NORMAL SURFACE SINGULARITIES OF SMALL DEGREES

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ABSTRACT. The notion of the Yau sequence was introduced by Tomaru, as an attempt to extend Yau's elliptic sequence for (weakly) elliptic singularities to normal surface singularities of higher fundamental genera. In this paper, we obtain the canonical cycle using the Yau cycle for certain surface singularities of degree two. Furthermore, we obtain a formula of arithmetic genera and an upper bound of geometric genera for these singularities. We also give some properties about the classification of weighted dual graphs of certain surface singularities of degree two.

Keywords. normal singularities, Yau cycle, canonical cycle.

MSC(2020). 14B05, 32S25.

1. INTRODUCTION

Let (V, o) be an isolated singular point on a surface V, and $\pi\colon X \to V$ be a resolution of V. Due to the negative definiteness of the intersection matrix of $\pi^{-1}(o)$, there exists a non-zero effective divisor with support $\pi^{-1}(o)$ that has a non-positive intersection number with every exceptional curve. We define Z as the unique minimal such divisor and refer to it as the *fundamental cycle*. Notably, $-Z^2$ represents one of the most fundamental invariants of (V, o) that remains independent of any choice in resolutions. We denote $-Z^2$ as the *degree* of (V, o).

In this paper, we investigate surface singularities by examining decompositions of various cycles. A primary focus is the *Yau sequence*, introduced by Tomaru [8], which extends the first author's elliptic sequence [9] to singularities with larger fundamental genera. Additionally, we consider the *Yau cycle* (see [4]), defined as the sum of all curves present in the Yau sequence (see Definition 2.15). Notably, the Yau cycles exhibit a strong connection with canonical cycles in certain types of singularities. Konno explores applications of the Yau cycle in degree one singularities in [4]. In this paper, we extend Konno's results to degree two singularities and establish additional properties within a more restrictive context.

Firstly, we discussed the relation between the canonical cycle and the Yau cycle of surface singularities of degree two. Compared to degree one situations, the degree two situations are more complicated. In the degree one situations, we know that the canonical cycle is obtained as a multiple of the Yau cycle, when Z is essentially irreducible. But the property doesn't hold in the degree two situations, so we consider a more restrictive condition $D_m = Z_{min}$ in Theorem 3.6 to generalize the property, where m denotes the length of the Yau sequence for Z and D_m is the smallest cycle in the Yau sequence. And we point out that the condition $D_m = Z_{min}$ holds when (V, o) are singularities of degree one.

Theorem A. *Let (V, o) be a normal surface singularity of degree two with $p_f(V, o) > 0$, and $\pi\colon X \to V$ be a minimal resolution of V. Assume that fundamental cycle Z is essentially irreducible and $D_m = Z_{min}$. Then (V, o) is numerically Gorenstein with canonical cycle $p_f(V, o)Y$.*

For surface singularities of degree one, Konno obtains a lower bound on the arithmetic genus by using the Yau cycle Y in [4]. For those with essentially irreducible Z (see Definition 3.1), we have found that the arithmetic genus is equal to this lower bound.

Theorem B. *Let (V, o) be a normal surface singularity of degree one with $p_f(V, o) > 0$, Z is the fundamental cycle on the minimal resolution. Assume that Z is essentially irreducible, then $p_a(V, o) = \frac{p(p-1)m}{2} + 1$, where $p = p_f(V, o)$ and m denotes the length of the Yau sequence for Z.*

We want to get a formula for the arithmetic genus of a singular point of degree 2 with essentially irreducible Z like in the Theorem B. For the degree two case, unlike the degree one case, if we add the additional condition that Z is essentially irreducible which still doesn't imply that $D_m = Z_{min}$. However, the condition $D_m = Z_{min}$ is necessary when the arithmetic genus is equal to the lower bound. In order to study singularities of degree two under the condition $D_m = Z_{min}$, In Theorem 3.9, we classify the weighted dual graphs of surface singularities of the degree two when $m > 1$. By classifying these graphs, we compute D_m and Z_{min} for each case in Theorem 3.9 and obtain all cases which satisfy the conditions of $m > 1$ and $D_m = Z_{min}$. From the classification of weighted dual graphs for these singularities, we obtain a formula in Theorem 3.14 (i.e., Theorem C) for their arithmetic genera.

Theorem C. *Let (V, o) be a normal surface singularity of degree two with $p_f(V, o) > 0$, and $\pi\colon X \to V$ be a minimal resolution of V. Assuming that the fundamental cycle Z is essentially irreducible and $D_m = Z_{min}$, then $p_a(V, o) = [\frac{p^2}{4}]m + 1$ and $p_g(V, o) \leq [\frac{(p+1)^2}{4}]m$.*

2. PRELIMINARIES

2.1. **Riemann-Roch and fundamental cycle.** Let (V, o) be an isolated singular point on a surface V, and $\pi\colon X \to V$ be a resolution of V. Let $\pi^{-1}(o) = \cup E_i$, $1 \leq i \leq n$, be the decomposition of the exceptional set $\pi^{-1}(o)$ into irreducible components.

The cycles are divisors of the form $D = \sum d_i E_i$ with $d_i \in \mathbb{Z}$, where E_i are irreducible exceptional curves. There is a natural partial ordering of the cycles: $D_1 = \sum_i m_i E_i \leq D_2 = \sum_i n_i E_i$ if and only if $m_i \leq n_i$ for all i. If $D_1 \leq D_2$ but $D_1 \neq D_2$ then we write $D_1 < D_2$. We let supp $D = \cup E_i$, $d_i \neq 0$, denote the support of D.

For a cycle $D = \sum d_i E_i$ on $\pi^{-1}(o)$, $\chi(D)$ is defined by

$$\chi(D) = \dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D),$$

where $\mathcal{O}_D = \mathcal{O}/\mathcal{O}(-D)$. Then by Riemann-Roch theorem [7, Proposition IV.4, p. 75], we have

$$\chi(D) = -\frac{1}{2}(D^2 + D \cdot K),$$

where K is the canonical divisor on X and $D \cdot K$ is the intersection number of D and K. For any irreducible curve E_i, the adjunction formula [7, Proposition IV, 5, p. 75] says

$$E_i \cdot K = -E_i^2 + 2g_i + 2\delta_i - 2,$$

where g_i is the genus of E_i and δ_i is the degree of the conductor of E_i. The arithmetic genus of $D \geq 0$ is defined by $p_a(D) = 1 - \chi(D)$. It follows immediately from Riemann-Roch theorem that if A and B are cycles, then

$$p_a(A + B) = p_a(A) + p_a(B) + A \cdot B - 1.$$

Definition 2.1. Let $\pi \colon X \to V$ be a resolution, then the intersection form is negative definite on the exceptional set $\pi^{-1}(o)$. Hence, there exists a cycle $D > 0$ with support $\pi^{-1}(o)$ such that $E_i \cdot D \leq 0$ for all E_i. We denote by Z the smallest one among such cycles and call it the fundamental cycle [1, 131-132].

Because the fundamental cycle Z is the smallest, for any proper subcycle D of Z, there exists an E_i such that $E_i \cdot D > 0$ and such that $E_i < Z - D$. In fact, the fundamental cycle Z can be computed from the intersection as follows via a computation sequence for Z in the sense of Laufer [5, Proposition 4.1].

$$Z_0 = 0, Z_1 = E_{i_1}, Z_2 = Z_1 + E_{i_2}, \ldots, Z_j = Z_{j-1} + E_{i_j}, \ldots,$$
$$Z_\ell = Z_{\ell-1} + E_{i_\ell} = Z,$$

where E_{i_1} is an irreducible component and $E_{i_j} \cdot Z_{j-1} > 0$, $1 < j \leq \ell$.

Consider the computation sequence for Z, we have

$$p_a(Z_j) = p_a(Z_{j-1}) + p_a(E_{i_j}) + Z_{j-1} \cdot E_{i_j} - 1 \geq p_a(Z_{j-1})$$

for $1 \leq j \leq \ell$. Then $p_a(Z) \geq p_a(D)$ for any subcycle $0 < D < Z$.

Definition 2.2. We define the arithmetic genus of Z the *fundamental genus* of (V, o) and denote it by $p_f(V, o)$. The *arithmetic genus* and *geometric genus* of (V, o) are respectively defined as $p_a(V, o) = \max\{p_a(D) | 0 < D\}$ and $p_g(V, o) = \dim_C H^1(X, \mathcal{O}_X)$.

2.2. **Chain-connected.** we recall the notion of chain-connected cycles introduced by Konno [3] and state the fundamental properties of the chain-connected cycles.

Definition 2.3. A line bundle on a cycle is nef if it is of non-negative degree on all irreducible components.

Definition 2.4. A cycle D is chain-connected if $\mathcal{O}_{D-C}(-C)$ is not nef for any proper subcycle $0 < C < D$.

According to the minimality of the fundamental cycle Z, the fundamental cycle is chain-connected cycle. In fact, it is the largest chain-connected cycle with support $\pi^{-1}(o)$.

