Let A be a finitely generated k-algebra. Let $\mathrm{Diff}_k^q(A)$ be the set of all q-th order differential operators on A over k, $Der_k^q(A)$ be the set of all q-th order derivations on A over k. It is known that $\mathrm{Diff}_k^q(A) = Der_k^q(A) \oplus A$, and there exist two natural filtrations $A = \mathrm{Diff}_k^0(A) \subset \mathrm{Diff}_k^1(A) \subset \mathrm{Diff}_k^2(A) \subset \cdots$, $0 = Der_k^0(A) \subset Der_k^1(A) \subset Der_k^2(A) \subset \cdots$. Moreover, the compositions of differential operators and derivations satisfy $\mathrm{Diff}_k^p(A)\,\mathrm{Diff}_k^q(A) \subset \mathrm{Diff}_k^{p+q}(A), Der_k^p(A)Der_k^q(A) \subset Der_k^{p+q}(A)$, the Lie brackets(commutators) of differential operators and derivations satisfy $[\mathrm{Diff}_k^p(A), \mathrm{Diff}_k^q(A)] \subset \mathrm{Diff}_k^{p+q-1}(A), [Der_k^p(A), Der_k^q(A)] \subset Der_k^{p+q-1}(A)$ ([12]).

Let $Der_k(A) = \underset{q\in\mathbb{N}}{\cup} Der_k^q(A), \mathrm{Diff}_k(A) = \underset{q\in\mathbb{N}}{\cup} \mathrm{Diff}_k^q(A)$, and denote by $der_k^q(A)$ the A-submodule of $Der_k^q(A)$ which consists of A-linear combinations of derivations of the form $\delta_1\delta_2\cdots\delta_j, 1 \leq j \leq q, \delta_i \in Der_k^1(A), \forall i$, and denote by $der_k(A)$ the A-submodule of $Der_k(A)$ generated by the compositions of elements in $Der_k^1(A)$. It is clear that $der_k(A) = \underset{q\in\mathbb{N}}{\cup} der_k^q(A)$, but $der_k^q(A) = der_k(A) \cap Der_k^q(A)$ does not necessarily hold. For simplicity, we omit the subscript k from now on.

Grothendieck [8] showed that $Der(A)$ is generated by $Der^1(A)$ when A is regular. Nakai conjectured that the converse is also true, it seems that he had not quoted the conjecture rigorously in [12], but his conjecture is often quoted in connection with his paper [12]. Since $der(A) = Der(A)$ may not be the same as $der^q(A) = Der^q(A), \forall q$ in general, his conjecture had different statements in previous works (see [4],[11],[17]). In this paper we deal with the following version:

Conjecture 1.1 (**Nakai** [4] [12])**.** *Let k be a field of characteristic zero and A be a finitely generated k-algebra, if $der^q(A) = Der^q(A)$ for each integer $q \geq 1$, then A is regular.*

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It has been proved by Becker [1] and Rego [13] that the Nakai Conjecture implies the well-known long-standing Zariski-Lipman conjecture, which asserts that if $Der^1(A)$ is A-projective, then A is regular.

The Nakai Conjecture has been proved for several cases and is still open in general cases. It is known to be true for algebraic curves [11], the case of monomial ideals [15], the case of hypersurface with two variables [16], the case where A is the invariant subring of $k[x_1, \cdots, x_n]$ by a finite subgroup of $GL(n,k)$ [20], the case of a cone on a Riemann surface of genus > 1 and other special cases ([3], [4], [17]). Moreover, Singh [16] conjectured that when A is the coordinate ring of a hypersurface, if $Der^2(A) = der^2(A)$ where A is the coordinate ring, then A is regular. This is the Singh Conjecture and it is obvious that the Singh Conjecture implies the Nakai Conjecture for hypersurface case.

For homogeneous hypersurface case, $Der(A)$ has been considered more concretely. Bernšteĭn, Gel'fand, and Gel'fand [2] analyzed $A = k[x,y,z]/(x^3+y^3+z^3)$, which is the coordinate ring of the cubic cone, showing that $Der(A)$ is not generated by any bounded order derivations. When $A = k[x,y,z]/(f)$, where $(V(f),0)$ is a homogeneous isolated hypersurface singularity, Vigué [18] showed that $Der(A)$ is not generated by any bounded order derivations when $deg(f) \geq 3$. Moreover, in [7], the authors studied the explicit generators of $Der^2(A)$ and $Der^3(A)$, which imply the Nakai Conjecture. However, their methods are very hard to be generalized to high dimensional homogeneous isolated hypersurface singularities.

For isolated singularities, there are few existing research results. In [5], the authors proved Nakai conjecture for homogeneous Brieskorn isolated hypersurface singularity, Xiao-Yau-Zuo [19] verified Nakai conjecture for weighted homogeneous Brieskorn case. Recently, Yau-Zhu-Zuo [22] proved the Nakai Conjecture for the homogeneous isolated hypersurface singularities. They introduced new ideas to analyse the necessary condition for $D \in Der^2(A)$ to be generated by $Der^1(A)$ and completed the proof by construction.

In this paper, we use an exact sequence (see theorem 2.8, [16]) as the main idea, to transfer the construction of an element in $Der^2(A)$ to an n-tuple of elements in $Der^1(A)$, and prove the Nakai Conjecture for the cases of weighted homogeneous hypersurface singularities and hypersurface cusp singularities.

More precisely, in section 3, we generalize part of the results obtained in [22] to weighted homogeneous hypersurface singularity cases (see theorem 3.2, theorem 3.7 and remark 3.8), and then follow the remark 3.5 to construct an n-tuple of first order derivations which cannot be the image of any $D \in der^2(A)$ under the map in theorem 2.8. The theorem is stated as following:

Theorem A. *Let $A = P/I$ where $P = k[x_1, \cdots, x_n]$, $I = (f)$ with $(V(f),0)$ a fewnomial weighted homogeneous isolated hypersurface singularity, then $der^2(A) \neq Der^2(A)$. The Nakai Conjecture holds for fewnomial weighted homogeneous isolated hypersurface singularities.*

In section 4, we move to the cases of some singularities which are not weighted homogeneous. We verify the Nakai Conjecture is true for hypersurface cusp singularities ($T_{p,q,r}$ singularities). The idea of proof is similar to the proof in section 3, by searching the necessary conditions for elements in $der^2(A)$ and doing concrete constructions.

Theorem B. *Let $A = \mathbb{C}[x,y,z]/(f)$ be the coordinate ring of the hypersurface cusp singularity $(\{f = x^p + y^q + z^r + xyz = 0\}, 0)$, where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, then $der^2(A) \neq Der^2(A)$. In particular, the Nakai Conjecture holds.*

2. Preliminaries

In the following two subsections we recall some basic definitions and theorems of higher order differential operators and derivations. Readers can refer to [12] and [16].

2.1. Higher order differential operators and derivations.

Let k, A be commutative rings with unit elements and let A be a k-algebra. Let F be an A-module. A q-th order differential operator Δ of A/k into F is, by definition, a k-homomorphism of A into F satisfying the following identity:

$$\Delta(x_0x_1, \cdots, x_q) = \sum_{s=1}^{q+1}(-1)^{s-1}\sum_{i_1<\cdots<i_s} x_{i_1}\cdots x_{i_s}\Delta(x_0\cdots\check{x}_{i_1}\cdots\check{x}_{i_s}\cdots x_q)$$

for any tuple $(x_0, x_1, \cdots, x_q)$ of $(q+1)$-elements in A. Denote by $\mathrm{Diff}^q(A, F)$ the set of q-th order differential operators of A into F, and $Der^q(A, F) = \{\Delta \in \mathrm{Diff}^q(A, F) | \Delta(1) = 0\}$ the set of q-th order derivations of A into F. When $F = A$, we will simply denote $\mathrm{Diff}^q(A, F)$ by $\mathrm{Diff}^q(A)$ and denote $Der^q(A, F)$ by $Der^q(A)$.

Let $\mathbb{N}$ be the set of all non-negative integers and put $V = \mathbb{N}^n$. For $\alpha = (\alpha_1, \cdots, \alpha_n) \in V$, we use the standard notation: $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1!\cdots\alpha_n!$, $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, etc. For $r \in \mathbb{Z}$ let $V_r = \{\alpha \in V | \, |\alpha| \le r\}$ and $W_r = \{\alpha \in V | \, |\alpha| = r\}$. For $1 \le i \le n$, let $e_i = (0, \cdots, 1, \cdots, 0) \in W_1$ with 1 at the i-th place.

Let $P = k[x_1, \cdots, x_n]$. For $\alpha \in V$ let ∂_α denote the derivation $(1/\alpha!)\partial^\alpha/\partial x^\alpha : P \to P$. The first order derivations of P is well-known as $Der^1(P) = P\langle\partial_{x_1}, \cdots, \partial_{x_n}\rangle$, meanwhile higher order derivations are generated by first order ones, i.e., $\partial_\alpha \in \mathrm{Diff}^{|\alpha|}(P)$. When $A = P/I$ with I proper ideal of P, the higher derivations are presented as follows:

Theorem 2.1. *Let $P = k[x_1, x_2, \cdots, x_n]$, I be a proper ideal of P and $A = P/I$. Then*

$$Der^m(A) \cong \frac{\{D \in Der^m(P), D(I) \in I\}}{I Der^m(P)} \cong \{D \in Der^m(P, A) | D(I) = 0\};$$

$$\mathrm{Diff}^m(A) \cong \frac{\{D \in \mathrm{Diff}^m(P), D(I) \in I\}}{I\,\mathrm{Diff}^m(P)} \cong \{D \in \mathrm{Diff}^m(P, A) | D(I) = 0\}.$$

By Theorem 2.1, we will identify derivations(differential operators) in $Der(A)(\mathrm{Diff}(A))$ with their lifts in $Der(P)(\mathrm{Diff}(P))$ or in $Der(P, A)(\mathrm{Diff}(P, A))$ throughout the later discussion. Then every $D \in \mathrm{Diff}(A)$ has a unique expression of the form in $\mathrm{Diff}(P, A)$:

$$D = \sum_{a\in V} c_a(D)\partial_a$$

with $c_\alpha(D) \in A$ for all α and $D(I) \in I$, and $c_\alpha(D) = 0$ for almost all α.

