THE WEIGHTS OF ISOLATED CURVE SINGULARITIES ARE DETERMINED BY HODGE IDEALS

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ABSTRACT. We calculate Hodge ideals and Hodge moduli algebras for three types of isolated quasi-homogeneous curve singularities. We show that Hodge ideals and Hodge moduli algebras of the singularities can determine the weights of the polynomials defining the singularities. We give some examples to explain why Hodge moduli algebras and Hodge moduli sequence are better invariants than characteristic polynomial (a topological invariant of the singularity) for non-degenerate quasi-homogeneous singularities, in the sense that characteristic polynomial cannot determine the weight type of the singularity.

Keywords. Hodge ideals, Hodge moduli algebras, Hodge moduli sequence, weight type, isolated quasi-homogeneous curve singularities

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1. INTRODUCTION

In [17] and [18], the authors ask whether the topology of the singularity determines the weights of the polynomial defining the singularity. They showed that this is valid in the category of isolated singularities of Brieskorn-Pham type and isolated quasi-homogeneous curve singularities.

Theorem 1.1 ([17])**.** The topology of a singularity of Brieskorn-Pham type determines the exponents (weight) of the polynomial defining the singularity.

Theorem 1.2 ([18])**.** Let $f_i(z_1, z_2), i = 1, 2$, be non-degenerate quasi-homogeneous polynomials of weight $(r_{i1}, r_{i2}; 1), 0 \leq r_{i1} \leq r_{i2} \leq \frac{1}{2}$, and let V_i be the germ of $f_i(z_1, z_2) = 0$ at the origin of $\mathbb{C}^2$. Then if $(\mathbb{C}^2, V_1, 0) \simeq (\mathbb{C}^2, V_2, 0)$, homeomorphically, we have $(r_{11}, r_{12}) = (r_{21}, r_{22})$.

For quasi-homogeneous surface singularities, there are some relevant results. Arnold [1], Orlik and Wagreich [13] showed that if $h(z_0, z_1, z_2)$ is a quasi-homogeneous polynomial in $\mathbb{C}^3$ and $V = \{h(z) = 0\}$ has an isolated singularity at the origin, then V can be deformed into one of the following seven classes below while keeping the differentiable structure of the link $K_V = S_\epsilon^{2n+1} \cap V$ constant.

$$\textbf{I.} \quad V(a_0, a_1, a_2; 1) = \{z_0^{a_0} + z_1^{a_1} + z_2^{a_2}\}, \qquad a_0, a_1, a_2 > 1,$$

$$\textbf{II.} \quad V(a_0, a_1, a_2; 2) = \{z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2}\}, \qquad a_0, a_1 > 1, a_2 > 0,$$

$$\textbf{III.} \quad V(a_0, a_1, a_2; 3) = \{z_0^{a_0} + z_1^{a_1} z_2 + z_1 z_2^{a_2}\}, \qquad a_0 > 1, a_1, a_2 > 0,$$

$$\textbf{IV.} \quad V(a_0, a_1, a_2; 4) = \{z_0^{a_0} + z_1^{a_1} z_2 + z_0 z_2^{a_2}\}, \qquad a_0 > 1, a_1, a_2 > 0,$$

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V. $V(a_0, a_1, a_2; 5) = \{z_0^{a_0} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2}\}, \qquad a_0, a_1, a_2 > 0,$

VI. $V(a_0, a_1, a_2; 6) = \{z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2}\}$

$a_0 > 1, a_1, a_2, b_1, b_2 > 0$ satisfies $(a_0 - 1)(a_1 b_2 + a_2 b_1) = a_0 a_1 a_2,$

VII. $V(a_0, a_1, a_2; 7) = \{z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2}\}$

$a_0, a_1, a_2, b_1, b_2 > 0$ satisfies $(a_0 - 1)(a_1 b_2 + a_2 b_1) = a_2(a_0 a_1 - 1).$

Xu and Yau [16] proved that the above deformation is actually a topological trivial deformation as a pair (S^{2n+1}, K_V). Therefore any isolated quasi-homogeneous surface singularity has the same topological type of one of the seven classes above. Let $\Delta_V(z)$ denote the characteristic polynomial of the **Milnor** fibration of $(V, 0)$.

Theorem 1.3 ([16])**.** If $(V, 0)$ and $(W, 0)$ are among the seven classes above, then $(\mathbb{C}^3, V, 0)$ is biholomorphic to $(\mathbb{C}^3, W, 0)$ if and only if $(\mathbb{C}^3, V, 0)$ is homeomorphic to $(\mathbb{C}^3, W, 0)$ with some exceptional cases. And $(\mathbb{C}^3, V, 0)$ is homeomorphic to $(\mathbb{C}^3, W, 0)$ if and only if $\pi_1(K_V) \simeq \pi_1(K_W)$ and $\Delta_V(z) = \Delta_W(z)$.

The following are direct corollaries of the above theorem:

Corollary 1.4 ([16])**.** Let $(V, 0)$ and $(W, 0)$ be two isolated quasi-homogeneous surface singularities in $\mathbb{C}^3$. Then $(\mathbb{C}^3, V, 0)$ is homeomorphic to $(\mathbb{C}^3, W, 0)$ if and only if $\pi_1(K_V) \simeq \pi_1(K_W)$ and $\Delta_V(z) = \Delta_W(z)$.

Corollary 1.5 ([16])**.** Let $(V, 0)$ be an isolated quasi-homogeneous surface singularity with weights (w_0, w_1, w_2). Then the topological type of $(V, 0)$ determines and is determined by its weights (w_0, w_1, w_2).

Corollary 1.6 ([16])**.** Let $(V, 0)$ be an isolated singularity defined by a quasi-homogeneous polynomial in $\mathbb{C}^3$ with weights (w_0, w_1, w_2). Then the fundamental group of the link $\pi_1(K_V)$ and the characteristic polynomial $\Delta_V(z)$ determine and are determined by the weights (w_0, w_1, w_2).

And the original motivation of their paper is to prove the **Zariski** conjecture (cf. [19]) for isolated quasi-homogeneous surface singularities in $\mathbb{C}^3$: multiplicity is an invariant of topological type. As a corollary, they proved:

Corollary 1.7 ([16])**.** Let $(V, 0)$ and $(W, 0)$ be two isolated quasi-homogeneous surface singularities in $\mathbb{C}^3$. If $(\mathbb{C}^3, V, 0)$ is homeomorphic to $(\mathbb{C}^3, W, 0)$, then V and W have the same multiplicity at the origin.

Recall that in our former paper [9], we proved that a series new invariants Hodge moduli algebras and Hodge moduli sequence of the singularity are complete contact invariants for simple surface singularities. And our final aim is to extend this result to isolated quasi-homogeneous surface singularities or even more general types of singularity. Note that in the proof of the above theorems, characteristic polynomial of the singularity plays a fundamental role, since characteristic polynomial is a topological invariant of the singularity. Motivated by these results and our former results, it is natural to ask whether we can replace characteristic polynomial by Hodge ideals and Hodge moduli algebras of the singularity to determine the weights of the polynomials defining the singularities. That is, we want to prove if the ith Hodge moduli algebras of two isolated quasi-homogeneous curve singularities are isomorphic, $\forall i \geq 0$, then the weights of these two singularities are the same.

If $h(x, y)$ is a quasi-homogeneous polynomial in $\mathbb{C}^2$ and $V = \{h(x, y) = 0\}$ has an isolated singularity at the origin, then V can be deformed into one of the following three classes below while keeping the differentiable structure of the link $K_V = S_\epsilon^{2n+1} \cap V$ constant.

$$\begin{aligned}
\mathbf{F}_1(x,y) &= x^a + y^b, \qquad a,b \geq 2,\\
\mathbf{F}_2(x,y) &= x^a + xy^b, \qquad a \geq 2, b \geq 1,\\
\mathbf{F}_3(x,y) &= x^a y + xy^b, \qquad a,b \geq 1.
\end{aligned}$$

After a tedious calculation for Hodge ideals and Hodge moduli algebras of isolated quasi-homogeneous curve singularities of the above three types, we obtain the following main theorem A.

Main Theorem A (0th and 1st Hodge moduli algebras determine weight type).

(1) For isolated quasi-homogeneous curve singularities

$$D_1^{(a_1,b_1)} = \{x^{a_1} + y^{b_1} = 0\}, \qquad 2 \leq a_1 \leq b_1,$$

and

$$D_2^{(a_2,b_2)} = \{x^{a_2} + xy^{b_2} = 0\}, \qquad 1 \leq a_2 - 1 \leq b_2,$$

if their 0th and 1st Hodge moduli algebras (taking $\alpha = 1$ in their Hodge ideals) are isomorphic, i.e.,

$$M_0(D_1^{(a_1,b_1)}) \simeq M_0(D_2^{(a_2,b_2)}) \qquad M_1(D_1^{(a_1,b_1)}) \simeq M_1(D_2^{(a_2,b_2)}),$$

then the weight types of $D_1^{(a_1,b_1)}$ and $D_2^{(a_2,b_2)}$ are the same, i.e.,

$$\mathbf{wt}(F_1^{(a_1,b_1)}) = \mathbf{wt}(F_2^{(a_2,b_2)}).$$

(2) For isolated quasi-homogeneous curve singularities

$$D_2^{(a_2,b_2)} = \{x^{a_2} + xy^{b_2} = 0\}, \qquad a_2 - 1 \geq b_2 \geq 1,$$

and

$$D_3^{(a_3,b_3)} = \{x^{a_3} y + xy^{b_3} = 0\}, \qquad 1 \leq a_3 \leq b_3,$$

if their 0th and 1st Hodge moduli algebras (taking $\alpha = 1$ in their Hodge ideals) are isomorphic, i.e.,

$$M_0(D_2^{(a_2,b_2)}) \simeq M_0(D_3^{(a_3,b_3)}) \qquad M_1(D_2^{(a_2,b_2)}) \simeq M_1(D_3^{(a_3,b_3)}),$$

then the weight type of $D_2^{(a_2,b_2)}$ and $D_3^{(a_3,b_3)}$ are the same, i.e.,

$$\mathbf{wt}(F_1^{(a_1,b_1)}) = \mathbf{wt}(F_3^{(a_3,b_3)}).$$

(3) For isolated quasi-homogeneous curve singularities

$$D_1^{(a_1,b_1)} = \{x^{a_1} + y^{b_1} = 0\}, \qquad a_1, b_1 \geq 2,$$

and

$$D_3^{(a_3,b_3)} = \{x^{a_3} y + xy^{b_3} = 0\}, \qquad a_3, b_3 \geq 1,$$

their ith Hodge moduli algebras (taking $\alpha = 1$ in their Hodge ideals) are not isomorphic, for $i = 0, 1$ respectively.

As a by-product, we obtain an inequality of the δ-invariant, 0th Hodge moduli number and multiplicity for isolated quasi-homogeneous curve singularities of the above three types:

Main Theorem B.