Proposition 2.5. *Let D be a chain-connected cycle, then $p_a(D) \geq p_a(C)$ for any subcycle $0 < C < D$.*

Proof. Since D is a chain-connected cycle, we have $\mathcal{O}_{D-A}(-A)$ is not nef for any subcycle $0 < A < D$. So we can find an irreducible component $E_i < D - A$ such that

$E_i \cdot A > 0$. For any subcycle $0 < C < D$, like Laufer's computation sequence, we can get an increasing sequence of cycles. Let

$$D_0 = C, D_1 = D_0 + E_{i_1}, D_2 = D_1 + E_{i_2}, \ldots, D_j = D_{j-1} + E_{i_j}, \ldots,$$
$$D_\ell = D_{\ell-1} + E_{i_\ell} = D,$$

where E_{i_j} is an irreducible component and $E_{i_j} \cdot D_{j-1} > 0$, $1 \le j \le \ell$.

We have $p_a(D_j) = p_a(D_{j-1}) + p_a(E_{i_j}) + E_{i_j} \cdot D_{j-1} - 1 \ge p_a(D_{j-1})$ for $1 \le j \le \ell$, since $p_a(E_{i_j}) \ge 0$ and $E_{i_j} \cdot D_{j-1} > 0$. We conclude that $p_a(D_\ell) \ge p_a(D_0)$, i.e., $p_a(D) \ge p_a(C)$.   Q.E.D.

We remark that, when $p_a(D) = p_a(C)$, we have $p_a(D_j) = p_a(D_{j-1})$ for $1 \le j \le \ell$. It means that $p_a(E_{i_j}) = 0$ and $E_{i_j} \cdot D_{j-1} = 1$ for $1 \le j \le \ell$.

Definition 2.6. Let D be a reducible curve. An irreducible component E of D is said to be a $(-m)_D - curve$ if $p_a(E) = 0$ and $E \cdot (D - E) = m$.

Proposition 2.7 ([3]). *Given a $(-1)_D - curve$ E of D. If D is chain-connected, then the subcycle $D' = D - E$ is also chain-connected.*

Corollary 2.8 ([3]). *Let D be a chain-connected cycle. If $p_a(C) = p_a(D)$ for a subcycle $C < D$, then the subcycle C is also chain-connected.*

Proposition 2.9 ([3]). *Let D be a chain-connected cycle. If $\mathcal{O}_D(-C)$ is nef for a cycle C, then either $D \le C$ or supp $C \cap$ supp $D = \emptyset$.*

Definition 2.10. If D is chain-connected and $p_a(D) > 0$, then there uniquely exists a minimal subcycle D_{min} of D such that $p_a(D_{min}) = p_a(D)$. We call D_{min} the minimal model of D.

Theorem 2.11 (Chain-connected component decomposition, [3]). *Let D be a cycle. Then there exists a sequence $D_1, D_2, \ldots, D_r$ of chain-connected subcycles of D and a sequence $m_1, \ldots, m_r$ of positive integers which satisfy*

(1) $D = m_1 D_1 + \cdots + m_r D_r$.
(2) *For $i < j$, the cycle $-D_i$ is nef on D_j.*
(3) *If $m_i \ge 2$, then $-D_i$ is nef on D_i.*
(4) *For $i < j$, either $D_i > D_j$ or supp $D_i \cap$ supp $D_j = \emptyset$.*

2.3. **The canonical cycle.**

Definition 2.12. The rational cycle Z_K is called the canonical cycle if $Z_K \cdot E_i = -KE_i$ for all i, i.e.

$$Z_K \cdot E_i = E_i^2 - 2\delta_i - 2g_i + 2 \text{ for all } i,$$

where δ_i is the "number" of nodes and cusps on E_i.

Definition 2.13. If the coefficients of Z_K are integers, then the singularity is called *numerical Gorenstein* singularity.

2.4. **Yau sequence and Yau cycle.** Assume that $p_f(V, o) = p_a(Z) > 0$. By Definition 2.10, there uniquely exists a minimal subcycle Z_{min} of Z such that $p_a(Z_{min}) = p_a(Z)$.

Lemma 2.14. *Assume that $-Z$ is numerically trivial on Z_{min}. Then there uniquely exists a maximal subcycle $D < Z$ such that $\mathcal{O}_D(-Z)$ is numerically trivial and $p_a(D) = p_f(V, o)$. Moreover, D is the fundamental cycle on its support.*

Proof. Let $S = \{0 < D < Z | \mathcal{O}_D(-Z)$ *is numerically trivial and* $p_a(D) = p_a(Z)\}$. By the assumption, $Z_{min} \in S$. Since the coefficients of subcycle D are integers, the nonempty set S contains a maximal element.

We show the maximal element of S is the fundamental cycle on its support. Let D be a maximal element of S and Z_1 be the fundamental cycle on supp D. Assume that there exists an irreducible component $C \le D$ satisfying $C \cdot D > 0$. Since $\mathcal{O}_D(-Z)$ is numerically trivial and $C \le D$, we have $C(Z - D) < 0$ and, hence, $C \le Z - D$. Then $C + D$ is a subcycle of Z and $\mathcal{O}_{C+D}(-Z)$ is numerically trivial. Furthermore, we have $p_a(Z) \ge p_a(C + D) = p_a(C) + p_a(D) + C \cdot D - 1 \ge p_a(D) = p_a(Z)$, hence $C + D \in S$. This contradicts the assumption that D is maximal. Then we have $\mathcal{O}_{Z_1}(-D)$ is nef and $Z_1 \le D$. Since $D < Z$ and $p_a(D) = p_a(Z)$, the cycle D is chain-connected according to Corollary 2.8. Furthermore, we have $D \le Z_1$ according to Proposition 2.9, since $\mathcal{O}_D(-Z_1)$ is nef and supp $D \cap$ supp $Z_1 \ne \emptyset$. Hence we get $D = Z_1$.

Assume D_1 and D_2 are different maximal elements in S. If $\mathcal{O}_{D_2}(-D_1)$ is not nef, then there exists an irreducible component $C < D_2$ such that $C \cdot D_1 > 0$. Since D_1 is the fundamental cycle on its support, this shows $C \not\le D_1$ and it follows $C + D_1 \le Z$. Then we have $p_a(Z) \ge p_a(C+D_1) = p_a(C) + p_a(D_1) + C \cdot D_1 - 1 \ge p_a(D_1) = p_a(Z)$ and $\mathcal{O}_{C+D_1}(-Z)$ is numerically trivial, hence $C + D \in S$. This contradicts the assumption that D_1 is maximal, hence we get $\mathcal{O}_{D_2}(-D_1)$ is nef. Since D_2 is chain-connected, either $D_2 \le D_1$ or supp $D_1 \cap$ supp $D_2 = \emptyset$. By the uniqueness of the minimal model Z_{min}, we get $Z_{min} \le D_1$ and $Z_{min} \le D_2$. It means that $D_2 \le D_1$, contradicting that D_1 and D_2 are different maximal elements in S. Hence there uniquely exists a maximal element in S. Q.E.D.

Definition 2.15 ([4])**.** We call D as in the Lemma 2.14 the *Tyurina component* of Z. Since D is the fundamental cycle on its support and Z_{min} is also the minimal model of D, we can get the Tyurina component of D when $-D$ is numerically trivial on Z_{min}. By the induction, we get the sequence of cycles

$$0 < D_m < D_{m-1} < \cdots < D_2 < D_1 = Z$$

such that D_{i+1} is the Tyurina component of D_i for $1 \le i \le m-1$ and $D_m \cdot Z_{min} < 0$. We call it the *Yau sequence* for Z and call $Y = \sum_{i=1}^m D_i$ the *Yau cycle*. The case $Z \cdot Z_{min} < 0$ is corresponds to $m = 1$.

Proposition 2.16 (Theorem 3.7, [9])**.** *If (V, o) is a numerically Gorenstein elliptic singular point and $\pi \colon X \to V$ is the minimal resolution, then the Yau cycle is the canonical cycle.*
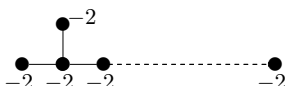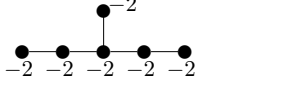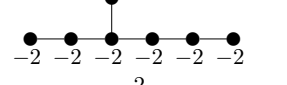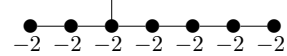
The length m of the Yau sequence is a numerical invariant of (V, o), and gives us the arithmetic genus for singular points of fundamental genus two in [4]. When $p_f(V, o) > 2$, length m also gives us a lower bound of $p_a(V, o)$ since $p_a(Y) = \sum_{i=1}^m (p_a(D_i) - 1) + \sum_{1 \le i < j \le m} D_i D_j + 1 = m(p_f(V, o) - 1) + 1$.

2.5. **Classfication of weighted dual graphs.** Let (V, o) be an isolated singular point on a surface V, and $\pi\colon X \to V$ be a minimal resolution of V. There are two beautiful results given by Artin in [1].

Definition 2.17. The singularity (V, o) is said to be rational if $\chi(Z) = 1$.

