ON THE k-TH HIGHER NASH BLOW-UP DERIVATION LIE ALGEBRAS OF ISOLATED HYPERSURFACE SINGULARITIES

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ABSTRACT. Many physical questions such as $4d$ $N = 2$ SCFTs, the Coulomb branch spectrum, and the Seiberg–Witten solution are related with singularities. In this paper, we introduce some new invariants $\mathcal{L}_n^k(V)$, ρ_n^k, and $d_n^k(V)$ to isolated hypersurface singularities $(V, 0)$. We give a new conjecture for the characterization of simple curve singularities using the k-th higher Nash blow-up derivation Lie algebra $\mathcal{L}_n^k(V)$, this conjecture is verified for small n and k. A new inequality conjecture for ρ_n^k and $d_n^k(V)$ are proposed, These two conjectures are verified for binomial singularities.

Keywords. Derivations, Nash blow-up, isolated hypersurface singularity.
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1. INTRODCTION

Many highly non-trivial physical questions such as the Coulomb branch spectrum and the Seiberg–Witten solution can be easily found by studying the mini-versal deformation of the isolated singularity [1, 2]. In [3], the authors classify threefold isolated quotient as Gorenstein singularities. These singularities are rigid, i.e., there is no non-trivial deformation, and we conjecture that they define $4d$ $N = 2$ SCFTs that do not have a Coulomb branch. In [4], the authors classify three-dimensional isolated weighted homogeneous rational complete intersection singularities, which define many new four-dimensional $N = 2$ superconformal field theories. In this article, we will introduce a series of new invariants for isolated singularities. These new invariants are very useful in the classification theory of isolated singularities.

This article has two purposes. On the one hand, we introduce a series of new local Artinian algebras associated with singularities. These algebras and their dimensions are natural new invariants of singularities. On the other hand, we investigate the derivation Lie algebras of these new local Artinian algebras. Again, these Lie algebras and their dimensions are new invariants of singularities. We propose some inequality conjecture and verify the conjecture in several particular cases.

In the classification theory of isolated singularities, one always wants to find various invariants associated to isolated singularities. Hopefully with enough invariants found, one can distinguish between different isolated singularities up to contact equivalence. However, not many effective invariants are known. Moreover, most known invariants, for example, the geometric genus, monodromy are hard to compute in general. In [5], we introduced some new invariants to isolated hypersurface singularities. i.e., the higher Nash blow-up local algebras. These invariants can be calculated easily compared with other invariants of isolated singularities. In this article, we introduce new invariants which are called k-th higher Nash blow-up local algebras. These new invariants are a natural generalization of higher Nash blow-up local algebras. We investigate some properties of these new invariants.

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The algebra of germs of holomorphic functions at the origin of $\mathbb{C}^n$ is denoted as $\mathcal{O}_n$. Clearly, $\mathcal{O}_n$ can be naturally identified with the algebra of convergent power series in n indeterminates with complex coefficients. As a ring $\mathcal{O}_n$ has a unique maximal ideal $\mathfrak{m}$, the set of germs of holomorphic functions which vanish at the origin. Let $(V,0)$ be an isolated hypersurface singularity defined by a holomorphic function $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$. The multiplicity $mult(f)$ of the singularity $(V,0)$ is defined to be the order of the lowest nonvanishing term in the power series expansion of f at 0.

For any isolated hypersurface singularity $(V,0)\subset(\mathbb{C}^n,0)$ defined by f, the second author considers the Lie algebra of derivations of moduli algebra

$$A(V):=\mathcal{O}_n/(f,\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n}),\text{i.e.},L(V)=\mathrm{Der}(A(V),A(V)).$$

It is known that $L(V)$ is a finite dimensional solvable Lie algebra ([6], [7]). $L(V)$ is called the Yau algebra of V in [8] and [9] in order to distinguish from Lie algebras of other types appearing in singularity theory ([10], [11]). The Yau algebra plays an important role in singularities [12]. In [5], we introduced a new derivation Lie algebra which is a generalization of Yau algebra.

Recall that the classical Nash blow-up of an algebraic variety can be viewed as the parameter space of the tangent spaces of smooth points and their limits. There is natural question to ask whether we can get a smooth variety by Nash blow-ups. There are lots of works on it, such as González-Sprinberg [13], Hironaka [14], Nobile [15], Rebassoo [16] and Spivakovsky [17], etc.

As we know, A. Noble has the following famous theorem.

Theorem 1.1 ([15]). *Let X be a variety, then the Nash blow-up of X is an isomorphism if and only if X is non-singular.*

The conception Nash blow-up was generalized which was called higher Nash blow-up. We just recall the basic definition and properties of higher Nash blow-ups, which can be found in [18].

Definition 1.2 (Higher Nash blow-up [18]). Let X be a variety of dimension d, $x\in X$, $x^{(n)}:=Spec(\mathcal{O}_{X,x}/m_x^{n+1})$ its n-th infinitesimal neighborhood and $\mathbf{Hilb}_{\binom{d+n}{n}}(X)$ is the Hilbert scheme of $\binom{d+n}{n}$ points of X. If X is smooth at x, then $x^{(n)}$ is an Artinian subscheme of X of length $\binom{d+n}{n}$. Therefore, it corresponds to a point

$$[x^{(n)}]\in\mathbf{Hilb}_{\binom{d+n}{n}}(X),$$

which induced the following morphism of schemes,

$$\sigma_n: X_{sm}\longrightarrow\mathbf{Hilb}_{\binom{d+n}{n}}(X).$$

The graph of σ_n is canonically isomorphic to X_{sm}, where X_{sm} denotes the smooth locus of X. We define the n-th Nash blow-up of X (called the higher Nash blow-up of order n of X), denoted by $\mathbf{Nash}_n(X)$, to be the closure of the graph Γ_{σ_n} with reduced scheme structure in $X\times_k\mathbf{Hilb}_{\binom{d+n}{n}}(X)$, together with the projection

$$\pi_n:\mathbf{Nash}_n(X)\longrightarrow X,$$

which is projective and birational. Moreover, it is an isomorphism over X_{sm}.

Definition 1.3 (Nash Blow-up associated to a coherent sheaf [18]). Let X be a reduced Noetherian scheme, $\mathcal{M}$ a coherent $\mathcal{O}_X$-module locally free of constant rank r on an open dense subscheme $U\subset X$, and $\mathbf{Grass}_r(\mathcal{M})$ the Grassmaniann of $\mathcal{M}$ of rank r, which is a projective X-scheme. Then the fiber product $\mathbf{Grass}_r(\mathcal{M})\times_X U$ is isomorphic to U by the projection. We define the Nash blow-up of X associated to $\mathcal{M}$, to be the closure of $\mathbf{Grass}_r(\mathcal{M}) \times_X U$, (denoted by $\mathbf{Nash}(X, \mathcal{M})$) together with the natural morphism

$$\pi_{\mathcal{M}} : \mathbf{Nash}(X, \mathcal{M}) \longrightarrow X,$$

which is projective and birational.

Remark 1.4 ([18])**.** When X is a variety and $\mathcal{M} = \Omega_{X/k}$, the $\mathbf{Nash}(X, \Omega_{X/k})$ is the classical Nash blow-up of X. Moreover,

$$\mathbf{Nash}_n(X) \cong \mathbf{Nash}(X, \mathcal{P}^n_X) \cong \mathbf{Nash}(X, \mathcal{P}^n_{X,+}),$$

where $\mathcal{P}^n_X := \mathcal{O}_{X \times_k X}/\mathcal{I}^{n+1}_\Delta$ (which is the structure sheaf of $\Delta^{(n)}$ and called the sheaf of principal parts of order n of X) and $\mathcal{P}^n_{X,+} := \mathcal{I}_\Delta/\mathcal{I}^{n+1}_\Delta$ ($\mathcal{I}_\Delta$ is the ideal sheaf defining the diagonal $\Delta \subset X \times_k X$, $\Delta^{(n)} \subset X \times_k X$ is the n-th infinitesimal neighborhood of the diagonal). When X is a variety, these are coherent. For more details, see [19] and [20].

Higher Nash blow-ups have lots of general properties. We just recall the following two theorems.

Theorem 1.5 (Compatibility with étale morphisms [18])**.** *Let $Y \longrightarrow X$ be an étale morphism of varieties. Then for every n, there exists a canonical isomorphism*

$$\mathbf{Nash}_n(Y) \cong \mathbf{Nash}_n(X) \times_X Y.$$

Theorem 1.6 (Stable under group actions [18])**.** *Let X be a variety of dimension d and G an algebraic group over k acting on X. The subscheme $\mathbf{Nash}_n(X) \subset X \times_k \mathbf{Hilb}_{\binom{d+n}{n}}(X)$ is stable under the induced natural action of G on $X \times_k \mathbf{Hilb}_{\binom{d+n}{n}}(X)$,*

$$G \times_k X \times_k \mathbf{Hilb}_{\binom{d+n}{n}}(X) \longrightarrow X \times_k \mathbf{Hilb}_{\binom{d+n}{n}}(X)$$
$$(g, x, [Z]) \longmapsto (gx, [gZ]).$$

Thus, the G-action on X naturally lifts to $\mathbf{Nash}_n(X)$.

Similarly, the higher version of Noble's result with some restrictions also holds.

Theorem 1.7 ([21] Theorem 4.13)**.** *Let $F \in \mathbb{C}[z_1, \dots, z_s]$ be an irreducible polynomial and $X = V(F) \subset \mathbb{C}^s$. Suppose X is normal. Let $(\mathbf{Nash}_n(X), \pi_n)$ be the n-th Nash blow-up of X. Then π_n is an isomorphism if and only if X is non-singular.*

However, the n-th Nash blow-up is hard to compute in general. To deal with this problem, especially in the hypersurface case, we recall the following definition which was given in [21], where readers can find more details.

We recall the definition of a higher-order Jacobian matrix of a polynomial. First recall the multi-index notation. Let $\alpha = (\alpha_1, \dots, \alpha_s), \beta = (\beta_1, \dots, \beta_s) \in \mathbb{N}^s$:

1). $\alpha \leq \beta \Leftrightarrow \alpha_i \leq \beta_i \quad \forall i \in \{1, \dots, s\}$.

2). $|\alpha| = \alpha_1 + \cdots + \alpha_s$.

3). $\alpha! = \alpha_1! \cdot \alpha_2! \cdots \alpha_s!$.

4). $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \cdots \begin{pmatrix} \alpha_s \\ \beta_s \end{pmatrix} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

5). $\partial^\alpha = \partial^{\alpha_1} \partial^{\alpha_2} \cdots \partial^{\alpha_s}$.

Using this notation, the general Leibniz rule states that

$$\partial^{\alpha}(g\cdot f)=\sum_{\{\beta|\beta\leq\alpha\}}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\partial^{\alpha-\beta}f\partial^{\beta}g$$

for any $f,g\in\mathbb{C}\left[x_1,\ldots,x_s\right]$. If we define $\partial^{\alpha-\beta}f=0$ when $\alpha_i<\beta_i$ for some $1\leq i\leq s$ then the general Leibniz rule can also be written as:

$$\partial^{\alpha}(g\cdot f)=\sum_{\{\beta|0\leq|\beta|\leq|\alpha|\}}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\partial^{\alpha-\beta}f\partial^{\beta}g. \tag{1}$$

Let $F\in\mathbb{C}\left[x_1,\ldots,x_s\right]$ and $p=(a_1,\ldots,a_s)\in X=\mathbf{V}(F)\subset\mathbb{C}^s$. Let $\mathfrak{a}_p=\langle x_1-a_1,\ldots,x_s-a_s\rangle\subset\mathbb{C}\left[x_1,\ldots,x_s\right]$. Fix $n\in\mathbb{N}$. Let $N=\begin{pmatrix}n+s\\ s\end{pmatrix}$ and consider the following linear map:

$$\begin{aligned}\theta:\mathfrak{a}_p&\to\mathbb{C}^{N-1}\\ f&\mapsto\left(\frac{\partial^{\alpha}f}{\alpha!}\mid 1\leq|\alpha|\leq n\right)_{|p}.\end{aligned}$$

We arrange this vector increasingly using graded lexicographical order, where $\alpha_1<\alpha_2<\cdots<\alpha_s$.

Let $\mathfrak{b}=\langle F\rangle$. Notice that $\mathfrak{b}\subset\mathfrak{a}_p$. Let $g\cdot F\in\mathfrak{b}$, where $g\in\mathbb{C}\left[x_1,\ldots,x_s\right]$. Using the general Leibniz rule (1) and the fact $F(p)=0$, we can write $\theta(gF)$ as follows (recall that we defined $\partial^{\alpha-\beta}F=0$ if $\alpha_i<\beta_i$ for some i):

$$\theta(gF)=\sum_{\{\beta|0\leq|\beta|\leq n-1\}}\partial^{\beta}g(p)\cdot\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\frac{\partial^{\alpha-\beta}F}{\alpha!}|1\leq|\alpha|\leq n\right)_{|p}.$$

Let $\beta\in\mathbb{N}^s$ be such that $0\leq|\beta|\leq n-1$. We define

$$r_{\beta}:=\beta!\cdot\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\frac{\partial^{\alpha-\beta}F}{\alpha!}|1\leq|\alpha|\leq n\right). \tag{2}$$

As before, we arrange r_{β} using graded lexicographical order on α. There are $M=\begin{pmatrix}n+s-1\\ s\end{pmatrix}$ such vectors.