(1) For isolated quasi-homogeneous curve singularities $D_1^{(a,b)} = \{x^a + y^b = 0\}, a, b \geq 2$, we have an inequality

$$0 \leq \delta_1(a,b) - m_0(D_1^{(a,b)}) \leq \mathbf{mt}(D_1^{(a,b)}),$$

where $\delta_1(a,b)$ is the δ-invariant of $D_1^{(a,b)}, m_0(D_1^{(a,b)})$ is the 0th Hodge moduli number of the divisor $D_1^{(a,b)}$ for $\alpha = 1$ and $\mathbf{mt}(D_1^{(a,b)})$ is the multiplicity of $D_1^{(a,b)}$.

(2) For isolated quasi-homogeneous curve singularities $D_2^{(a,b)} = \{x^a + xy^b = 0\}, a \geq 2, b \geq 1$, we have an inequality

$$1 \leq \delta_2(a,b) - m_0(D_2^{(a,b)}) \leq \mathbf{mt}(D_2^{(a,b)}),$$

where $\delta_2(a,b)$ is the δ-invariant of $D_2^{(a,b)}, m_0(D_2^{(a,b)})$ is the 0th Hodge moduli number of the divisor $D_2^{(a,b)}$ for $\alpha = 1$ and $\mathbf{mt}(D_2^{(a,b)})$ is the multiplicity of $D_2^{(a,b)}$.

(3) For isolated quasi-homogeneous curve singularities $D_3^{(a,b)} = \{x^a y + xy^b = 0\}, a, b \geq 1$, we have an inequality

$$2 \leq \delta_3(a,b) - m_0(D_3^{(a,b)}) \leq \mathbf{mt}(D_3^{(a,b)}),$$

where $\delta_3(a,b)$ is the δ-invariant of $D_3^{(a,b)}, m_0(D_3^{(a,b)})$ is the 0th Hodge moduli number of the divisor $D_3^{(a,b)}$ for $\alpha = 1$ and $\mathbf{mt}(D_3^{(a,b)})$ is the multiplicity of $D_3^{(a,b)}$.

In the second section, we recall a number of classical results on the Hodge ideals of effective $\mathbb{Q}$-divisors and the δ-invariants of curve singularities. We also collect some important lemmas and theorems that will be used in the following parts. In the third section, we explicitly calculate Hodge ideals and Hodge moduli algebras of isolated quasi-homogeneous curve singularities of three types. In the fourth and fifth sections we prove our main theorems A and B by the results in the third section. Finally, in the last section, we give some examples to explain Hodge moduli algebras and Hodge moduli sequence are better invariants than characteristic polynomial (topological invariant of singularity) for non-degenerate quasi-homogeneous singularities. Furthermore, from the observation of some examples, we raise a conjecture that the Hodge moduli numbers of isolated quasi-homogeneous curve singularities remain constant under semi-quasihomogeneous deformation. That is, Hodge moduli numbers of isolated quasi-homogeneous curve singularities only depend on the weights of the singularities.

2. Preliminaries

2.1. **Hodge ideals.** In [7] and [8], the authors extend the notion of Hodge ideals to the case when D is an arbitrary effective $\mathbb{Q}$-divisor on X, where X is a smooth complex variety. Hodge ideals $\{I_k(D)\}_{k\in\mathbb{N}}$ are defined in terms of the Hodge filtration $F_\bullet$ on some $\mathscr{D}_X$-module associated with D (cf. [7], §2 − §4 for more details). When D is an integral and blackuced divisor, this recovers the definition of Hodge ideals $I_k(D)$ in [5].

Let X be a smooth complex variety, and $\mathscr{D}_X$ be the sheaf of differential operators on X. If H is an integral and blackuced effective divisor on X, $D = \alpha H, \alpha \in \mathbb{Q} \cap (0,1]$, let $\mathcal{O}_X(*D)$ be the sheaf of rational functions with poles along D. It is also a left $\mathscr{D}_X$-module underlying the mixed Hodge module $j_*\mathbb{Q}_U^H[n]$, where $U = X\backslash D$ and $j : U \hookrightarrow X$ is the inclusion map. Any $\mathscr{D}_X$-module associated with a mixed Hodge module has a good filtration $F_\bullet$, the Hodge filtration of the mixed Hodge module [14].

To study the Hodge filtration of $\mathcal{O}_X(*D)$, it seems easier to consider a series of ideal sheaves, defined by Mustaţă and Popa [5], which can be consideblack to be a generalization of multiplier ideals of divisors. The Hodge ideals $\{I_k(D)\}_{k\in\mathbb{N}}$ of the divisor D are defined by:

$$F_k\mathscr{O}_X(*D) = I_k(D)\otimes\mathscr{O}_X\big((k+1)D\big),\quad \text{for all } k\in\mathbb{N}.$$

These are coherent sheaves of ideals. See [5] for details and an extensive study of the ideals $I_k(D)$. Hodge ideals are indexed by the non-negative integers; at the 0-th step, they essentially coincide with multiplier ideals. It turns out that $I_0(D) = \mathscr{J}((1-\epsilon)D)$, the multiplier ideal of the divisor $(1-\epsilon)D,\ 0<\epsilon\ll 1$. The multiplier ideal sheaves are ubiquitous objects in birational geometry, encoding local numerical invariants of singularities, and satisfying Kodaira-type vanishing theorems in the global setting. The Hodge ideals are interesting invariants of the singularities, they have similar properties as multiplier ideals.

We summarize the properties and results (cf. [6] and[7]) of Hodge ideals as follows:

Given a blackuced effective divisor H on a smooth complex variety X, $D=\alpha H, \alpha\in\mathbb{Q}\cap(0,1]$, we also denote by Z the support of D. The sequence of Hodge ideals $I_k(D)$, with $k\geq 0$, satisfies:

- $I_0(D)$ is the multiplier ideal $\mathcal{I}((1-\epsilon)D)$, so in particular $I_0(D)=\mathscr{O}_X$ if and only if the pair(X,D) is log canonically.
- When Z has simple normal crossings, then
$$I_k(D) = I_k(Z)\otimes\mathscr{O}_X(Z-\lceil D\rceil),$$
where $I_k(Z)$ can be computed explicitly as in [5]. If Z is smooth, then $I_k(D)=\mathscr{O}_X(Z-\lceil D\rceil)$.
- The Hodge filtration is generated at level $n-1$, where $n=\dim X$, i.e.,
$$F_\ell\mathscr{D}_X\cdot\big(I_k(D)\otimes\mathscr{O}_X(kZ)h^{-\alpha}\big) = I_{k+\ell}(D)\otimes\mathscr{O}_X((k+\ell)Z)h^{-\alpha}$$
for all $k\geqslant n-1$ and $\ell\geqslant 0$.
- There are non-triviality criteria for $I_k(D)$ at a point $x\in D$ in terms of the multiplicity of D at x.
- If X is projective, $I_k(D)$ satisfy a vanishing theorem analogous to Nadel Vanishing for multiplier ideals.
- If Y is a smooth divisor in X such that $Z|_Y$ is blackuced, then $I_k(D)$ satisfy
$$I_k\left(D|_Y\right)\subseteq I_k(D)\cdot\mathscr{O}_Y,$$
with equality when Y is general.
- If $X\to T$ is a smooth family with a section $s: T\to X$, and D is a relative divisor on X that satisfies a suitable condition then
$$\left\{t\in T\mid I_k\left(D_t\right)\nsubseteq\mathfrak{m}_{s(t)}^q\right\}$$
is an open subset of T, for each $q\geqslant 1$.
- If D_1 and D_2 are Q-divisors with supports Z_1 and Z_2, such that Z_1+Z_2 is also blackuced, then the subadditivity property
$$I_k\left(D_1+D_2\right)\subseteq I_k\left(D_1\right)\cdot I_k\left(D_2\right)$$
holds.

For comparison, the list of properties of Hodge ideals in the case when D is blackuced is summarized in [11]. The setting of Q-divisors is more intricate. For instance, the bounds for the generation level of the Hodge filtration can become worse. Moreover, it is not known whether the inclusions $I_k(D)\subseteq I_{k-1}(D)$ continue to hold for arbitrary Q-divisors. New phenomena appear as well: given two rational numbers $\alpha_1 < \alpha_2$, usually the ideals $I_k(\alpha_1 Z)$ and $I_k(\alpha_2 Z)$ cannot be compablack for $k \geqslant 1$, unlike in the case of multiplier ideals.

We recall the following definition.

Definition 2.1. Let $f, g \in R = \mathbb{C}\{x_1, \ldots, x_n\}$ which is the convergent power series ring. We say f and g are contact equivalent if the local $\mathbb{C}$-algebras $R/(f)$ and $R/(g)$ are isomorphic.

Definition 2.2. Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0), n \geq 2$, be an isolated hypersurface singularity. Let $H = \{f = 0\}$ be an integral and blackuced effective divisor defined by $f, D^\alpha = \alpha H, \alpha \in \mathbb{Q} \cap (0, 1]$. We define the i-th Hodge moduli algebra of D^α to be the moduli algebra of the ideal $J_i(D^\alpha) := (f) + I_i(D^\alpha)$ (or J_i for short)

$$M_i(D^\alpha) := \mathbb{C}\{x_1, \ldots, x_n\}/J_i(D^\alpha)$$

for $i \geq 0$ (or M_i for short), where $I_i(D^\alpha)$ be the i-th Hodge ideal (or I_i for short). The i-th Hodge moduli number of D^α is defined to be

$$m_i(D^\alpha) := \dim_{\mathbb{C}}(M_i(D^\alpha))$$

for $i \geq 0$ (or m_i for short). We define the Hodge moduli sequence of D to be the sequence

$$\{m_i\} := \{m_0, m_1, m_2, \ldots\}.$$

Definition 2.3. A polynomial $f \in \mathbb{C}[x_1, \cdots, x_n]$ is called weighted homogeneous if there exists positive rational numbers $w_1, \cdots, w_n$ (i.e., weights of $x_1, \cdots, x_n$) and d such that, $\sum a_i w_i = d$ for each monomial $\prod x_i^{a_i}$ appearing in f with a non-zero coefficient. The number d is called the weighted homogeneous degree (w-deg) of f for weights $w_j, 1 \leq j \leq n$. These $w_j, 1 \leq j \leq n$ called the weight type of f.