If (V, o) is a rational singularity, then π is also a minimal good resolution, i.e., exceptional set with nonsingular E_i and normal crossings. Moreover, each E_i is a rational curve and $E_i^2 = -2$.

Theorem 2.18 ([1])**.** *If (V, o) is a hypersurface rational singularity, then (V, o) is a rational double point. Moreover the set of weighted dual graphs of hypersurface rational singularities consists of the following graphs:*

To each such weighted dual graph is associated an intersection matrix whose (i, j)th entry is $E_i \cdot E_j$.

These graphs $(1) - (5)$ in Theorem 2.18 are called ADE graphs in the literature. This theorem completely classifies the weighted dual graphs with all $E_i^2 = -2$. In general, according to [1] and [2], to classify the weighted dual graphs we need to classify the corresponding negative-definite matrices:

Proposition 2.19 ([1])**.** *Let $\{E_i\}_{i=1,\cdots,n}$ be a connected bunch of complete curves on a regular two-dimensional scheme:*
  (i) *Suppose that $E_i \cdot E_j$ is negative-definite, then there exists positive cycles $Z = \sum r_i E_i$ such that $Z \cdot E_i \leq 0$ for all i.*
  (ii) *Conversely, if there exists a positive cycle $Z = \sum r_i E_i$ such that $Z \cdot E_i \leq 0$ for all i, then $E_i \cdot E_j$ is negative semi-definite. If in addition $Z^2 < 0$, then $E_i \cdot E_j$ is negative-definite.*

3. Singularities of degree two

3.1. **The relation between the canonical cycle and the Yau cycle.** Let (V, o) be an isolated singular point on a surface V, and $\pi\colon X \to V$ be a minimal resolution of V. We know the Yau cycle is the canonical cycle if (V, o) is a numerically Gorenstein elliptic singular point. This property doesn't hold when $p_f(V, o) > 0$. But if the

degree of (V, o) is small, we can get a similar property under restrictive conditions. If the fundamental cycle Z satisfying $Z^2 = -1$, the relation between the canonical cycle and the Yau cycle is given by Konno in [4].

Definition 3.1. Since $p_f(V, o) > 0$, there exists an irreducible component $A \leq Z$ such that A is not a (-2)-curve. Let k be the coefficient of cycle A of Z. We say that Z is essentially irreducible if either $Z = kA$ or $Z - kA$ consists of (-2)-curves.

Proposition 3.2 (Lemma 3.4, [4])**.** *Let (V, o) be a normal surface singularity with $p_f(V, o) > 0$ and $Z^2 = -1$, and $\pi \colon X \to V$ be a minimal resolution of V. Assume that Z is essentially irreducible. Then (V, o) is numerically Gorenstein with canonical cycle $(2p_f(V, o) - 1)Y$.*

For the case with $Z^2 = -2$, we consider a similar property: (V, o) is numerically Gorenstein with canonical cycle $p_f(V, o)Y$. However this property doesn't hold when only Z is essentially irreducible. So we need a more restrictive situation.

Lemma 3.3. *Let (V, o) be a normal surface singularity with $p_f(V, o) > 0$ and $Z^2 = -2$, and $\pi \colon X \to V$ be a minimal resolution of V. Assume that Z is essentially irreducible, then either $m = 1$ or $Z - D_m$ consists of (-2)-curves, where m denotes the length of the Yau sequence for Z and D_m is the smallest term in the Yau sequence for Z.*

Proof. If $m > 1$, we can get an increasing sequence of cycles like the proof of proposition 2.4:

$$Z_0 = D_m, Z_1 = Z_0 + E_{i_1}, Z_2 = Z_1 + E_{i_2}, \ldots, Z_j = Z_{j-1} + E_{i_j}, \ldots,$$
$$Z_\ell = Z_{\ell-1} + E_{i_\ell} = Z,$$

where E_{i_j} is an irreducible component and $E_{i_j} \cdot Z_{j-1} > 0$, $1 \leq j \leq \ell$.

By the definition of Yau sequence, we have $p_a(D_m) = p_a(Z)$. Since $p_a(Z_j) = p_a(Z_{j-1}) + p_a(E_{i_j}) + E_{i_j} \cdot Z_{j-1} - 1 \geq p_a(Z_{j-1})$ for $1 \leq j \leq \ell$, we can get $p_a(E_{i_j}) = 0$ and $E_{i_j} \cdot Z_{j-1} = 1$ for $1 \leq j \leq \ell$.

Consider that $Z_{j-1}^2 = Z_j^2 - 2E_{i_j} \cdot Z_{j-1} - E_{i_j}^2 = Z_j^2 - 2 - E_{i_j}^2$ and π is minimal, we have that $-2 = Z_\ell^2 \leq Z_{\ell-1}^2 \leq \cdots \leq Z_1^2 \leq Z_0^2 \leq -1$. Assume $Z - D_m$ does not consists of (-2)-curve, then the E_{i_j} that is not (-2)-curve is (-3)-curve and only one. Since $D_m^2 = -1$, we have the unique non-multiple component A with $A \cdot D_m = -1$ and $\mathcal{O}_{D_m - A}(-D_m)$ is numerically trivial. Since $D_m \cdot Z_{min} < 0$, A is not (-2)-curve. Thus we can get that $\pi^{-1}(o) = \cup E_i$ satisfying all E_i are (-2)-curves except for one (-3)-curve A, and the coefficient of cycle A of Z is 2. Thus $p_a(Z) = (2A \cdot K - 2)/2 + 1 = 1$ by the Riemann-Roch theorem. These weighted dual graphs consisting of (-2)-curves and exactly one (-3)-curve are classified in [10], there is no situation where the conditions are met. This means $Z - D_m$ consists of (-2)-curves.   Q.E.D.

Remark 3.4. If we don't use the classification results of weighted dual graphs in [10], we can still prove Lemma 3.3 by discussing the irreducible components with a negative intersection number with Z. This is the main approach to proving Theorem 3.9 later on, however, the process may be more cumbersome. Here, let's briefly explain the train of thought.

Continuing from the process of Lemma 3.3, we now need to prove that there are no singularities where the coefficient of A in Z is 2 and the coefficient in D_m is 1. Since $D_m \neq Z$ and Z is essentially irreducible, we know that the irreducible component with a negative intersection number with Z is a (-2)-curve. We assert that such an irreducible component is unique. Otherwise, there exists an irreducible component B such that $B \cdot Z = -1$, and the coefficient of cycle B of Z is 1. It means there exists a unique irreducible component B_1 connected with B and the coefficient of cycle B_1 of Z is 1. If B_1 is a (-2)-curve, then we can find the unique irreducible component B_2 connected with B_1, excluding B. We can continue repeating this step until B_n is not longer a (-2)-curve. Since Z is essentially irreducible, we have $B_n = A$, but the coefficient of A of Z is 2, contradicting.

Then we have the unique irreducible component C such that $C \cdot Z = -1$, and the coefficient of cycle C of Z is 2. After removing A, we observe that Z can be divided into several connected branches consisting of (-2)-curves. We will only consider the branch where C is located. Its weighted dual graph is an ADE. The number of the irreducible components connected with C is either 1 or 2. By using a similar method as mentioned above, we can determine the weighted dual graph of the branch where C is located is A_n for some n when the number is 1, and it is D_n for some n when the number is 2. In these cases, we can conclude that the coefficient of A of D_m is 2. The specific process can be referred to case (2) and case (5) of Theorem 3.9.

Remark 3.5. It is easy to see that when $Z - D_m$ consists of (-2)-curves, then $D_i^2 = Z^2$ for $1 \leq i \leq m$.

Theorem 3.6 (i.e. Theorem A). *Let (V, o) be a normal surface singularity with $p_f(V, o) > 0$ and $Z^2 = -2$, and $\pi \colon X \to V$ be a minimal resolution of V. Assume that Z is essentially irreducible and $D_m = Z_{min}$. Then (V, o) is numerically Gorenstein with canonical cycle $p_f(V, o)Y$.*

Proof. Assume A is an irreducible component $A \leq Z$ such that A is not a (-2)-curve, and k is the coefficient of cycle A of Z. Since $-2 = Z_{min} \cdot (kA) = kY \cdot A$, we have $K_X \cdot A = \frac{1}{k} K_X \cdot Z = \frac{2p_f(V,o)}{k} = -p_f(V, o)Y \cdot A$. We claim that $B \cdot Y = 0$ for any component $B < Z - kA$. Let j be the largest index such that $B \leq D_j$. If $j = m$, then $\mathcal{O}_B(-D_i)$ is numerically trivial for $1 \leq i \leq m - 1$ by the definition of the Yau sequence and $B \cdot D_m = 0$ by $D_m = Z_{min}$, hence $B \cdot Y = 0$. If $j < m$, then D_{j+1} is the Tyurina component of D_j. We have $\mathcal{O}_B(-D_i)$ is numerically trivial for $1 \leq i \leq j - 1$ and $\mathcal{O}_{D_k}(-(D_j - D_{j+1}))$ is numerically trivial for $j + 2 \leq k \leq m$. Since D_k is chain-connected, we have either $D_k \leq D_j - D_{j+1}$ or supp $D_k \cap$ supp $(D_j - D_{j+1}) = \emptyset$. However, $D_k \leq D_j - D_{j+1}$ is impossible, since $p_a(D_k + D_{j+1}) = p_a(D_k) + p_a(D_{j+1}) - 1 \geq p_a(D_{j+1})$ and $\mathcal{O}_{D_k + D_{j+1}}(-D_j)$ is numerically trivial. Thus supp $D_k \cap$ supp $(D_j - D_{j+1}) = \emptyset$ and $C \cdot D_k = 0$ for any irreducible component $C \leq D_j - D_{j+1}$. In particular, $B \cdot D_k = 0$ for $j + 2 \leq k \leq m$. In sum, we get $B \cdot Y = B \cdot D_j + B \cdot D_{j+1}$.