Definition 2.2. For $D \in \mathrm{Diff}(P)$ and $\beta \in V$, define

$$\langle D, x^\beta\rangle = \sum_{\alpha\in V} c_{\alpha+\beta}(D)\partial_\alpha.$$

Note that if $D \in \mathrm{Diff}^r(P)$ then $\langle D, x^\beta\rangle \in \mathrm{Diff}^{r-|\beta|}(P)$.

Lemma 2.3. *Let $D \in Der^1(P)$ and $D' \in \mathrm{Diff}^r(P)$. Then for every $\alpha \in W_{r+1}$ we have*

$$c_\alpha(DD') = \sum_{i=1}^{n} \alpha_i D(x_i)c_{\alpha-e_i}(D').$$

2.2. Descent of Higher Order Derivations.

Definition 2.4. [16] Let $\Phi : \mathrm{Diff}(P,A) \times V \to \mathrm{Der}(P,A)$ be the pairing defined by $\Phi(D,\beta) = \left\langle D, x^\beta \right\rangle - \left(\left\langle D, x^\beta \right\rangle (1)\right)_R = \left\langle D, x^\beta \right\rangle - c_\beta(D)\partial_0$. Note that Φ is the direct limit of the pairings

$$\Phi_{m,r} : \mathrm{Diff}^m(P,A) \times W_r \to \mathrm{Der}^{m-r}(P,A)$$

given by

$$\Phi_{m,r}(D,\beta) = \langle D, x^\beta \rangle - c_\beta(D)\partial_0.$$

Proposition 2.5. [16] *For $r \le m$, we have an exact sequence*

$$0 \to \mathrm{Diff}^r(P,A) \longrightarrow \mathrm{Diff}^m(P,A) \xrightarrow{\theta_{m,r}} \bigoplus_{\beta \in W_r} \mathrm{Der}^{m-r}(P,A),$$

where $\theta_{m,r}(D) = (\Phi_{m,r}(D,\beta))_{\beta \in W_r}$, and $\mathrm{Diff}^r(P,A) \hookrightarrow \mathrm{Diff}^m(P,A)$ is the natural inclusion.

Corollary 2.6. [16] *For $D \in \mathrm{Diff}^m(P,A)$ the following three conditions are equivalent:*
(i) $D \in \mathrm{Diff}^m(A)$.
(ii) $\left\langle D, x^\beta \right\rangle \in \mathrm{Diff}^{m-|\beta|}(A)$ for every $\beta \in V$.
(iii) $\left\langle D, x^\beta \right\rangle \in \mathrm{Diff}^{m-|\beta|}(A)$ for every $\beta \in V_{m-1}$.

In view of the above corollary, the pairings $\Phi_{m,r}$ induce pairings

$$\varphi_{m,r} : \mathrm{Diff}^m(A) \times W_r \longrightarrow \mathrm{Der}^{m-r}(A).$$

It follows from Proposition 2.5 that for $r \le m$, we have an exact sequence

$$0 \to \mathrm{Diff}^r(A) \longrightarrow \mathrm{Diff}^m(A) \xrightarrow{\theta_{m,r}} \bigoplus_{\beta \in W_r} \mathrm{Der}^{m-r}(A),$$

where $\theta_{m,r}(D) = (\varphi_{m,r}(D,\beta))_{\beta \in W_r}$, and $\mathrm{Diff}^r(A) \hookrightarrow \mathrm{Diff}^m(A)$ is the natural inclusion.

Definition 2.7. For $m \in \mathbb{Z}$ define

$$\begin{aligned} \mathscr{D}^m(A) = &\{(d_\beta)_{\beta \in W_{m-1}} \in \bigoplus_{\beta \in W_{m-1}} \mathrm{Der}^1(A) \mid d_\beta(x_i) = d_\gamma(x_j) \quad \text{whenever} \\ &\beta + e_i = \gamma + e_j, \beta, \gamma \in W_{m-1}, 1 \leqq i, j \leqq n\}. \end{aligned}$$

If $D \in \mathrm{Diff}^m(A)$ and $\theta_{m,m-1}(D) = (d_\beta)_{\beta \in W_{m-1}}$, then $d_\beta(x_i) = c_{\beta+e_i}(D)$. It follows that

$$\mathrm{Im}\,(\theta_{m,m-1}) \subset \mathscr{D}^m(A).$$

We write $\theta_m = \theta_{m,m-1}$ for simplicity.

It is easy to see that $\mathscr{D}^2(A) := \{(d_1, \cdots, d_n) \in \oplus_{i=1}^n Der^1(A) | d_i(x_j) = d_j(x_i) \text{ for all } i,j\}$.

Theorem 2.8. *Suppose $A = P/I$ and I is principal. Then the sequence*

$$0 \to \mathrm{Der}^1(A) \to \mathrm{Der}^2(A) \xrightarrow{\theta_2} \mathscr{D}^2(A) \to 0$$

is exact.

Proof. See Theorem 2.13 in [16]. □

2.3. Weighted homogeneous fewnomial isolated singularities.

In this subsection, we recall definitions related to weighted homogeneous fewnomial isolated singularities.

Definition 2.9. A polynomial $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \in k[x_1, x_2, \cdots, x_n]$ is called weighted homogeneous of weight type $(w_1, w_2, \cdots, w_n; d)$, if $w_1\alpha_1 + w_2\alpha_2 + \cdots + w_n\alpha_n = d$ holds for each multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $a_\alpha \neq 0$. We call w_i the weight of x_i and d the weighted degree of f, denoted by $wt(x_i) = w_i$ and $wt(f) = d$.

Definition 2.10. [9] We say that a polynomial $f \in k[x_1, x_2, \cdots, x_n]$ is fewnomial if the number of monomials appearing in f does not exceed n.

Obviously, the number of monomials in f may depend on the system of coordinates. In order to obtain a rigorous concept we shall only allow linear changes of coordinates and say that f (or rather its germ at the origin) is a m-nomial if m is the smallest natural number such that f becomes a m-nomial after (possibly) a linear change of coordinates. An isolated hypersurface singularity V is called m-nomial if there exists an isolated hypersurface singularity Y analytically isomorphic to V which can be defined by a m-nomial and m is the smallest such number. It was shown that a singularity defined by a fewnomial f can be isolated only if f is a n-nomial in n variables when its multiplicity at least 3 [6].

Definition 2.11. We say that an isolated hypersurface singularity $(V, 0)$ is fewnomial if it can be defined by a fewnomial polynomial f. $(V, 0)$ is called weighted homogeneous fewnomial isolated singularity if it can be defined by a weighted homogeneous fewnomial f. 2-nomial (resp. 3-nomial) isolated hypersurface singularity is also called binomial (resp. trinomial) singularity.

The following proposition tells us that each simple singularity belongs to one of the following three types of series.

Proposition 2.12. [21] *Let $(V(f), 0)$ be a weighted homogeneous fewnomial isolated hypersurface singularity with multiplicity at least 3. Then f is analytically equivalent to a linear combination of the following three series:*
Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}, n \geq 1,$
Type B. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}, n \geq 2,$
Type C. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1, n \geq 2.$

3. The weighted homogeneous fewnomial isolated singularity case

From now on we consider the case of $P = k[x_1, x_2, \cdots, x_n]$, $I = (f) \subset P$, where f is a weighted homogeneous polynomial of weight type $(w_1, \cdots, w_n; 1)$ and $A = P/I$. Denote by f_i the partial derivative $\frac{\partial f}{\partial x_i}$, and $J(f) = (f_1, f_2, \cdots, f_n)$ the Jacobian ideal of f throughout later discussion.

Proposition 3.1. *$Der^1(A)$ is generated by the Euler derivation $E = \sum_{i=1}^n w_i x_i \partial_{x_i}$ and Hamiltonian derivations $D_{ij} := f_i\partial_{x_j} - f_j\partial_{x_i}$.*

For the proposition 3.1, one can refer to [5] for a simple proof, the key point is that $f_1, f_2, \cdots, f_n$ form a regular sequence in P.

Theorem 3.2. *Let $D \in Der^2(A)$ and $\theta_2(D) = (d_1, \cdots, d_n)$. If D is in $der^2(A)$, then*

$$d_i(x_i) \in (f_1, \cdots, f_{i-1}, x_i, f_{i+1}, \cdots, f_n)^2.$$

Proof. By Proposition 3.1, we know $Der^1(A)$ is generated by E and D_{ij}'s where $1 \leq i < j \leq n$. Therefore the generators of $Der^1(A)Der^1(A)$ as an A-module are the followings: $D_{ij}D_{kl}$, $D_{ij}E$, ED_{ij} and E^2. As $[Der^1(A), Der^1(A)] \subset Der^1(A)$, $\theta_2(D_1D_2) = \theta_2(D_2D_1)$ for $D_1, D_2 \in Der^1(A)$.

Therefore, we only need to consider the image of the generators $E^2, ED_{ij}, D_{ij}D_{kl}$ under θ_2. Without loss of generality, we can only consider $d_1(x_1)$. For $D = D_{ij}D_{kl}$, if $i, j, k, l \neq 1$, then $d_1 = 0$. Therefore, we only need to consider the following cases:
If $D = \frac{1}{2}E^2$,

$$d_1 = w_1x_1E, \ d_1(x_1) = w_1^2x_1^2.$$

If $D = D_{1j}E$,

$$d_1 = -f_jE + w_1x_1D_{1j}, \ d_1(x_1) = -2w_1x_1f_j.$$

If $D = D_{ij}E$, $i, j \neq 1$,

$$d_1 = w_1x_1D_{ij}, \ d_1(x_1) = 0.$$

If $D = D_{1j}D_{1l}$,

$$d_1 = -f_jD_{1l} - f_lD_{1j}, \ d_1(x_1) = 2f_jf_l.$$

If $D = D_{1j}D_{kl}$, $k, l \neq 1$.

$$d_1 = -f_jD_{kl}, \ d_1(x_1) = 0.$$

Immediately we get $d_1(x_1) \in (x_1, f_2, \cdots, f_n)^2$. □

Corollary 3.3. *For $D \in Der^2(A)$ with $\theta_2(D) = (d_1, \cdots, d_n)$, if*

$$d_i(x_i) \notin (f_1, \cdots, f_{i-1}, x_i, f_{i+1}, \cdots, f_n)^2$$

for some i, then D does not belong to $der^2(A)$ and the Nakai Conjecture holds for the ring A.