The Hodge filtration $F_\bullet$ of $\mathcal{O}_X(*D)$ is usually hard to describe. However, it does have an explicit formula in the case when D is defined by a blackuced weighted homogeneous polynomial f which has an isolated singularity at the origin, which is proved by M. Saito [15]. To state Saito's result, we first clarify the notations as follows. We denote

- $\mathcal{O} = \mathbb{C}\{x_1, \ldots, x_n\}$ the ring of germs of holomorphic function for local coordinates $x_1, \ldots, x_n$.
- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ a germ of holomorphic function that is quasi-homogeneous, i.e., $f \in \mathcal{J}(f) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$, and with an isolated singularity at the origin. Kyoji Saito [12] showed that after a biholomorphic coordinate change, we can assume f is a weighted homogeneous polynomial with an isolated singularity at the origin. We will keep this assumption for f unless otherwise stated.
- $w = w(f) = (w_1, \ldots, w_n)$ the weights of the weighted homogeneous polynomial f.
- $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ a germ of a holomorphic function, and we write

$$g = \sum_{A \in \mathbb{N}^n} g_A x^A,$$

  where $A = (a_1, \ldots, a_n), g_A \in \mathbb{C}$ and $x^A = x_1^{a_1} \cdots x_n^{a_n}$.
- $\rho(g)$ the weight of an element $g \in \mathcal{O}$ defined by

$$\rho(g) = \left(\sum_{i=1}^{m} w_i\right) + \inf\{\langle w, A\rangle : g_A \neq 0\}.$$

The weight function ρ defines a filtration on $\mathcal{O}$ as

$$\mathcal{O}^{>k} = \{u \in \mathcal{O} : \rho(u) > k\},$$
$$\mathcal{O}^{\geq k} = \{u \in \mathcal{O} : \rho(u) \geq k\}.$$

Since we consider $\mathscr{D}_X$-modules locally around the isolated singularity, so we can assume $X = \mathbb{C}^n$ and identify the stalk at the singularity to be that of $\mathscr{D}_X$-modules on $\mathbb{C}^n$. For example, we replace $F_k\mathcal{O}_{X,0}(*D)$ with $F_k\mathcal{O}_X(*D)$. Now we can state the formula proved by M. Saito (see [15], Theorem 0.7):

$$F_k\mathcal{O}_X(*D) = \sum_{i=0}^{k} F_{k-i}\mathscr{D}_X \left(\frac{\mathcal{O}^{\geq i+1}}{f^{i+1}}\right), \forall k \in \mathbb{N}. \tag{1}$$

Since the Hodge filtration can be constructed on analogous $\mathscr{D}_X$-modules associated with any effective $\mathbb{Q}$-divisor D, so it satisfies a similar formula in the case when D is supported on a hypersurface defined by such a polynomial f.

Assume that the divisor is $D = \alpha Z$, where $0 < \alpha \leq 1$ and $Z = \{f = 0\}$ is an integral and blackuced effective divisor defined by f, a weighted homogeneous polynomial with an isolated singularity at the origin. In this case, the associated $\mathscr{D}_X$-module is the well-known twisted localization $\mathscr{D}_X$-module $\mathcal{M}\left(f^{1-\alpha}\right) := \mathcal{O}_X(*Z)f^{1-\alpha}$ (see more details in [7] about how to construct the Hodge filtration $F_\bullet\mathcal{M}\left(f^{1-\alpha}\right)$). With new ingblackients from Mustaţă and Popa's [8], where this Hodge filtration is compablack to the V-filtration on $\mathcal{M}\left(f^{1-\alpha}\right)$, M. Zhang generalized Saito's formula and proved the following theorem:

Theorem 2.4. (Zhang, [20]) If $D = \alpha Z$, where $0 < \alpha \leq 1$ and $Z = \{f = 0\}$ is an integral and blackuced effective divisor defined by f, a weighted homogeneous polynomial with an isolated singularity at the origin, then we have

$$F_k\mathcal{M}\left(f^{1-\alpha}\right) = \sum_{i=0}^{k} F_{k-i}\mathscr{D}_X \left(\frac{\mathcal{O}^{\geq \alpha+i}}{f^{i+1}} f^{1-\alpha}\right),$$

where the action $\cdot$ of $\mathscr{D}_X$ on the right hand side is the action on the left $\mathscr{D}_X$-module $\mathcal{M}\left(f^{1-\alpha}\right)$ defined by

$$D \cdot \left(wf^{1-\alpha}\right) := \left(D(w) + w\frac{(1-\alpha)D(f)}{f}\right) f^{1-\alpha}, \text{ for any } D \in \operatorname{Der}_{\mathbb{C}} \mathcal{O}_X.$$

Notice that if we set $\alpha = 1$, Theorem 2.4 recovers Saito's formula (1) mentioned above. For any polynomial f with an isolated singularity at the origin, it is well-known that the **Milnor** algebra

$$\mathcal{A}_f := \mathbb{C}\{x_1, \ldots, x_n\} / (\partial_1 f, \ldots, \partial_n f)$$

is a finite-dimensional $\mathbb{C}$-vector space. Fix a monomial basis $\{v_1, \ldots, v_\mu\}$ for this vector space, where μ is the dimension of $\mathcal{A}_f$ (i.e., **Milnor** number). The following theorem follows from Theorem 2.4.

Theorem 2.5. (Zhang, [20]) If $D = \alpha Z$, where $0 < \alpha \leq 1$ and $Z = \{f = 0\}$ is an integral and blackuced effective divisor defined by f, a weighted homogeneous polynomial with an isolated singularity at the origin, then we have

$$F_0\mathcal{M}(f^{1-\alpha}) = f^{-1} \cdot \mathcal{O}^{\geq \alpha} f^{1-\alpha}$$

and

$$F_k\mathcal{M}(f^{1-\alpha}) = (f^{-1} \cdot \sum_{v_j \in \mathcal{O}^{\geq k+1+\alpha}} \mathcal{O}_X \cdot v_j) f^{1-\alpha} + F_1\mathfrak{D}_X \cdot F_{k-1}\mathcal{M}(f^{1-\alpha}).$$

Alternatively, in terms of Hodge ideals, these formulas say that

$$I_0(D) = \mathcal{O}^{\geq \alpha}$$

and

$$I_{k+1}(D) = \sum_{v_j \in \mathcal{O}^{\geq k+1+\alpha}} \mathcal{O}_X \cdot v_j + \sum_{1 \leq i \leq n, a \in I_k(D)} \mathcal{O}_X(f\partial_i a - (\alpha + k)a\partial_i f).$$

2.2. **Delta invariant of curve singularities.**

Definition 2.6 (**δ-invariant**)**.** Let $f \in \mathbb{C}\{x, y\}$ be a blackuced convergent power series, and let

$$\mathcal{O} = \mathbb{C}\{x, y\}/\langle f \rangle \hookrightarrow \overline{\mathcal{O}}$$

denote the normalization. Then we call

$$\delta(f) := \dim_{\mathbb{C}} \overline{\mathcal{O}}/\mathcal{O}$$

the δ-invariant of f.

Although We can explicitly calculate the δ-invariants of isolated quasi-homogeneous singularities of three types F_1, F_2, F_3, by blowing up singularities and using the above theorem. We use lemma 2.7, which a very useful equality of the **Milnor** number,the δ-invariant and the number of irblackucible factors of a curve singularity $\{f = 0\}$, to show the following lemmas 2.8, 2.9, and 2.10. And we only give proof for lemma 2.8 for simplicity, since the proof for lemmas 2.9, and 2.10 are similar.

Lemma 2.7 ([2], proposition 3.35)**.** Let $f \in \mathfrak{m} \subseteq \mathbb{C}\{x, y\}$ be blackuced. Then

$$\mu(f) = 2\delta(f) - r(f) + 1,$$

where $\mu(f)$ is the **Milnor** number of $f, \delta(f)$ is the δ-invariant of f and $r(f)$ is the number of irblackucible factors of f.

Lemma 2.8. For isolated quasi-homogeneous curve singularities $D_1^{(a,b)} = \{x^a + y^b = 0\}$, defined by $F_1^{(a,b)} = x^a + y^b, a, b \geq 2$, its δ-invariant is

$$\delta_1(a, b) = \frac{(a-1)(b-1) + \gcd(a, b) - 1}{2}.$$

In particular, $\delta_1(a, b) = \frac{(a-1)(b-1)}{2}$, if $\gcd(a, b) = 1$.

Proof. Since $\mu(f) = (a-1)(b-1)$ and $r(f) = \gcd(a, b)$. We have

$$\delta_1(a, b) = \frac{1}{2}(\mu(f) + r(f) - 1) = \frac{(a-1)(b-1) + \gcd(a, b) - 1}{2}.$$

□

Lemma 2.9. For isolated quasi-homogeneous curve singularities $D_2^{(a,b)} = \{x^a + xy^b = 0\}$, defined by $F_2^{(a,b)} = x^a + xy^b, a \geq 2, b \geq 1$, its δ-invariant is

$$\delta_2(a, b) = \frac{a(b-1) + \gcd(a-1, b) + 1}{2}.$$

In particular, $\delta_2(a, b) = \frac{a(b-1)+2}{2}$, if $\gcd(a-1, b) = 1$.

Lemma 2.10. For isolated quasi-homogeneous curve singularities $D_3^{(a,b)} = \{x^a y + xy^b = 0\}$, defined by $F_3^{(a,b)} = x^a y + xy^b, a, b \geq 1$, its δ-invariant is

$$\delta_3(a, b) = \frac{ab + \gcd(a-1, b-1) + 1}{2}.$$

In particular, $\delta_3(a, b) = \frac{ab+2}{2}$, if $\gcd(a-1, b-1) = 1$.

3. The first two Hodge ideals of three types isolated quasi-homogeneous curve singularities

In this section, we compute Hodge ideals of three types isolated quasi-homogeneous curve singularities for $\alpha = 1$ in theorem 2.5. And $\mathcal{O}_X = \mathbb{C}\{x, y\}$ in the following computation. The following lemma is used in the computation of dimensions of Hodge moduli algebras.