By Lemma 3.3 and Remark 3.5, we have that $D_j^2 = -2$ and $D_j - D_{j+1}$ consists of (-2)-curves. Since D_j is the fundamental cycle on its support, there either exist two components C_1 and C_2 of D_j such that $C_1 \cdot D_j = C_2 \cdot D_j = -1$ and the coefficient of cycle C_i of D_j is 1 for $i = 1, 2$, or exists a component C_3 of D_j such that $C_3 \cdot D_j = -1$ and the coefficient of cycle C_3 of D_j is 2. If the first alternative happens, then

$\mathcal{O}_{D_j-C_1-C_2}(-D_j)$ is numerically trivial and $C_1 \cdot C_2 = 0$. This implies that $D_j - C_1 - C_2$ is the Tyurina component of Z, and $B \in \{C_1, C_2\}$. We have $B \cdot D_j = -1$ and $B \cdot D_{j+1} = B \cdot (D_j - C_1 - C_2) = B \cdot D_j - B^2 = 1$, hence $B \cdot Y = B \cdot D_j + B \cdot D_{j+1} = 0$. If the last alternative happens, we can get an increasing sequence of cycles like the proof of Proposition 2.5:

$$Z_0 = D_{j+1}, Z_1 = Z_0 + E_{i_1}, Z_2 = Z_1 + E_{i_2}, \ldots, Z_j = Z_{j-1} + E_{i_j}, \ldots,$$
$$Z_\ell = Z_{\ell-1} + E_{i_\ell} = D_j,$$

where E_{i_j} is an irreducible component and $E_{i_j} \cdot Z_{j-1} = 1$ for $1 \le j \le \ell$ by $p_a(D_j) = p_a(D_{j+1})$. Then we have $E_{i_1} = C_3$ and $C_3 \cdot Y = C_3 \cdot D_j + C_3 \cdot D_{j+1} = 0$. If $B \ne C_3$, it means that $B \cdot D_j = 0$. Since $B \not\le D_{j+1}$ and $\mathcal{O}_{D_{j+1}+B}(-D_j)$ is numerically trivial, we have $B \cdot D_{j+1} \ge 0$ and $B \cdot D_{j+1} \le 0$, furthermore, $B \cdot Y = B \cdot D_j + B \cdot D_{j+1} = 0$.

In sum, we have shown that (V, o) is numerically Gorenstein with canonical cycle $p_f(V, o)Y$.

Q.E.D.

We remark that, when $Z^2 = -1$, the condition that Z is essentially irreducible assuming that $D_m = Z_{min}$. However when $Z^2 = -2$, there exists a singularity (V, o) satisfies that Z is essentially irreducible and $D_m \ne Z_{min}$.

Proposition 3.7. *Let (V, o) be a normal surface singularity with $p_f(V, o) > 0$, and $\pi \colon X \to V$ be a minimal resolution of V. Assume that Z is essentially irreducible with $K_X \cdot A + Z^2 \ge 0$ and $D_m = Z_{min}$, where A is the unique cycle in Z that is not a (-2)-curve. Then we have $Z_K = (\frac{2-2p_f(V,o)}{Z^2} + 1)Y$.*

Proof. Since this property is not relevant to the main conclusions of this paper, only a brief outline of the proof is provided here.

First, let us retain the notations used in Lemma 3.3. We have that $-Z^2 = Z_\ell^2 \le Z_{\ell-1}^2 \le \cdots \le Z_1^2 \le Z_0^2 = D_m^2 \le -1$. Since Z is essentially irreducible, assume k is the coefficient of cycle A of Z, and k' is the coefficient of cycle A of D_m. Then we have that $p_a(Z) = 1 + \frac{1}{2}(Z^2 + kA \cdot K)$ and $p_a(D_m) = 1 + \frac{1}{2}(D_m^2 + k'A \cdot K)$ by the Riemann-Roch theorem. Since $Z^2 + A \cdot K \ge 0$, we have $k' = k$. It means that $Z - D_m$ consists of (-2)-curves and $D_i^2 = Z^2$ for $1 \le i \le m$.

To prove $Z_K = (\frac{2-2p_f(V,o)}{Z^2} + 1)Y$, we need to demonstrate $(\frac{2-2p_f(V,o)}{Z^2} + 1)Y \cdot A = -K_X \cdot A$ and $Y \cdot B = 0$ for any (-2)-curve B. The former can be derived from that $Z^2 = Z_{min} \cdot (kA) = kY \cdot A$ and $p_a(Z) = 1 + \frac{1}{2}(Z^2 + kA \cdot K)$. Similar to the proof process of Theorem 3.6, the latter can be converted to $B \cdot (D_j + D_{j+1}) = 0$, where j is the largest index such that $B \le D_j$. We can assume $B \cdot D_j = -1$, since the case when $B \cdot D_j = 0$ is similar to the last part of the proof of Theorem 3.6.

After removing A, we observe that D_j can be divided into several connected branches consisting of (-2)-curves; we will only consider the branch where B is located, and claim that B is the only irreducible component in this branch satisfying $B \cdot D_j < 0$. Otherwise, if we can find a path connecting two irreducible components with a negative intersection number with D_j, subtracting 1 from the coefficients of the curves in this path of D_j would result in a new curve with an arithmetic genus larger than that of D_j, contradicts the chain-connectivity of D_j. Additionally, as shown in the proof of Proposition 2.5, we can transform D_j into D_{j+1} by progressively removing irreducible components. We further claim that during this process,

B is the last irreducible component to be removed in this branch. This implies that $B \cdot D_{j+1} = 1$. Thus, we have proven this remark.   Q.E.D.

3.2. **Classification of weighted dual graphs of the singularities of degree two with the fundamental cycle Z being essentially irreducible.** Let (V, o) be a normal surface singularity with $p_f(V, o) > 0$ and $Z^2 = -2$, and $\pi \colon X \to V$ be a minimal resolution of V. Assume that Z is essentially irreducible and the irreducible componet $A \leq Z$ is not a (-2)-curve. Let k be the coefficient of cycle A of Z and m be the length of the Yau sequence for Z. If $m = 1$, the condition $D_m = Z_{min}$ means $\mathcal{O}_{Z-kA}(-Z)$ is numerically trivial. In order to study the relation between Z and D_m, we aim to classify the weighted dual graphs of the singularities of degree two with the fundamental cycle Z being essentially irreducible. We use Γ to denote the weighted dual tree graph of the exceptional set $\pi^{-1}(o)$. After removing the point corresponding to the A, the remaining connected graphs are denoted as $\Gamma_1, \cdots, \Gamma_n$.

Lemma 3.8. *With the notations as above, we have Γ_i must be ADE for any $1 \leq i \leq n$. Let $Z|_{\Gamma_i}$ be the fundamental cycle Z restricted to each Γ_i. If $m > 1$, we have $\{i | (Z|_{\Gamma_i} \cdot Z) < 0\} = \{i | \operatorname{supp} \Gamma_i \cap \operatorname{supp} (Z - D_m) \neq \emptyset\}$, where m denotes the length of the Yau sequence for Z and D_m denotes the smallest component of the Yau sequence.*

Proof. Firstly, we have $Z|_{\Gamma_i} \cdot E_j \leq 0$, $\forall E_j \in \operatorname{supp} \Gamma_i$. Assume cycle $E_{j_0} \in \operatorname{supp} \Gamma_i$ connected with A, then $Z|_{\Gamma_i} \cdot E_{j_0} < 0$. By Proposition 2.19, we conclude that Γ_i is negative-definite. Hence Γ_i must be ADE.

According to Lemma 3.3, we have an increasing sequence of cycles when $m > 1$:

$$Z_0 = D_m, Z_1 = Z_0 + E_{i_1}, Z_2 = Z_1 + E_{i_2}, \ldots, Z_j = Z_{j-1} + E_{i_j}, \ldots,$$
$$Z_\ell = Z_{\ell-1} + E_{i_\ell} = Z,$$

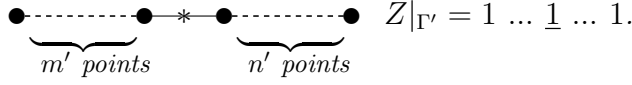where E_{i_j} is a (-2)-curve and $E_{i_j} \cdot Z_{j-1} = 1$, $1 \leq j \leq \ell$.