Theorem 3.4. *([22]) For $d_1, \cdots, d_n \in Der^1(A)$, if $d_i(x_j) - d_j(x_i) \in J(f)$ $\forall i, j$, then $\exists$ $(d_1', \cdots, d_n') \in \mathscr{D}^2(A)$, such that $d_i' - d_i$ is an A-linear combination of D_{kl}'s, $(k, l = 1, \cdots, n)$ for each i.*

Proof. The idea of proof is to adjust d_i by adding or deleting D_{kl}. The $n = 2$ case is easy to adjust. However when n grows larger, the latter adjustment may break the equality constructed by former adjustments. We begin with $n = 2$.
Step 1: $n = 2$.
In the case of $n = 2$, the diagram of (d_1, d_2) can be presented as follows:

$$d_1 = d_1(x_1)\partial_1 + d_1(x_2)\partial_2,$$
$$d_2 = d_2(x_1)\partial_1 + d_2(x_2)\partial_2.$$

Now assume $d_1(x_2) - d_2(x_1) = a_1f_1 + a_2f_2$. By adding $-a_1D_{12}$ to d_1 and $-a_2D_{12}$ to d_2, we obtain

$$d_1' = d_1 - a_1D_{12} = (d_1(x_1) + a_1f_2)\partial_1 + (d_1(x_2) - a_1f_1)\partial_2,$$
$$d_2' = d_2 - a_2D_{12} = (d_2(x_1) + a_2f_2)\partial_1 + (d_2(x_2) - a_2f_1)\partial_2.$$

Therefore $d_1'(x_2) = d_2'(x_1)$.
Step 2: $n = 3$.
Now we consider $n = 3$, the diagram of d_i becomes:

$$d_1 = d_1(x_1)\partial_1 + d_1(x_2)\partial_2 + d_1(x_3)\partial_3,$$
$$d_2 = d_2(x_1)\partial_1 + d_2(x_2)\partial_2 + d_2(x_3)\partial_3,$$
$$d_3 = d_3(x_1)\partial_1 + d_3(x_2)\partial_2 + d_3(x_3)\partial_3.$$

We can first assume that

$$d_1(x_2) - d_2(x_1) = a_3 f_3,$$

because the part of the difference with respect to f_1 and f_2 can be diminished by the operation in case of $n = 2$.
Now adding $a_3 D_{23}$ to d_1 we get:

$$\begin{aligned} d_1' &= d_1(x_1)\partial_1 + (d_1(x_2) - a_3 f_3)\partial_2 + (d_1(x_3) + a_3 f_2)\partial_3, \\ d_2' &= d_2(x_1)\partial_1 + d_2(x_2)\partial_2 + d_2(x_3)\partial_3, \\ d_3' &= d_3(x_1)\partial_1 + d_3(x_2)\partial_2 + d_3(x_3)\partial_3. \end{aligned}$$

Note that $d_1'(x_2) = d_2'(x_1)$ and $d_i'(x_j) - d_j'(x_i) \in J(f)$. Therefore, in later discussion we can assume $d_1(x_2) = d_2(x_1)$, meanwhile we abuse the notation of d_i' and d_i.
Now the diagram of d_i becomes:

$$\begin{aligned} d_1 &= (*)\partial_1 + (*)\partial_2 + d_1(x_3)\partial_3, \\ d_2 &= (*)\partial_1 + (*)\partial_2 + d_2(x_3)\partial_3, \\ d_3 &= d_3(x_1)\partial_1 + d_3(x_2)\partial_2 + (*)\partial_3. \end{aligned}$$

We aim to adjust $d_1(x_3)$ and $d_3(x_1)$. By adding D_{13} to d_1 and d_3 as in Step 1 we can assume

$$d_1(x_3) - d_3(x_1) \in (f_2).$$

Assume that $d_1(x_3) - d_3(x_1) = a_2 f_2$. By adding $-a_2 D_{12}$ to d_3 we get

$$d_3' = (d_3(x_1) + a_2 f_2)\partial_1 + (d_3(x_2) - a_2 f_1)\partial_2 + (*)\partial_3.$$

In this case $d_3'(x_1) = d_1(x_3)$ and $d_3'(x_2) - d_2(x_3) \in J(f)$. So without loss of generality we can assume $d_3(x_1) = d_1(x_3)$.
We are left with the difference between $d_2(x_3)$ and $d_3(x_2)$:

$$\begin{aligned} d_1 &= (*)\partial_1 + (*)\partial_2 + (*)\partial_3, \\ d_2 &= (*)\partial_1 + (*)\partial_2 + d_2(x_3)\partial_3, \\ d_3 &= (*)\partial_1 + d_3(x_2)\partial_2 + (*)\partial_3. \end{aligned}$$

Similarly we can assume $d_3(x_2) - d_2(x_3) = a_1 f_1$. However we should take care not to influence the equality of $d_1(x_2) = d_2(x_1)$ and $d_1(x_3) = d_3(x_1)$. The adjustment is as follows:

$$\begin{aligned} d_1' &= (*)\partial_1 + ((*) + \frac{1}{2}a_1 f_3)\partial_2 + ((*) - \frac{1}{2}a_1 f_2)\partial_3, \\ d_2' &= ((*) + \frac{1}{2}a_1 f_3)\partial_1 + (*)\partial_2 + (d_2(x_3) - \frac{1}{2}a_1 f_1)\partial_3, \\ d_3' &= ((*) - \frac{1}{2}a_1 f_2)\partial_1 + (d_3(x_2) + \frac{1}{2}a_1 f_1)\partial_2 + (*)\partial_3. \end{aligned}$$

Note this adjustment makes $d_i'(x_j) = d_j'(x_i)$ hold for all i, j.
Step 4: $n = 4$.
Now we consider $n = 4$, which will be helpful for general n. The diagram of d_i is as follows:

$$\begin{aligned} d_1 &= (*)\partial_1 + d_1(x_2)\partial_2 + d_1(x_3)\partial_3 + d_1(x_4)\partial_4, \\ d_2 &= d_2(x_1)\partial_1 + (*)\partial_2 + d_2(x_3)\partial_3 + d_2(x_4)\partial_4, \\ d_3 &= d_3(x_1)\partial_1 + d_3(x_2)\partial_2 + (*)\partial_3 + d_3(x_4)\partial_4, \\ d_4 &= d_4(x_1)\partial_1 + d_4(x_2)\partial_2 + d_4(x_3)\partial_3 + (*)\partial_4. \end{aligned}$$

By operation in the case $n = 2$ we can assume

$$d_1(x_2) - d_2(x_1) \in (f_3, f_4).$$

By operation in the case $n = 3$ we can assume $d_1(x_2) = d_2(x_1)$ by adding D_{23}, D_{24} to d_1. Similarly, for $d_i(x_j) - d_j(x_i)$ containing f_l with $l \geq i$ or $l \geq j$, we can always add D_{il} or D_{jl} to d_i or d_j to diminish the f_l part. Therefore, the diagram exchanges to

$$\begin{aligned}
d_1 &= (*)\partial_1 + (*)\partial_2 + (*)\partial_3 + (*)\partial_4,\\
d_2 &= (*)\partial_1 + (*)\partial_2 + d_2(x_3)\partial_3 + d_2(x_4)\partial_4,\\
d_3 &= (*)\partial_1 + d_3(x_2)\partial_2 + (*)\partial_3 + d_3(x_4)\partial_4,\\
d_4 &= (*)\partial_1 + d_4(x_2)\partial_2 + d_4(x_3)\partial_3 + (*)\partial_4,
\end{aligned}$$

with

$$d_2(x_3) - d_3(x_2) \in (f_1),\ d_2(x_4) - d_4(x_2) \in (f_1),\ d_3(x_4) - d_4(x_3) \in (f_1, f_2).$$

Following the last adjustment in Step 3, we can diminish the difference between $d_2(x_3) - d_3(x_2)$, $d_2(x_4) - d_4(x_2)$ and $d_3(x_4) - d_4(x_3)$. To avoid occupying too much, we illustrate the adjustment for $d_3(x_4) - d_4(x_4) = 2f_1 + 2f_2$:

$$\begin{aligned}
d_1 &= (*)\partial_1 + (*)\partial_2 + ((*) + f_4)\partial_3 + ((*) - f_3)\partial_4,\\
d_2 &= (*)\partial_1 + (*)\partial_2 + ((*) + f_4)\partial_3 + ((*) - f_3)\partial_4,\\
d_3 &= ((*) + f_4)\partial_1 + ((*) + f_4)\partial_2 + (*)\partial_3 + (d_3(x_4) - f_1 - f_2)\partial_4,\\
d_4 &= ((*) - f_3)\partial_1 + ((*) - f_3)\partial_2 + (d_4(x_3) + f_1 + f_2)\partial_3 + (*)\partial_4.
\end{aligned}$$

Notice f_1 and f_2 are independent in this adjustment, that is to say, we can first diminish the difference in (f_1) then (f_2). Therefore it provides the proof for general n case.

Step 5: General n.

For $d_i(x_j) - d_j(x_i) \in J(f)$, by Step 2 we can assume

$$d_i(x_j) - d_j(x_i) \in (f_1, \cdots, f_{i-1}, f_{i+1}, \cdots, f_{j-1}, f_{j+1}, \cdots, f_n).$$

Step 3 and 4 tell us that each part in the difference with respect to f_k is independent and can be diminished without changing other equalities. Therefore we just need to do adjustment repeatedly as the last one in Step 4, and we will obtain equality of all $d_i(x_j)$ and $d_j(x_i)$ in the end. □

Remark 3.5. We can transfer the Nakai Conjecture to the construction of the tuple $(d_1, \cdots, d_n) \in \oplus^n(Der^1(A))$, satisfying $d_i(x_j) - d_j(x_i) \in J(f), \forall 1 \leq i < j \leq n$, and moreover $d_i(x_i)$ is not in $(f_1, \cdots, f_{i-1}, x_i^2, f_{i+1}, \cdots, f_n)$ for some i.