Lemma 3.1. For $n, m \in \mathbb{N}, n, m \geq 1$,

$$\sum_{i=1}^{n-1}[\frac{mi}{n}] = \frac{(m-1)(n-1) + \gcd(m,n) - 1}{2},$$

For isolated quasi-homogeneous curve singularities $D_1^{(a,b)} = \{x^a + y^b = 0\}$, defined by $F_1^{(a,b)} = x^a + y^b$. If $a \leq b$, let $r = \frac{a}{\gcd(a,b)}$, then $1 \leq r \leq a$. Since $\frac{1+a-1}{a} + \frac{1}{b} \geq 1$, we have $x^{a-1} \in I_0(D_1)$. And We have

$$x^{k-1}y^{[\frac{b(a-k)}{a}]} \in I_0(D_1), \qquad \forall 1 \leq k \leq a-1, r \nmid k,$$

and

$$x^{ir-1}y^{b-\frac{ibr}{a}-1} \in I_0(D_1), \qquad \forall 1 \leq i \leq \gcd(a,b) - 1.$$

So the 0th Hodge ideal for D_1 is

$$\begin{aligned} J_0(D_1) = I_0(D_1) =&(x^{a-1}, x^{a-2}y^{[\frac{b}{a}]}, \ldots, x^{a-r}y^{[\frac{b(r-1)}{a}]}, \\ &\vdots \\ &x^{ir-1}y^{b-\frac{ibr}{a}-1}, x^{ir-2}y^{[\frac{b(a-ir+1)}{a}]}, \ldots, x^{(i-1)r}y^{[\frac{b(a-(i-1)r-1)}{a}]}, \\ &\vdots \\ &x^{r-1}y^{b-\frac{br}{a}-1}, x^{r-2}y^{[\frac{b(a-r+1)}{a}]}, \ldots, y^{[\frac{b(a-1)}{a}]}), \end{aligned}$$

where $1 \leq i \leq \gcd(a,b) - 1$. It's multiplicity $\mathbf{mt}(J_0(D_1)) = a - 1$. Using lemma 3.1, we obtain the dimension of the 0th Hodge moduli algebra $M_0(D_1) = \mathcal{O}_X / J_0(D_1)$

$$\begin{aligned} m_0(D_1) =& \sum_{i=1}^{a-1}[\frac{bi}{a}] - (\gcd(a,b) - 1) \\ =& \frac{(a-1)(b-1) + \gcd(a,b) - 1}{2} - (\gcd(a,b) - 1) \\ =& \frac{(a-1)(b-1) - \gcd(a,b) + 1}{2}. \end{aligned}$$

And the 1st Hodge ideal of D_1 is

$$\begin{aligned} J_1(D_1) =&(f) + I_0(D_1) \cdot (Jf) \\ =&(x^a + y^b, \\ &x^{a-2}y^b, x^{a-3}y^{[\frac{b}{a}]+b}, \ldots, x^{a-r-1}y^{[\frac{b(r-1)}{a}]+b}, \\ &\vdots \\ &x^{ir-2}y^{2b-\frac{ibr}{a}-1}, \ldots, x^{(i-1)r-1}y^{[\frac{b(a-(i-1)r-1)}{a}]+b}, \\ &\vdots \\ &x^{r-2}y^{2b-\frac{br}{a}-1}, \ldots, y^{[\frac{b(a-2)}{a}]+b}, x^{a-1}y^{[\frac{b(a-1)}{a}]}), \end{aligned}$$

where $2 \leq i \leq \gcd(a,b)$. It's multiplicity $\mathbf{mt}(J_1(D_1)) = a$. And by lemma 3.1, the dimension of the 1st Hodge moduli algebra $M_1(D_1) = \mathcal{O}_X/J_1(D_1)$ is

$$\begin{aligned} m_1(D_1) &= \sum_{i=1}^{a-2}([\frac{bi}{a}] + b) - (\gcd(a,b) - 1) + b + [\frac{b(a-1)}{a}] \\ &= \sum_{i=1}^{a-1}[\frac{bi}{a}] + (a-1)b - (\gcd(a,b) - 1) \\ &= \frac{(a-1)(b-1) + \gcd(a,b) - 1}{2} + (a-1)b - (\gcd(a,b) - 1) \\ &= \frac{(a-1)(3b-1) - \gcd(a,b) + 1}{2}. \end{aligned}$$

If $a \geq b$, let $r = \frac{b}{\gcd(a,b)}$, then $1 \leq r \leq b$. By symmetry of a, b, we obtain the 0th Hodge ideal for D_1

$$\begin{aligned} J_0(D_1) = I_0(D_1) =&(y^{b-1}, y^{b-2}x^{[\frac{a}{b}]}, \ldots, y^{b-r}x^{[\frac{a(r-1)}{b}]}, \\ &\vdots \\ &y^{ir-1}x^{a-\frac{iar}{b}-1}, y^{ir-2}x^{[\frac{a(b-ir+1)}{b}]}, \ldots, y^{(i-1)r}x^{[\frac{a(b-(i-1)r-1)}{b}]}, \\ &\vdots \\ &y^{r-1}x^{a-\frac{ar}{b}-1}, y^{r-2}x^{[\frac{a(b-r+1)}{b}]}, \ldots, x^{[\frac{a(b-1)}{b}]}), \end{aligned}$$

where $1 \leq i \leq \gcd(a,b) - 1$. It's multiplicity $\mathbf{mt}(J_0(D_1)) = b - 1$. And by lemma 3.1, the dimension of the 1st Hodge moduli algebra $M_0(D_1) = \mathcal{O}_X/J_0(D_1)$ is

$$\begin{aligned} m_0(D_1) &= \sum_{i=1}^{b-1}[\frac{ai}{b}] - (\gcd(a,b) - 1) \\ &= \frac{(a-1)(b-1) + \gcd(a,b) - 1}{2} - (\gcd(a,b) - 1) \\ &= \frac{(a-1)(b-1) - \gcd(a,b) + 1}{2}. \end{aligned}$$

And the 1st Hodge ideal for D_1 is

$$\begin{aligned} J_1(D_1) =&(f) + I_0(D_1) \cdot (Jf) \\ =&(x^a + y^b, \\ &y^{b-2}x^a, y^{b-3}x^{[\frac{a}{b}]+a}, \ldots, y^{b-r-1}x^{[\frac{a(r-1)}{b}]+a}, \\ &\vdots \\ &y^{ir-2}x^{2a-\frac{iar}{b}-1}, \ldots, y^{(i-1)r-1}x^{[\frac{a(b-(i-1)r-1)}{b}]+a}, \\ &\vdots \\ &y^{r-2}x^{2a-\frac{ar}{b}-1}, \ldots, x^{[\frac{a(b-2)}{b}]+a}, y^{b-1}x^{[\frac{a(b-1)}{b}]}), \end{aligned}$$

where $2 \leq i \leq \gcd(a,b)$. It's multiplicity $\mathbf{mt}(J_1(D_1)) = b$. And by lemma 3.1, the dimension of the 1st Hodge moduli algebra $M_1(D_1) = \mathcal{O}_X/J_1(D_1)$ is

$$m_1(D_1) = \sum_{i=1}^{b-2}([\frac{ai}{b}] + a) - (\gcd(a,b) - 1) + a + [\frac{a(b-1)}{b}]$$

$$
\begin{aligned}
&= \sum_{i=1}^{b-1}[\frac{ai}{b}] + (b-1)a - (\gcd(a,b)-1) \\
&= \frac{(a-1)(b-1)+\gcd(a,b)-1}{2} + (b-1)a - (\gcd(a,b)-1) \\
&= \frac{(3a-1)(b-1)-\gcd(a,b)+1}{2}.
\end{aligned}
$$

For isolated quasi-homogeneous curve singularities $D_2^{(a,b)} = \{x^a + xy^b = 0\}$, defined by $F_2^{(a,b)} = x^a + xy^b$. If $a-1 \le b$, let $r = \frac{a-1}{\gcd(a-1,b)}$, then $1 \le r \le a-1$. Since $\frac{1+a-1}{a} + \frac{a-1}{ab} \ge 1$, we have $x^{a-1} \in I_0(D_2)$. And We have

$$
x^k y^{[\frac{b(a-k-1)}{a-1}]} \in I_0(D_2), \qquad \forall 1 \le k \le a-1, r \nmid k,
$$

and

$$
x^{ir} y^{b-\frac{ibr}{a-1}-1} \in I_0(D_2), \qquad \forall 1 \le i \le \gcd(a-1,b)-1.
$$

Since $\frac{1}{a} + \frac{(a-1)(1+b-1)}{ab} \ge 1$, we have $y^{b-1} \in I_0(D_2)$. So the 0th Hodge ideal for D_2 is

$$
\begin{aligned}
J_0(D_2) =& I_0(D_2) \\
=& (x^{a-1}, x^{a-2}y^{[\frac{b}{a-1}]}, \dots, x^{a-r}y^{[\frac{b(r-1)}{a-1}]}, \\
& \vdots \\
& x^{ir}y^{b-\frac{ibr}{a-1}-1}, x^{ir-1}y^{[\frac{b(a-ir)}{a-1}]}, \dots, x^{(i-1)r+1}y^{[\frac{b(a-((i-1)r+1)-1)}{a-1}]}, \\
& \vdots \\
& x^r y^{b-\frac{br}{a-1}-1}, x^{r-1}y^{[\frac{b(a-r)}{a-1}]}, \dots, xy^{[\frac{b(a-2)}{a-1}]}, \\
& y^{b-1}),
\end{aligned}
$$

where $1 \le i \le \gcd(a-1,b)-1$. It's multiplicity $\mathbf{mt}(J_0(D_2)) = a-1$. And by lemma 3.1, the dimension of the 0th Hodge moduli algebra $M_0(D_2) = \mathcal{O}_X/J_0(D_2)$ is

$$
\begin{aligned}
m_0(D_2) &= \sum_{i=1}^{a-2}[\frac{bi}{a-1}] - (\gcd(a-1,b)-1) + b - 1 \\
&= \frac{(a-2)(b-1)+\gcd(a-1,b)-1}{2} - (\gcd(a,b)-1) + b - 1 \\
&= \frac{a(b-1)-\gcd(a-1,b)+1}{2}.
\end{aligned}
$$

And the 1st Hodge ideal of D_2 is

$$
\begin{aligned}
J_1(D_2) =& (f) + I_0(D_2)\cdot(Jf) \\
=& (x^a + xy^b, ax^{a-1}y^{b-1} + y^{2b-1} \\
& x^{a-1}y^b, x^{a-2}y^{[\frac{b}{a-1}]+b}, \dots, x^{a-r}y^{[\frac{b(r-1)}{a-1}]+b}, \\
& \vdots \\
& x^{ir}y^{2b-\frac{ibr}{a-1}-1}, \dots, x^{(i-1)r+1}y^{[\frac{b(a-((i-1)r+1)-1)}{a-1}]+b}, \\
& \vdots \\
& x^r y^{2b-\frac{br}{a-1}-1}, \dots, xy^{[\frac{b(a-2)}{a-1}]+b}),
\end{aligned}
$$

where $1 \leq i \leq \gcd(a-1,b)-1$. It's multiplicity $\mathbf{mt}(J_1(D_2)) = a$. And by lemma 3.1, the dimension of the 1st Hodge moduli algebra $M_1(D_2) = \mathcal{O}_X/J_1(D_2)$ is

$$\begin{aligned} &m_1(D_2)\\ &= 2b-1+\sum_{i=1}^{a-2}([\frac{bi}{a-1}]+b)-(\gcd(a-1,b)-1)+b\\ &= \frac{(a-2)(b-1)+\gcd(a-1,b)-1}{2}+(a-2)b+3b-1-(\gcd(a-1,b)-1)\\ &= \frac{a(3b-1)-\gcd(a-1,b)+1}{2}. \end{aligned}$$

If $a-1 \geq b$, let $r = \frac{b}{\gcd(a-1,b)}$, then $1 \leq r \leq b$. Since $\frac{1}{a}+\frac{(a-1)(1+b-1)}{ab} \geq 1$, we have $y^{b-1} \in I_0(D_2)$. And we have

$$x^{[\frac{(a-1)(b-k)}{b}]+1}y^{k-1} \in I_0(D_2), \qquad \forall 1 \leq k \leq b, r \nmid k,$$

and

$$x^{a-1-\frac{i(a-1)r}{b}} \in I_0(D_2), \qquad \forall 1 \leq i \leq \gcd(a-1,b).$$