Definition 1.8. [21] Let $\mathrm{Jac}_n(F)$ be the matrix whose rows are the M vectors r_{β} defined in (2). We arrange these rows using graded lexicographical order on β, where $\beta_1<\beta_2<\ldots<\beta_s$. In particular, $\mathrm{Jac}_n(F)$ is a $M\times(N-1)$-matrix. We call $\mathrm{Jac}_n(F)$ the Jacobian matrix of order n or the higher-order Jacobian matrix.

The following example gives an intuitional illustration of the above definition of higher-order version of the Jacobian matrix of a polynomial. Since we only care about isolated hypersurface singularities in this paper, it is important to show the computation for hypersurface case.

Example 1.9. [21] Let $F(x,y)=x^3-y^2\in\mathbb{C}[x,y]$. Let $p=(a,b)\in X=\mathbf{V}(F)$. The Jacobian matrix of F evaluated at p is defined as:

$$\mathrm{Jac}(F)_{|p}:=\begin{pmatrix}3x^2 & -2y\end{pmatrix}_{|p},$$

and the following $\mathrm{Jac}_2(F)$ is the Jacobian matrix of order 2 of F.

$$\mathrm{Jac}_2(F) := \begin{pmatrix} 3x^2 & -2y & 3x & 0 & -1 \\ F & 0 & 3x^2 & -2y & 0 \\ 0 & F & 0 & 3x^2 & -2y \end{pmatrix}.$$

Theorem 1.10 (Generalized Jacobian criterion [21])**.** *Let $F \in \mathbb{C}[z_1, \ldots, z_s]$ be a reduced non-constant polynomial. Let $p \in V(F) \subset \mathbb{C}^s$. For $n \in \mathbb{N}$, let $M = \binom{n+s-1}{s}$. Then p is non-singular if and only if $rank(Jac_n(F))|_p = M$.*

The higher-order Jacobian matrix $\mathrm{Jac}_n(F)$ is useful to make explicit computations concerning the higher Nash blow-up of hypersurfaces. The ideal whose blow-up is the higher Nash blow-up of order n of a hypersurface, correspond to the ideal generated by the maximal minors of $\mathrm{Jac}_n(F)$ [21]. From this fact, in [5], we have introduced the following higher Nash blow-up local algebras for isolated hypersurface singularities.

Definition 1.11. With the notations as above. Let $(V, 0)$ be an isolated hypersurface singularity defined by a polynomial $F(x_1, \cdots, x_s)$. Let $\mathcal{J}_n(F) \subset \mathcal{O}_s$ be the ideal generated by the $(M \times M)$-minors of $\mathrm{Jac}_n(F)$. Then we define a new higher Nash blow-up local algebra of V to be: $\mathcal{M}_n(V) := \mathcal{O}_s/\langle F, \mathcal{J}_n(F)\rangle$. We use $d_n(V)$ to denote the dimension of $\mathcal{M}_n(V)$.

In this article, we introduce the following definition of k-th higher Nash blow-up local algebras of order n which is generalization of higher Nash blow-up local algebras in Definition 1.11.

Definition 1.12. With the notations as above. Let $(V, 0)$ be an isolated hypersurface singularity defined by a polynomial $F(x_1, \cdots, x_s)$. Let $\mathcal{J}_n^k(F) \subset \mathcal{O}_s$ be the ideal generated by the $(k \times k)$-minors, $1 \le k \le M$ of $\mathrm{Jac}_n(F)$. Then we define a new higher Nash blow-up local algebra of V to be: $\mathcal{M}_n^k(V) := \mathcal{O}_s/\langle F, \mathcal{J}_n^k(F)\rangle$. We use $d_n^k(V)$ to denote the dimension of $\mathcal{M}_n^k(V)$.

Remark 1.13. When $k = M$, then $\mathcal{J}_n^M(F) = \mathcal{J}_n(F)$ and $\mathcal{M}_n^M(V) = \mathcal{M}_n(V)$.

Remark 1.14. If $(V, 0)$ is an isolated hypersurface singularity, then it is easy to see that the $\mathcal{M}_1(V)$ is exactly the moduli algebra $A(V)$, moreover, $\mathcal{M}_n(V)$ is Artinian (cf. Corollary 2.2, [21]). Thus $\mathcal{M}_n^k(V)$ is also Artinian.

In [22], Fan-Yau-Zuo have proved the following highly nontrivial theorem.

Theorem 1.15. *With the notations as above. Suppose*

$$(V, 0) = \{(x_1, \cdots, x_s) \in \mathbb{C}^s \ : \ F(x_1, \cdots, x_s) = 0\}$$

and $(W, 0) = \{(x_1, \cdots, x_s) \in \mathbb{C}^s \ : \ G(x_1, \cdots, x_s) = 0\}$ are isolated hypersurface singularities. If $(V, 0)$ is biholomorphically equivalent to $(W, 0)$, then $\mathcal{M}_n(V)$ (resp. $\mathcal{M}_n^k(V)$) is isomorphic to $\mathcal{M}_n(W)$ (resp. $\mathcal{M}_n^k(V)$).

Based on Theorem 1.15, it is natural for us to introduce the following new derivation Lie algebras.

Definition 1.16. The derivation Lie algebras $\mathcal{L}_n(V)$ (resp. $\mathcal{L}_n^k(V)$) which is defined to be the Lie algebra of derivations of the local Artinian algebra $\mathcal{M}_n(V)$ (resp. $\mathcal{M}_n^k(V)$), i.e., $\mathcal{L}_n(V) = Der(\mathcal{M}_n(V))$ (resp. $\mathcal{L}_n^k(V) = Der(\mathcal{M}_n^k(V))$). Its dimension is denoted as $\rho_n(V)$ (resp. $\rho_n^k(V)$).

Since $L(V)$ can not distinguish the simple singularities completely [27], it is interesting to ask whether these simple singularities (or which classes of more general singularities) can be distinguished completely by the Lie algebra $\mathcal{L}_n(V)$ (resp. $\mathcal{L}_n^k(V)$)? We proposed the following conjectures.

Among Arnold's most famous results in the singularity theory is his classification of simple singularities [23]. Simple singularities can be considered in arbitrary dimension. Recall that simple curve singularities, consist of two series $A_k : \{x^{k+1} + y^2 = 0\}, k \geq 1$, $D_k : \{x^{k-1} + xy^2 = 0\}, k \geq 4$ and three exceptional singularities E_6, E_7, E_8 defined by polynomials $x^3 + y^4, x^3 + xy^3, x^3 + y^5$ respectively. These singularities play an important role in algebraic geometry and singularity theory (cf. [11], [24]). Simple surface singularities (i.e., rational double points) can be characterized in many ways [25], all of which involve some form of finiteness. These characterizations that build on the work of Artin, Brieskorn, Du Val, Arnold, Tjurina and many others, form a very interesting subject in singularity theory. In this paper, we show that our newly introduced invariants (i.e., k-th higher Nash blow-up derivation Lie algebras) can be used to distinguish between different simple curve singularities. In a subsequence paper, we will show that this result can be generalized to more general singularities including the simple surface singularities. For recent progress in distinguishing between different simple singularities by using several derivation Lie algebras, the interested readers can refer to [26], [27], [28], and [29].

Conjecture 1.17. *If X and Y are two simple curve singularities, then $\mathcal{L}_n^k(X) \cong \mathcal{L}_n^k(Y)$ if and only if X and Y are analytically isomorphic.*

Theorem A. *If X and Y are two simple curve singularities, then $\mathcal{L}_2^k(X) \cong \mathcal{L}_2^k(Y), k = 2, 3$, as Lie algebras, if and only if X and Y are analytically isomorphic.*

Remark 1.18. It is easy to see that the Lie algebra $\mathcal{L}_2^1(X)$ for simple curve singularities are trivial.

For the estimation of the numerical invariant $\rho_n^k(V)$, we proposed the following new conjecture.

Conjecture 1.19. *Assume that $\rho_n^k(\{x_1^{a_1} + \cdots + x_s^{a_s} = 0\}) = h_n^k(a_1, \cdots, a_s)$. Let $(V, 0) = \{(x_1, x_2, \cdots, x_s) \in \mathbb{C}^s : f(x_1, x_2, \cdots, x_s) = 0\}, (s \geq 2)$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \cdots, x_s)$ of weight type $(w_1, w_2, \cdots, w_s; 1)$. Then $\rho_n^k(V) \leq h_n^k(1/w_1, \cdots, 1/w_s)$.*

The Theorem B verifies Conjecture 1.19 partially.

Theorem B. *Let $(V, 0)$ be a binomial singularity defined by the weighted homogeneous polynomial $f(x_1, x_2)$ (see corollary 2.6) with weight type $(w_1, w_2; 1)$. Then*

$$\rho_2^2(V) \leq h_2^2(\frac{1}{w_1}, \frac{1}{w_2}) = \begin{cases} \frac{3}{w_1 w_2} - 4(\frac{1}{w_1} + \frac{1}{w_2}) + 8; & w_1 \leq \frac{1}{4}, w_2 \leq \frac{1}{4} \\ \frac{5}{w_2} - 5; & w_1 = \frac{1}{3}, w_2 \leq \frac{1}{3} \\ \frac{1}{w_1} - 2; & w_1 \leq \frac{1}{2}, w_2 = \frac{1}{2}. \end{cases}$$

Remark 1.20. Conjecture 1.19 was proved in [5] for $k = 3, n = 2$, and in [30] for $k = 1, n = 2$.

Conjecture 1.21. *Assume that $d_n^k(\{x_1^{a_1} + \cdots + x_s^{a_s} = 0\}) = \ell_n^k(a_1, \cdots, a_s)$. Let $(V, 0) = \{(x_1, x_2, \cdots, x_s) \in \mathbb{C}^s : f(x_1, x_2, \cdots, x_s) = 0\}, (s \geq 2)$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \cdots, x_s)$ of weight type $(w_1, w_2, \cdots, w_s; 1)$. Then*

$$d_n^k(V) \leq \ell_n^k(1/w_1, \cdots, 1/w_s).$$

Remark 1.22. Conjecture 1.21 was verified in [31] for $k = 1, n = 2$. The following theorem verify the Conjecture 1.21 for $k = 2, 3, n = 2$.

Theorem C. *Let $(V,0)$ be a binomial singularity defined by the weighted homogeneous polynomial $f(x_1,x_2)$ (see Corollary 2.6) with weight type $(w_1,w_2;1)$. Then*

$$d_2^2(V)\leq \ell_2^2(\frac{1}{w_1},\frac{1}{w_2})=\begin{cases}\frac{2}{w_1w_2}-3(\frac{1}{w_1}+\frac{1}{w_2})+6; & w_1\leq\frac{1}{3}, w_2\leq\frac{1}{3},\\ \frac{1}{w_1}-1; & w_1\leq\frac{1}{2}, w_2=\frac{1}{2}.\end{cases}$$

$$d_2^3(V)\leq \ell_2^3(\frac{1}{w_1},\frac{1}{w_2})=\begin{cases}\frac{3}{w_1w_2}-3(\frac{1}{w_1}+\frac{1}{w_2})+6; & w_1\leq\frac{1}{3},\ w_2\leq\frac{1}{3} \text{ and } w_1=w_2,\\ 5; & w_1=2, w_2=2\\ \frac{3}{w_2}-2; & w_1=2, w_2\leq\frac{1}{3}\\ \frac{3}{w_1w_2}-3(\frac{1}{w_1}+\frac{1}{w_2})+5; & w_1\leq\frac{1}{3},\ w_2\leq\frac{1}{3} \text{ and } w_1>w_2 \text{ or } w_1<w_2.\end{cases}$$

Conjecture 1.23. *With the above notations, let $(V,0)$ be an isolated hypersurface singularity defined by $f\in\mathcal{O}_s$, $s\geq 2$. Then*

$$\rho_n^{k+1}(V)>\rho_n^k(V), 1\leq k\leq M.$$

Theorem D. *Let $(V,0)$ be a binomial singularity defined by the weighted homogeneous polynomial $f(x_1,x_2)$ with weight type $(w_1,w_2;1)$. Then*

$$\rho_2^3(V)>\rho_2^2(V)>\rho_2^1(V).$$

2. Derivation Lie algebras and fewnomial singularities

In this section we shall briefly introduce the basic definitions and important results which are helpful to solve the problem. The following basic concepts and results will be used to compute the derivation Lie algebras of isolated hypersurface singularities.

Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab)=D(a)b+aD(b)$. Thus for such an algebra A one can consider the Lie algebra of its derivations $\mathrm{Der}(A,A)$ with the bracket defined by the commutator of linear endomorphisms.

Definition 2.1. Let J be an ideal in an analytic algebra S. Then $\mathrm{Der}_J S\subseteq \mathrm{Der}_{\mathbb{C}}S$ is Lie subalgebra of all $\sigma\in\mathrm{Der}_{\mathbb{C}}S$ for which $\sigma(J)\subset J$.

We shall use the following well-known result to compute the derivations.