As if we find such a tuple $(d_1, d_2, \cdots, d_n)$, by theorem 3.4 and theorem 2.8, there exists $D \in Der^2(A)$, $\theta_2(D) = (d_1', d_2', \cdots, d_n')$, such that for each $1 \leq i \leq n$, $d_i - d_i'$ is A-linear combination of D_{kl}'s, then $d_i(x_i) - d_i'(x_i) \in (f_1, \cdots, f_{i-1}, f_{i+1}, \cdots, f_n)$. If $D \in der^2(A)$, theorem 3.2 tells that $d_i'(x_i) \in (f_1, \cdots, f_{i-1}, x_i, f_{i+1}, \cdots, f_n)^2$, then $d_i(x_i) \in (f_1, \cdots, f_{i-1}, x_i^2, f_{i+1}, \cdots, f_n)$ for each i, which leads to a contradiction with the hypothesis for $(d_1, d_2, \cdots, d_n)$.

Definition 3.6. For $Hess(f) = (\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j=1,\cdots,n} = (f_{ij})_{i,j=1,\cdots,n}$ the Hessian matrix of f, let M_{ij} be the complementary minor of f_{ij} and H_{ij} be the algebraic co-factor of f_{ij}. Let $M_{i_1 j_1 i_2 j_2 \cdots i_k j_k}$ be the complementary minor of the submatrix $Hess(f)\begin{bmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{bmatrix}$ and

$H_{i_1j_1i_2j_2\cdots i_kj_k}$ be the algebraic co-factor of $Hess(f)\begin{bmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{bmatrix}$. More precisely, let $\sigma_1 = (i_1, \cdots, i_n)$, $\sigma_2 = (j_1, \cdots, j_n)$ be permutations of $(1, 2, \cdots, n)$, then $H_{i_1j_1i_2j_2\cdots i_kj_k} = (sgn(\sigma_1)sgn(\sigma_2)) \cdot det\left(Hess(f)\begin{bmatrix} i_{k+1} & \cdots & i_n \\ j_{k+1} & \cdots & j_n \end{bmatrix}\right)$. It is easy to see that this expresion is independent of the choice of orders of $(i_{k+1}, \cdots, i_n)$ and $(j_{k+1}, \cdots, j_n)$, in particular, when $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$, then $H_{i_1j_1i_2j_2\cdots i_kj_k} = (-1)^{\sum_{l=1}^{k} i_l + j_l} \cdot M_{i_1j_1i_2j_2\cdots i_kj_k}$. And $H_{i_1j_1i_2j_2\cdots i_kj_k}$ is anti-symmetric for $\{i_1, \cdots, i_k\}$ and $\{j_1, \cdots, j_k\}$, for example, $H_{1224} = -H_{1422}$.

Theorem 3.7. *With notations as above, for any i, j, k, one holds the following identity*

$$w_ix_iH_{jk} - w_kx_kH_{ji} = \sum_{l\neq j}(1 - w_l)f_lH_{jkli}.$$

Proof. Notice that both sides are anti-symmetric for i and k, we may assume $i < k$ in the following proof.

As f is weighted homogeneous of weight type $(w_1, \cdots, w_n; 1)$, f_l is also weighted homogeneous of weight type $(w_1, w_2, \cdots, w_n; 1 - w_l)$, then $(1 - w_l)f_l = \sum_{s=1}^{n} w_sx_sf_{ls}$, and

$$RHS = \sum_{l\neq j}\left(\sum_{s=1}^{n} w_sx_sf_{ls}\right)H_{jkli} = \sum_{s=1}^{n} w_sx_s\left(\sum_{l\neq j} f_{ls}H_{jkli}\right).$$

For $s \neq i, k$, we have

$$\begin{aligned}
\sum_{l\neq j} f_{ls}H_{jkli} =& \sum_{l=1}^{j-1} f_{ls}(-H_{lkji}) + \sum_{l=j+1}^{n} f_{ls}H_{jkli} \\
=& \sum_{l=1}^{j-1} f_{ls}(-1)^{i+k+j+l}M_{lijk} - \sum_{l=j+1}^{n} f_{ls}(-1)^{i+k+j+l}M_{lijk} \\
=&(-1)^{i+k+j+1}det\left(Hess(f)\begin{bmatrix} 1 & 2 & \cdots & j-2 & j-1 & j+1 & j+2 & \cdots & n-1 & n \\ s & 1 & \cdots & i-1 & i+1 & \cdots & k-1 & k+1 & \cdots & n \end{bmatrix}\right) \\
=&0.
\end{aligned}$$

Similarly for $s = i$, $\sum_{l\neq j} f_{ls}H_{jkli} = \sum_{l\neq j} f_{li}H_{jkli} = H_{jk}$; for $s = k$, $\sum_{l\neq j} f_{ls}H_{jkli} = \sum_{l\neq j} f_{lk}H_{jkli} = -\sum_{l\neq j} f_{lk}H_{jilk} = -H_{ji}$, therefore

$$RHS = \sum_{s=1}^{n} w_sx_s\left(\sum_{l\neq j} f_{ls}H_{jkli}\right) = w_ix_iH_{jk} - w_kx_kH_{ji} = LHS.$$

□

Remark 3.8. If we take $d_i = H_{ji} \cdot E$ and $d_k = H_{jk} \cdot E$, then $d_i(x_k) = w_kx_kH_{ji}$ and $d_k(x_i) = w_ix_iH_{jk}$. By Theorem 3.7, $d_i(x_k) - d_k(x_i) \in J(f)$.

To finish proof of Thm.A, we need a theorem by Saito.

Theorem 3.9. *Let $f \in k[x_1, \cdots, x_n]$ be a weighted homogeneous polynomial, defining an isolated singularity $(V(f), 0)$ at the origin, then*

$$\det(Hess(f)) \notin J(f).$$

Proof. Since f is weighted homogeneous, $f \in J(f)$, and the common zero locus of the partial derivatives $f_1, f_2, \cdots, f_n$ is a single point at the origin. Consider the germs of $f_1, f_2, \cdots, f_n$ at the origin, by lemma 3.4 in [14],

$$\det\left(\frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)}\right) \notin (f_1, \cdots, f_n)\mathcal{O}_{\mathbb{C}^n,0}.$$

Therefore, $\det(Hess(f)) \notin J(f)\mathcal{O}_{\mathbb{C}^n,0}$, and necessarily $\det(Hess(f)) \notin J(f)$ in $k[x_1, \cdots, x_n]$. □

Now we begin to prove the main theorem A.

Proof. (of main theorem A)

By remark 3.5, it is enough to construct derivaions $d_1, \cdots, d_n \in Der^1(A)$, such that $d_i(x_j) - d_j(x_i) \in J(f)$, and $d_i(x_i) \notin (f_1, \cdots, f_{i-1}, x_i^2, f_{i+1}, \cdots, f_n)$ for some i.

Consider

$$d_i = H_{ij} \cdot E, 1 \leq i \leq n$$

for some fixed j. Since $d_i(x_i) = w_i x_i \cdot H_{ij}$, by Theorem 3.7, $d_i(x_j) - d_j(x_i) \in J(f)$, we only need to find $x_i H_{ij} \notin (f_1, \cdots, f_{i-1}, x_i^2, f_{i+1}, \cdots, f_n)$ for some $i, j \in \{1, \cdots, n\}$.

First, we treat the special cases: f is of Type A,B or C in proposition 2.12.
For Type A, it is obvious $x_1 H_{11} \notin (x_1^2, f_2, \cdots, f_n)$.
For Type B, $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$. In this case we have

$$f_1 = a_1 x_1^{a_1-1}x_2,\ f_2 = x_1^{a_1} + a_2 x_2^{a_2-1}x_3,\ \cdots,\ f_n = x_{n-1}^{a_{n-1}} + a_n x_n^{a_n-1}.$$

For any $g \in k[x_1, \cdots, x_n]$, we denote by $\overline{g}$ the image of g in $k[x_1, x_2, \cdots, x_n]/(x_1)$. Since

$$\overline{f_2} = x_1^{a_1} + a_2 x_2^{a_2-1}x_3 = a_2 x_2^{a_2-1}x_3 \in k[x_1, x_2, \cdots, x_n]/(x_1),$$

$(\overline{f_2}, \cdots, \overline{f_n})$ is the Jacobian ideal of $\overline{f} = x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$, which defines an isolated singularity in hyperplane $\{x_1 = 0\}$. Therefore by theorem 3.9, $\overline{H_{11}} = \det(Hess(\overline{f})) \notin (\overline{f_2}, \cdots, \overline{f_n})$, which implies

$$H_{11} \notin (x_1, f_2, \cdots, f_n).$$

As x_1 is regular in $k[x_1, \cdots, x_n]/(f_2, \cdots, f_n)$, $x_1 H_{11} \notin (x_1^2, f_2, \cdots, f_n)$.
For Type C, $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1$. Then $Hess(f) =$

$$\begin{pmatrix} a_1(a_1-1)x_1^{a_1-2}x_2 & a_1x_1^{a_1-1} & 0 & \cdots & 0 & a_nx_n^{a_n-1} \\ a_1x_1^{a_1-1} & a_2(a_2-1)x_2^{a_2-2}x_3 & a_2x_2^{a_2-1} & 0 & \cdots & 0 \\ 0 & a_2x_2^{a_2-1} & a_3(a_3-1)x_3^{a_3-2}x_4 & a_3x_3^{a_3-1} & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ a_nx_n^{a_n-1} & 0 & \cdots & 0 & a_{n-1}x_{n-1}^{a_{n-1}-1} & a_n(a_n-1)x_n^{a_n-2}x_1 \end{pmatrix}$$