So the 0th Hodge ideal for D_2 is

$$\begin{aligned} J_0(D_2) =& I_0(D_2)\\ =& (x^{[\frac{(a-1)(b-1)}{b}]+1}, x^{[\frac{(a-1)(b-2)}{b}]+1}y, \dots, x^{a-1-\frac{(a-1)r}{b}}y^{r-1},\\ &\vdots\\ &x^{[\frac{(a-1)(b-(i-1)r-1)}{b}]+1}y^{(i-1)r}, x^{[\frac{(a-1)(b-(i-1)r-2)}{b}]+1}y^{(i-1)r+1}, \dots, x^{a-1-\frac{i(a-1)r}{b}}y^{ir-1},\\ &\vdots\\ &x^{[\frac{(a-1)(r-1)}{b}]+1}y^{b-r}, x^{[\frac{(a-1)(r-2)}{b}]+1}y^{b-r+1}, \dots, y^{b-1}), \end{aligned}$$

where $1 \leq i \leq \gcd(a-1,b)$. It's multiplicity $\mathbf{mt}(J_0(D_2)) = b-1$. And by lemma 3.1, the dimension of the 0th Hodge moduli algebra $M_0(D_2) = \mathcal{O}_X/J_0(D_2)$ is

$$\begin{aligned} m_0(D_2) &= \sum_{i=1}^{b-1}([\frac{(a-1)i}{b}]+1)-(\gcd(a-1,b)-1)\\ &= \frac{(a-2)(b-1)+\gcd(a-1,b)-1}{2}+b-1-(\gcd(a-1,b)-1)\\ &= \frac{a(b-1)-\gcd(a-1,b)+1}{2}. \end{aligned}$$

And the 1st Hodge ideal of D_2 is

$$\begin{aligned} J_1(D_2) =& (f)+I_0(D_2)\cdot(Jf)\\ =& (x^a+xy^b, ax^{a-1}y^{b-1}+y^{2b-1},\\ &x^{[\frac{(a-1)(b-1)}{b}]+2}y^{b-1}, x^{[\frac{(a-1)(b-2)}{b}]+2}y^{b}, \dots, x^{a-\frac{(a-1)r}{b}}y^{b+r-2},\\ &\vdots\\ &x^{[\frac{(a-1)(b-(i-1)r-1)}{b}]+2}y^{b+(i-1)r-1}, x^{[\frac{(a-1)(b-(i-1)r-2)}{b}]+2}y^{b+(i-1)r}, \dots, x^{a-\frac{i(a-1)r}{b}}y^{b+ir-2},\\ &\vdots \end{aligned}$$

$$x^{[\frac{(a-1)(r-1)}{b}]+2}y^{2b-r-1}, x^{[\frac{(a-1)(r-2)}{b}]+2}y^{2b-r}, \dots, xy^{2b-2}),$$

where $1 \le i \le \gcd(a-1, b)$. It's multiplicity $\mathbf{mt}(J_1(D_2)) = b+1$. And by lemma 3.1, the dimension of the 1st Hodge moduli algebra $M_1(D_2) = \mathcal{O}_X/J_1(D_2)$ is

$$\begin{aligned} m_1(D_2) &= a(b-1) + \sum_{i=1}^{b-1}([\frac{(a-1)i}{b}]+2) - (\gcd(a-1,b)-1)+1 \\ &= (a+2)(b-1) + \frac{(a-2)(b-1)+\gcd(a-1,b)-1}{2} - (\gcd(a-1,b)-1)+1 \\ &= \frac{(3a+2)(b-1)-\gcd(a-1,b)+3}{2}. \end{aligned}$$

For isolated quasi-homogeneous curve singularities $D_3^{(a,b)} = \{x^a y + xy^b = 0\}$, defined by $F_3^{(a,b)} = x^a y + xy^b$. If $a \le b$, let $r = \frac{a-1}{\gcd(a-1,b-1)}$, then $1 \le r \le a-1$. Since $\frac{(b-1)(1+a-1)}{ab-1} + \frac{a-1}{ab-1} \ge 1$, we have $x^{a-1} \in I_0(D_3)$. And we have

$$x^k y^{[\frac{(b-1)(a-1-k)}{a-1}]+1} \in I_0(D_3), \qquad \forall 1 \le k \le a-2, r \nmid k,$$

and

$$x^{ir}y^{b-1-\frac{i(b-1)r}{a-1}} \in I_0(D_3), \qquad \forall 1 \le i \le \gcd(a-1,b-1)-1.$$

So the 0th Hodge ideal for D_3 is

$$\begin{aligned} J_0(D_3) =& I_0(D_3) \\ =& (x^{a-1}, x^{a-2}y^{[\frac{b-1}{a-1}]+1}, \dots, x^{a-r}y^{[\frac{(b-1)(r-1)}{a-1}]+1}, \\ & \vdots \\ & x^{ir}y^{b-1-\frac{i(b-1)r}{a-1}}, x^{ir-1}y^{[\frac{(b-1)(a-ir)}{a-1}]+1}, \dots, x^{(i-1)r+1}y^{[\frac{(b-1)(a-1-(i-1)r-1)}{a-1}]+1}, \\ & \vdots \\ & x^r y^{b-1-\frac{(b-1)r}{a-1}}, x^{r-1}y^{[\frac{(b-1)(a-r)}{a-1}]+1}, \dots, xy^{[\frac{(b-1)(a-2)}{a-1}]+1}, \\ & y^{b-1}), \end{aligned}$$

where $1 \le i \le \gcd(a-1, b-1)$. It's multiplicity $\mathbf{mt}(J_0(D_3)) = a-1$. And by lemma 3.1, the dimension of the 0th Hodge moduli algebra $M_0(D_3) = \mathcal{O}_X/J_0(D_3)$ is

$$\begin{aligned} & m_0(D_3) \\ &= \sum_{i=1}^{a-2}([\frac{(b-1)i}{a-1}]+1) - (\gcd(a-1,b-1)-1) + b-1 \\ &= \frac{ab-\gcd(a-1,b-1)-1}{2}. \end{aligned}$$

And the 1st Hodge ideal of D_3 is

$$\begin{aligned} J_1(D_3) =& (f) + I_0(D_3)\cdot(Jf) \\ =& (x^a y + xy^b, x^{2a-1}, y^{2b-1} \\ & x^{2a-2}y, \dots, x^{2a-r-1}y^{[\frac{(b-1)(r-1)}{a-1}]+2}, \\ & \vdots \\ & x^{a-1+ir}y^{b-\frac{i(b-1)r}{a-1}}, \dots, x^{a+(i-1)r}y^{[\frac{(b-1)(a-1-(i-1)r-1)}{a-1}]+2}, \end{aligned}$$

$$\vdots$$

$$x^{a-1+r}y^{b-\frac{(b-1)r}{a-1}},\dots,x^a y^{[\frac{(b-1)(a-2)}{a-1}]+2}),$$

where $1\le i\le \gcd(a-1,b-1)$. It's multiplicity $\mathbf{mt}(J_1(D_3))=a+1$. And by lemma 3.1, the dimension of the 1st Hodge moduli algebra $M_1(D_3)=\mathcal{O}_X/J_1(D_3)$ is

$$\begin{aligned}&m_1(D_3)\\&=(2b-1)+(a-2)b+b+\sum_{i=1}^{a-2}([\frac{(b-1)i}{a-1}]+2)-(\gcd(a-1,b-1)-1)+1\\&=\frac{a(3b+2)-\gcd(a-1,b-1)-3}{2}.\end{aligned}$$

If $a\ge b$, let $r=\frac{b-1}{\gcd(a-1,b-1)}$, then $1\le r\le b-1$. By symmetry of a,b, we obtain the 0th Hodge ideal

$$\begin{aligned}J_0(D_3)=&I_0(D_3)\\=&(y^{b-1},y^{b-2}x^{[\frac{a-1}{b-1}]+1},\dots,y^{b-r}x^{[\frac{(a-1)(r-1)}{b-1}]+1},\\&\vdots\\&y^{ir}x^{a-1-\frac{i(a-1)r}{b-1}},y^{ir-1}x^{[\frac{(a-1)(b-ir)}{b-1}]+1},\dots,y^{(i-1)r+1}x^{[\frac{(a-1)(b-1-(i-1)r-1)}{b-1}]+1},\\&\vdots\\&y^r x^{a-1-\frac{(a-1)r}{b-1}},y^{r-1}x^{[\frac{(a-1)(b-r)}{b-1}]+1},\dots,yx^{[\frac{(a-1)(b-2)}{b-1}]+1},\\&x^{a-1}),\end{aligned}$$

where $1\le i\le \gcd(a-1,b-1)$. It's multiplicity $\mathbf{mt}(J_0(D_3))=b-1$. And by lemma 3.1, the dimension of the 0th Hodge moduli algebra $M_0(D_3)=\mathcal{O}_X/J_0(D_3)$ is

$$\begin{aligned}&m_0(D_3)\\&=\sum_{i=1}^{b-2}([\frac{(a-1)i}{b-1}]+1)-(\gcd(a-1,b-1)-1)+a-1\\&=\frac{ab-\gcd(a-1,b-1)-1}{2}.\end{aligned}$$

And the 1st Hodge ideal of D_3 is

$$\begin{aligned}J_1(D_3)=&(f)+I_0(D_3)\cdot(Jf)\\=&(x^a y+xy^b,x^{2a-1},y^{2b-1}\\&y^{2b-2}x,\dots,y^{2b-r-1}x^{[\frac{(a-1)(r-1)}{b-1}]+2},\\&\vdots\\&y^{b-1+ir}x^{a-\frac{i(a-1)r}{b-1}},\dots,y^{b+(i-1)r}x^{[\frac{(a-1)(b-1-(i-1)r-1)}{b-1}]+2},\\&\vdots\\&y^{b-1+r}x^{a-\frac{(a-1)r}{b-1}},\dots,y^a x^{[\frac{(a-1)(b-2)}{b-1}]+2}),\end{aligned}$$

where $1 \leq i \leq \gcd(a-1, b-1)$. It's multiplicity $\mathbf{mt}(J_1(D_3)) = b+1$. And by lemma 3.1, the dimension of the 1st Hodge moduli algebra $M_0(D_3) = \mathcal{O}_X/J_1(D_3)$ is

$$\begin{aligned} & m_1(D_3) \\ & = (2a-1) + (b-2)a + a + \sum_{i=1}^{b-2}([\frac{(a-1)i}{b-1}] + 2) - (\gcd(a-1, b-1) - 1) + 1 \\ & = \frac{(3a+2)b - \gcd(a-1, b-1) - 3}{2}. \end{aligned}$$

4. Proof of the Main Theorem A

I. Compare singularities of types $\mathbf{F}_1$ and $\mathbf{F}_2$:

(1) Suppose for singularities

$$\begin{aligned} D_1 &= \{x^{a_1} + y^{b_1} = 0\}, \qquad 2 \leq a_1 \leq b_1, \\ D_2 &= \{x^{a_2} + xy^{b_2} = 0\}, \qquad 1 \leq a_2 - 1 \leq b_2, \end{aligned}$$

their 0th and 1st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_1) \simeq M_0(D_2), \qquad M_1(D_1) \simeq M_1(D_2).$$