Theorem 2.2. (cf. [32]) *Let J be an ideal in $R=\mathbb{C}\{x_1,\cdots,x_n\}$. Then there is a natural isomorphism of Lie algebras*

$$(Der_J R)/(J\cdot Der_{\mathbb{C}}R)\cong Der_{\mathbb{C}}(R/J).$$

Definition 2.3. A polynomial $f\in\mathbb{C}[x_1,x_2,\cdots,x_n]$ is called quasi-homogeneous (or weighted homogeneous) if there exist positive rational numbers $w_1,\dots,w_n$ (called weights of indeterminates x_j) and d such that, for each monomial $\prod x_j^{k_j}$ appearing in f with non-zero coefficient, one has $\sum w_jk_j=d$. The number d is called the quasi-homogeneous degree (w-degree) of f with respect to weights w_j and is denoted by $\deg f$. The collection $(w;d)=(w_1,\cdots,w_n;d)$ is called the quasi-homogeneity type (qh-type) of f.

Definition 2.4. An isolated hypersurface singularity in $\mathbb{C}^n$ is fewnomial if it can be defined by a n-nomial in n variables and it is a weighted homogeneous fewnomial isolated singularity if it can be defined by a weighted homogeneous fewnomial. 3-nomial isolated hypersurface singularity is also called trinomial singularity.

Proposition 2.5. *Let f be a weighted homogeneous fewnomial isolated singularity with mult$(f) \geq 3$. Then f analytically equivalent to a linear combination of the following three series:*

Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$, $n \geq 1$,

Type B. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$, $n \geq 2$,

Type C. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1$, $n \geq 2$.

Proposition 2.5 has an immediate corollary.

Corollary 2.6. *Each binomial isolated singularity is analytically equivalent to one from the three series: A) $x_1^{a_1} + x_2^{a_2}$, B) $x_1^{a_1}x_2 + x_2^{a_2}$, C) $x_1^{a_1}x_2 + x_2^{a_2}x_1$.*

The following proposition will be use to prove main theorems.

Proposition 2.7. [30] *Let $(V,0)$ be a binomial isolated singularity of type A which is defined by $f = x_1^{a_1} + x_2^{a_2}$ $(a_1 \geq 3, a_2 \geq 3)$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}; 1)$. Then*

$$\rho_2^1(V) = 2a_1a_2 - 5(a_1 + a_2) + 12.$$

Proposition 2.8. [30] *Let $(V,0)$ be a binomial isolated singularity of type B which is defined by $f = x_1^{a_1} + x_2^{a_2}x_1$ $(a_1 \geq 3,\ a_2 \geq 2)$ with weight type $(\frac{1}{a_1}, \frac{a_1-1}{a_1a_2}; 1)$. Then*

$$\rho_2^1(V) = \begin{cases} 2a_1a_2 - 5(a_1 + a_2) + 15; & a_1 \geq 4, a_2 \geq 3 \\ a_2 - 2; & a_1 = 3, a_2 \geq 3 \\ 0; & a_1 \geq 3, a_2 = 2. \end{cases}$$

Proposition 2.9. [30] *Let $(V,0)$ be a binomial isolated singularity of type C which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ $(a_1 \geq 2, a_2 \geq 2)$ with weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$. Then*

$$\rho_2^1(V) = \begin{cases} 2a_1a_2 - 5(a_1 + a_2) + 19; & a_1 \geq 5, a_2 \geq 5 \\ a_2 + 1; & a_1 = 3, a_2 \geq 3 \\ 3a_2 - 2; & a_1 = 4, a_2 \geq 5 \\ 9; & a_1 = 4, a_2 = 4 \\ 0; & a_1 = 2, a_2 \geq 2. \end{cases}$$

Proposition 2.10. [5] *Let $(V,0)$ be a binomial isolated singularity of type A which is defined by $f = x_1^{a_1} + x_2^{a_2}$ $(a_1 \geq 2, a_2 \geq 2)$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}; 1)$. Then*

$$\rho_2^3(V) = \begin{cases} 4a_1a_2 - 4(a_1 + a_2) + 7; & a_1 \geq 3,\ a_2 \geq 3 \text{ and } a_1 = a_2 \\ 6; & a_1 = 2, a_2 = 2 \\ 4a_2 - 3; & a_1 = 2, a_2 \geq 3 \\ 4a_1 - 3; & a_1 \geq 3, a_2 = 2 \\ 4a_1a_2 - 4(a_1 + a_2) + 6; & a_1 \geq 3,\ a_2 \geq 3 \text{ and } a_1 \neq a_2. \end{cases}$$

Proposition 2.11. [5] *Let* $(V,0)$ *be a binomial isolated singularity of type B which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(a_1 \geq 1, a_2 \geq 2)$ *with weight type* $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. *Then*

$$\rho_2^3(V) = \begin{cases} 4a_1a_2 - 4a_2 + 7; & a_1 \geq 2,\ a_2 \geq 3 \text{ and } a_1 = a_2 - 1 \\ 6; & a_1 = 1, a_2 \geq 2 \\ 8a_1 - 3; & a_1 \geq 2, a_2 = 2 \\ 4a_1a_2 - 4a_2 + 6; & a_1 \geq 2,\ a_2 \geq 3 \text{ and } a_1 \neq a_2 - 1. \end{cases}$$

Proposition 2.12. [5] *Let* $(V,0)$ *be a binomial isolated singularity of type C which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ $(a_1 \geq 1, a_2 \geq 1)$ *with weight type* $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$. *Then*

$$\rho_2^3(V) = \begin{cases} 6a_1; & a_1 \geq 1, a_2 = 1 \\ 4a_1a_2 + 3; & a_1 \geq 2,\ a_2 \geq 2 \text{ and } a_1 = a_2 \\ 4a_1a_2 + 2; & a_1 \geq 2,\ a_2 \geq 2 \text{ and } a_1 \neq a_2. \end{cases}$$

3. Proof of theorems

In order to prove the Theorems we need to use Lemma and propositions.

Lemma 3.1. *Let* $(V,0)$ *be an isolated hypersurface singularity defined by a polynomial* $f(x_1,x_2)$. *Then*

$$\mathcal{M}_2^2(V) = \mathcal{O}_2 \Big/ \Big\langle f, \left(\frac{\partial f}{\partial x_1}\right)^2, \left(\frac{\partial f}{\partial x_2}\right)^2, \frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_1^2}, \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}, \frac{\partial^2 f}{\partial x_1^2}\frac{\partial f}{\partial x_2} - 2\frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_2^2}, \frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_2^2}, \frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_1 \partial x_2} \Big\rangle.$$

Proof. Note that

$$Jac_2(f) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2} \\ 0 & 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & 0 \\ 0 & 0 & 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}.$$

After simple calculations, we get following $2*2$ minors

$$\Big\{ \left(\frac{\partial f}{\partial x_1}\right)^2, \left(\frac{\partial f}{\partial x_2}\right)^2, \frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_1^2}, \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}, \frac{\partial^2 f}{\partial x_1^2}\frac{\partial f}{\partial x_2} - 2\frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_2^2}, \frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_2^2}, \frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_1 \partial x_2} \Big\}$$

Hence , $\mathcal{J}_2(f)$ is generated by above 9 generators. That implies that

$$(f, \mathcal{J}_2(f)) = \Big\langle f, \left(\frac{\partial f}{\partial x_1}\right)^2, \left(\frac{\partial f}{\partial x_2}\right)^2, \frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_1^2}, \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}, \frac{\partial^2 f}{\partial x_1^2}\frac{\partial f}{\partial x_2} - 2\frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_2^2}, \frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_2^2}, \frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_1 \partial x_2} \Big\rangle.$$

Thus the lemma is proved. □

Proposition 3.2. *Let $(V,0)$ be a binomial isolated singularity of type A which is defined by $f = x_1^{a_1} + x_2^{a_2}$ $(a_1 \geq 2, a_2 \geq 2)$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}; 1)$. Then*

$$\rho_2^2(V) = \begin{cases} 3a_1a_2 - 4(a_1+a_2) + 8; & a_1 \geq 4,\ a_2 \geq 4 \\ 5a_2 - 5; & a_1 = 3, a_2 \geq 3 \\ a_1 - 2; & a_1 \geq 2, a_2 = 2. \end{cases}$$

Proof. It follows Lemma 3.1 that the local algebra

$$\mathcal{M}_2^2(V) = \mathcal{O}_2 \Big/ \Big\langle f, \left(\frac{\partial f}{\partial x_1}\right)^2, \left(\frac{\partial f}{\partial x_2}\right)^2, \frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_1^2}, \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}, \frac{\partial^2 f}{\partial x_1^2}\frac{\partial f}{\partial x_2} - 2\frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_1 \partial x_2},$$
$$\frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_2^2}, \frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_2^2}, \frac{\partial f}{\partial x_1}\frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_1 \partial x_2}\Big\rangle.$$

has a monomial basis of the form:

(1) if $a_1 \geq 3,\ a_2 \geq 3$,
$\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 4; 0 \leq i_2 \leq 2a_2 - 4; x_1^{a_1-3}x_2^{i_2}, 0 \leq i_2 \leq a_2 - 1; x_1^{a_1-2}x_2^{i_2}, 0 \leq i_2 \leq a_2 - 2$
$x_1^{a_1-1}x_2^{i_2}, 0 \leq i_2 \leq a_2 - 3\}$,
(2) if $a_1 \geq 2, a_2 = 2$, $\{x_1^{i_1}, 0 \leq i_1 \leq a_1 - 2\}$.

Case (1): When $a_1 \geq 4,\ a_2 \geq 4$, then we obtain following a basis of Lie algebra $\mathcal{L}_2^2(V)$:

$$x_2^{i_2}\partial_1, a_2 - 3 \leq i_2 \leq 2a_2 - 4; x_1^{i_1}x_2^{i_2}\partial_1, 1 \leq i_1 \leq a_1 - 4; 0 \leq i_2 \leq 2a_2 - 4;$$

$$x_1^{a_1-3}x_2^{i_2}\partial_1, 0 \leq i_2 \leq a_2 - 1; x_1^{a_1-2}x_2^{i_2}\partial_1, 0 \leq i_2 \leq a_2 - 2; x_1^{a_1-1}x_2^{i_2}\partial_1, 0 \leq i_2 \leq a_2 - 3;$$

$$x_2^{i_2}\partial_2, a_2 - 2 \leq i_2 \leq 2a_2 - 4; x_1^{i_1}x_2^{i_2}\partial_2, 1 \leq i_1 \leq a_1 - 4; a_2 - 2 \leq i_2 \leq 2a_2 - 4;$$

$$x_1^{a_1-3}x_2^{i_2}\partial_2, 1 \leq i_2 \leq a_2 - 1; x_1^{a_1-2}x_2^{i_2}\partial_2, 0 \leq i_2 \leq a_2 - 2; x_1^{a_1-1}x_2^{i_2}\partial_2, 0 \leq i_2 \leq a_2 - 3.$$

Therefore we have the following formula

$$\rho_2^2(V) = 3a_1a_2 - 4(a_1 + a_2) + 8.$$

Case (2): when $a_1 = 3,\ a_2 \geq 3$, then we obtain following a basis of Lie algebra $\mathcal{L}_2^2(V)$:

$$x_2^{i_2}\partial_1, a_2 - 2 \leq i_2 \leq a_2 - 1; x_1x_2^{i_2}\partial_1, 0 \leq i_2 \leq a_2 - 2; x_1^2x_2^{i_2}\partial_1, 0 \leq i_2 \leq a_2 - 3;$$

$$x_2^{i_2}\partial_2, 1 \leq i_2 \leq a_2 - 1; x_1x_2^{i_2}\partial_2, 0 \leq i_2 \leq a_2 - 2; x_1^2x_2^{i_2}\partial_2, 0 \leq i_2 \leq a_2 - 3.$$

Therefore we have the following formula

$$\rho_2^2(V) = 5a_2 - 5.$$

Case (3): when $a_1 \geq 2,\ a_2 = 2$, then we obtain following a basis of Lie algebra $\mathcal{L}_2^2(V)$:

$$x_1^{i_1}\partial_1, 1 \leq i_1 \leq a_1 - 2.$$

Therefore we have the following formula

$$\rho_2^2(V) = a_1 - 2.$$

□

Proposition 3.3. *Let $(V,0)$ be a binomial isolated singularity of type B which is defined by $f=x_1^{a_1}x_2+x_2^{a_2}$ $(a_1\geq 1, a_2\geq 2)$ with weight type $(\frac{a_2-1}{a_1a_2},\frac{1}{a_2};1)$. Then*

$$\rho_2^2(V)=\begin{cases} 3a_1a_2-4a_2-2a_1+11; & a_1\geq 4,\ a_2\geq 5\\ 10a_1-6; & a_1\geq 3, a_2=4\\ a_2+7; & a_1=2, a_2\geq 3\\ 7a_1-4; & a_1\geq 3,\ a_2=3\\ 2a_1-2; & a_1\geq 2,\ a_2=2\\ 0; & a_1=1,\ a_2\geq 2\end{cases}$$

Furthermore, we need to show that

$$\rho_2^2(V)\leq\begin{cases} \frac{3a_1a_2^2}{a_2-1}-4(\frac{a_1a_2}{a_2-1}+a_2)+8; & \frac{a_1a_2}{a_2-1}\geq 4,\ a_2\geq 4\\ 5a_2-5; & \frac{a_1a_2}{a_2-1}=3, a_2\geq 3\\ \frac{a_1a_2}{a_2-1}-2; & \frac{a_1a_2}{a_2-1}\geq 2, a_2=2.\end{cases}$$

Proof. It follows Lemma 3.1 that the local algebra $\mathcal{M}_2^2(V)$ has a monomial basis of the form:

(1) if $a_1\geq 3,\ a_2\geq 4,$

$\{x_1^{i_1}x_2^{i_2}, 0\leq i_1\leq a_1-4; 0\leq i_2\leq 2a_2-4; x_1^{a_1-3}x_2^{i_2}, 0\leq i_2\leq a_2; x_1^{a_1-2}x_2^{i_2}; 0\leq i_2\leq a_2-1;$
$x_1^{a_1-1}x_2^{i_2}, 0\leq i_2\leq a_2-2; x_1^{i_1}, a_1\leq i_1\leq 2a_1-2\},$

(2) if $a_1\geq 2, a_2=3,$ $\{x_1^{i_1}x_2^{i_2}, 0\leq i_1\leq a_1-2; 0\leq i_2\leq 2; x_1^{a_1-1}; x_1^{a_1-1}x_2; x_1^{i_1}, a_1\leq i_1\leq 2a_1-2\},$

(3) if $a_1\geq 2,\ a_2=2, \{x_1^{i_1}x_2^{i_2}, 0\leq i_1\leq a_1-2; 0\leq i_2\leq 1; x_1^{a_1-1}\},$

(4) if $a_1=2,\ a_2\geq 3, \{x_2^{i_2}, 0\leq i_2\leq a_2-1; x_1; x_1x_2; x_1^2\},$

(5) if $a_1=1,\ a_2\geq 2, \{1\}.$

Case (1): When $a_1\geq 4,\ a_2\geq 5$, then we obtain following a basis of Lie algebra $\mathcal{L}_2^2(V)$:

$$\begin{gathered} x_2^{i_2}\partial_1, a_2-4\leq i_2\leq 2a_2-4; x_1^{i_1}x_2^{i_2}\partial_1, 1\leq i_1\leq a_1-4, 0\leq i_2\leq 2a_2-4;\\ x_1^{a_1-3}x_2^{i_2}\partial_1; 0\leq i_2\leq a_2; x_1^{a_1-2}x_2^{i_2}\partial_1; 0\leq i_2\leq a_2-1; x_1^{a_1-1}x_2^{i_2}\partial_1; 0\leq i_2\leq a_2-2;\\ x_1^{i_1}\partial_1, a_1\leq i_1\leq 2a_1-2; x_1^{i_1}\partial_2, a_1\leq i_1\leq 2a_1-2; x_1^{i_1}x_2^{i_2}\partial_2, 0\leq i_1\leq a_1-4,\\ a_2-2\leq i_2\leq 2a_2-4; x_1^{a_1-3}x_2^{i_2}\partial_2; 2\leq i_2\leq a_2; x_1^{a_1-2}x_2^{i_2}\partial_2; 1\leq i_2\leq a_2-1;\\ x_1^{a_1-1}x_2^{i_2}\partial_2; 0\leq i_2\leq a_2-2.\end{gathered}$$

Therefore we have the following formula

$$\rho_2^2(V)=3a_1a_2-2a_1-4a_2+11. \tag{3}$$

Case (2): When $a_1\geq 3,\ a_2=4$, then we obtain following a basis of Lie algebra $\mathcal{L}_2^2(V)$:

$$\begin{gathered} x_2^{i_2}\partial_1, 0\leq i_2\leq 4; x_1^{i_1}x_2^{i_2}\partial_1, 1\leq i_1\leq a_1-3, 0\leq i_2\leq 4; x_1^{a_1-2}x_2^{i_2}\partial_1, 0\leq i_2\leq 3; x_1^{a_1-1}x_2^{i_2}\partial_1,\\ 0\leq i_2\leq 2; x_1^{i_1}\partial_1, a_1\leq i_1\leq 2a_1-2; x_1^{i_1}x_2^{i_2}\partial_2, 0\leq i_1\leq a_1-3, 2\leq i_2\leq 4; x_1^{a_1-2}x_2^{i_2}\partial_2, 1\leq i_2\leq 3;\\ x_1^{a_1-1}x_2^{i_2}\partial_2, 0\leq i_2\leq 2; x_1^{i_1}\partial_2, a_1\leq i_1\leq 2a_1-2.\end{gathered}$$

Therefore we have the following formula

$$\rho_2^2(V)=10a_1-6.$$

Case (3): when $a_1=2,\ a_2\geq 3$, then we obtain following a basis of Lie algebra $\mathcal{L}_2^2(V)$:

$$x_2^{i_2}\partial_1; a_2-2\leq i_2\leq a_2-1; x_1\partial_1; x_1x_2\partial_1; x_1^2\partial_1; x_2^{i_2}\partial_2; 1\leq i_2\leq a_2-1; x_1\partial_2; x_1x_2\partial_2; x_1^2\partial_2.$$

Therefore we have the following formula

$$\rho_2^2(V) = a_2 + 7.$$

Case (4): when $a_1 \geq 3$, $a_2 = 3$, then we obtain following a basis of Lie algebra $\mathcal{L}_2^2(V)$:

$$x_2\partial_1; x_2^2\partial_1; x_1^{i_1}x_2^{i_2}\partial_1, 1 \leq i_1 \leq a_1 - 2, 0 \leq i_2 \leq 2; x_1^{a_1-1}x_2^{i_2}\partial_1; 0 \leq i_2 \leq 1; x_1^{i_1}\partial_1; a_1 \leq i_1 \leq 2a_1 - 2;$$
$$x_1^{i_1}x_2^{i_2}\partial_2, 0 \leq i_1 \leq a_1 - 2, 1 \leq i_2 \leq 2; x_1^{a_1-1}x_2^{i_2}\partial_2; 0 \leq i_2 \leq 1; x_1^{i_1}\partial_2; a_1 \leq i_1 \leq 2a_1 - 2.$$

Therefore we have the following formula

$$\rho_2^2(V) = 7a_1 - 4.$$

Case (5): When $a_1 \geq 2$, $a_2 = 2$, then we obtain following a basis of Lie algebra $\mathcal{L}_2^2(V)$:

$$x_1^{i_1}x_2^{i_2}\partial_1, 1 \leq i_1 \leq a_1 - 2, 0 \leq i_2 \leq 1; x_2\partial_1; x_1^{a_1-1}\partial_2.$$

we have the following formula

$$\rho_2^2(V) = 2a_1 - 2.$$

Case (6): When $a_1 = 1$, $a_2 \geq 2$, then we have the following formula

$$\rho_2^2(V) = 0.$$

It is follows from Proposition 3.2 we have

$$h_2^2(a_1, a_2) = \begin{cases} 3a_1a_2 - 4(a_1 + a_2) + 8; & a_1 \geq 4,\ a_2 \geq 4 \\ 5a_2 - 5; & a_1 = 3, a_2 \geq 3 \\ a_1 - 2; & a_1 \geq 2, a_2 = 2. \end{cases}$$

After putting the weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$ of binomial isolated singularity of type B we have

$$h_2^2(\frac{1}{w_1}, \frac{1}{w_2}) = \begin{cases} \frac{3a_1a_2^2}{a_2-1} - 4(\frac{a_1a_2}{a_2-1} + a_2) + 8; & \frac{a_1a_2}{a_2-1} \geq 4,\ a_2 \geq 4 \\ 5a_2 - 5; & \frac{a_1a_2}{a_2-1} = 3, a_2 \geq 3 \\ \frac{a_1a_2}{a_2-1} - 2; & \frac{a_1a_2}{a_2-1} \geq 2, a_2 = 2. \end{cases}$$

From Conjecture 1.19 and proposition 3.3 we have

$$h_2^2(\frac{1}{w_1}, \frac{1}{w_2}) \geq \begin{cases} 3a_1a_2 - 4a_2 - 2a_1 + 11; & a_1 \geq 4,\ a_2 \geq 5 \\ 10a_1 - 6; & a_1 \geq 3, a_2 = 4 \\ a_2 + 7; & a_1 = 2, a_2 \geq 3 \\ 7a_1 - 4; & a_1 \geq 3,\ a_2 = 3 \\ 2a_1 - 2; & a_1 \geq 2,\ a_2 = 2 \\ 0; & a_1 = 1,\ a_2 \geq 2 \end{cases} \tag{4}$$

It is easy to see that the above inequality (4) hold true.

□

Proposition 3.4. *Let $(V, 0)$ be a binomial isolated singularity of type C which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ $(a_1 \geq 1, a_2 \geq 1)$ with weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$. Then*

$$\rho_2^2(V) = \begin{cases} 3a_1a_2 - 2(a_1 + a_2) + 12; & a_1 \geq 5,\ a_2 \geq 5 \\ 10a_2 + 3; & a_1 = 4,\ a_2 \geq 4 \\ 7a_2 + 3; & a_1 = 3, a_2 \geq 3 \\ 2a_2 + 6; & a_1 = 2, a_2 \geq 2 \\ 0; & a_1 = 1,\ a_2 \geq 1 \end{cases}$$

Furthermore, we need to show that

$$\rho_2^2(V)\le\begin{cases}\frac{3(a_1a_2-1)^2}{(a_1-1)(a_2-1)}-4(\frac{a_1a_2-1}{a_2-1}+\frac{a_1a_2-1}{a_1-1})+8; & \frac{a_1a_2-1}{a_2-1}\ge 4,\ \frac{a_1a_2-1}{a_1-1}\ge 4\\ \frac{5(a_1a_2-1)}{a_1-1}-5; & \frac{(a_1a_2-1)}{a_2-1}=3,\ \frac{(a_1a_2-1)}{a_1-1}\ge 3\\ \frac{(a_1a_2-1)}{a_2-1}-2; & \frac{(a_1a_2-1)}{a_2-1}\ge 2,\ \frac{4(a_1a_2-1)}{a_1-1}=2.\end{cases}$$

Proof. It follows Lemma 3.1 that the local algebra $\mathcal{M}_2^2(V)$ has a monomial basis of the form:

(1) if $a_1\ge 4,\ a_2\ge 4$,
$\{x_1^{i_1}x_2^{i_2}, 0\le i_1\le a_1-3; 0\le i_2\le 2a_2-4; x_2^{i_2}, 2a_2-3\le i_2\le 2a_2-2; x_1x_2^{2a_2-3}; x_1^{a_1-2}x_2^{i_2}$, $0\le i_2\le a_2; x_1^{a_1-1}x_2^{i_2}, 0\le i_2\le a_2-1; x_1^{i_1}, a_1\le i_1\le 2a_1-2\}$,

(2) if $a_1=3,\ a_2\ge 3$, $\{x_2^{i_2}, 0\le i_2\le 2a_2-2; x_1x_2^{i_2}, 0\le i_2\le a_2; x_1^3; x_1^4; x_1^2x_2^{i_2}; 0\le i_2\le a_2-1\}$,

(3) if $a_1=2,\ a_2\ge 2$, $\{x_1^{i_1}x_2^{i_2}, 0\le i_2\le 2; 0\le i_2\le a_2-1; x_1x_2^{a_2}; x_1^3; x_2^4; x_2^{i_2}; a_2\le i_2\le 2a_2-2\}$,

(4) if $a_1=1,\ a_2\ge 1$, $\{1\}$.

Case (1): when $a_1\ge 5,\ a_2\ge 5$, then we obtain following a basis of Lie algebra $\mathcal{L}_2^2(V)$:

$$x_1^{i_1}x_2^{i_2}\partial_1, 1\le i_1\le a_1-3, 0\le i_2\le 2a_2-4; x_2^{i_2}\partial_1, a_2-4\le i_2\le 2a_2-2; x_1x_2^{2a_2-3}\partial_1;$$
$$x_1^{a_1-2}x_2^{i_2}\partial_1; 0\le i_2\le a_2; x_1^{a_1-1}x_2^{i_2}\partial_1, 0\le i_2\le a_2-1; x_1^{i_1}\partial_1, a_1\le i_1\le 2a_1-2; x_1^{i_1}x_2^{i_2}\partial_2, 0\le i_1\le a_1-3;$$
$$a_2-1\le i_2\le 2a_2-4; x_1^{i_1}x_2^{a_2-2}\partial_2, 1\le i_1\le a_1-3; x_1^{a_1-2}x_2^{i_2}\partial_2, 1\le i_2\le a_2;$$
$$x_1^{a_1-1}x_2^{i_2}\partial_2, 0\le i_2\le a_2-1; x_1^{i_1}\partial_2; a_1\le i_1\le 2a_1-2; x_1x_2^{2a_2-3}\partial_2; x_2^{i_2}\partial_2, 2a_2-3\le i_2\le 2a_2-2;$$
$$x_1^{a_1-3}x_2^{i_2}\partial_2; 2\le i_2\le a_2-3$$

Therefore we have the following formula

$$\rho_2^2(V)=3a_1a_2-2(a_1+a_2)+12.$$

Case (2): when $a_1=4,\ a_2\ge 4$, then we obtain following a basis of Lie algebra $\mathcal{L}_2^2(V)$:

$$x_2^{i_2}\partial_1, a_2-3\le i_2\le 2a_2-2; x_1x_2^{i_2}\partial_1, 0\le i_2\le 2a_2-3; x_1^2x_2^{i_2}\partial_1; 0\le i_2\le a_2; x_1^3x_2^{i_2}\partial_1; 0\le i_2\le a_2-1;$$
$$x_1^{i_1}\partial_1; 4\le i_1\le 6; x_2^{i_2}\partial_2; a_2-1\le i_2\le 2a_2-2; x_1x_2^{i_2}\partial_2; 2\le i_2\le 2a_2-3; x_1^2x_2^{i_2}\partial_2; 1\le i_2\le a_2;$$
$$x_1^3x_2^{i_2}\partial_2; 0\le i_2\le a_2-1; x_1^{i_1}\partial_2; 4\le i_1\le 6.$$