We claim $H_{1n} \notin (x_1, f_2, \cdots, f_n)$.
Proof of the claim:
Expanding M_{1n} on first column, there exist two parts. The first is $f_{21}f_{32}\cdots f_{n(n-1)}$ and the second is $(-1)^n f_{n1}M_{11nn}$.
We have $f_{21}f_{32}\cdots f_{n(n-1)} \in (x_1, f_2, \cdots, f_n)$ for $f_{21} = a_1x_1^{a_1-1}$. We only need to show $f_{n1}M_{11nn} \notin (x_1, f_2, \cdots, f_n)$. Consider

$$g = x_2^{a_2}x_3 + x_3^{a_3}x_4 + \cdots + x_{n-2}^{a_{n-2}}x_{n-1} + x_{n-1}^{a_{n-1}}x_n,$$

then $g_i = f_i$ for all $3 \leq i \leq n-1$. And $g|_{\{x_n=0\}} = x_2^{a_2}x_3 + x_3^{a_3}x_4 + \cdots + x_{n-2}^{a_{n-2}}x_{n-1}$ defines an isolated singulariy in $\{x_1 = x_n = 0\}$. Then by theorem 3.9, $\det(Hess(g|_{\{x_n=0\}})) = \det(\frac{\partial(g_2,\cdots,g_{n-1})}{\partial(x_2,\cdots,x_{n-1})})|_{\{x_n=0\}} \notin (g_2, g_3\cdots, g_{n-1})$ in $k[x_2, x_3, \cdots, x_{n-1}]$. Necessarily,

$$M_{11nn} = \det(\frac{\partial(f_2,\cdots,f_{n-1})}{\partial(x_2,\cdots,x_{n-1})}) = \det(\frac{\partial(g_2,\cdots,g_{n-1})}{\partial(x_2,\cdots,x_{n-1})}) \notin (g_2, g_3\cdots, g_{n-1})$$

holds in $k[x_1, \cdots, x_n]$. We also see that

$$(x_1, f_2, \cdots, f_n) = (x_1, a_2x_2^{a_2-1}x_3+x_1^{a_1}, g_3, \cdots, g_{n-1}, x_{n-1}^{a_{n-1}}+a_nx_n^{a_n-1}x_1) = (x_1, g_2, g_3, \cdots, g_{n-1}, x_{n-1}^{a_{n-1}}).$$

Since $f_{1n}M_{11nn} = a_nx_n^{a_n-1}M_{11nn}$ is independent of the variable x_1 and any term containing x_{n-1} has degree less than a_{n-1}, quotient by x_1 and $x_{n-1}^{a_{n-1}}$ will lead to

$$a_nx_n^{a_n-1}M_{11nn} \notin (x_1, g_2, g_3, \cdots, g_{n-1}, x_{n-1}^{a_{n-1}}) = (x_1, f_2, \cdots, f_n).$$

Now we get $H_{1n} \notin (x_1, f_2, \cdots, f_n)$, as x_1 is regular in $k[x_1, \cdots, x_n]/(f_2, \cdots, f_n)$, $x_1H_{1n} \notin (x_1^2, f_2, \cdots, f_n)$.

For general case, since f is fewnomial, f is a direct sum of polynomials of Type A,B or C. We can write

$$f = h_1 + \cdots + h_l,$$

with $h_i \in k[x_{j_{i-1}+1}, \cdots, x_{j_i}]$ of Type A,B or C, where $j_0 = 0$ and $j_l = n$. We consider h_1, by above arguments, there exists a polynomial Q_1, a minor of $Hess(h_1)$ by removing the s-th row and the t-th column, such that

$$x_sQ_1 \notin (\frac{\partial h_1}{\partial x_1}, \cdots, x_s^2, \cdots, \frac{\partial h_1}{\partial x_{j_1}})= (f_1, \cdots, f_{s-1}, x_s^2, f_{s+1}, \cdots, f_{j_1}),$$

Then from $Hess(f) = \text{diag}(Hess(f_1), \cdots, Hess(f_l))$, $M_{st} = Q_1 \cdot \prod_{i=2}^n hess(h_i)$, and $x_sH_{st} = (-1)^{s+t}x_sM_{st} \notin (f_1, \cdots, f_{s-1}, x_s^2, f_{s+1}, \cdots, f_n)$, which finishes our proof. □

Remark 3.10. Indeed, the proof for fewnomial case can be applied to general weighted homogeneous cases, the problem is how to find some $H_{ij} \notin I_i := (f_1, \cdots, f_{i-1}, x_i, f_{i+1}, \cdots, f_n)$. Theorem 3.9 cannot be used when $k[x_1, \cdots, x_n]/(f_1, \cdots, f_n)$ is not Artinian. To see this, one can check for f of Type C, in this case, $H_{11} \in I_1$.

Here is another example for $H_{11} \notin I_1$.

Example 3.11. Let $f = x^6 + y^3 + z^2 + tx^4y$, which defines the $\widetilde{E}_8$ simple elliptic singularity. Then

$$Hess(f) = \begin{pmatrix} 30x^4 + 12tx^2y & 4tx^3 & 0 \\ 4tx^3 & 6y & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

as $\mathbb{C}[x, y, z]/I_1 = \mathbb{C}[x, y, z]/(x, 3y^2+tx^4, 2z) \simeq \mathbb{C}[y]/(y^2)$, $H_{11} = 12y \notin I_1$. Since $\mathbb{C}[x, y, z]/(f_y, f_z) = \mathbb{C}[x, y, z]/(tx^4 + 3y^2, 2z) \simeq \mathbb{C}[x, y]/(tx^4 + 3y^2)$, x is regular in $\mathbb{C}[x, y, z]/(f_y, f_z)$, and we have $xH_{11} \notin (x^2, f_y, f_z)$, the Nakai Conjecture holds for $\mathbb{C}[x, y, z]/(f)$.

4. The hypersurface cusp singularity case

Dimension two hypersurface cusp singularities are almost classical (see [10]), and are locally isomorphic to the so called $T_{p,q,r}$ singularities, where $T_{p,q,r}$ is the isomorphism class of the hypersurface singularities $(\{x^p + y^q + z^r + xyz = 0\}, 0)$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. In this section, we will verify the Nakai Conjecture for these types of singularities.

With notations in section 2 , let $D_1 = \sum_{i=1}^n \alpha_i \partial_{x_i}$, $D_2 = \sum_{i=1}^n \beta_i \partial_{x_i}$ $(\alpha_i, \beta_i \in P)$ induce two first order derivations on A, then $D_1 D_2 = \sum_{i,j=1}^n (\alpha_i \beta_j \partial_{x_i} \partial_{x_j} + \alpha_i \frac{\partial \beta_j}{\partial x_i} \partial_{x_j})$, and by definition the i-th term of $\theta_2(D_1 D_2)$ is $2\alpha_i \beta_i \partial_{x_i} + \sum_{j \neq i} (\alpha_i \beta_j + \alpha_j \beta_i) \partial_{x_j}$, $\forall 1 \leq i \leq n$. As $D_1(f) = \sum_{i=1}^n \alpha_i f_i \in (f)$, by taking the j-th partial derivatives on both sides, we have $\sum_{i=1}^n \alpha_i f_{ij} \in (f, J(f))$, $\forall 1 \leq j \leq n$, where $J(f) = (f_1, \cdots, f_n)$ is the Jacobi ideal of f. Express it in the matrix form, we have

$$Hess(f)(\alpha_1, \cdots, \alpha_n)^T = 0 \text{ in } R = \mathbb{C}[x_1, \cdots, x_n]/(f, J(f)) = A/(J(f)).$$

Notice that R is not the Tjurina algebra of the singularity $(V(f), 0)$, but we will see R is also Artinian later, then we can translate this equation in R to several linear equations in $\mathbb{C}$.

Now for the hypersurface cusp singularity case $(V(f), 0)$, $f = x^p + y^q + z^r + xyz$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, then $f_1 := \frac{\partial f}{\partial x} = px^{p-1} + yz$, $f_2 := \frac{\partial f}{\partial y} = qy^{q-1} + zx$, $f_3 := \frac{\partial f}{\partial z} = rz^{r-1} + xy$. As $xyz = (\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1)^{-1} (\frac{x}{p} f_1 + \frac{y}{q} f_2 + \frac{z}{r} f_3 - f)$, $xyz = 0$ in $R = \mathbb{C}[x, y, z]/(f, f_1, f_2, f_3)$ and $x^p = y^q = z^r = 0$ in R, so the unique maximal ideal $\mathfrak{m} = (x, y, z)$ is nilpotent and R is Artinian. Next we deal with it by dividing to smaller cases.

4.1. Cases of $p, q, r \geq 3$.

When $p, q, r \geq 3$, $x^2 y = x(f_3 - rz^{r-1}) = xf_3 - rz^{r-2}(f_2 - qy^{q-1}) = xf_3 - rz^{r-2} f_2 + qry^{q-2} z^{r-3} (f_1 - px^{p-1}) = xf_3 - rz^{r-2} f_2 + qry^{q-2} z^{r-3} f_1 - pqrx^{p-3} y^{q-3} z^{r-3} \cdot x^2 y$, as $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, $p + q + r > 9$, $x^{p-3} y^{q-3} z^{r-3} \in \mathfrak{m}$. Thus $x^2 y \in x^2 y \cdot \mathfrak{m}$, and $x^2 y \in x^2 y \cdot \mathfrak{m}^k$ for any integer k. Since $\mathfrak{m}$ is nilpotent in R, $x^2 y = 0$ in R, similarly, $xy^2, x^2 z, xz^2, y^2 z, yz^2$ all equal to 0 in R. So R has a $\mathbb{C}$-basis $\{1, x, \cdots, x^{p-1}, y, \cdots, y^{q-1}, z, \cdots, z^{r-1}\}$, $dim_{\mathbb{C}} R = p + q + r - 2$.

For the equation $Hess(f)(\alpha_1, \cdots, \alpha_n)^T = 0$ in R, it is written as

$$\begin{pmatrix} p(p-1)x^{p-2} & z & y \\ z & q(q-1)y^{q-2} & x \\ y & x & r(r-1)z^{r-2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0 \text{ in } R.$$

By some calculations, we see that

$$\begin{aligned} &(\alpha_1, \alpha_2, \alpha_3) \in Span_{\mathbb{C}} \{(x^2, 0, 0), (x^3, 0, 0) \cdots (x^{p-1}, 0, 0), (y^{q-1}, 0, 0), (z^{r-1}, 0, 0), (0, y^2, 0), (0, y^3, 0) \\ &\quad \cdots (0, y^{q-1}, 0), (0, x^{p-1}, 0), (0, z^{r-1}, 0), (0, 0, z^2), (0, 0, z^3) \cdots (0, 0, z^{r-1}), (0, 0, x^{p-1}), (0, 0, y^{q-1})\} \\ &= R < (x^2, 0, 0), (xy, 0, 0), (xz, 0, 0), (0, y^2, 0), (0, xy, 0), (0, yz, 0), (0, 0, z^2), (0, 0, xz), (0, 0, yz) > . \end{aligned}$$

Thus $\alpha_i \in \mathfrak{m}^2, \forall 1 \leq i \leq 3$, together with previous calculations, write $\theta_2(D_1 D_2) = (d_1, d_2, d_3)$, then $d_i(x_i) = 2D_1(x_i) D_2(x_i)$, and $D_1(x_i), D_2(x_i) \in \mathfrak{m}^2$ $(mod\ J(f))$ in A, we obtain the following lemma immediately.