By our computation in the third section, we have

$$\begin{aligned} & \mathbf{mt}(J_0(D_1)) = a_1 - 1, \qquad \mathbf{mt}(J_1(D_1)) = a_1, \\ & m_0(D_1) = \frac{(a_1-1)(b_1-1) - \gcd(a_1, b_1) + 1}{2}, \\ & m_1(D_1) = \frac{(a_1-1)(3b_1-1) - \gcd(a_1, b_1) + 1}{2}. \end{aligned}$$

And

$$\begin{aligned} & \mathbf{mt}(J_0(D_2)) = a_2 - 1, \qquad \mathbf{mt}(J_1(D_2)) = a_2, \\ & m_0(D_2) = \frac{a_2(b_2-1) - \gcd(a_2-1, b_2) + 1}{2}, \\ & m_1(D_2) = \frac{a_2(3b_2-1) - \gcd(a_2-1, b_2) + 1}{2}. \end{aligned}$$

Hence we obtain the following equations

$$\begin{cases} a_1 - 1 = a_2 - 1 \\ a_1 = a_2 \\ \frac{(a_1-1)(b_1-1) - \gcd(a_1,b_1)+1}{2} = \frac{a_2(b_2-1) - \gcd(a_2-1,b_2)+1}{2} \\ \frac{(a_1-1)(3b_1-1) - \gcd(a_1,b_1)+1}{2} = \frac{a_2(3b_2-1) - \gcd(a_2-1,b_2)+1}{2} \end{cases},$$

that is,

$$\begin{cases} a_1 = a_2 \\ (a_1-1)b_1 = a_2 b_2 \\ \gcd(a_1, b_1) - \gcd(a_2-1, b_2) = a_2 - (a_1-1) \end{cases}.$$

Its solutions are $(a_1, b_1) = (a_2, a_2m), (a_2, b_2) = (a_2, (a_2-1)m)$, where $a_2, m \in \mathbb{N}, a_2 \geq 2, m \geq 1$. And we have

$$\mathbf{wt}(F_1) = \{\frac{1}{a_1}, \frac{1}{b_1}\} = \{\frac{1}{a_2}, \frac{1}{a_2m}\},$$
$$\mathbf{wt}(F_2) = \{\frac{1}{a_2}, \frac{a_2-1}{a_2b_2}\} = \{\frac{1}{a_2}, \frac{1}{a_2m}\}.$$

It follows that $\mathbf{wt}(F_1) = \mathbf{wt}(F_2)$. Under these conditions, we obtain

$$J_0(D_1) = (x^{a_2-1}, x^{a_2-2}y^{m-1}, \ldots, y^{(a_2-1)m-1}),$$
$$J_0(D_2) = (x^{a_2-1}, x^{a_2-2}y^{m-1}, \ldots, y^{(a_2-1)m-1}),$$

i.e., $J_0(D_1) = J_0(D_2)$, which shows $M_0(D_1) \simeq M_0(D_2)$ directly.

(2) Suppose for singularities

$$D_1 = \{x^{a_1} + y^{b_1} = 0\}, \qquad 2 \leq a_1 \leq b_1,$$
$$D_2 = \{x^{a_2} + xy^{b_2} = 0\}, \qquad a_2 - 1 \geq b_2 \geq 1,$$

their 0th and 1st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_1) \simeq M_0(D_2), \qquad M_1(D_1) \simeq M_1(D_2).$$

By our computation in the third section, we have

$$\mathbf{mt}(J_0(D_1)) = a_1 - 1, \qquad \mathbf{mt}(J_1(D_1)) = a_1,$$
$$m_0(D_1) = \frac{(a_1-1)(b_1-1) - \gcd(a_1, b_1) + 1}{2},$$
$$m_1(D_1) = \frac{(a_1-1)(3b_1-1) - \gcd(a_1, b_1) + 1}{2}.$$

And

$$\mathbf{mt}(J_0(D_2)) = b_2 - 1, \qquad \mathbf{mt}(J_1(D_2)) = b_2 + 1,$$
$$m_0(D_2) = \frac{a_2(b_2-1) - \gcd(a_2-1, b_2) + 1}{2},$$
$$m_1(D_2) = \frac{(3a_2+2)(b_2-1) - \gcd(a_2-1, b_2) + 3}{2}.$$

Hence we obtain the following equations

$$\begin{cases} a_1 - 1 = b_2 - 1 \\ a_1 = b_2 + 1 \\ \frac{(a_1-1)(b_1-1) - \gcd(a_1,b_1)+1}{2} = \frac{a_2(b_2-1)-\gcd(a_2-1,b_2)+1}{2} \\ \frac{(a_1-1)(3b_1-1)-\gcd(a_1,b_1)+1}{2} = \frac{(3a_2+2)(b_2-1)-\gcd(a_2-1,b_2)+3}{2} \end{cases}.$$

It has no solution.

II. Compare singularities of types $\mathbf{F}_2$ and $\mathbf{F}_3$:

(1) Suppose for singularities

$$D_2 = \{x^{a_2} + xy^{b_2} = 0\}, \qquad a_2 - 1 \geq b_2 \geq 1,$$
$$D_3 = \{x^{a_3}y + xy^{b_3} = 0\}, \qquad 1 \leq a_3 \leq b_3,$$

their 0th and 1st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_2) \simeq M_0(D_3), \qquad M_1(D_2) \simeq M_1(D_3)$$

By our computation in the third section, we have

$$\begin{aligned}
&\mathbf{mt}(J_0(D_2)) = b_2 - 1, \qquad \mathbf{mt}(J_1(D_2)) = b_2 + 1,\\
&m_0(D_2) = \frac{a_2(b_2-1) - \gcd(a_2-1, b_2) + 1}{2},\\
&m_1(D_2) = \frac{(3a_2+2)(b_2-1) - \gcd(a_2-1, b_2) + 3}{2}.
\end{aligned}$$

And

$$\begin{aligned}
&\mathbf{mt}(J_0(D_3)) = a_3 - 1, \qquad \mathbf{mt}(J_1(D_3)) = a_3 + 1,\\
&m_0(D_3) = \frac{a_3b_3 - \gcd(a_3-1, b_3-1) - 1}{2},\\
&m_1(D_3) = \frac{a_3(3b_3+2) - \gcd(a_3-1, b_3-1) - 3}{2}.
\end{aligned}$$

Hence we obtain the following equations

$$\begin{cases}
b_2 - 1 = a_3 - 1\\
b_2 + 1 = a_3 + 1\\
\frac{a_2(b_2-1)-\gcd(a_2-1,b_2)+1}{2} = \frac{a_3b_3-\gcd(a_3-1,b_3-1)-1}{2}\\
\frac{(3a_2+2)(b_2-1)-\gcd(a_2-1,b_2)+3}{2} = \frac{a_3(3b_3+2)-\gcd(a_3-1,b_3-1)-3}{2}
\end{cases},$$

that is,

$$\begin{cases}
b_2 = a_3\\
a_2b_2 + b_2 - a_2 = a_3b_3 + a_3 - 1\\
\gcd(a_2-1, b_2) - \gcd(a_3-1, b_3-1) = a_3 - (b_2 - 1)
\end{cases}.$$

Its solutions are $(a_2, b_2) = (mb_2+1, b_2), (a_3, b_3) = (b_2, m(b_2-1)+1)$, where $b_2, m \in \mathbb{N}, b_2 \geq 2, m \geq 1$. And we have

$$\begin{aligned}
\mathbf{wt}(F_2) &= \{\frac{1}{a_2}, \frac{a_2-1}{a_2b_2}\} = \{\frac{1}{mb_2+1}, \frac{m}{mb_2+1}\},\\
\mathbf{wt}(F_3) &= \{\frac{b_3-1}{a_3b_3-1}, \frac{a_3-1}{a_3b_3-1}\} = \{\frac{m}{mb_2+1}, \frac{1}{mb_2+1}\}.
\end{aligned}$$

It follows that $\mathbf{wt}(F_2) = \mathbf{wt}(F_3)$. Under these conditions, we obtain

$$\begin{aligned}
J_0(D_2) &= (x^{m(b_2-1)}, x^{m(b_2-2)}y, \ldots, y^{b_2-1})\\
J_0(D_3) &= (y^{m(b_2-1)}, y^{m(b_2-2)}x, \ldots, x^{b_2-1}),
\end{aligned}$$

i.e., $J_0(D_2) \simeq J_0(D_3), x \mapsto y$, which shows $M_0(D_1) \simeq M_0(D_2)$ directly.

(2) Suppose for singularities

$$\begin{aligned}
D_2 &= \{x^{a_2} + xy^{b_2} = 0\}, \qquad 1 \leq a_2 - 1 \leq b_2,\\
D_3 &= \{x^{a_3}y + xy^{b_3} = 0\}, \qquad 1 \leq a_3 \leq b_3,
\end{aligned}$$

their 0th and 1st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_2) \simeq M_0(D_3), \qquad M_1(D_2) \simeq M_1(D_3)$$

By our computation in the third section, we have

$$\begin{aligned} &\mathbf{mt}(J_0(D_2)) = a_2 - 1, \qquad \mathbf{mt}(J_1(D_2)) = a_2, \\ &m_0(D_2) = \frac{a_2(b_2-1) - \gcd(a_2-1, b_2) + 1}{2}, \\ &m_1(D_2) = \frac{a_2(3b_2-1) - \gcd(a_2-1, b_2) + 1}{2}. \end{aligned}$$

And

$$\begin{aligned} &\mathbf{mt}(J_0(D_3)) = a_3 - 1, \qquad \mathbf{mt}(J_1(D_3)) = a_3 + 1, \\ &m_0(D_3) = \frac{a_3b_3 - \gcd(a_3-1, b_3-1) - 1}{2}, \\ &m_1(D_3) = \frac{a_3(3b_3+2) - \gcd(a_3-1, b_3-1) - 3}{2}. \end{aligned}$$

Hence we obtain the following equations

$$\begin{cases} a_2 - 1 = a_3 - 1 \\ a_2 = a_3 + 1 \\ \frac{a_2(b_2-1) - \gcd(a_2-1,b_2)+1}{2} = \frac{a_3b_3 - \gcd(a_3-1,b_3-1)-1}{2} \\ \frac{a_2(3b_2-1) - \gcd(a_2-1,b_2)+1}{2} = \frac{a_3(3b_3+2) - \gcd(a_3-1,b_3-1)-3}{2} \end{cases}.$$

It has no solution.