Therefore we have the following formula

$$\rho_2^2(V)=10a_2+3.$$

Case (3): When $a_1=3,\ a_2\ge 3$, then we obtain following a basis of Lie algebra $\mathcal{L}_2^2(V)$:

$$x_2^{i_2}\partial_1, a_2-2\le i_2\le 2a_2-2; x_1^{i_1}x_2^{i_2}\partial_1, 1\le i_1\le 2, 0\le i_2\le a_2-1; x_2^{i_2}\partial_2, 2\le i_2\le 2a_2; x_1x_2^{a_2}\partial_1;$$
$$x_1^3\partial_1; x_1^4\partial_1; x_1x_2^{i_2}\partial_2, 1\le i_2\le a_2; x_1^2x_2^{i_2}\partial_2, 0\le i_2\le a_2-1; x_1^3\partial_2; x_1^4\partial_2.$$

Therefore we have the following formula

$$\rho_2^2(V)=7a_2+3.$$

Case (4): When $a_1=2,\ a_2\ge 2$, then we obtain following a basis of Lie algebra $\mathcal{L}_2^2(V)$:

$$x_2^{i_2+a_2-2}\partial_1-\frac{2}{a_2-1}x_2^{i_2}\partial_2, 1\le i_2\le a_2-2; x_2^{i_2}\partial_1, 2a_2-3\le i_2\le 2a_2-2; x_1^2\partial_2$$
$$x_1\partial_1+\frac{1}{a_2-1}x_2\partial_2; x_1x_2\partial_1+\frac{1}{a_2-1}x_2^2\partial_2; x_1^2\partial_1; x_2^{i_2}\partial_2; a_2-1\le i_2\le 2a_2-2; x_1\partial_2; x_1x_2\partial_2.$$

Therefore we have the following formula

$$\rho_2^2(V)=2a_2+6.$$

Case (5): when $a_1 = 1$, $a_2 \geq 1$, then we get

$$\rho_2^2(V) = 0.$$

It is follows from Proposition 3.2 we have

$$h_2^2(a_1, a_2) = \begin{cases} 3a_1a_2 - 4(a_1 + a_2) + 8; & a_1 \geq 4,\ a_2 \geq 4 \\ 5a_2 - 5; & a_1 = 3, a_2 \geq 3 \\ a_1 - 2; & a_1 \geq 2, a_2 = 2. \end{cases}$$

After putting the weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$ of binomial isolated singularity of type C we have

$$h_2^2(\frac{1}{w_1}, \frac{1}{w_2}) = \begin{cases} \frac{3(a_1a_2-1)^2}{(a_1-1)(a_2-1)} - 4(\frac{a_1a_2-1}{a_2-1} + \frac{a_1a_2-1}{a_1-1}) + 8; & \frac{a_1a_2-1}{a_2-1} \geq 4,\ \frac{a_1a_2-1}{a_1-1} \geq 4 \\ \frac{5(a_1a_2-1)}{a_1-1} - 5; & \frac{(a_1a_2-1)}{a_2-1} = 3,\ \frac{(a_1a_2-1)}{a_1-1} \geq 3 \\ \frac{(a_1a_2-1)}{a_2-1} - 2; & \frac{(a_1a_2-1)}{a_2-1} \geq 2,\ \frac{4(a_1a_2-1)}{a_1-1} = 2. \end{cases}$$

From Conjecture 1.19 and Proposition 3.4 we have

$$h_2^2(\frac{1}{w_1}, \frac{1}{w_2}) \geq \begin{cases} 3a_1a_2 - 2(a_1 + a_2) + 12; & a_1 \geq 5,\ a_2 \geq 5 \\ 10a_2 + 3; & a_1 = 4,\ a_2 \geq 4 \\ 7a_2 + 3; & a_1 = 3, a_2 \geq 3 \\ 2a_2 + 6; & a_1 = 2, a_2 \geq 2 \\ 0; & a_1 = 1,\ a_2 \geq 1 \end{cases} \tag{5}$$

It is easy to see that the above inequality (5) holds true. □

Proposition 3.5. *Let $(V, 0)$ be a binomial isolated singularity of type A which is defined by $f = x_1^{a_1} + x_2^{a_2}$ $(a_1 \geq 2, a_2 \geq 2)$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}; 1)$. Then*

$$d_2^3(V) = \begin{cases} 3a_1a_2 - 3(a_1 + a_2) + 6; & a_1 \geq 3,\ a_2 \geq 3 \text{ and } a_1 = a_2 \\ 5; & a_1 = 2, a_2 = 2 \\ 3a_2 - 2; & a_1 = 2,\ a_2 \geq 3 \\ 3a_1a_2 - 3(a_1 + a_2) + 5; & a_1 \geq 3,\ a_2 \geq 3 \text{ and } a_1 < a_2 \text{ or } a_1 > a_2. \end{cases}$$

Proof. It follows that the local algebra

$$\mathcal{M}_2^3(V) = \mathcal{O}_2 \Big/ \Big\langle f, \left(\frac{\partial f}{\partial x_1}\right)^3, \left(\frac{\partial f}{\partial x_2}\right)^3, \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right)^2,$$
$$\left(\frac{\partial f}{\partial x_2}\right)^2 \frac{\partial^2 f}{\partial x_1^2} + \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial^2 f}{\partial x_2^2} - 2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_1 \partial x_2}\Big\rangle.$$

has a monomial basis of the form:

(1) if $a_1 \geq 3$, $a_2 \geq 3$ and $a_1 = a_2$,

$\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 1; 0 \leq i_2 \leq 2a_2 - 3; x_1^{a_1-3}x_2^{i_2}, 2a_2 - 2 \leq i_2 \leq 2a_2 - 1; x_1^{a_1-2}x_2^{2a_2-2};$

$x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 4, 2a_2 - 2 \leq i_2 \leq 3a_2 - 4\}$,

(2) if $a_1 = 2, a_2 = 2$, $\{1, x_1, x_2, x_2^2, x_1x_2\}$,

(3) if $a_1 = 2$, $a_2 \geq 3$, $\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq 1; 0 \leq i_2 \leq a_2 - 1; x_2^{i_2}, a_2 \leq i_2 \leq 2a_2 - 3\}$,

(4) if $a_1 \geq 3$, $a_2 \geq 3$ and $a_1 < a_2$ or $a_1 > a_2$,

$\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 1; 0 \leq i_2 \leq 2a_2 - 3; x_1^{a_1-3}x_2^{i_2}, 2a_2 - 2 \leq i_2 \leq 2a_2 - 1;$

$x_1^{i_1}x_2^{i_2}, 0 \le i_1 \le a_1-4, 2a_2-2 \le i_2 \le 3a_2-4\}$.

□

Proposition 3.6. *Let $(V,0)$ be a binomial isolated singularity of type B which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(a_1 \ge 1, a_2 \ge 2)$ with weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. Then*

$$d_2^3(V) = \begin{cases} 3a_1a_2 - 3a_2 + 6; & a_1 \ge 2,\ a_2 \ge 3 \text{ and } a_1 = a_2 - 1 \\ 5; & a_1 = 1, a_2 \ge 2, \\ 6a_1 - 2; & a_1 \ge 2,\ a_2 = 2 \\ 3a_1a_2 - 3a_2 + 5; & a_1 \ge 3, a_2 \ge 3 \text{ and } a_1 = a_2 \text{ or } a_1 < a_2 - 1 \text{ or } a_1 > a_2 - 1 \end{cases}$$

Furthermore, we need to show that

$$d_2^3(V) \ge \begin{cases} 5; & a_1 = 1, a_2 = 2 \\ 3a_2 - 2; & \frac{a_1a_2}{a_2-1} = 2, a_2 \ge 3 \\ \frac{3a_1a_2^2}{a_2-1} - 3(\frac{a_1a_2}{a_2-1} + a_2) + 6; & \frac{a_1a_2}{a_2-1} \ge 3,\ a_2 \ge 3 \text{ and } \frac{a_1a_2}{a_2-1} = a_2 \\ \frac{3a_1a_2^2}{a_2-1} - 3(\frac{a_1a_2}{a_2-1} + a_2) + 5; & \frac{a_1a_2}{a_2-1} \ge 3,\ a_2 \ge 3 \text{ and } \frac{a_1a_2}{a_2-1} < a_2 \text{ or } \frac{a_1a_2}{a_2-1} > a_2. \end{cases}$$

Proof. It follows that the local algebra

$$\mathcal{M}_2^3(V) = \mathcal{O}_2 \Big/ \Big\langle f, \left(\frac{\partial f}{\partial x_1}\right)^3, \left(\frac{\partial f}{\partial x_2}\right)^3, \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right)^2, \left(\frac{\partial f}{\partial x_2}\right)^2 \frac{\partial^2 f}{\partial x_1^2} + \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial^2 f}{\partial x_2^2} - 2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_1 \partial x_2}\Big\rangle.$$

has a monomial basis of the form:

(1) if $a_1 \ge 2,\ a_2 \ge 3$ and $a_1 = a_2 - 1$,
$\{x_1^{i_1}x_2^{i_2}, 1 \le i_1 \le a_1-1; 0 \le i_2 \le 2a_2-2; x_1^{a_1-3}x_2^{i_2}, 2a_2-1 \le i_2 \le 2a_2; x_1^{a_1-2}x_2^{2a_2-1};$
$x_2^{i_2}, 0 \le i_2 \le 2a_2-2; x_1^{i_1}x_2^{i_2}, 0 \le i_1 \le a_1-4, 2a_2-1 \le i_2 \le 3a_2-3; x_1^{i_1}, a_1 \le i_1 \le 3a_1-1\}$,
(2) if $a_1 = 1, a_2 \ge 2$, $\{1, x_1, x_2, x_2^2, x_1^2\}$,
(3) if $a_1 \ge 2,\ a_2 = 2$, $\{x_1^{i_1}x_2^{i_2}, 1 \le i_1 \le a_1-1; 0 \le i_2 \le 2; x_2^{i_2}, 0 \le i_2 \le 2; x_1^{i_1}x_2^3, 0 \le i_1 \le a_1-3;$
$x_1^{i_1}, a_1 \le i_1 \le 3a_1-1\}$,
(4) if $a_1 = 2,\ a_2 \ge 4$, $\{x_1^{i_1}x_2^{i_2}, 0 \le i_1 \le 1; 0 \le i_2 \le a_2+1; x_2^{i_2}, a_2+2 \le i_2 \le 2a_2-2; x_1^{i_1}, 2 \le i_1 \le 5\}$,
(5) if $a_1 \ge 3,\ a_2 \ge 3$ and $a_1 = a_2$ or $a_1 < a_2 - 1$ or $a_1 > a_2 - 1$,
$\{x_1^{i_1}x_2^{i_2}, 1 \le i_1 \le a_1-1; 0 \le i_2 \le 2a_2-2; x_1^{a_1-3}x_2^{i_2}, 2a_2-1 \le i_2 \le 2a_2;$
$x_2^{i_2}, 0 \le i_2 \le 2a_2-2; x_1^{i_1}x_2^{i_2}, 0 \le i_1 \le a_1-4, 2a_2-1 \le i_2 \le 3a_2-3; x_1^{i_1}, a_1 \le i_1 \le 3a_1-1\}$.

It is follows from proposition 3.5 we have

$$\ell_2^3(a_1, a_2) = \begin{cases} 3a_1a_2 - 3(a_1+a_2) + 6; & a_1 \ge 3,\ a_2 \ge 3 \text{ and } a_1 = a_2 \\ 5; & a_1 = 2, a_2 = 2 \\ 3a_2 - 2; & a_1 = 2,\ a_2 \ge 3 \\ 3a_1a_2 - 3(a_1+a_2) + 5; & a_1 \ge 3,\ a_2 \ge 3 \text{ and } a_1 < a_2 \text{ or } a_1 > a_2. \end{cases}$$

After putting the weight type $(\frac{a_2-1}{a_1a_2},\frac{1}{a_2};1)$ of binomial isolated singularity of type B we have

$$\ell_2^3(\frac{1}{w_1},\frac{1}{w_2})=\begin{cases}5; & a_1=1, a_2=2\\ 3a_2-2; & \frac{a_1a_2}{a_2-1}=2, a_2\geq 3\\ \frac{3a_1a_2^2}{a_2-1}-3(\frac{a_1a_2}{a_2-1}+a_2)+6; & \frac{a_1a_2}{a_2-1}\geq 3,\ a_2\geq 3 \text{ and } \frac{a_1a_2}{a_2-1}=a_2\\ \frac{3a_1a_2^2}{a_2-1}-3(\frac{a_1a_2}{a_2-1}+a_2)+5; & \frac{a_1a_2}{a_2-1}\geq 3,\ a_2\geq 3 \text{ and } \frac{a_1a_2}{a_2-1}<a_2 \text{ or } \frac{a_1a_2}{a_2-1}>a_2.\end{cases}$$

From conjecture 1.21 and proposition 3.6 we have

$$\ell_2^3(\frac{1}{w_1},\frac{1}{w_2})\leq\begin{cases}3a_1a_2-3a_2+6; & a_1\geq 2,\ a_2\geq 3 \text{ and } a_1=a_2-1\\ 5; & a_1=1, a_2\geq 2,\\ 6a_1-2; & a_1\geq 2,\ a_2=2\\ 3a_1a_2-3a_2+5; & \mathrm{a}_1\geq 3, a_2\geq 3 \text{ and } a_1=a_2 \text{ or } a_1<a_2-1 \text{ or } a_1>a_2-1\end{cases} \tag{6}$$

It is easy to see that the above inequality (6) hold true.