Lemma 4.1. *For $A = \mathbb{C}[x, y, z]/(f)$, where $f = x^p + y^q + z^r + xyz$ with $p, q, r \geq 3, \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, let $D \in der^2(A)$ and $\theta_2(D) = (d_1, d_2, d_3)$, then*

$$d_1(x), d_2(y), d_3(z) \in \mathfrak{m}^4 + J(f) \text{ in } A,$$

which is equivalent to say, they belong to $\mathfrak{m}^4$ in R.

Now we begin to prove Nakai's conjecture for these cases. We need to do some concrete calculations. In $\mathbb{C}[x, y, z]$, we have

$$\begin{aligned} x^2 yz &= xzf_3 + xyf_2 - f_2 f_3 + (f_2 - xz)(f_3 - xy) \\ &= xzf_3 + xyf_2 - f_2 f_3 + qry^{q-1} z^{r-1} \\ &= xzf_3 + xyf_2 - f_2 f_3 + qry^{q-2} z^{r-2} (f_1 - px^{p-1}) \\ &= xzf_3 + xyf_2 - f_2 f_3 + qry^{q-3} z^{r-3} f_1 (f_1 - px^{p-1}) - pqrx^{p-1} y^{q-2} z^{r-2} \end{aligned}$$

$$=xzf_3+xyf_2-f_2f_3+qry^{q-3}z^{r-3}f_1^2-pqrx^{p-1}y^{q-3}z^{r-3}f_1-pqrx^{p-1}y^{q-2}z^{r-2}.$$

As $xyz=(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1)^{-1}(\frac{x}{p}f_1+\frac{y}{q}f_2+\frac{z}{r}f_3-f)$, then $x(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1)^{-1}(\frac{x}{p}f_1+\frac{y}{q}f_2+\frac{z}{r}f_3-f)$ $=xzf_3+xyf_2-f_2f_3+qry^{q-3}z^{r-3}f_1^2-pqrx^{p-1}y^{q-3}z^{r-3}f_1-pqrx^{p-2}y^{q-3}z^{r-3}(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1)^{-1}(\frac{x}{p}f_1+\frac{y}{q}f_2+\frac{z}{r}f_3-f)$. Since $x^{p-2}q^{q-2}z^{r-3},x^{p-2}y^{q-3}z^{r-2}=0$ in R, $x^{p-2}q^{q-2}z^{r-3},x^{p-2}y^{q-3}z^{r-2}\in J(f)$ in A, and we have

$$(\frac{x^2}{p}+pqr(\frac{2}{p}+\frac{1}{q}+\frac{1}{r}-1)x^{p-1}y^{q-3}z^{r-3})f_1+(1-\frac{1}{p}-\frac{1}{r})xyf_2+(1-\frac{1}{p}-\frac{1}{q})xzf_3\equiv 0\ (mod\ J(f)^2)$$

in A. So there exist $\alpha_{11},\alpha_{12},\alpha_{13}\in A$, such that $\alpha_{11}f_1+\alpha_{12}f_2+\alpha_{13}\alpha_3=0$ in A, where

$$\alpha_{11}\equiv(1-\frac{1}{p}-\frac{1}{r})^{-1}(1-\frac{1}{p}-\frac{1}{q})^{-1}(\frac{x^2}{p}+pqr(\frac{2}{p}+\frac{1}{q}+\frac{1}{r}-1)x^{p-1}y^{q-3}z^{r-3})\ (mod\ J(f)),$$

$$\alpha_{12}\equiv(1-\frac{1}{p}-\frac{1}{q})^{-1}xy\ (mod\ J(f)),$$

$$\alpha_{13}\equiv(1-\frac{1}{p}-\frac{1}{r})^{-1}xz\ (mod\ J(f)).$$

Similarly, there exist $\alpha_{21},\alpha_{22},\alpha_{23}\in A$, $\alpha_{31},\alpha_{32},\alpha_{33}\in A$, such that $\alpha_{21}f_1+\alpha_{22}f_2+\alpha_{23}f_3=0$ in A, $\alpha_{31}f_1+\alpha_{32}f_2+\alpha_{33}f_3=0$ in A, satisfying

$$\alpha_{21}\equiv(1-\frac{1}{p}-\frac{1}{q})^{-1}xy\ (mod\ J(f)),$$

$$\alpha_{22}\equiv(1-\frac{1}{q}-\frac{1}{r})^{-1}(1-\frac{1}{q}-\frac{1}{p})^{-1}(\frac{y^2}{q}+pqr(\frac{2}{q}+\frac{1}{p}+\frac{1}{r}-1)x^{p-3}y^{q-1}z^{r-3})\ (mod\ J(f)),$$

$$\alpha_{23}\equiv(1-\frac{1}{q}-\frac{1}{r})^{-1}yz\ (mod\ J(f));$$

$$\alpha_{31}\equiv(1-\frac{1}{p}-\frac{1}{r})^{-1}xz\ (mod\ J(f)),$$

$$\alpha_{32}\equiv(1-\frac{1}{q}-\frac{1}{r})^{-1}yz\ (mod\ J(f)),$$

$$\alpha_{33}\equiv(1-\frac{1}{p}-\frac{1}{r})^{-1}(1-\frac{1}{q}-\frac{1}{r})^{-1}(\frac{z^2}{r}+pqr(\frac{2}{r}+\frac{1}{p}+\frac{1}{q}-1)x^{p-3}y^{q-3}z^{r-1})\ (mod\ J(f)).$$

From these calculations, we obtain the following proposition immediately.

Proposition 4.2. *Let $A=\mathbb{C}[x,y,z]/(f)$, where $f=x^p+y^q+z^r+xyz$ with $p,q,r\geq 3$, $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$ defines a hypersurface cusp singularity, then there exists $D\in Der^2(A)$ does not lie in $der^2(A)$.*

Proof. With notations as above, we have seen that $d_i:=\sum_{j=1}^{3}\alpha_{ij}\partial_{x_j}$, $1\leq i\leq 3$ are three derivations in $Der^1(A)$. Since for any $1\leq i,j\leq 3$, $d_i(x_j)-d_j(x_i)=\alpha_{ij}-\alpha_{ji}\in J(f)$, by theorem 3.4 and theorem 2.8, there exists $D\in Der^2(A)$, $\theta_2(D)=(d_1',d_2',d_3')$ such that $d_1(x)-d_1'(x)\in J(f)$ in A. And $d_1(x)=\alpha_{11}\notin\mathfrak{m}^4+J(f)$ forces $d_1'(x)\notin\mathfrak{m}^4+J(f)$. So D does not lie in $der^2(A)$ follows from lemma 4.1. □

4.2. **Cases of** $p, q \geq 4, r = 2$.

When $p, q \geq 4, r = 2$, $R = \mathbb{C}[x,y,z]/(f, J(f)) = \mathbb{C}[x,y,z]/(px^{p-1} + yz, qy^{q-1} + zx, 2z + xy, xyz) = \mathbb{C}[x,y]/(x^2y^2, 2px^{p-1} - xy^2, 2qy^{q-1} - x^2y)$, and $x^3y = x^2(f_3 - 2z) = x^2f_3 - 2x(f_2 - qy^{q-1}) = x^2f_3 - 2xf_2 + 2qy^{q-2}(f_3 - 2z) = x^2f_3 - 2xf_2 + 2qy^{q-2}f_3 - 4qy^{q-3}(f_1 - px^{p-1}) \equiv 4pqx^3y \cdot x^{p-4}y^{q-4}$ $(mod\ (f, J(f)))$, as $\frac{1}{p} + \frac{1}{q} + \frac{1}{2} < 1$, $p + q > 8$, $x^{p-4}y^{q-4} \in \mathfrak{m}$. Thus $x^3y \in x^3y \cdot \mathfrak{m}$, and $x^3y \in x^3y \cdot \mathfrak{m}^k$ for any integer k, hence $x^3y = 0$ in R from the nilpotency of $\mathfrak{m}$, similarly, $xy^3 = 0$ in R. So R has a $\mathbb{C}$-basis $\{1, x, \cdots, x^{p-1}, y, \cdots, y^{q-1}, xy\}$, $dim_{\mathbb{C}}R = p + q$.

For the equations $Hess(f)(\alpha_1, \alpha_2, \alpha_3)^T = 0$ in R, we calculated that

$$(\alpha_1, \alpha_2, \alpha_3) \in R < (x^3, 0, 0), (y^{q-1}, 0, 0), (y^{q-2}, \frac{1}{2q}xy, 0), (y^{q-2}, 0, -\frac{1}{2}y^{q-1}), (x^2, -xy, 0), (x^2, 0, -qy^{q-1}),$$
$$(0, y^3, 0), (0, x^{p-1}, 0), (\frac{1}{2p}xy, x^{p-2}, 0), (0, x^{p-2}, -\frac{1}{2}x^{p-1}), (-xy, y^2, 0), (0, y^2, -px^{p-1}) > .$$

We see that each $\alpha_i \in \mathfrak{m}^2$, lemma 4.1 still holds for this case.

Lemma 4.3. *For $A = \mathbb{C}[x,y,z]/(f)$, where $f = x^p + y^q + z^2 + xyz$ with $p, q \geq 4$, $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, let $D \in der^2(A)$ and $\theta_2(D) = (d_1, d_2, d_3)$, then*

$$d_1(x), d_2(y), d_3(z) \in \mathfrak{m}^4 + J(f) \text{ in } A.$$

Similar as the construction of the $p, q, r \geq 3$ case, we just need to construct a matrix $Q = (\alpha_{ij})_{3\times 3}$ in A, such that $Q \cdot (f_1, f_2, f_3)^T = 0$ and $\alpha_{ij} - \alpha_{ji} \in J(f)$, and one of the $\alpha_{11}, \alpha_{22}, \alpha_{33}$ is not in $\mathfrak{m}^4 + J(f)$.