III. Compare singularities of types $\mathbf{F}_1$ and $\mathbf{F}_3$:

(1) Suppose for singularities

$$\begin{aligned} &D_1 = \{x^{a_1} + y^{b_1} = 0\}, \qquad 2 \le a_1 \le b_1, \\ &D_3 = \{x^{a_3}y + xy^{b_3} = 0\}, \qquad 1 \le a_3 \le b_3, \end{aligned}$$

their 0th and 1st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_1) \simeq M_0(D_3), \qquad M_1(D_1) \simeq M_1(D_3)$$

By our computation in the third section, we have

$$\begin{aligned} &\mathbf{mt}(J_0(D_1)) = a_1 - 1, \qquad \mathbf{mt}(J_1(D_1)) = a_1, \\ &m_0(D_1) = \frac{(a_1-1)(b_1-1) - \gcd(a_1, b_1) + 1}{2}, \\ &m_1(D_1) = \frac{(a_1-1)(3b_1-1) - \gcd(a_1, b_1) + 1}{2}. \end{aligned}$$

And

$$\begin{aligned} &\mathbf{mt}(J_0(D_3)) = a_3 - 1, \qquad \mathbf{mt}(J_1(D_3)) = a_3 + 1, \\ &m_0(D_3) = \frac{a_3b_3 - \gcd(a_3-1, b_3-1) - 1}{2}, \\ &m_1(D_3) = \frac{a_3(3b_3+2) - \gcd(a_3-1, b_3-1) - 3}{2}. \end{aligned}$$

Hence we obtain the following equations

$$\begin{cases} a_1 - 1 = a_3 - 1 \\ a_1 = a_3 + 1 \\ \frac{(a_1-1)(b_1-1)-\gcd(a_1,b_1)+1}{2} = \frac{a_3b_3-\gcd(a_3-1,b_3-1)-1}{2} \\ \frac{(a_1-1)(3b_1-1)-\gcd(a_1,b_1)+1}{2} = \frac{a_3(3b_3+2)-\gcd(a_3-1,b_3-1)-3}{2} \end{cases}.$$

It has no solutions.

(2) Suppose for singularities

$$D_1 = \{x^{a_1} + y^{b_1} = 0\}, \qquad 2 \le a_1 \le b_1,$$
$$D_3 = \{x^{a_3}y + xy^{b_3} = 0\}, \qquad a_3 \ge b_3 \ge 1,$$

their 0th and 1st Hodge moduli algebras are isomorphic, i.e.,

$$M_0(D_1) \simeq M_0(D_3), \qquad M_1(D_1) \simeq M_1(D_3)$$

By our computation in the third section, we have

$$\mathbf{mt}(J_0(D_1)) = a_1 - 1, \qquad \mathbf{mt}(J_1(D_1)) = a_1,$$
$$m_0(D_1) = \frac{(a_1-1)(b_1-1) - \gcd(a_1,b_1) + 1}{2},$$
$$m_1(D_1) = \frac{(a_1-1)(3b_1-1) - \gcd(a_1,b_1) + 1}{2}.$$

And

$$\mathbf{mt}(J_0(D_3)) = b_3 - 1, \qquad \mathbf{mt}(J_1(D_3)) = b_3 + 1,$$
$$m_0(D_3) = \frac{a_3b_3 - \gcd(a_3-1, b_3-1) - 1}{2},$$
$$m_1(D_3) = \frac{(3a_3+2)b_3 - \gcd(a_3-1, b_3-1) - 3}{2}.$$

Hence we obtain the following equations

$$\begin{cases} a_1 - 1 = b_3 - 1 \\ a_1 = b_3 + 1 \\ \frac{(a_1-1)(b_1-1)-\gcd(a_1,b_1)+1}{2} = \frac{a_3b_3-\gcd(a_3-1,b_3-1)-1}{2} \\ \frac{(a_1-1)(3b_1-1)-\gcd(a_1,b_1)+1}{2} = \frac{(3a_3+2)b_3-\gcd(a_3-1,b_3-1)-3}{2} \end{cases}.$$

It has no solution.

5. Proof of the Main Theorem B

(1) For isolated quasi-homogeneous curve singularity

$$D_1^{(a,b)} = \{x^a + y^b = 0\}, \qquad a, b \ge 2,$$

since the 0th Hodge moduli number is

$$m_0(D_1^{(a,b)}) = \frac{1}{2}((a-1)(b-1) - \gcd(a,b) + 1),$$

we have

$$\delta_1(a,b) - m_0(D_1^{(a,b)}) = \gcd(a,b) - 1 \ge 0.$$

And we also have

$$\delta_1(a,b) - m_0(D_1^{(a,b)}) = \gcd(a,b) - 1 \le \min\{a,b\} - 1 = \mathbf{mt}(D_1^{(a,b)}) - 1.$$

The equality holds iff $\min\{a,b\} = \gcd(a,b)$, i.e., $(a,b) = (a, am)$ or $(a,b) = (bm', b)$ for some $m, m' \in \mathbb{N}$.

(2) For isolated quasi-homogeneous curve singularity

$$D_2^{(a,b)} = \{x^a + xy^b = 0\}, \qquad a \geq 2, b \geq 1,$$

since the 0th Hodge moduli number is

$$m_0(D_2^{(a,b)}) = \frac{1}{2}(a(b-1) - \gcd(a-1,b) + 1),$$

we have

$$\delta_2(a,b) - m_0(D_2^{(a,b)}) = \gcd(a-1,b) \geq 1.$$

And we also have

$$\delta_2(a,b) - m_0(D_2^{(a,b)}) = \gcd(a-1,b) \leq \min\{a-1,b\} = \mathbf{mt}(D_2^{(a,b)}) - 1.$$

The equality holds iff $\min\{a-1,b\} = \gcd(a-1,b)$, i.e., $(a,b) = (a,(a-1)m)$ or $(a,b) = (bm'+1, b)$ for some $m, m' \in \mathbb{N}$.

(3) For isolated quasi-homogeneous curve singularity

$$D_3^{(a,b)} = \{x^a y + xy^b = 0\}, \qquad a, b \geq 1,$$

since the 0th Hodge moduli number is

$$m_0(D_3^{(a,b)}) = \frac{1}{2}(ab - \gcd(a-1,b-1) + 1),$$

we have

$$\delta_3(a,b) - m_0(D_3^{(a,b)}) = \gcd(a-1,b-1) + 1 \geq 2.$$

And we also have

$$\delta_3(a,b) - m_0(D_3^{(a,b)}) = \gcd(a-1,b-1) + 1 \leq \min\{a-1,b-1\} + 1 = \min\{a,b\} = \mathbf{mt}(D_3^{(a,b)}) - 1.$$

The equality holds iff $\min\{a-1,b-1\} = \gcd(a-1,b-1)$, i.e., $(a,b) = (a,(a-1)m+1)$ or $(a,b) = ((b-1)m'+1, b)$ for some $m, m' \in \mathbb{N}$.

6. Some examples and conjectures

Example 6.1. Let curve singularities $H_1 = \{x^2y + xy^6 = 0\}$ defined by a polynomial $f(x,y) = x^2y + xy^6$, and $H_2 = \{x^3y + xy^4 = 0\}$ defined by a polynomial $g(x,y) = x^3y + xy^4$. Then f is quasi-homogeneous of weight type $(\frac{5}{11}, \frac{1}{11}; 1)$ and g is quasi-homogeneous of weight type $(\frac{3}{11}, \frac{2}{11}; 1)$. In [18], the characteristic polynomials of f and g coincide:

$$\Delta_f(t) = (t-1)(t^{11}-1) = \Delta_g(t).$$

So this tells us that the characteristic polynomial does not determine the weights of the non-degenerate quasi-homogeneous polynomial defining the singularity.

However, their ith Hodge moduli algebras $M_i(D^\alpha)$ are not isomorphic for $i \geq i_0(\alpha)$, for a enough big $i_0(\alpha)$. Precisely speaking,

$$M_i(D_1^\alpha) \not\cong M_i(D_2^\alpha), \qquad \forall i \geq 1,$$

where $D_j^\alpha = \alpha H_j, j = 1, 2$, for $\alpha = 1$. In fact, we just simply observe this result by their Hodge moduli numbers are different for $i \geq 1$ as follows:

Table 1. Comparison of Hodge moduli sequences

singularity	weight type	$m_0(D)$	$m_1(D)$	$m_2(D)$	$m_3(D)$	$m_4(D)$	$m_5(D)$
x^2y+xy^6	$(\frac{5}{11},\frac{1}{11};1)$	5	18	32	46	60	74
x^3y+xy^4	$(\frac{3}{11},\frac{2}{11};1)$	5	19	39	49	64	79

Example 6.2. Let $f(z_1,\cdots,z_n,w_1,w_2)=z_1^2+\cdots+z_n^2+w_1^3+w_2^{2p}$, be a quasi-homogeneous polynomial of weight type $(\frac{1}{2},\cdots,\frac{1}{2},\frac{1}{3},\frac{1}{2p};1)$ with an isolated singularity at the origin for any $p\in\mathbb{N}$. Let $n\geq 0$, even and $\gcd(3,p)=1$. Then we know their characteristic polynomials are

$$\Delta_f(t)=\frac{t^{4p}+t^{2p}+1}{t^2+t+1},\qquad \forall n\geq 0,\text{even}.$$

Hence $\Delta_f(1)=1$. By Theorem 8.5 in [4], each of their links $K_f=S_\epsilon\cap\{f(z,w)=0\}$ is a topological sphere. Thus all K_f for all p, $(3,p)=1$, are homeomorphic each other though $f(z_1,\cdots,z_n,w_1,w_2)$ are of the different quasi-homogeneous types for all p.