□

Proposition 3.7. *Let $(V,0)$ be a binomial isolated singularity of type C which is defined by $f=x_1^{a_1}x_2+x_2^{a_2}x_1$ $(a_1\geq 1, a_2\geq 1)$ with weight type $(\frac{a_2-1}{a_1a_2-1},\frac{a_1-1}{a_1a_2-1};1)$. Then*

$$d_2^3(V)=\begin{cases}3a_1a_2+3; & a_1\geq 2,\ a_2\geq 2 \textit{ and } a_1=a_2\\ 3a_1a_2+2; & a_1\geq 2,\ a_2\geq 2 \textit{ and } a_1<a_2 \textit{ or } a_1>a_2\\ 5a_1; & a_1\geq 1,\ a_2=1\end{cases}$$

Furthermore, we need to show that

$$d_2^3(V)\geq\begin{cases}5; & \frac{a_1a_2-1}{a_2-1}=2, \frac{a_1a_2-1}{a_1-1}=2\\ \frac{3(a_1a_2-1)}{a_1-1}-2; & \frac{a_1a_2-1}{a_2-1}=2, \frac{a_1a_2-1}{a_1-1}\geq 3\\ \frac{3(a_1a_2-1)^2}{(a_1-1)(a_2-1)}-3(\frac{a_1a_2-1}{a_2-1}+\frac{a_1a_2-1}{a_1-1})+6; & \frac{a_1a_2-1}{a_2-1}\geq 3, \frac{a_1a_2-1}{a_1-1}\geq 3\\ & \textit{and } \frac{a_1a_2-1}{a_2-1}=\frac{a_1a_2-1}{a_1-1}\\ \frac{3(a_1a_2-1)^2}{(a_1-1)(a_2-1)}-3(\frac{a_1a_2-1}{a_2-1}+\frac{a_1a_2-1}{a_1-1})+5; & \frac{a_1a_2-1}{a_2-1}\geq 3, \frac{a_1a_2-1}{a_1-1}\geq 3\\ & \textit{and } \frac{a_1a_2-1}{a_2-1}<\frac{a_1a_2-1}{a_1-1} \textit{ or } \frac{a_1a_2-1}{a_2-1}>\frac{a_1a_2-1}{a_1-1}\ .\end{cases}$$

Proof. It follows that the local algebra

$$\mathcal{M}_2^3(V)=\mathcal{O}_2\Big/\Big\langle f,\left(\frac{\partial f}{\partial x_1}\right)^3,\left(\frac{\partial f}{\partial x_2}\right)^3,\left(\frac{\partial f}{\partial x_1}\right)^2\frac{\partial f}{\partial x_2},\frac{\partial f}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right)^2,$$
$$\left(\frac{\partial f}{\partial x_2}\right)^2\frac{\partial^2 f}{\partial x_1^2}+\left(\frac{\partial f}{\partial x_1}\right)^2\frac{\partial^2 f}{\partial x_2^2}-2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_1\partial x_2}\Big\rangle.$$

has a monomial basis of the form:

(1) if $a_1\geq 3,\ a_2\geq 3$ and $a_1=a_2$,

$\{x_1^{i_1}x_2^{i_2}, 0\leq i_1\leq a_1-1; 0\leq i_2\leq 2a_2; x_1^{i_1}, a_1\leq i_1\leq 3a_1-1; x_2^{i_2}, 2a_2+1\leq i_2\leq 4a_2-4;$
$x_1^{i_1}x_2^{i_2}, 1\leq i_1\leq a_1-2, 2a_2+1\leq i_2\leq 3a_2-3; x_1x_2^{3a_2-2}\}$,

(2) if $a_1=2,\ a_2=2$,

$\{x_1^{i_1}x_2^{i_2}, 0\leq i_1\leq 1; 0\leq i_2\leq 4; x_1^{i_1}, 2\leq i_1\leq 5; x_2^5\}$,

(3) if $a_1\geq 1,\ a_2=1, \{x_1^{i_1}x_2^{i_2}, 0\leq i_1\leq a_1-1; 0\leq i_2\leq 2; x_1^{i_1}, a_1\leq i_1\leq 3a_1-1\}$,

(4) if $a_1 \geq 2$, $a_2 \geq 3$ and $a_1 < a_2$ or $a_1 > a_2$,
$\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 1; 0 \leq i_2 \leq 2a_2; x_1^{i_1}, a_1 \leq i_1 \leq 3a_1 - 1; x_2^{i_2},$
$2a_2 + 1 \leq i_2 \leq 4a_2 - 4; x_1^{i_1}x_2^{i_2}, 1 \leq i_1 \leq a_1 - 2, 2a_2 + 1 \leq i_2 \leq 3a_2 - 3\}$
(5) if $a_1 \geq 3$, $a_2 = 2$,
$\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 1; 0 \leq i_2 \leq 3; x_1^{i_1}, a_1 \leq i_1 \leq 3a_1 - 1; x_2^{i_2}, 4 \leq i_2 \leq 5\}$.

It is follows from proposition 3.5 we have

$$\ell_2^3(a_1, a_2) = \begin{cases} 3a_1a_2 - 3(a_1 + a_2) + 6; & a_1 \geq 3,\ a_2 \geq 3 \text{ and } a_1 = a_2 \\ 5; & a_1 = 2, a_2 = 2 \\ 3a_2 - 2; & a_1 = 2,\ a_2 \geq 3 \\ 3a_1a_2 - 3(a_1 + a_2) + 5; & a_1 \geq 3,\ a_2 \geq 3 \text{ and } a_1 < a_2 \text{ or } a_1 > a_2. \end{cases}$$

After putting the weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$ of binomial isolated singularity of type C we have

$$\ell_2^3(\frac{1}{w_1}, \frac{1}{w_2}) = \begin{cases} 5; & \frac{a_1a_2-1}{a_2-1} = 2,\ \frac{a_1a_2-1}{a_1-1} = 2 \\ \frac{3(a_1a_2-1)}{a_1-1} - 2; & \frac{a_1a_2-1}{a_2-1} = 2,\ \frac{a_1a_2-1}{a_1-1} \geq 3 \\ \frac{3(a_1a_2-1)^2}{(a_1-1)(a_2-1)} - 3(\frac{a_1a_2-1}{a_2-1} + \frac{a_1a_2-1}{a_1-1}) + 6; & \frac{a_1a_2-1}{a_2-1} \geq 3,\ \frac{a_1a_2-1}{a_1-1} \geq 3 \\ & \text{and } \frac{a_1a_2-1}{a_2-1} = \frac{a_1a_2-1}{a_1-1} \\ \frac{3(a_1a_2-1)^2}{(a_1-1)(a_2-1)} - 3(\frac{a_1a_2-1}{a_2-1} + \frac{a_1a_2-1}{a_1-1}) + 5; & \frac{a_1a_2-1}{a_2-1} \geq 3,\ \frac{a_1a_2-1}{a_1-1} \geq 3 \\ & \text{and } \frac{a_1a_2-1}{a_2-1} < \frac{a_1a_2-1}{a_1-1} \text{ or } \frac{a_1a_2-1}{a_2-1} > \frac{a_1a_2-1}{a_1-1}. \end{cases}$$

From conjecture 1.21 and proposition 3.7 we have

$$\ell_2^3(\frac{1}{w_1}, \frac{1}{w_2}) \leq \begin{cases} 3a_1a_2 + 3; & a_1 \geq 2,\ a_2 \geq 2 \text{ and } a_1 = a_2 \\ 3a_1a_2 + 2; & a_1 \geq 2,\ a_2 \geq 2 \text{ and } a_1 < a_2 \text{ or } a_1 > a_2 \\ 5a_1; & a_1 \geq 1,\ a_2 = 1. \end{cases} \tag{7}$$

It is easy to see that the above inequality (7) holds true.

□

Proposition 3.8. *Let* $V = \{(x_1, x_2) \in \mathbb{C}^2 : x_1^{k+1} + x_2^2 = 0\}$ *be the* A_k *singularity,* $k \geq 1$ *and* $\mathcal{L}_2^2(V)$ *be a derivation Lie algebra. Then*

$$\rho_2^2(V) = k - 1.$$

Proof. It follows that the local algebra $\mathcal{M}_2^2(V)$ has a monomial basis of the form:

$$x_1^{i_1}, 0 \leq i_1 \leq k - 1$$

The Lie algebra $\mathcal{L}_2^2(V)$ has following basis:

$$e_1 = x_1\partial_1 + 2x_1^2\partial_2, \quad e_2 = x_1^2\partial_1 + 2x_1^3\partial_2, \quad e_3 = x_1^4\partial_1 + 2x_1^5\partial_2, \cdots, e_{k-1} = x_1^k\partial_1.$$

It is easy to check that for $k = 1$ the Lie algebra dimension are zero and for $k = 2$ the dimension of nilradical $g(V)$ are zero. It is also note that for $k \geq 3$ the nilradical $g(V)$ of Lie algebra $\mathcal{L}_2^2(V)$ generated by $< e_2, e_3, e_4, \cdots, e_{k-1} >$. For A_4 singularity, the nilradical $g(V) =< e_2, e_3 >$ have following multiplication table $[e_2, e_3] = 0$.

Case 1. When k is odd and $k \geq 5$, then nilradical $g(V) =< e_2, e_3, e_4, \cdots, e_{k-1} >$ have following multiplication table:

$$\begin{aligned}
&[e_2, e_3] = -e_4, \quad [e_2, e_4] = -2e_5, \quad [e_2, e_5] = -3e_6, \cdots, \ [e_2, e_{k-2}] = -(k-4)e_{k-1},\\
&[e_3, e_4] = -e_6, \quad [e_3, e_5] = -2e_7, \quad [e_3, e_6] = -3e_8, \cdots, \ [e_3, e_{k-3}] = -(k-6)e_{k-1},\\
&[e_4, e_5] = -e_8, \quad [e_4, e_6] = -2e_9, \quad [e_4, e_7] = -3e_{10}, \cdots, \ [e_4, e_{k-4}] = -(k-8)e_{k-1},\\
&\quad \vdots\\
&[e_{\frac{k-1}{2}}, e_{\frac{k-1}{2}+1}] = -e_{k-1}.
\end{aligned}$$

Case 2. When k is even and $k \geq 6$, then nilradical $g(V) =< e_2, e_3, e_4, \cdots, e_{k-1} >$ have following multiplication table:

$$\begin{aligned}
&[e_2, e_3] = -e_4, \quad [e_2, e_4] = -2e_5, \quad [e_2, e_5] = -3e_6, \cdots, \ [e_2, e_{k-2}] = -(k-4)e_{k-1},\\
&[e_3, e_4] = -e_6, \quad [e_3, e_5] = -2e_7, \quad [e_3, e_6] = -3e_8, \cdots, \ [e_3, e_{k-3}] = -(k-6)e_{k-1},\\
&[e_4, e_5] = -e_8, \quad [e_4, e_6] = -2e_9, \quad [e_4, e_7] = -3e_{10}, \cdots, \ [e_4, e_{k-4}] = -(k-8)e_{k-1},\\
&\quad \vdots\\
&[e_{\frac{k-2}{2}}, e_{\frac{k-2}{2}+1}] = -e_{k-2}, \quad [e_{\frac{k-2}{2}}, e_{\frac{k-2}{2}+2}] = -2e_{k-1}.
\end{aligned}$$

□

Proposition 3.9. *Let $V = \{(x_1, x_2) \in \mathbb{C}^2 : x_1^2 x_2 + x_2^{k-1} = 0\}$ be the D_k singularity, $k \geq 4$ and $\mathcal{L}_2^2(V)$ be a derivation Lie algebra. Then*

$$\rho_2^2(V) = k + 6.$$

Proof. It follows that the local algebra $\mathcal{M}_2^2(V)$ has a monomial basis of the form:

$$\{x_2^{i_2}, 0 \leq i_2 \leq k-2; x_1; x_1 x_2; x_1^2\}.$$

When $k = 4$, then we obtain following bases of Lie algebra $\mathcal{L}_2^2(V)$:

$$\begin{aligned}
&e_1 = -x_1\partial_1 + x_2\partial_2, \quad e_2 = -x_2\partial_1, \quad e_3 = -x_1\partial_2, \quad e_4 = -x_2\partial_2, \quad e_5 = -x_1^2\partial_2,\\
&e_6 = -x_1x_2\partial_2, \quad e_7 = -x_2^2\partial_2, \quad e_8 = -x_1^2\partial_1, \quad e_9 = -x_1x_2\partial_1, \quad e_{10} = -x_2^2\partial_1.
\end{aligned}$$

For D_4 singularity, the nilradical $g(V) =< e_2, e_5, e_6, e_7, \cdots, e_{10} >$ have following multiplication table:

$$[e_2, e_5] = 2e_6 - e_8, \quad [e_2, e_6] = e_7 - e_9, \quad [e_2, e_7] = -e_{10}, \quad [e_2, e_8] = 2e_9, \quad [e_2, e_9] = e_{10}.$$