For any $1 \leq i \leq n$, if $Z|_{\Gamma_i} \cdot Z = 0$, we prove that $\mathcal{O}_{Z|_{\Gamma_i}}(-Z_j)$ is numerically trivial for each j by induction on j. Firstly, since $Z|_{\Gamma_i} \cdot Z = 0$, we have $\mathcal{O}_{Z|_{\Gamma_i}}(-Z_\ell)$ is numerically trivial. Assume $\mathcal{O}_{Z|_{\Gamma_i}}(-Z_j)$ is numerically trivial for any $1 \leq j \leq \ell$, notice that $E_{i_j} \cdot Z_j = E_{i_j} \cdot Z_{j-1} + E_{i_j}^2 = -1$, we have $E_{i_j} \notin \operatorname{supp} \Gamma_i$ and $E_{i_j} \neq A$. It means that E_{i_j} is disjoint from $Z|_{\Gamma_i}$, so that $-Z_{j-1} = -(Z_j - E_{i_j})$ is numerically trivial $Z|_{\Gamma_i}$. By induction, we have $\mathcal{O}_{Z|_{\Gamma_i}}(-Z_j)$ is numerically trivial for each j, hence $\operatorname{supp} \Gamma_i \cap \operatorname{supp} (Z - D_m) = \emptyset$.

Since $\mathcal{O}_{D_m}(-Z)$ is numerically trivial when $m > 1$, it is obvious that $\operatorname{supp} \Gamma_i \cap \operatorname{supp} (Z - D_m) \neq \emptyset$ when $Z|_{\Gamma_i} \cdot Z < 0$. Hence, we have $\{i | (Z|_{\Gamma_i} \cdot Z) < 0\} = \{i | \operatorname{supp} \Gamma_i \cap \operatorname{supp} (Z - D_m) \neq \emptyset\}$.   Q.E.D.
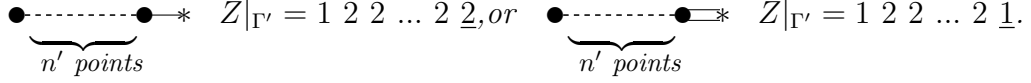
Let $S = \{i | (Z|_{\Gamma_i} \cdot Z) < 0\}$, by the proof of the Lemma 3.8, in order to study the relation between Z and D_m, we only need to observe that the subgraph Γ' of Γ by removing all the Γ_i, $i \notin S$. In a dual graph, the $*$ represents the point corresponding to cycle A. We still call it the cycle A later. The others are the points corresponding to the (-2)-curves, denoted by $\bullet$, we call them (-2)-curves later.

Theorem 3.9. *With the notations as above, when $m > 1$ case, Γ' and $Z|_{\Gamma'}$ must be one of the following(the underlined number represents the A):*
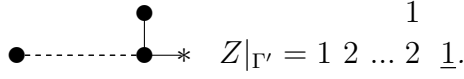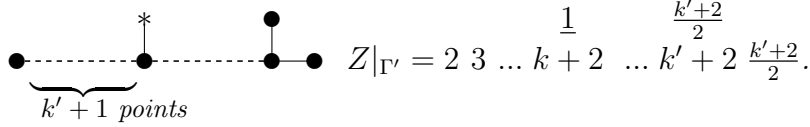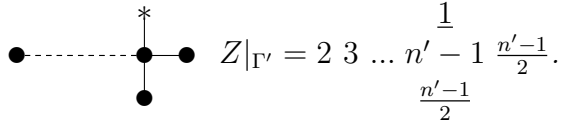
(1) $A_{m'} + A + A_{n'}$.

 $Z|_{\Gamma'} = 1 \ ... \ \underline{1} \ ... \ 1.$

(2) $A + A_{n'}$: $n' \geq 3$.

 $Z|_{\Gamma'} = 1 \ 2 \ 2 \ ... \ 2 \ \underline{2},$ or $Z|_{\Gamma'} = 1 \ 2 \ 2 \ ... \ 2 \ \underline{1}.$

(3) $A + (1 - A_{n'})$: $n' \geq 3$.

 $Z|_{\Gamma'} = 1 \ 2 \ ... \ 2 \ \overset{1}{} \ \underline{1}.$

(4) $A + (k' - D_{n'})$: k' is an even number and $0 \leq k' \leq n - 3$.

 $Z|_{\Gamma'} = 2 \ 3 \ ... \ k + 2 \ \overset{\underline{1}}{} \ ... \ k' + 2 \ \overset{\frac{k'+2}{2}}{\frac{k'+2}{2}}.$

In particular, when n' is an odd number and $k' = n' - 3$, we have:

 $Z|_{\Gamma'} = 2 \ 3 \ ... \ n' - 1 \ \overset{\underline{1}}{\underset{\frac{n'-1}{2}}{\frac{n'-1}{2}}}.$

In additional, when $k' = 0$, we have:

 $Z|_{\Gamma'} = \underline{1} \ 2 \ 2 \ ... \ 2 \ 1.$

(5) $A + (D'_{n'})$: n' is an odd number.

 $Z|_{\Gamma'} = 2 \ 3 \ ... \ n' - 1 \ \overset{\frac{n'-1}{2}}{} \ \frac{n'+1}{2} \ \underline{2},$

or $Z|_{\Gamma'} = 2 \ 3 \ ... \ n' - 1 \ \overset{\frac{n'-1}{2}}{} \ \frac{n'+1}{2} \ \underline{1}.$

(6) $A + (D''_{n'})$: n' is an even number.

 $Z|_{\Gamma'} = 2 \ 3 \ ... \ n' - 1 \ \overset{\frac{n'}{2}}{\frac{n'}{2}} \ \overset{1}{}.$

(7) $A + E_6$.

 $Z|_{\Gamma'} = 2 \ 3 \ 4 \ 3 \ 2 \ \underline{1}.$

(8) $A + D'''_5$:

 $Z|_{\Gamma'} = 1 \ 2 \ 3 \ 2 \ \underline{1}.$
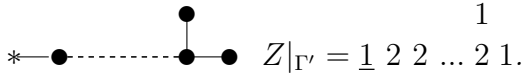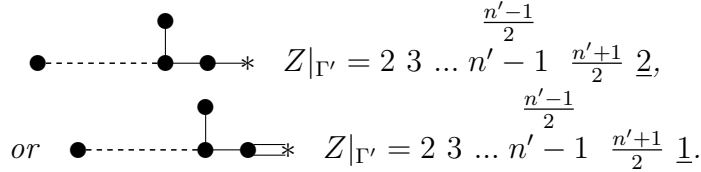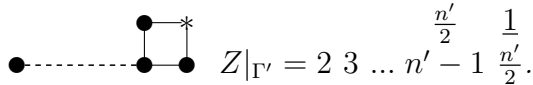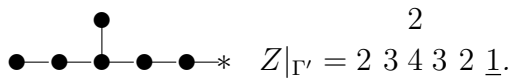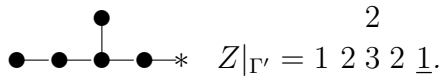
The weighted dual graph of case (8) is the same as case (5), but $Z|_{\Gamma'}$ differs.

Proof. Since $m > 1$, $Z \cdot A = 0$ and $-2 = Z^2 = \sum_{i=1}^{n}(Z \cdot Z|_{\Gamma_i})$, hence, the number of elements in S is either 1 or 2. If S has two elements, we can assume $S = \{1, 2\}$. Then we know that $Z \cdot Z|_{\Gamma_1} = -1$ and, hence, there exists an irreducible componet $C_1 \leq Z|_{\Gamma_1}$ such that $C_1 \cdot Z = -1$ and the coefficient of C_1 of Z is 1. Since C_1 is a (-2)-curve, there exists a unique cycle in Γ connected with C_1. If the cycle isn't A, we call it C_2. Then we have $C_2 \cdot Z = -1$ and the coefficient of C_1 of Z is 1. Except for C_1, there exists a unique cycle in Γ connected with C_2. Now, by the obvious induction, we can show that the weighted dual graph of Γ_1 is $A_{m'}$ for any m' and similarly the weighted dual graph of Γ_2 is $A_{n'}$ for any n', and the weighted dual graph of Γ' and $Z|_{\Gamma'}$ is the same as in case (1).

Then we consider that S has only one element, we can assume $S = \{1\}$. Then we know that $Z \cdot Z|_{\Gamma_1} = -2$ and there exists a unique cycle C in Γ_1 such that $C \cdot Z < 0$. That is because if there exist two cycles C and D, we can find a connection way of C and D in Γ_1. We denote the points of the connection way by $C_1 = C, C_2, \ldots, C_{k'} = D$, then $\mathcal{O}_Z(-(Z - \sum_{i=1}^{k'} C_i))$ is nef, contradicting the minimality of Z. Since $C \cdot Z = p_a(Z) - p_a(Z - C) - 1 \geq -1$, the coefficient of C of Z is 2.

Referring to [6], for an ADE graph, with abuse of notations, E_i denotes exceptional curve, and $E = \sum E_i$ is the exceptional cycle. Let $\delta_i = (E - E_i) \cdot E_i$ be the number of irreducible components of E connected with E_i. A cycle E_i is called an end (resp. a node), if $\delta_i = 1$ (resp. $\delta_i \geq 3$).