However, their ith Hodge moduli algebras $M_i(D^\alpha)$ are not isomorphic for $i\geq 1,\forall n\geq 2$, where $D^\alpha=\{f(z_1,\cdots,z_n,w_1,w_2)=0\}$ for $\alpha=1$. In fact, we have their 0th Hodge ideal

$$I_0(D)=\begin{cases}(1), & n\geq 2,\text{or } n=0,p=1,2\\ (w_1,w_2^{i_0}), & n=0,p\geq 4\end{cases}$$

where $i_0=\lceil\frac{4p}{3}\rceil-1$, is the smallest integer bigger than or equal to $\frac{4p}{3}-1$. Then we compute their 1st Hodge ideal as follows. For example, if $n=2$, we have

$$\begin{aligned}I_1^{(2)}(D)&=\sum_{v_j\in\mathcal{O}^{\geq 2}}\mathcal{O}_X\cdot v_j+\sum_{1\leq i\leq 4,a\in I_0(D)}\mathcal{O}_X(f\partial_i a-\alpha a\partial_i f)\\&=(w_2^{i_1},w_1w_2^{j_1})+(z_1,z_2,w_1^2,w_2^{2p-1})\\&=(z_1,z_2,w_1^2,w_1w_2^{j_1},w_2^{i_1}),\\J_1^{(2)}(D)&=(f)+I_1^{(2)}(D)\\&=(z_1,z_2,w_1^2,w_1w_2^{j_1},w_2^{i_1}),\end{aligned}$$

where $i_1=\lceil\frac{4p}{3}\rceil-1$ and $j_1=\lceil\frac{2p}{3}\rceil-1$. And if $n=4$, we have

$$\begin{aligned}I_1^{(4)}(D)&=\sum_{v_j\in\mathcal{O}^{\geq 2}}\mathcal{O}_X\cdot v_j+\sum_{1\leq i\leq 6,a\in I_0(D)}\mathcal{O}_X(f\partial_i a-\alpha a\partial_i f)\\&=(z_1,z_2,z_3,z_4,w_1^2,w_2^{2p-1}),\\J_1^{(4)}(D)&=(f)+I_1^{(4)}(D)\\&=(z_1,z_2,z_3,z_4,w_1^2,w_2^{2p-1}).\end{aligned}$$

So we obtain their corresponding Hodge moduli algebras

$$\begin{aligned}M_1^{(2)}(D)&=\mathbb{C}\{z_1,z_2,w_1,w_2\}/I_1^{(2)}(D)=\mathbb{C}\{w_1,w_2\}/(w_1^2,w_1w_2^{j_1},w_2^{i_1}),\\M_2^{(4)}(D)&=\mathbb{C}\{z_1,z_2,z_3,z_4,w_1,w_2\}/I_1^{(4)}(D)=\mathbb{C}\{w_1,w_2\}/(w_1^2,w_2^{2p-1}),\end{aligned}$$

which are not isomorphic obviously, since one can verify

$$\dim_{\mathbb{C}}M_1^{(2)}(D)=i_1+j_1<2(2p-1)=\dim_{\mathbb{C}}M_1^{(4)}(D).$$

Thus these examples imply that Hodge moduli algebras and Hodge moduli numbers (or Hodge moduli sequence) are better invariants than characteristic polynomial (a topological invariant of the singularity) for non-degenerate quasi-homogeneous singularities.

It is interesting that whether Hodge ideals and Hodge moduli algebras of singularities remain constant or isomorphic under some deformations, like quasi-homogeneous deformation or semi-quasihomogeneous (or μ-constant deformation more generally). Unfortunately, it is false since even in the case of quasi-homogeneous deformations we can find some counterexamples:

Example 6.3. For quasi-homogeneous polynomial $f = x^2 + y^4$ of weight $\mathbf{wt}(f) = (\frac{1}{2}, \frac{1}{4}; 1)$, let divisor $D_1^\alpha = \{f = 0\}$, where $\alpha = 1$. Then its 1st Hodge ideal and Hodge moduli algebra are

$$J_1(D_1) = (x^2, xy, y^4)$$
$$M_1(D_1) = \mathbb{C}\{x, y\}/(x^2, xy, y^4).$$

And for quasi-homogeneous polynomial $g = x^2 + y^4 + xy^2$ of weight $\mathbf{wt}(g) = (\frac{1}{2}, \frac{1}{4}; 1)$, which is a quasi-homogeneous deformation of f. Let divisor $D_2^\alpha = \{g = 0\}$, where $\alpha = 1$. Then its 1st Hodge ideal and Hodge moduli algebra are

$$J_1(D_2) = (x^2, xy^2, 2xy + y^3, y^4)$$
$$M_1(D_2) = \mathbb{C}\{x, y\}/(x^2, xy^2, 2xy + y^3, y^4).$$

Although as $\mathbb{C}$-vector spaces the $\mathbb{C}$-basis of $M_1(D_1)$ and $M_1(D_2)$ are the same, both are

$$1, x, y, y^2, y^3.$$

But as $\mathbb{C}$-algebras, the structures of $M_1(D_1)$ and $M_1(D_2)$ are different, since in $M_1(D_1), xy = 0$, but in $M_1(D_2), xy = -\frac{1}{2}y^3 \neq 0$. However, the Hodge moduli numbers of D_1 and D_2 are the same.

So we raise a conjecture from the above example.

Conjecture 6.4. Suppose F_i is one of the three types[1] of quasi-homogeneous polynomial in $\mathbb{C}^2, 1 \leq i \leq 3$. Let $H_{i,t} = F_i + tG_i$ be a semi-quasihomogeneous deformation of $F_i, t \in \mathbb{C}$, $1 \leq i \leq 3$. Then the kth Hodge moduli algebras of the divisors $D_H^\alpha = \{H_{i,t} = 0\}$ for $\alpha = 1$, and $D_F^\alpha = \{F_i = 0\}$ for $\alpha = 1$, have the same basis over $\mathbb{C}, \forall k \geq 0$. Hence, their dimensions, i.e., their kth Hodge moduli numbers are the same,

$$m_k(D_H^\alpha) = m_k(D_F^\alpha), \qquad \forall k \geq 0, \forall 1 \leq i \leq 3, \forall t \in \mathbb{C}.$$

And we can also ask whether the inequalities in our main theorem B can be extended to more general singularities. Suppose F_i is one of the three types of quasi-homogeneous curve singularities, $1 \leq i \leq 3$. Consider a μ-constant deformation $H_{i,t} = F_i + tG_i$ of F_i, $t \in \mathbb{C}$. If we furthermore assume $H_{i,t}$ is blackuced, i.e., all distinct irblackucible factors of $H_{i,t}$ have multiplicity 1, we have

$$\mu(H_{i,t}) = \mu(F_i),$$
$$r(H_{i,t}) \leq r(F_i).$$

By lemma 2.7, we have

$$\delta(H_{i,t}) \leq \delta(F_i),$$

where μ, δ and r are the same notations as in lemma 2.7. So we have a corollary:

[1] see pages 2-3 in introduction

Corollary 6.5. Suppose the above Conjecture 6.4 is true. For a semi-quasihomogeneous deformation $H_{i,t}, t \in \mathbb{C}$, of F_i, we have the following inequalities

$$\delta(D_{i,t}) - m_0(D_{i,t}) \leq \mathbf{mt}(D_{i,t}),$$

for any $t \in \mathbb{C}$ s.t. $H_{i,t}$ is a blackuced polynomial, $1 \leq i \leq 3$, where $D_{i,t} = \{H_{i,t} = 0\}$ is the corresponding singularity, $\delta(D_{i,t})$ is the δ-invariant of $D_{i,t}$, $m_0(D_{i,t})$ is the 0th Hodge moduli number of $D_{i,t}$, and $\mathbf{mt}(D_{i,t})$ is the multiplicity of $D_{i,t}$.

Proof. In fact, one can verify the multiplicity of F_i is not decreasing under the above semi-quasihomogeneous deformation $\forall 1 \leq i \leq 3$, i.e.,

$$\mathbf{mt}(D_{i,t}) \geq \mathbf{mt}(D_{i,0})$$

for any $t \in \mathbb{C}$ s.t. $H_{i,t}$ is a blackuced polynomial, $1 \leq i \leq 3$. And by Conjecture 6.4, we have

$$m_0(D_{i,t}) = m_0(D_{i,0})$$

for any $t \in \mathbb{C}$ s.t. $H_{i,t}$ is a blackuced polynomial, $1 \leq i \leq 3$. Finally, by the discussion after Conjecture 6.4, we have

$$\delta(D_{i,t}) \leq \delta(D_{i,0})$$

for any $t \in \mathbb{C}$ s.t. $H_{i,t}$ is a blackuced polynomial, $1 \leq i \leq 3$. So we have

$$\delta(D_{i,t}) - m_0(D_{i,t}) \leq \delta(D_{i,0}) - m_0(D_{i,0}) \leq \mathbf{mt}(D_{i,0}) \leq \mathbf{mt}(D_{i,t}),$$

for any $t \in \mathbb{C}$ s.t. $H_{i,t}$ is a blackuced polynomial, $1 \leq i \leq 3$. □

References

[1] V. I. Arnold, *Normal forms of functions in neighborhoods of degenerate critical points* Lecture Notes of London Math. Soc 53 (1981), 91-131.

[2] G. M. Greuel, C. Lossen, and E.I. Shustin , *Introduction to Singularities and Deformations*, Springer Monographs in Mathematics, Springer, Berlin, 2007.

[3] S. Ishii, *Introduction to singularities*, Second edition, Springer, Tokyo, 2018. x+236 pp. ISBN: 978-4-431-56836-0; 978-4-431-56837-7.

[4] J. Milnor, *Singular Points of Complex Hypersurfaces*, (AM-61), Volume 61. Vol. 61. Princeton University Press, 2016.

[5] M. Mustaţă and M. Popa, *Hodge ideals*, Mem. Amer. Math. Soc. 262 (2019), no. 1268, v+80pp.

[6] M. Mustaţă and M. Popa, *Restriction, subadditivity, and semicontinuity theorems for Hodge ideals*, Int. Math. Res. Not. (11) 2018, 3587-3605.

[7] M. Mustaţă and M. Popa, *Hodge ideals for $\mathbb{Q}$-divisors: birational approach*, J. Éc. polytech. Math. 6 (2019), 283-328.

[8] M. Mustaţă and M. Popa, *Hodge ideals for $\mathbb{Q}$-divisors, V-filtration, and minimal exponent*, Forum Math. Sigma 8 (2020), Paper No.e19, 41pp.

[9] G.R. Ma, Y. Wang, S. S.-T.Yau and H.Q. Zuo, *Hodge moduli algebras and complete Invariants of singularities*, Asian J. Math. 28, 1(2024), 1-46.

[10] J. Mather and S. S.-T.Yau, *Classification of isolated hypersurface singularities by their moduli algebras*, Invent. Math. 69 (1982), 243-251.

[11] M. Popa, *$\mathscr{D}$-modules in birational geometry*, Proceedings of the ICM, Rio de Janeiro (2018). Vol. II. Invited lectures, 781-806, World Sci. Publ., Hackensack, NJ, 2018.

[12] K. Saito, *Quasihomogene isolierte singularitäten von hyperflächen*, Invent. Math. 14 (1971), 123-142.

[13] O. Peter and P. Wagreich. *Isolated singularities of algebraic surfaces with $\mathbb{C}^*$-action* Annals of Mathematics (1971), 205-228.

[14] M. Saito, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. Kyoto Univ. 26, 2 (1990), 221-333.

[15] M. Saito, *On the Hodge filtration of Hodge modules*, Mosc. Math. J. 9.1 (2009), 151-181.

[16] Y.J. Xu and S. S.-T.Yau *Classification of topological types of isolated quasi-homogeneous two dimensional hypersurface singularities*, manuscripta mathematica 64.4 (1989), 445-469.

[17] E. Yoshinaga *Topological types of isolated singularities defined by weighted homogeneous polynomials*, Journal of the Mathematical Society of Japan 35.3 (1983), 431-436.

[18] E. Yoshinaga and S. Masahiko *Topological types of quasi-homogeneous singularities in* $\mathbb{C}^2$, Topology 18.2 (1979), 113-116.

[19] O. Zariski, *Some open questions in the theory of singularities*, Bull. Amer. Math. Soc. 77(1971), 481-491.

[20] M.Y. Zhang *Hodge filtration and Hodge ideals for* $\mathbb{Q}$*-divisors with weighted homogeneous isolated singularities*, Asian J. Math. 25 (2021), 641-664.