Case 2: When $k \geq 5$, then we obtain following bases of Lie algebra $\mathcal{L}_2^2(V)$:

$$\begin{aligned}
&e_1 = x_1\partial_1 - x_2\partial_2, \quad e_2 = x_1\partial_2, \quad e_3 = x_2\partial_2, \quad e_4 = x_2^{k-3}\partial_1, \quad e_5 = x_2^2\partial_2,\\
&e_6 = x_2^3\partial_2, \quad e_7 = x_2^4\partial_2, \cdots, e_k = x_2^{k-3}\partial_2, \quad e_{k+1} = x_1^2\partial_2, \quad e_{k+2} = x_1x_2\partial_2, \quad e_{k+3} = x_2^{k-2}\partial_2,\\
&e_{k+4} = x_1^2\partial_1, \quad e_{k+5} = x_1x_2\partial_1, \quad e_{k+6} = x_2^{k-2}\partial_1.
\end{aligned}$$

For D_5 singularity, the nilradical $g(V) =< e_2, e_4, e_5, e_6, \cdots, e_{11} >$ have following multiplication table:

$$\begin{aligned}
&[e_2, e_4] = -2e_{10} + e_5, \quad [e_2, e_5] = -2e_7, \quad [e_2, e_7] = -e_6, \quad [e_2, e_9] = e_6, \quad [e_4, e_5] = 2e_{11},\\
&[e_4, e_7] = -e_8, \quad [e_4, e_{10}] = -e_{11}.
\end{aligned}$$

For D_6 singularity, the nilradical $g(V) =< e_2, e_4, e_5, e_6, \cdots, e_{12} >$ have following multiplication table:

$$[e_2, e_4] = e_6, \quad [e_2, e_5] = -2e_8, \quad [e_2, e_8] = -e_7, \quad [e_2, e_{10}] = e_7, \quad [e_2, e_{11}] = -e_{10} + e_8,$$
$$[e_2, e_{12}] = e_9, \quad [e_4, e_5] = 3e_{12}, \quad [e_4, e_8] = -e_9, \quad [e_4, e_{11}] = -e_{12}, \quad [e_5, e_6] = -e_9.$$

For D_7 singularity, the nilradical $g(V) =< e_2, e_4, e_5, e_6, \cdots, e_{13} >$ have following multiplication table:

$$[e_2, e_4] = e_7, \quad [e_2, e_5] = -2e_9, \quad [e_2, e_9] = -e_8, \quad [e_2, e_{11}] = e_8, \quad [e_2, e_{12}] = -e_{11} + e_9,$$
$$[e_2, e_{13}] = e_{10}, \quad [e_4, e_5] = 4e_{13}, \quad [e_4, e_9] = -e_{10}, \quad [e_4, e_{12}] = -e_{13}, \quad [e_5, e_6] = -e_7,$$
$$[e_5, e_7] = -2e_{10}.$$

Case 1. When k is even and $k \geq 8$, then nilradical $g(V) =< e_2, e_4, e_5, e_6, \cdots, e_{k+6} >$ have following multiplication table:

$$[e_2, e_4] = e_k, \quad [e_2, e_5] = -2e_{k+2}, \quad [e_2, e_{k+2}] = -e_{k+1}, \quad [e_2, e_{k+4}] = e_{k+1}, \quad [e_2, e_{k+5}]$$
$$= e_{k+2} - e_{k+4}, \quad [e_2, e_{k+6}] = e_{k+3}, \quad [e_4, e_5] = (k-3)e_{k+6}, \quad [e_4, e_{k+2}] = -e_{k+3}, \quad [e_4, e_{k+5}]$$
$$= -e_{k+6}, \quad [e_5, e_6] = -e_7, \quad [e_5, e_7] = -2e_8, \quad [e_5, e_8] = -3e_9, \cdots, [e_5, e_{k-1}] = -(k-6)e_k,$$
$$[e_5, e_k] = -(k-5)e_{k+3}, \quad [e_6, e_7] = -e_9, [e_6, e_8] = -2e_{10}, \quad [e_6, e_9] = -3e_{11}, \cdots,$$
$$[e_6, e_{k-2}] = -(k-8)e_k, \quad [e_6, e_{k-1}] = -(k-7)e_{k+3},$$
$$[e_7, e_8] = -e_{11}, \quad [e_7, e_9] = -2e_{12}, \quad [e_7, e_{10}] = -3e_{13}, \cdots, [e_7, e_{k-3}] = -(k-10)e_k,$$
$$[e_7, e_{k-2}] = -(k-9)e_{k+3},$$
$$\vdots$$
$$[e_{\frac{k+2}{2}}, e_{\frac{k+2}{2}+1}] = -e_{k-1}, \quad [e_{\frac{k+2}{2}}, e_{\frac{k+2}{2}+2}] = -2e_k, \quad [e_{\frac{k+2}{2}}, e_{\frac{k+2}{2}+3}] = -3e_{k+3},$$
$$[e_{\frac{k+4}{2}}, e_{\frac{k+4}{2}+1}] = -e_{k+3}.$$

Case 2. When k is odd and $k \geq 9$, then nilradical $g(V) =< e_2, e_4, e_5, e_6, \cdots, e_{k+6} >$ have following multiplication table:

$$[e_2, e_4] = e_k, \quad [e_2, e_5] = -2e_{k+2}, \quad [e_2, e_{k+2}] = -e_{k+1}, \quad [e_2, e_{k+4}] = e_{k+1}, \quad [e_2, e_{k+5}] = e_{k+2}$$
$$- e_{k+4}, [e_2, e_{k+6}] = e_{k+3}, \quad [e_4, e_5] = (k-3)e_{k+6}, \quad [e_4, e_{k+2}] = -e_{k+3}, \quad [e_4, e_{k+5}] = -e_{k+6},$$
$$[e_5, e_6] = -e_7, \quad [e_5, e_7] = -2e_8, \quad [e_5, e_8] = -3e_9, \cdots, [e_5, e_{k-1}] = -(k-6)e_k,$$
$$[e_5, e_k] = -(k-5)e_{k+3}, \quad [e_6, e_7] = -e_9, [e_6, e_8] = -2e_{10}, \quad [e_6, e_9] = -3e_{11}, \cdots,$$
$$[e_6, e_{k-2}] = -(k-8)e_k, \quad [e_6, e_{k-1}] = -(k-7)e_{k+3},$$
$$[e_7, e_8] = -e_{11}, \quad [e_7, e_9] = -2e_{12}, \quad [e_7, e_{10}] = -3e_{13}, \cdots, [e_7, e_{k-3}] = -(k-10)e_k,$$
$$[e_7, e_{k-2}] = -(k-9)e_{k+3},$$
$$\vdots$$
$$[e_{\frac{k+1}{2}}, e_{\frac{k+1}{2}+1}] = -e_{k-2}, \quad [e_{\frac{k+1}{2}}, e_{\frac{k+1}{2}+2}] = -2e_{k-1}, \quad [e_{\frac{k+1}{2}}, e_{\frac{k+1}{2}+3}] = -3e_k,$$
$$[e_{\frac{k+1}{2}}, e_{\frac{k+1}{2}+4}] = -4e_{k+3}, \quad [e_{\frac{k+3}{2}}, e_{\frac{k+3}{2}+1}] = -e_k, \quad [e_{\frac{k+3}{2}}, e_{\frac{k+3}{2}+2}] = -2e_{k+3}.$$

□

Proposition 3.10. *Let $V = \{(x_1, x_2) \in \mathbb{C}^2 : x_1^3 + x_2^4 = 0\}$ be the E_6 singularity and $\mathcal{L}_2^2(V)$ be a derivation Lie algebra. Then*

$$\rho_2^2(V) = 15.$$

Proof. It follows that the local algebra $\mathcal{M}_2^2(V)$ has a monomial basis of the form:

$$\{1, x_1, x_2, x_2^2, x_2^3, x_1x_2, x_1x_2^2, x_1^2, x_1^2x_2\}.$$

We obtain following bases of Lie algebra $\mathcal{L}_2^2(V)$:

$$e_1 = x_2\partial_2; \quad e_2 = x_1x_2\partial_1 - x_2^2\partial_2, \quad e_3 = x_1\partial_1 - 2x_2\partial_2, \quad e_4 = x_2^2\partial_1, \quad e_5 = x_1\partial_2, \quad e_6 = x_1^2\partial_1,$$
$$e_7 = x_2^2\partial_2, \quad e_8 = x_1^2\partial_2, \quad e_9 = -x_1^2\partial_1 + x_1x_2\partial_2, \quad e_{10} = x_1^2x_2\partial_2, \quad e_{11} = x_1x_2^2\partial_2, \quad e_{12} = x_2^3\partial_2,$$
$$e_{13} = x_1^2x_2\partial_1, \quad e_{14} = x_1x_2^2\partial_1, \quad e_{15} = x_2^3\partial_1.$$

The nilradical $g(V) =< e_2, e_4, e_5, e_6, \cdots, e_{15} >$ have following multiplication table:

$$[e_2, e_4] = 3e_{15}, \quad [e_2, e_5] = -2e_6 - 3e_9, \quad [e_2, e_6] = -e_{13}, \quad [e_2, e_7] = e_{14}, \quad [e_2, e_8] = -4e_{10},$$
$$[e_2, e_9] = -2e_{11} + 2e_{13}, \quad [e_4, e_5] = 2e_2 + e_7, \quad [e_4, e_6] = -2e_{14}, \quad [e_4, e_7] = 2e_{15},$$
$$[e_4, e_8] = -2e_{11} + 2e_{13}, \quad [e_4, e_9] = -e_{12} + 4e_{14}, \quad [e_5, e_6] = e_8, \quad [e_5, e_7] = -2e_6 - 2e_9,$$
$$[e_5, e_9] = -2e_8, \quad [e_5, e_{11}] = -2e_{10}, \quad [e_5, e_{12}] = -3e_{11}, \quad [e_5, e_{13}] = e_{10}, \quad [e_5, e_{14}] = e_{11} - 2e_{13},$$
$$[e_5, e_{15}] = e_{12} - 3e_{14}, \quad [e_6, e_9] = -e_{10}, \quad [e_7, e_8] = 2e_{10}, \quad [e_7, e_9] = e_{11}.$$

□

Proposition 3.11. *Let $V = \{(x_1, x_2) \in \mathbb{C}^2 : x_1^3 + x_2^3x_1 = 0\}$ be the E_7 singularity and $\mathcal{L}_2^2(V)$ be a derivation Lie algebra. Then*

$$\rho_2^2(V) = 17.$$

Proof. It follows that the local algebra $\mathcal{M}_2^2(V)$ has a monomial basis of the form:

$$\{1, x_1, x_2, x_2^2, x_2^3, x_2^4, x_1x_2, x_1x_2^2, x_1^2, x_1^2x_1\}.$$

We obtain following bases of Lie algebra $\mathcal{L}_2^2(V)$:

$$e_1 = x_2\partial_2, \quad e_2 = x_1x_2\partial_1 - x_2^2\partial_2, \quad e_3 = x_1\partial_1 - 2x_2\partial_2, \quad e_4 = x_2^2\partial_1, \quad e_5 = x_1\partial_2, \quad e_6 = x_1^2\partial_1,$$
$$e_7 = x_2^2\partial_2, \quad e_8 = x_2^3\partial_1, \quad e_9 = x_1^2\partial_2, \quad e_{10} = -x_1^2\partial_1 + x_1x_2\partial_2, \quad e_{11} = x_2^3\partial_2, \quad e_{12} = x_1^2x_2\partial_2,$$
$$e_{13} = x_1x_2^2\partial_2, \quad e_{14} = x_2^4\partial_2, \quad e_{15} = x_1^2x_2\partial_1, \quad e_{16} = x_1x_2^2\partial_1, \quad e_{17} = x_2^4\partial_1.$$

The nilradical $g(V) =< e_2, e_4, e_5, e_6, \cdots, e_{17} >$ have following multiplication table:

$$[e_2, e_5] = -3e_{10} - 2e_6, \quad [e_2, e_6] = -e_{15}, \quad [e_2, e_7] = e_{16}, \quad [e_2, e_8] = 4e_{17}, \quad [e_2, e_9] = -4e_{12},$$
$$[e_2, e_{10}] = -2e_{13} + 2e_{15}, \quad [e_2, e_{11}] = e_{14}, \quad [e_4, e_5] = 2e_2 + 2e_7, \quad [e_4, e_6] = -2e_{16},$$
$$[e_4, e_7] = 2e_8, \quad [e_4, e_9] = -2e_{13} + 2e_{15}, \quad [e_4, e_{10}] = -e_{11} + 4e_{16}, \quad [e_4, e_{11}] = 2e_{17},$$
$$[e_4, e_{13}] = -e_{14}, \quad [e_4, e_{16}] = -e_{17}, \quad [e_5, e_6] = e_9, \quad [e_5, e_7] = -2e_{10} - 2e_6, \quad [e_5, e_8] = e_{11} - 3e_{16},$$
$$[e_5, e_{10}] = -2e_9, \quad [e_5, e_{11}] = -3e_{13}, \quad [e_5, e_{13}] = -2e_{12}, \quad [e_5, e_{15}] = e_{12}, \quad [e_5, e_{16}] = e_{13} - 2e_{15},$$
$$[e_5, e_{17}] = e_{14} \quad [e_6, e_{10}] = -e_{12}, \quad [e_7, e_8] = -3e_{17}, \quad [e_7, e_9] = 2e_{12}, \quad [e_7, e_{10}] = e_{13},$$
$$[e_7, e_{11}] = -e_{14}, \quad [e_8, e_{10}] = -e_{14}.$$