In $\mathbb{C}[x,y,z]$, $y^{q-1}z = (f_1 - px^{p-1})y^{q-2} = y^{q-2}f_1 - px^{p-1}y^{q-2} = y^{q-2}f_1 - px^{p-2}y^{q-3}(f_3 - 2z)$, $x^{p-2}y^{q-3} = x^2y \cdot x^{p-4}y^{q-4} \in (f, J(f))$, $x^{p-2}y^{q-3}z = x^{p-3}y^{q-4}(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1)^{-1}(\frac{x}{p}f_1 + \frac{y}{q}f_2 + \frac{z}{r}f_3 - f)$, and $x^{p-3}y^{q-4}z \in (f, J(f))$, so we have

$$\begin{aligned}\frac{x^2}{p}f_1 + \frac{xy}{q}f_2 + \frac{xz}{r}f_3 \equiv& (\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1)x^2yz\\ \equiv& (\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1)(xyf_2 + xzf_3 - f_2f_3 + 2qy^{q-1}z)\\ \equiv& (\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1)(2qy^{q-2}f_1 + xyf_2 + xzf_3) + 4qx^{p-2}y^{q-4}f_1 + 4px^{p-3}y^{q-3}f_2\end{aligned}$$
$$(mod\ (f, J(f)^2)). \quad (*)$$

From the symmetry of (x, p) and (y, q), we also have

$$\frac{xy}{p}f_1 + \frac{y^2}{q}f_2 + \frac{yz}{r}f_3 \equiv (\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1)(2px^{p-2}f_2 + xyf_1 + yzf_3) + 4px^{p-4}y^{q-2}f_2 + 4qx^{p-3}y^{q-3}f_1$$
$$(mod\ (f, J(f)^2)) \quad (*'),$$

(1) If $p + q \geq 10$, $x^{p-3}y^{q-3} \in (f, J(f))$, we can choose $\alpha_{11}, \alpha_{12}, \alpha_{13} \in A$ and $\alpha_{21}, \alpha_{22}, \alpha_{23} \in A$, such that

$$\alpha_{11} \equiv (1 - \frac{1}{p} - \frac{1}{r})^{-1}(1 - \frac{1}{p} - \frac{1}{q})^{-1}(\frac{x^2}{p} - 2q(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1)y^{q-2} - 4qx^{p-2}y^{q-4}) \ (mod\ J(f)),$$

$$\alpha_{12} \equiv (1 - \frac{1}{p} - \frac{1}{q})^{-1}xy \ (mod\ J(f)),$$

$$\alpha_{13} \equiv (1 - \frac{1}{p} - \frac{1}{r})^{-1}xz \ (mod\ J(f));$$

$$\alpha_{21} \equiv (1 - \frac{1}{p} - \frac{1}{q})^{-1}xy \ (mod\ J(f)),$$

$$\alpha_{22} \equiv (1-\frac{1}{q}-\frac{1}{r})^{-1}(1-\frac{1}{p}-\frac{1}{q})^{-1}(\frac{y^2}{q}-2p(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1)x^{p-2}-4px^{p-4}y^{q-2})\ (mod\ J(f)),$$

$$\alpha_{23} \equiv (1-\frac{1}{q}-\frac{1}{r})^{-1}yz\ (mod\ J(f)).$$

For α_{31}, α_{32} and α_{33}, we use the following identity in $\mathbb{C}[x, y, z]$,

$$\begin{aligned} xyz^2 &= yzf_2 + xzf_1 - f_1f_2 + pqx^{p-1}y^{q-1} \\ &= yzf_2 + xzf_1 - f_1f_2 + pqx^{p-2}y^{q-2}(f_3 - rz^{r-1}) \\ &= yzf_2 + xzf_1 - f_1f_2 + pqx^{p-3}y^{q-3}f_3(f_3 - rz^{r-1}) - pqrx^{p-2}y^{q-2}z^{r-1} \\ &\equiv yzf_2 + xzf_1 - pqrx^{p-3}y^{q-3}z^{r-1}f_3 - pqrx^{p-3}y^{q-3}z^{r-2}(\frac{x}{p}f_1 + \frac{y}{q}f_2 + \frac{z}{r}f_3)(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1)^{-1} \\ &\quad (mod\ (f, J(f)^2)), \end{aligned}$$

as $r = 2, p, q \geq 4, p + q > 8$, $x^{p-2}y^{q-3}z^{r-2} = x^{p-2}y^{q-3} = 0$ in R, similarly $x^{p-3}y^{q-2}z^{r-2} = x^{p-3}y^{q-2} = 0$ in R and $x^{p-3}y^{q-3}z^{r-1} = -\frac{1}{2}x^{p-2}y^{q-2} = 0$ in R, so

$$z(\frac{x}{p}f_1 + \frac{y}{q}f_2 + \frac{z}{r}f_3)(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1)^{-1} \equiv yzf_2 + xzf_1\ (mod\ (f, J(f)^2)).$$

We can choose $\alpha_{31} \equiv (1-\frac{1}{p}-\frac{1}{r})^{-1}xz\ (mod\ J(f))$, $\alpha_{32} \equiv (1-\frac{1}{q}-\frac{1}{r})^{-1}xy\ (mod\ J(f))$, and $\alpha_{33} \equiv 0\ (mod\ J(f))$. We see that $\alpha_{11} \equiv (1-\frac{1}{p}-\frac{1}{r})^{-1}(1-\frac{1}{p}-\frac{1}{q})^{-1}(\frac{x^2}{p}-2q(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1)y^{q-2})\ (mod\ (\mathfrak{m}^4, J(f)))$, so $\alpha_{11} \notin (\mathfrak{m}^4, J(f))$.

(2) If $p+q < 10$, we may assume $p = 5, q = 4$. Multiplying x on both sides of $(*)$, and notice that $x^2z, x^{p-2}y^{q-3} = x^3y \in (f, J(f))$, we have

$$(\frac{x^3}{5}+\frac{2}{5}xy^2-16x^4)f_1 + \frac{3}{10}x^2yf_2 \equiv 0\ (mod\ (f, J(f)^2)).$$

Multiplying x on both sides of $(*')$, and notice that $xyz \in (f, J(f))$, $x^{p-2}y^{q-3} = x^3y \in (f, J(f))$, $x^{p-3}y^{q-2} = x^2y^2 \in (f, J(f))$, we have

$$\frac{1}{4}x^2yf_1 + \frac{3}{10}xy^2f_2 \equiv 0\ (mod\ (f, J(f)^2)).$$

Now there exists $Q = (\alpha_{ij})$ in A, $Q \cdot (f_1, f_2, f_3)^T = 0$, and satisfies $\alpha_{11} \equiv \frac{10}{3}(\frac{x^3}{5}+\frac{2}{5}xy^2-16x^4)\ (mod\ J(f))$, $\alpha_{12} \equiv x^2y\ (mod\ J(f))$, $\alpha_{21} \equiv x^2y\ (mod\ J(f))$, $\alpha_{22} \equiv \frac{6}{5}xy^2\ (mod\ J(f))$, and $\alpha_{ij} \equiv 0\ (mod\ J(f))$ if $i = 3$ or $j = 3$. We see that $\alpha_{11} \notin (\mathfrak{m}^4, J(f))$.

From this concrete constructions, we get the following property.

Proposition 4.4. *Let $A = \mathbb{C}[x, y, z]/(f)$, where $f = x^p + y^q + z^2 + xyz$ with $p, q \geq 4, \frac{1}{p}+\frac{1}{q} < \frac{1}{2}$ defines a hypersurface cusp singularity, then there exists $D \in Der^2(A)$ does not lie in $der^2(A)$.*

Proof. With notations and calculations as above, there exists a matrix $Q = (\alpha_{ij})_{3\times 3}$ in A, satisfying $Q \cdot (f_1, f_2, f_3)^T = 0$, $Q^T \equiv Q\ (mod\ J(f))$, and $\alpha_{11} \notin (\mathfrak{m}^4, J(f))$. Then $d_i := \sum_{j=1}^{3} \alpha_{ij}\partial_{x_j}$, $1 \leq i \leq 3$ are three derivations in $Der^1(A)$, and $d_i(x_j) - d_j(x_i) = \alpha_{ij} - \alpha_{ji} \in J(f)$. By theorem 3.4 and theorem 2.8, there exists $D \in Der^2(A)$, $\theta_2(D) = (d_1', d_2', d_3')$ such that $d_1(x) - d_1'(x) \in J(f)$ in A. And $d_1(x) = \alpha_{11} \notin \mathfrak{m}^4 + J(f)$ forces $d_1'(x) \notin \mathfrak{m}^4 + J(f)$. So D does not lie in $der^2(A)$ follows from lemma 4.3. ☐

4.3. **Cases of $p \geq 7, q = 3, r = 2$.**

When $p > 6, q = 3, r = 2$, $R = \mathbb{C}[x,y,z]/(f, J(f)) = \mathbb{C}[x,y,z]/(px^{p-1}+yz, 3y^2+xz, 2z+xy) \backsimeq \mathbb{C}[x,y]/(x^2y^2, 2px^{p-1} - xy^2, 6y^2 - x^2y)$, then $x^3y = 6y^2x = 12px^{p-1}$ in R, R has a $\mathbb{C}$-basis $\{1, x, \cdots, x^{p-1}, y, y^2, xy\}$, $dim_{\mathbb{C}} R = p+3$.

For the equation $Hess(f) \cdot (\alpha_1, \alpha_2, \alpha_3)^T = 0$ in $\mathbb{R}$, we calculated that

$$(\alpha_1, \alpha_2, \alpha_3) \in R < (x^4, 0, 0), (x^3, -12px^{p-2}, 0), (x^3, -6y^2, 0), (x^3, 0, -6px^{p-1}), (x^2, 2xy, -9y^2), (y^2, 0, 0), (y, \frac{1}{3}xy, -\frac{3}{2}y^2), (xy, -y^2, 0), (xy, -2px^{p-2}, 0), (xy, 0, -px^{p-1}), (0, x^4, -\frac{1}{2}x^5) > .$$

We see that each $\alpha_2, \alpha_3 \in \mathfrak{m}^2$, $\alpha_1 \in (x^2, y)$, lemma 4.1 can be adjusted as following.