If C isn't an end in Γ_1, we claim that Γ_1 is $A_{n'}$ for any n'. If not, then Γ_1 has a node D, we can find a connection way of C and D. We denote the points of the connection way by $C_1 = C, C_2, \ldots, C'_k = D$(If $C = D$, denote $k' = 1$). If $C \neq D$, we denote the cycles connected with D in Γ_1 by $C_{k'-1}, B_1, B_2$, and the another cycle connected with C in Γ_1 as B_3. If $C = D$, we denote the cycles connected with D in Γ_1 as B_1, B_2, B_3. Then we have $\mathcal{O}_Z(-(Z - 2\sum_{i=1}^{k'} C_i - \sum_{i=1}^{3} B_i))$ is nef, contradicting the minimality of Z. Since the coefficient of C of Z is 2 and C isn't an end in Γ_1, we have there exists a cycle D connected with C in Γ_1 such that the coefficient of D of Z is 1, hence, D is an end. Then either C is connected with A or the coefficient of the another cycle C_2 that connected with C in Γ_1 of Z is 2. If the first alternative happens, there are two situations. One is that C is connected with another cycle C_2 and C_2 is an end when $kA \cdot C = 1$, the other is that C is an end in Γ_1 when $kA \cdot C = 2$(which contradicts the assumption that C isn't an end). If the last alternative happens, for C_2, either C_2 is connected with A, or the coefficient of the another cycle C_3 that is connected with C_2 in Γ_1 of Z is 2. By induction, we can get case (2) and case (3).

If C is an end, we have Γ_1 isn't $A_{n'}$ for any n' by $m > 1$. Then Γ_1 has a node called D, we can find a connection way of C and D, and denote the points of the connection way by $C_1 = C, C_2, \ldots, C'_k = D$, and denote B_1, B_2 as the other cycles connected with D. Assume c_i(resp. b_i) is the coefficient of C_i(resp. B_i) of Z. By induction, we have $c_j = j + 1 - \sum_{i=1}^{j-1}((j-i)kA \cdot C_i)$ for $2 \leq j \leq k'$. Since $b_1, b_2 \geq \frac{c'_k}{2}$,

we have $0 \le c_{k'} - c_{k'-1} - kA \cdot C_{k'} = 1 - \sum_{i=1}^{k'}(kA \cdot C_i)$. If $A \cdot \sum_{i=1}^{k'} C_i > 0$, we have B_1, B_2 are the ends and $k = 1$, then we get case (4).

If $A \cdot \sum_{i=1}^{k'} C_i = 0$, we have $c_j = j + 1$ for $1 \le j \le k'$. If k' is an odd number, then we can assume $b_1 = \frac{k'+1}{2}$ and $b_2 = \frac{k'+3}{2}$, hence, B_1 is an end and $kA \cdot B_2 \le 2$. We can get case (5) when $kA \cdot B_2 = 2$, and get case (7) when $kA \cdot B_2 = 0$. If $kA \cdot B_2 = 1$, there exists a cycle B_3 connected with B_2, and the coefficient of B_3 of Z is 1. But $B_3 \cdot Z \ge -2 + \frac{k'+3}{2} > 0$, it's impossible.

If k' is an even number, then we can assume $b_1 = b_2 = \frac{k'+2}{2}$. We can get case (6) when $kA \cdot B_1 > 0$ and $kA \cdot B_2 > 0$, and get case (8) when $kA \cdot B_1 = 0$ or $kA \cdot B_2 = 0$. If $kA \cdot B_1 = 0$ and $kA \cdot B_2 = 0$, we can prove that $A \cdot Z|_{\Gamma_1} = 0$, it's impossible.   Q.E.D.

Proposition 3.10. *With the notations as above, when $m > 1$ and $D_m = Z_{min}$ case, Γ' must be one of the following:*

(1) $A_{n'} + A + A_{n'}$.
(2) $A + A_{n'}$: n' *is an odd number and $n' \ge 3$.*
(3) $A + (1 - A_{n'})$: n' *is an odd number and $n' \ge 3$.*
(4) $A + (k' - D_{n'})$: k' *is an even number and $0 \le k' \le n - 3$.*
(5) $A + (D'_{n'})$: n' *is an odd number.*
(6) $A + (D''_{n'})$: n' *is an even number.*
(7) $A + E_6$.

remark: The weighted dual graphs Γ' of cases (1)-(7) in the Proposition 3.10 are the same as those of cases (1)-(7) mentioned in the Theorem 3.9.

Proof. We only need to compare $D_m|_{\Gamma'}$ with $Z_{min}|_{\Gamma'}$, since supp $(Z - D_m) \subset$ supp $(Z - Z_{min}) \subset$ supp Γ'. We can compute the $D_m|_{\Gamma'}$ and $Z_{min}|_{\Gamma'}$ for each case in the Theorem 3.9. Computations for cases (1)-(8) are simple, so let us take the computation of the case (1) as an example here.

In case (1), there exist two different irreducible components B_1 and C_1 in $D_1 = Z$ such that $B_1 \cdot D_1 = -1$, $C_1 \cdot D_1 = -1$, and $B_1 \cdot C_1 = 0$. In fact, the points corresponding to the B_1 and C_1 are the ends of $A_{m'}$ and $A_{n'}$ that are not connected with the point corresponding to the A. Then we have $D_2 = D_1 - B_1 - C_1$. We can assume $m' \ge n'$. By induction, we know that $m = n' + 1$ and the weighted dual graph of $D_m|_{\Gamma'}$ consists of $A_{m'-n'}$ and the point corresponding to the A. If $m' - n' \ne 0$, since the cycle C corresponding to the end of $A_{m'-n'}$ that is not connected with the point corresponding to the A, and it satisfies $C \cdot D_m = -1$, we have $p_a(D_m) = p_a(D_m - C)$. By induction, we have $Z_{min}|_{\Gamma'} = A$.

Now we give the $D_m|_{\Gamma'}$ and $Z_{min}|_{\Gamma'}$ for each case in the Theorem 3.9.

(1) $A_{m'} + A + A_{n'}$: assume $m' \ge n'$.

(2) $A + A_{n'}$: $n' \geq 3$.

$D_m|_{\Gamma'}$(when n' is an even number): •—•—∗ 1 2 $\underline{2}$ or •—•—▭∗ 1 2 $\underline{1}$.

$D_m|_{\Gamma'}$(when n' is an odd number): •—∗ 1 $\underline{2}$ or •—▭∗ 1 $\underline{1}$.

$Z_{min}|_{\Gamma'}$: •—∗ 1 $\underline{2}$ or •—▭∗ 1 $\underline{1}$.

(3) $A + (1 - A_{n'})$: $n' \geq 3$.

$D_m|_{\Gamma'}$(when n' is an even number): (graph) 1 •—∗ 1 $\underline{1}$.

$D_m|_{\Gamma'}$(when n' is an odd number): ∗ $\underline{1}$.

$Z_{min}|_{\Gamma'}$: ∗ $\underline{1}$.

(4) $A + (k' - D_{n'})$: k' is an even number and $0 \leq k' \leq n - 3$.

$D_m|_{\Gamma'}$: (graph with k' points) 1 2 ... k ... $\overset{\frac{1}{}}{k'}$ $\overset{\frac{k'}{2}}{\frac{k'}{2}}$.

$Z_{min}|_{\Gamma'}$: (graph with k' points) 1 2 ... k ... $\overset{\frac{1}{}}{k'}$ $\overset{\frac{k'}{2}}{\frac{k'}{2}}$.

(5) $A + (D'_{n'})$: n' is an odd number.

$D_m|_{\Gamma'}$: (graph with $n' - 3$ points) ∗ 1 ... $n' - 3$ $\overset{\frac{n'-3}{2}}{\frac{n'-1}{2}}$ $\underline{2}$

or (graph with $n' - 3$ points) ▭∗ 1 ... $n' - 3$ $\overset{\frac{n'-3}{2}}{\frac{n'-1}{2}}$ $\underline{1}$.

$Z_{min}|_{\Gamma'}$: (graph with $n' - 3$ points) ∗ 1 ... $n' - 3$ $\overset{\frac{n'-3}{2}}{\frac{n'-1}{2}}$ $\underline{2}$

or (graph with $n' - 3$ points) ▭∗ 1 ... $n' - 3$ $\overset{\frac{n'-3}{2}}{\frac{n'-1}{2}}$ $\underline{1}$.

(6) $A + (D''_{n'})$: n' is an even number.

$D_m|_{\Gamma'}$: (graph with $n' - 3$ points) 1 ... $n' - 3$ $\overset{\frac{n'-2}{2}}{\frac{n'-2}{2}}$ $\overset{\frac{1}{}}{}$.

$$Z_{min}|_{\Gamma'}: \quad \text{(diagram)} \quad 1 \ \dots \ n'-3 \ \tfrac{n'-2}{2} \overset{\tfrac{n'-2}{2} \ \ \tfrac{1}{1}}{}.$$

$$\underbrace{}_{n'-3 \text{ points}}$$

(7) $A + E_6$.

$D_m|_{\Gamma'}: \quad * \quad \underline{1}.$

$Z_{min}|_{\Gamma'}: \quad * \quad \underline{1}.$

(8) $A + D_5'''$:

$D_m|_{\Gamma'}: \quad \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!* \quad 1\ 1\ 1\ 1\ \underline{1}.$

$Z_{min}|_{\Gamma'}: \quad * \quad \underline{1}.$

Comparing the $D_m|_{\Gamma'}$ with $Z_{min}|_{\Gamma'}$, we can get this proposition.   Q.E.D.

3.3. **Arithmetic genus of a singularity in the essentially irreducible case.**
Let i be a non-negative integer and $p_f(V,o) = p > 0$. We have $p_a(iY) - 1 = i(p_a(Y) - 1) + \frac{i(i-1)}{2}Y^2 = m(i(p-1) + \frac{i(i-1)}{2}Z^2)$, where m denotes the length of the Yau sequence for Z. In the degree one case, we can get a lower bound for $p_a(V,o)$ by $\max\limits_{i}\{p_a(iY)\} = p_a(pY) = \frac{p(p-1)m}{2} + 1$.

Lemma 3.11 (Lemma 3.2, [4]). $p_a(V,o) \geq \frac{p(p-1)m}{2} + 1$ holds for a normal surface singularity (V,o) of degree one, where $p = p_f(V,o)$ and m denotes the length of the Yau sequence for Z.

In fact, we have the equality sign holds when Z is essentially irreducible.

Theorem 3.12 (i.e. Theorem B). *Let (V,o) be a normal surface singularity of degree one with $p_f(V,o) > 0$, Z be the fundamental cycle on the minimal resolution. Assume that Z is essentially irreducible, then $p_a(V,o) = \frac{p(p-1)m}{2} + 1$, where $p = p_f(V,o)$ and m denotes the length of the Yau sequence for Z.*

Proof. By Lemma 3.1 in [4], we have that $A_i = D_i - D_{i+1}$ is a (-2)-curve with $A_i \cdot D_i = -1$ for $1 \leq i < m$ and $D_m = Z_{min}$. Let C be a cycle whose support is in $\pi^{-1}(o)$ such that $p_a(C) = p_a(V,o)$. Let $C = C_1 + \cdots + C_n$ be a chain-connected component decomposition, where C_i is a chain-connected cycle and $\mathcal{O}_{C_j}(-C_i)$ is nef for $i < j$.

Let $A \leq Z$ be an irreducible component such that A is not a (-2)-curve. Since Z is essentially irreducible, we have $p_a(C - C_i) = p_a(C) - p_a(C_i) - C_i \cdot (C_1 + \cdots + C_{i-1} + C_{i+1} + \cdots + C_n) + 1 \geq p_a(C) - p_a(C_i) + 1 \geq p_a(C) + 1$ when $A \not\leq C_i$. So, we can get that $A \leq C_i$ for any i.

We prove $p_a(C) \leq \frac{p(p-1)m}{2} + 1$ by induction on the length m of the Yau sequence. When $m = 1$, we have $Z = Z_{min}$ and $Z \cdot A = -1$. Since the coefficient of A in Z is 1 and $A \leq C_i \leq Z$, we have $A \cdot C_i \leq A \cdot Z = -1$. Furthermore, since $\mathcal{O}_{C_j}(-C_i)$ is nef,

we have $C_i \cdot C_j = A \cdot C_j + (C_i - A) \cdot C_j \leq A \cdot C_j \leq -1$ for $i < j$. Then

$$p_a(C) - 1 = \sum_{i=1}^{n}(p_a(C_i) - 1) + \sum_{i<j} C_i \cdot C_j \leq n(p-1) - \frac{n(n-1)}{2} \leq \frac{p(p-1)}{2}.$$

When $m > 1$, assume the inequality holds for $m - 1$, let us proceed to show that it is true for m. We consider all the chain-connected components C_i such that $A_1 \leq C_i$ and assume that n_0 is the number of these cycles. Since the coefficient of A_1 in Z is 1 and $C_i \leq Z$, we have $A_1 \cdot C_i \leq A_1 \cdot Z = -1$ when $A_1 \leq C_i$. Furthermore, since $\mathcal{O}_{C_j}(-C_i)$ is nef, we have $C_i \cdot C_j = A_1 \cdot C_j + (C_i - A_1) \cdot C_j \leq -1$ when $A_1 \leq C_i$ and $A_1 \leq C_j (i < j)$. Then

$$p_a(C) - 1 = (p_a(C - \sum_{A_1 \leq C_i} C_i) - 1) + (p_a(\sum_{A_1 \leq C_i} C_i) - 1) + (C - \sum_{A_1 \leq C_i} C_i) \cdot (\sum_{A_1 \leq C_i} C_i)$$

$$\leq (p_a(C - \sum_{A_1 \leq C_i} C_i) - 1) + \sum_{A_1 \leq C_i} (p_a(C_i) - 1) + \sum_{A_1 \leq C_i, A_1 \leq C_j, i<j} C_i \cdot C_j$$

$$\leq (p_a(C - \sum_{A_1 \leq C_i} C_i) - 1) + n_0(p-1) - \frac{n_0(n_0-1)}{2}$$

$$\leq (p_a(C - \sum_{A_1 \leq C_i} C_i) - 1) + \frac{p(p-1)}{2}.$$

Notice that $A_1 \not\leq C - \sum_{A_1 \leq C_i} C_i$ and $D_2 = Z - A_1$, we know that supp $(C - \sum_{A_1 \leq C_i} C_i) \subseteq$ supp (D_2). Since D_2 is the fundamental cycle on its support and the length of the Yau sequence for D_2 is $m - 1$, by the induction hypothesis, we have $p_a(C - \sum_{A_1 \leq C_i} C_i) \leq \frac{p(p-1)(m-1)}{2} + 1$. It means that $p_a(C) \leq p_a(C - \sum_{A_1 \leq C_i} C_i) + \frac{p(p-1)}{2} \leq \frac{p(p-1)m}{2} + 1$.

It follows from Lemma 3.11 that we have $p_a(V, o) = \frac{p(p-1)m}{2} + 1$.   Q.E.D.

We want to get a similar formula for the higher degree case:

$$p_a(V, o) = p_a(([\frac{p-1}{d}] + 1)Y) = \frac{dm}{2}(\frac{2p-2}{d} - [\frac{p-1}{d}])([\frac{p-1}{d}] + 1) + 1,$$

where $d = -Z^2$ and $[a] := \max\{n \in \mathbb{Z} | n \leq a\}$ for real number a(Gauss symbol). However this formula doesn't hold in the general case. For example, if $m = 1$, $Z \neq Z_{min}$ and $p \geq d + 1$, then

$$p_a(([\frac{p-1}{d}] + 1)Y) = p_a(([\frac{p-1}{d}] + 1)Z) < p_a([\frac{p-1}{d}]Z + Z_{min}) \leq p_a(V, o).$$

Notice that, the condition $Z^2 = -1$ implies that $D_m = Z_{min}$, so we need the restrictive condition $D_m = Z_{min}$ when $d > 1$.

Lemma 3.13. *Let (V, o) be a normal surface singularity of degree two or degree three, Z is the fundamental cycle on the minimal resolution. Assume that Z is essentially irreducible and $Z = Z_{min}$, then*

$$p_a(V, o) = p_a(([\frac{p-1}{d}] + 1)Z) = \frac{d}{2}(\frac{2p-2}{d} - [\frac{p-1}{d}])([\frac{p-1}{d}] + 1) + 1.$$

Proof. There exists a cycle D such that $p_a(D) = p_a(V, o)$ and $\mathcal{O}_Z(-D)$ is nef since the negative definiteness of the intersection matrix. Assume A is the irreducible component $A \leq Z$ such that A is not a (-2)-curve and k is the coefficient of cycle A of Z. Assume $ak + b$ is the coefficient of cycle A of D, where $a, b \in \mathbb{Z}$ and $0 \leq b < k$. Then we claim that $aZ \leq D' < (a+1)Z$.

If $D \not\leq (a+1)Z$, let $B = \max(D - (a+1)Z, 0) > 0$, where $\max(D_1, D_2) := \sum \max(n_i, m_i)E_i$ for $D_1 = \sum n_i E_i$ and $D_2 = \sum m_i E_i$, then we have $D - B < (a+1)Z$ by the definition of B. For any irreducible components $C \leq B$, we have the coefficient of C of $D - B$ is equal to the coefficient of C of $(a+1)Z$. Therefore, $C \cdot (D - B) \leq C \cdot (a+1)Z$. Notice that Z is essentially irreducible and $A \not\leq B$ and $Z = Z_{min}$, we have that C is a (-2)-curve and $C \cdot Z = 0$. It means that B consists of (-2)-curve and $B \cdot (D - B) \leq 0$. Then we have $p_a(D - B) = p_a(D) - p_a(B) - B \cdot (D - B) + 1 > p_a(D)$, contradicting the maximality of $p_a(D)$.