Lemma 4.5. *Let $A = \mathbb{C}[x,y,z]/(f)$ where $f = x^p + y^3 + z^2, p \geq 7$, $D \in der^2(A)$, and $\theta_2(D) = (d_1, d_2, d_3)$, then*

$$d_1(x) \in (x^2y, x^4) + J(f), \text{ and } d_2(y), d_3(z) \in \mathfrak{m}^4 + J(f).$$

We begin to do some concrete calculations, in $\mathbb{C}[x,y,z]$, $x^2(\frac{x}{p}f_1 + \frac{y}{3}f_2 + \frac{z}{2}f_3 - f)(\frac{1}{p} + \frac{1}{3} + \frac{1}{2} - 1)^{-1} = x^3yz = x^2zf_3 + x^2yf_2 - xf_2f_3 + 6xy^2z = x^2zf_3 + x^2yf_2 - xf_2f_3 + 6y(\frac{x}{p}f_1 + \frac{y}{3}f_2 + \frac{z}{2}f_3 - f)(\frac{1}{p} + \frac{1}{3} + \frac{1}{2} - 1)^{-1}$, then we have

$$\frac{x^3 - 6xy}{p} f_1 + (\frac{1}{6} - \frac{1}{p})x^2yf_2 + 6p(\frac{1}{p} - \frac{1}{6})x^{p-1}f_3 \equiv 0 \ (mod\ (f, J(f)^2)).$$

As $xy(\frac{x}{p}f_1 + \frac{y}{3}f_2 + \frac{z}{2}f_3 - f)(\frac{1}{p} + \frac{1}{3} + \frac{1}{2} - 1)^{-1} = x^2y^2z = x(xyf_1 + yzf_3 - f_1f_3 + 2px^{p-1}z) = x^2yf_1 + xyzf_3 - xf_1f_3 + 2px^pz = x^2yf_1 + xyzf_3 - xf_1f_3 + 2px^{p-1}(f_2 - 3y^2) = x^2yf_1 + xyzf_3 - xf_1f_3 + 2px^{p-1}f_2 - 6px^{p-2}y(f_3 - 2z) = x^2yf_1 + xyzf_3 - xf_1f_3 + 2px^{p-1}f_2 - 6px^{p-2}yf_3 + 12px^{p-2}yz = x^2yf_1 + xyzf_3 - xf_1f_3 + 2px^{p-1}f_2 - 6px^{p-2}yf_3 + 12px^{p-3}(\frac{x}{p}f_1 + \frac{y}{3}f_2 + \frac{z}{2}f_3 - f)(\frac{1}{p} + \frac{1}{3} + \frac{1}{2} - 1)^{-1}$, and moreover $x^{p-3}y \in (f, J(f))$, we have

$$(\frac{1}{6}x^2y - 12x^{p-2})f_1 + (\frac{xy^2}{3} - 2p(\frac{1}{p} - \frac{1}{6})x^{p-1})f_2 \equiv 0 \ (mod\ (f, J(f)^2)).$$

Multiplying x on both sides leading to

$$x^{p-1}f_1 \equiv 0 \ (mod\ (f, J(f)^2)).$$

$x^{p-2}f_2 = x^{p-2}(3y^2 + zx) = 3x^{p-3}y(f_3 - 2z) + x^{p-1}z \equiv -6x^{p-3}yz + x^{p-1}\frac{f_3 - xy}{2} \equiv -6x^{p-4}(\frac{x}{p}f_1 + \frac{y}{3}f_2 + \frac{z}{2}f_3)(\frac{1}{p} - \frac{1}{6})^{-1} + \frac{x^{p-1}}{2}f_3 - \frac{x^py}{2}$ $(mod\ (f, J(f)^2))$.

(1)If $p \geq 8$, $x^{p-4}y, x^{p-4}z \in (f, J(f))$, so we have

$$(\frac{1}{p} - \frac{1}{6})(x^{p-2}f_2 - \frac{x^{p-1}}{2}f_3) \equiv -\frac{6}{p}x^{p-3}f_1 - \frac{y}{2p}((\frac{1}{p} - \frac{1}{6})xf_1 - \frac{x}{p}f_1 - \frac{y}{3}f_2 - \frac{z}{2}f_3) \ (mod\ (f, J(f)^2)).$$

Using the standard $\mathbb{C}$-basis of R, the above equations can be written as

$$\begin{pmatrix} \frac{x^3 - 6xy}{p} & (1 - \frac{6}{p})y^2 & (6-p)x^{p-1} \\ (\frac{6}{p}x^{p-3} - \frac{1}{12p}xy) & (\frac{1}{p} - \frac{1}{6})x^{p-2} - \frac{1}{6p}y^2 & (\frac{1}{3} - \frac{1}{2p})x^{p-1} \\ y^2 - 12x^{p-2} & (p-2)x^{p-1} & 0 \\ x^{p-1} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \equiv 0 \ (mod\ (f, J(f)^2)).$$

Multiplying the matrix $\begin{pmatrix} \frac{p}{p-6}\cdot(\frac{1}{12p}+\frac{1}{72}) & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{12}(\frac{1}{p}-\frac{1}{6}) & 0 \\ 0 & 0 & 0 & -\frac{p}{72}+\frac{1}{4}-\frac{1}{2p} \end{pmatrix}$ on both sides, and modulo f, we have

$$\begin{pmatrix} \frac{6}{p}x^{p-3}+\frac{p+6}{72p(p-6)}x^3-\frac{(p-2)(p-3)}{12p(p-6)}xy & -\frac{1}{12}(\frac{1}{p}-\frac{1}{6})(y^2-12x^{p-2}) & (-\frac{p}{72}+\frac{1}{4}-\frac{1}{2p})x^{p-1} \\ -\frac{1}{12}(\frac{1}{p}-\frac{1}{6})(y^2-12x^{p-2}) & -\frac{1}{12}(\frac{1}{p}-\frac{1}{6})(p-2)x^{p-1} & 0 \\ (-\frac{p}{72}+\frac{1}{4}-\frac{1}{2p})x^{p-1} & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}\equiv 0$$

$(mod\ J(f)^2)$ in A.

Therefore, we can take $Q=(\alpha_{ij})$ in A such that $Q\cdot(f_1,f_2,f_3)^T=0$ and

$$Q\equiv\begin{pmatrix} \frac{6}{p}x^{p-3}+\frac{p+6}{72p(p-6)}x^3-\frac{(p-2)(p-3)}{12p(p-6)}xy & -\frac{1}{12}(\frac{1}{p}-\frac{1}{6})(y^2-12x^{p-2}) & (-\frac{p}{72}+\frac{1}{4}-\frac{1}{2p})x^{p-1} \\ -\frac{1}{12}(\frac{1}{p}-\frac{1}{6})(y^2-12x^{p-2}) & -\frac{1}{12}(\frac{1}{p}-\frac{1}{6})(p-2)x^{p-1} & 0 \\ (-\frac{p}{72}+\frac{1}{4}-\frac{1}{2p})x^{p-1} & 0 & 0 \end{pmatrix}\ (mod\ J(f)),$$

we can see that $\alpha_{11}\notin(x^2y,x^4)+J(f)$.

(2)If $p=7$, similar calculations lead to the following equations:

$$\begin{pmatrix} \frac{x^3-6xy}{7} & \frac{1}{7}y^2 & -x^6 \\ \frac{6}{7}x^4-\frac{1}{84}xy & -\frac{1}{42}x^5+168x^6-\frac{1}{42}y^2 & \frac{11}{42}x^6 \\ y^2-12x^5 & 5x^6 & 0 \\ x^6 & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}\equiv 0\ (mod\ (f,J(f)^2)).$$

We can take $Q=(\alpha_{ij})$ in A such that $Q\cdot(f_1,f_2,f_3)^T=0$ and

$$Q\equiv\begin{pmatrix} -\frac{91}{12} & -42 & 42\times\frac{168}{5} & 0 \\ 0 & 0 & -\frac{1}{12} & 0 \\ 0 & 0 & 0 & -\frac{41}{12} \end{pmatrix}\cdot\begin{pmatrix} \frac{x^3-6xy}{7} & \frac{1}{7}y^2 & -x^6 \\ \frac{6}{7}x^4-\frac{1}{84}xy & -\frac{1}{42}x^5+168x^6-\frac{1}{42}y^2 & \frac{11}{42}x^6 \\ y^2-12x^5 & 5x^6 & 0 \\ x^6 & 0 & 0 \end{pmatrix}$$

$$\equiv\begin{pmatrix} -\frac{42\times168\times12}{5}x^5-36x^4-\frac{13}{12}x^3+7xy+\frac{42\times168}{5}y^2 & -\frac{1}{12}y^2+x^5 & -\frac{41}{12}x^6 \\ -\frac{1}{12}y^2+x^5 & -\frac{5}{12}x^6 & 0 \\ -\frac{41}{12}x^6 & 0 & 0 \end{pmatrix}\ (mod\ J(f)),$$

we can see that $\alpha_{ij}\equiv\alpha_{ji}\ (mod\ J(f))$, and $\alpha_{11}\notin(x^2y,x^4)+J(f)=(y^2,x^4)+J(f)$.

These constructions together with lemma 4.5 lead to the following proposition.

Proposition 4.6. *Let $A=\mathbb{C}[x,y,z]/(f)$ where $f=x^p+y^3+z^2, p\geq 7$ defines a hypersurface cusp singularity, then there exists $D\in Der^2(A)$ does not lie in $der^2(A)$.*

Proof. With notations and calculations as above, there exists a matrix $Q=(\alpha_{ij})_{3\times3}$ in A, satisfying $Q\cdot(f_1,f_2,f_3)^T=0$, $Q^T\equiv Q\ (mod\ J(f))$, and $\alpha_{11}\notin(x^2y,x^4)+J(f)$. Then $d_i:=\sum_{j=1}^{3}\alpha_{ij}\partial_{x_j}$, $1\leq i\leq 3$ are three derivations in $Der^1(A)$, and $d_i(x_j)-d_j(x_i)=\alpha_{ij}-\alpha_{ji}\in J(f)$. By theorem 3.4 and theorem 2.8, there exists $D\in Der^2(A)$, $\theta_2(D)=(d_1',d_2',d_3')$ such that $d_1(x)-d_1'(x)\in J(f)$ in A. And $d_1(x)=\alpha_{11}\notin(x^2y,x^4)+J(f)$ forces $d_1'(x)\notin(x^2y,x^4)+J(f)$. So D does not lie in $der^2(A)$ follows from lemma 4.5. □

Now we can complete the proof of the main theorem B

Proof. (of main theorem B) Thm.B follows from propositions 4.2, 4.4 and 4.6. □

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