If $aZ \not\leq D$, let $B = \max(aZ - D, 0) > 0$, then we have $aZ \leq D + B$ by the definition of B. There exists an irreducible component $C \leq B$ such that $C \cdot B < 0$ by the negative definiteness of the intersection matrix. Since $C \leq B$, we know that the coefficient of C of $D + B$ is equal to the coefficient of C of aZ. Therefore, $C \cdot (D + B) \geq C \cdot aZ = 0$. But $C \cdot B < 0$, contradicting that $\mathcal{O}_Z(-D)$ is nef. So we get $aZ \leq D' < (a+1)Z$.

If $D \neq aZ$, assume $D = aZ + B$, where $0 < B < Z$. We claim that $p_a(B) - 1 \leq \frac{b}{k}(p_a(Z) - 1)$. By the Riemann-Roch theorem, $p_a(B) - 1 = \frac{1}{2}(B^2 + B \cdot K) = \frac{1}{2}(B^2 + bA \cdot K)$ and $p_a(Z) - 1 = \frac{1}{2}(Z^2 + kA \cdot K)$. Hence the claim is equivalent to $B^2 \leq \frac{b}{k}Z^2$. When degree two or degree three case, we know that $1 \leq k \leq 3$ by $Z^2 = kA \cdot Z$. If $k = 1$, the claim is obvious by $0 \leq b < k$. If $k = 2$, it means that $d = 2$ and $\frac{b}{k}Z^2 \geq -1$, the claim is obvious too. If $k = 3$, it means that $d = 3$ and $0 \leq b \leq 2$. The claim is obvious when $b \leq 1$, so we only need to observe $b = 2$ case. Since $B^2 + 2A \cdot K = 2p_a(B) - 2$, we know that B^2 is even number. Then $B^2 \leq -2 = \frac{b}{k}Z^2$.

By $p_a(D) = p_a(aZ) + p_a(B) + aZ \cdot B - 1 \geq p_a(aZ)$, we have $p_a(V, o) = p_a(D) = p_a(aZ) + p_a(B) - 1 + abZ \cdot A \leq p_a(aZ) + \frac{b}{k}(p_a(Z) - 1 + akZ \cdot A) \leq p_a(aZ) + (p_a(Z) - 1 + akZ \cdot A) = p_a((a+1)Z)$. So we know that $\max_i\{p_a(iZ)\} \leq p_a(V, o) = p_a(D) \leq p_a(a+1)Z$, hence

$$p_a(V, o) = \max_i\{p_a(iZ)\} = p_a(([\frac{p-1}{d}] + 1)Z) = \frac{d}{2}(\frac{2p-2}{d} - [\frac{p-1}{d}])([\frac{p-1}{d}] + 1) + 1.$$

Q.E.D.

When the singular point is of degree two, according to Proposition 3.10, we can generalized Lemma 3.13 to $m > 1$ case.

Theorem 3.14 (i.e. Theorem C). *Let (V, o) be a normal surface singularity of degree two with $p_f(V, o) > 0$, Z is the fundamental cycle on the minimal resolution. Assume that Z is essentially irreducible and $D_m = Z_{min}$, then*

$$p_a(V, o) = p_a(([\frac{p-1}{2}] + 1)Y) = m(p - 1 - [\frac{p-1}{2}])[\frac{p+1}{2}] + 1 = [\frac{p^2}{4}]m + 1.$$

Proof. According to Proposition 3.10, we only need to prove that the theorem holds in each case. There exists a cycle C such that $p_a(C) = p_a(V, o)$ and $\mathcal{O}_Z(-D)$ is nef since the negative definiteness of the intersection matrix. Assume A is the irreducible

component $A \leq Z$ such that A is not a (-2)-curve and k is the coefficient of cycle A of Z.

In case (1), let $C = C_1 + \cdots + C_n$ be a chain-connected component decomposition, where C_i is a chain-connected cycle and $\mathcal{O}_{C_j}(-C_i)$ is nef for $i < j$. As in the proof of Theorem 3.12, we have $A \leq C_i$ for $1 \leq i \leq n$. Since C_i is chain-connected, we have that supp C_i is connected. With the notations as in Theorem 3.9, we know the coefficient of cycle B of Z is 1 for each cycle B in Γ'. We denote the number of the cycle B that $B \leq C_i$ and B in the first $A_{n'}$(resp. the last $A_{n'}$) by a_i(resp. b_i) for $1 \leq i \leq n$. Let c_j(resp. d_j) be the number of i that $a_i = j$(resp. $b_i = j$) for $0 \leq j \leq n'$. Then we have $\sum_{j=0}^{n'} c_j = \sum_{j=0}^{n'} c_j = n$ and $p_a(C) - 1 = \sum_{i=1}^{n}(p_a(C_i) - 1) + \sum_{i<j} C_i \cdot C_j \leq n(p-1) - \sum_{j=0}^{n'} \frac{c_j^2 - c_j + d_j^2 - d_j}{2} \leq np - \sum_{j=0}^{n'} \frac{c_j^2 + d_j^2}{2}$. It is easy to see that $p_a(C)$ reaches its maximum value when $c_0 = c_1 = c_2 = \cdots = c_{n'} = d_0 = d_1 = \cdots = d_{n'} = [\frac{p-1}{d}]$ and $m = n' + 1$, so the theorem holds in case (1).

In case (2) or case (3), there exists a unique cycle $B \leq Z$ such that $B \cdot Z = -1$. We consider the coefficient of cycle B of C, denote by $2a + b(a, b \in \mathbb{Z}, 0 \leq b \leq 1)$. As in the proof of Lemma 3.13, $aZ \leq C$. If $b = 0$, we have $\mathcal{O}_{C-aZ}(-Z)$ is numerically trivial, hence, $p_a(C) - 1 = (p_a(C - aZ) - 1) + a(p-1) - \frac{a^2 - a}{2} \leq (p_a(C - aZ) - 1) + [\frac{p^2}{4}]$ and supp $(C - aZ) \subset$ supp (D_2). If $b = 1$, there exists an end D in Γ' that connected with B. Denote the coefficient of cycle D of C by c, we have $c = a$ and $D \cdot C = -1$ since $-1 \leq D \cdot C = -2c + (2a + b) \leq 0$, hence, $\mathcal{O}_{C-aZ-B-D}(-Z)$ is numerically trivial and $p_a(C) - 1 \leq p_a(C - B - D) - 1 = (p_a(C - aZ - B - D) - 1) + a(p-1) - \frac{a^2 - a}{2} \leq (p_a(C - aZ - B - D) - 1) + [\frac{p^2}{4}]$. By induction, the theorem holds in case (2), case (3).

In case (4), case (5), case (6), we have $m = 2$. There exists a unique cycle $B \leq Z$ such that $B \cdot Z = -1$, denote the coefficient of cycle B of C by $2a + b(a, b \in \mathbb{Z}, 0 \leq b \leq 1)$. As in the proof of Lemma 3.13, $aZ \leq C$. The theorem is trivial when $b = 0$, so we only need to prove the theorem when $b = 1$. We have $p_a(C) - 1 = (p_a(aZ) - 1) + (p_a(C - aZ - B) - 1) + (B \cdot (C - B) - 1)$. Notice that $B \cdot (C - B) - 1 \leq 1$, $p_a(aZ) - 1 \leq [\frac{p^2}{4}]$, and $p_a(C - aZ - B) - 1 \leq [\frac{p^2}{4}]$, we have $p_a(C) - 1 \leq 2[\frac{p^2}{4}] + 1$. Whatmore, $p_a(aZ) - 1 = [\frac{p^2}{4}]$ means $a \geq [\frac{p}{2}]$, and $p_a(C - aZ - B) - 1 = [\frac{p^2}{4}]$ means $[\frac{p}{2}]D_2 \leq C - aZ - B \leq [\frac{p+3}{2}]D_2$, then $B \cdot (C - B) - 1 = B \cdot aZ + B \cdot (C - B - aZ) - 1 < -[\frac{p}{2}] + [\frac{p+3}{2}] - 1 \leq 1$. So the theorem holds in case (4), case (5), case (6).

In case (7), we have $m = 3$. There exists a unique cycle $B \leq Z$ such that $B \cdot Z = -1$, denote the coefficient of cycle B of C by $2a + b(a, b \in \mathbb{Z}, 0 \leq b \leq 1)$. Similarly, we have $p_a(C) = p_a(aZ) + p_a(C - aZ - bB) - 1 \leq p_a(C - aZ - bB) + [\frac{p^2}{4}]$ when $b = 0$, and $p_a(C) = p_a(aZ) + p_a(C - aZ - bB) - 1 + (B \cdot (C - B) - 1)$ when $b = 1$. Since D_2 is the same as case (4) with $k' = 0$ and $n' = 5$, we have $p_a(C - aZ - bB) \leq 2[\frac{p^2}{4}] + 1$, and $p_a(C - aZ - bB) = 2[\frac{p^2}{4}] + 1$ means $C - aZ - bB \leq [\frac{p+3}{2}](D_2 + D_3)$. Hence, similarly as case (4)-(6), we have the theorem holds in case (7).   Q.E.D.

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