□

Proposition 3.12. *Let $V = \{(x_1, x_2) \in \mathbb{C}^2 : x_1^3 + x_2^5 = 0\}$ be the E_8 singularity and $\mathcal{L}_2^2(V)$ be a derivation Lie algebra. Then*

$$\rho_2^2(V) = 20.$$

Proof. It follows that the local algebra $\mathcal{M}_2^2(V)$ has a monomial basis of the form:

$$\{1, x_1, x_2, x_2^2, x_2^3, x_2^4, x_1x_2, x_1x_2^2, x_1x_2^3, x_1^2, x_1^2x_2, x_1^2x_2^2\}.$$

We obtain following basis of Lie algebra $\mathcal{L}_2^2(V)$:

$e_1 = x_1\partial_1 - 2x_2\partial_2$, $e_2 = -x_2^2\partial_2$, $e_3 = -x_1\partial_1 + x_2\partial_2$, $e_4 = x_1x_2^2\partial_1 + x_2^3\partial_2$, $e_5 = x_1x_2\partial_1 + 2x_2^2\partial_2$, $e_6 = -x_2^3\partial_1$, $e_7 = -x_1\partial_2$, $e_8 = x_1^2\partial_1 - x_1x_2\partial_2$, $e_9 = -x_1^2x_2\partial_1$, $e_{10} = -2x_1^2\partial_1 + x_1x_2\partial_2$, $e_{11} = -x_2^3\partial_2$, $e_{12} = -x_1^2\partial_2$, $e_{13} = -x_1^2x_2\partial_2$, $e_{14} = x_1^2x_2\partial_1 - x_1x_2^2\partial_2$, $e_{15} = -x_1^2x_2^2\partial_2$, $e_{16} = -x_1x_2^3\partial_2$, $e_{17} = -x_2^4\partial_2$, $e_{18} = -x_1^2x_2^2\partial_1$, $e_{19} = -x_1x_2^3\partial_1$, $e_{20} = -x_2^4\partial_1$.

The nilradical $g(V) =< e_2, e_4, e_5, e_6, \cdots, e_{20} >$ have following multiplication table:

$[e_2, e_4] = -e_{17} + 2e_{19}$, $[e_2, e_5] = e_{11} + e_4$, $[e_2, e_6] = 3e_{20}$, $[e_2, e_7] = -2e_{10} - 4e_8$, $[e_2, e_8] = -e_{14} - e_9$, $[e_2, e_9] = e_{18}$, $[e_2, e_{10}] = e_{14} + e_9$, $[e_2, e_{11}] = e_{17}$, $[e_2, e_{12}] = -2e_{13}$, $[e_2, e_{13}] = -e_{15}$, $[e_2, e_{14}] = -e_{18}$, $[e_4, e_5] = -2e_{17} + 3e_{19}$, $[e_4, e_7] = 4e_{14} + 2e_9$, $[e_4, e_8] = 3e_{16} - 3e_{18}$, $[e_4, e_{10}] = -3e_{16} + 4e_{18}$, $[e_4, e_{12}] = 5e_{15}$, $[e_5, e_6] = -7e_{20}$, $[e_5, e_7] = 4e_{10} + 9e_8$, $[e_5, e_8] = 3e_{14} + e_9$, $[e_5, e_9] = -e_{18}$, $[e_5, e_{10}] = -3e_{14}$, $[e_5, e_{11}] = -2e_{17} - e_{19}$, $[e_5, e_{12}] = 6e_{13}$, $[e_5, e_{13}] = 4e_{15}$, $[e_5, e_{14}] = e_{16}$, $[e_6, e_7] = -2e_{11} - 3e_4$, $[e_6, e_8] = e_{17} - 5e_{19}$, $[e_6, e_{10}] = -e_{17} + 7e_{19}$, $[e_6, e_{12}] = 2e_{16} - 3e_{18}$, $[e_7, e_8] = 2e_{12}$, $[e_7, e_9] = -e_{13}$, $[e_7, e_{10}] = -3e_{12}$, $[e_7, e_{11}] = 3e_{14} + 3e_9$, $[e_7, e_{14}] = 3e_{13}$, $[e_7, e_{16}] = 3e_{15}$, $[e_7, e_{17}] = 4e_{16}$, $[e_7, e_{18}] = -e_{15}$, $[e_7, e_{19}] = -e_{16} + 3e_{18}$, $[e_7, e_{20}] = -e_{17} + 4e_{19}$, $[e_8, e_9] = -e_{15}$, $[e_8, e_{10}] = -e_{13}$, $[e_8, e_{11}] = 2e_{16}$, $[e_8, e_{14}] = e_{15}$, $[e_9, e_{10}] = -e_{15}$, $[e_{10}, e_{11}] = -2e_{16}$, $[e_{11}, e_{12}] = -3e_{15}$.

□

Proposition 3.13. *Let $V = \{(x_1, x_2) \in \mathbb{C}^2 : x_1^{k+1} + x_2^2 = 0\}$ be the A_k singularity, $k \geq 1$ and $\mathcal{L}_2(V)$ be a derivation Lie algebra. Then*

$$\rho_2^3(V) = \begin{cases} 6; & k = 1 \\ 4k + 1; & k \geq 2. \end{cases}$$

Proof. It follows that the local algebra

$$\mathcal{M}_2^3(V) = \mathcal{O}^2 \Big/ \Big\langle f, \left(\frac{\partial f}{\partial x_1}\right)^3, \left(\frac{\partial f}{\partial x_2}\right)^3, \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right)^2, \left(\frac{\partial f}{\partial x_2}\right)^2 \frac{\partial^2 f}{\partial x_1^2} + \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial^2 f}{\partial x_2^2} - 2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_1 \partial x_2}\Big\rangle.$$

has a monomial basis of the form:

(1) if $k = 1$, $\{1, x_1, x_2, x_2^2, x_1x_2\}$,

(2) if $k \geq 2$, $\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq k-2, 0 \leq i_2 \leq 2; x_1^{i_1}x_2^{i_2}, k-1 \leq i_1 \leq k, 0 \leq i_2 \leq 1\}$.

Case 1: When $k = 1$, then we obtain following bases of Lie algebra $\mathcal{L}_2^3(V)$:

$e_0 = x_1\partial_1 + x_2\partial_2$, $e_1 = x_2\partial_1$, $e_2 = x_2^2\partial_1$, $e_3 = x_1x_2\partial_1$, $e_4 = x_2^2\partial_2$, $e_5 = x_1x_2\partial_2$.

For A_1 singularity, the nilradical $g(V)=<e_1,e_2,e_3,e_4,e_5>$ have following multiplication table:

$$[e_1,e_3]=e_2,\quad [e_1,e_4]=-e_2,\quad [e_1,e_5]=-e_3+e_4.$$

When $k=2$, then we obtain following bases of Lie algebra $\mathcal{L}_2^3(V)$:

$$e_0=x_1\partial_1+\frac{3x_2}{2}\partial_2,\quad e_1=x_2\partial_1,\quad e_2=x_2^2\partial_1,\quad e_3=x_1x_2\partial_1\quad e_4=x_1^2\partial_1,\quad e_5=x_1^2x_2\partial_1,$$
$$e_6=x_2^2\partial_2,\quad e_7=x_1x_2\partial_2,\quad e_8=x_1^2x_2\partial_2.$$

For A_2 singularity, the nilradical $g(V)=<e_1,e_2,e_3,\cdots,e_8>$ have following multiplication table:

$$[e_1,e_3]=e_2,\quad [e_1,e_4]=2e_3,\quad [e_1,e_6]=-e_2,\quad [e_1,e_7]=-e_3+e_6,\quad [e_1,e_8]=-e_5,$$
$$[e_3,e_4]=e_5,\quad [e_3,e_7]=-e_5,\quad [e_4,e_7]=e_8.$$

When $k=3$, then we obtain following bases of Lie algebra $\mathcal{L}_2^3(V)$:

$$e_0=x_1\partial_1+2x_2\partial_2,\quad e_1=x_2\partial_1,\quad e_2=x_2^2\partial_1,\quad e_3=x_1x_2\partial_1,\quad e_4=x_1x_2^2\partial_1,\quad e_5=x_1^2\partial_1,$$
$$e_6=x_1^2x_2\partial_1,\quad e_7=x_1^3\partial_1,\quad e_8=x_1^3x_2\partial_1,\quad e_9=x_2^2\partial_2,\quad e_{10}=x_1x_2^2\partial_2,\quad e_{11}=x_1^2x_2\partial_2,$$
$$e_{12}=x_1^3x_2\partial_2.$$

For A_3 singularity, the nilradical $g(V)=<e_1,e_2,e_3,\cdots,e_{12}>$ have following multiplication table:

$$[e_1,e_3]=e_2,\quad [e_1,e_5]=2e_3,\quad [e_1,e_6]=2e_4,\quad [e_1,e_7]=3e_6,\quad [e_1,e_9]=-e_2,\quad [e_1,e_{10}]=-e_4,$$
$$[e_1,e_{11}]=-e_6+2e_{10},\quad [e_1,e_{12}]=-e_8,\quad [e_2,e_5]=2e_4,\quad [e_3,e_5]=e_6,\quad [e_3,e_7]=2e_8,$$
$$[e_3,e_9]=-e_4,\quad [e_3,e_{11}]=-e_8,\quad [e_5,e_7]=-e_2,\quad [e_5,e_{11}]=2e_{12}.$$

Case 2: When $k\geq 4$, then we obtain following bases of Lie algebra $\mathcal{L}_2^3(V)$:

$$e_0=x_1\partial_1+\frac{(k+1)x_2}{2}\partial_2,\quad e_1=x_2\partial_1,\quad e_2=x_2^2\partial_1,\quad e_3=x_1x_2\partial_1,\quad e_4=x_1x_2^2\partial_1,\quad e_5=x_1^2\partial_1,$$
$$e_6=x_1^2x_2\partial_1,\quad e_7=x_1^2x_2^2\partial_1,\quad e_8=x_1^3\partial_1,\quad e_9=x_1^3x_2\partial_1,\quad e_{10}=x_1^3x_2^2\partial_1,\quad e_{11}=x_1^4\partial_1,$$
$$e_{12}=x_1^4x_2\partial_1,\quad e_{13}=x_1^4x_2^2\partial_1,\cdots,e_{3k-7}=x_1^{k-2}\partial_1,\quad e_{3k-6}=x_1^{k-2}x_2\partial_1,\quad e_{3k-5}=x_1^{k-2}x_2^2\partial_1,$$
$$e_{3k-4}=x_1^{k-1}\partial_1,\quad e_{3k-3}=x_1^{k-1}x_2\partial_1,\quad e_{3k-2}=x_1^{k}\partial_1,\quad e_{3k-1}=x_1^{k}x_2\partial_1,\quad e_{3k}=x_2^2\partial_2,$$
$$e_{3k+1}=x_1x_2^2\partial_2,\quad e_{3k+2}=x_1^2x_2^2\partial_2,\quad e_{3k+3}=x_1^3x_2^2\partial_2,\cdots,e_{4k-2}=x_1^{k-2}x_2^2\partial_2,\quad e_{4k-1}=x_1^{k-1}x_2\partial_2,$$
$$e_{4k}=x_1^{k}x_2\partial_2.$$

For A_4 singularity, the nilradical $g(V)=<e_1,e_2,e_3,\cdots,e_{16}>$ have following multiplication table:

$$[e_1,e_3]=e_2,\quad [e_1,e_5]=2e_3,\quad [e_1,e_6]=2e_4,\quad [e_1,e_8]=3e_6,\quad [e_1,e_9]=3e_7,\quad [e_1,e_{10}]=4e_9,$$
$$[e_1,e_{12}]=-e_2,\quad [e_1,e_{13}]=-e_4,\quad [e_1,e_{14}]=-e_7,\quad [e_1,e_{15}]=-e_9+3e_{14},\quad [e_1,e_{16}]=-e_{11},$$
$$[e_2,e_5]=2e_4,\quad [e_2,e_8]=3e_7,\quad [e_3,e_5]=e_6,\quad [e_3,e_6]=e_7,\quad [e_3,e_8]=2e_9,\quad [e_3,e_{10}]=3e_{11},$$
$$[e_3,e_{12}]=-e_4,\quad [e_3,e_{13}]=-e_7,\quad [e_3,e_{15}]=-e_{11},\quad [e_4,e_5]=e_7,\quad [e_5,e_8]=e_{10},$$
$$[e_5,e_9]=e_{11},\quad [e_5,e_{10}]=-2e_2,\quad [e_5,e_{13}]=e_{14},\quad [e_5,e_{15}]=3e_{16},\quad [e_6,e_8]=e_{11},$$
$$[e_6,e_{12}]=-e_7,\quad [e_8,e_{10}]=-e_4.$$

For A_5 singularity, the nilradical $g(V)=<e_1,e_2,e_3,\cdots,e_{20}>$ have following multiplication table:

$$[e_1,e_3]=e_2,\quad [e_1,e_5]=2e_3,\quad [e_1,e_6]=2e_4,\quad [e_1,e_8]=3e_6,\quad [e_1,e_9]=3e_7,\quad [e_1,e_{11}]=4e_9,$$

$[e_1, e_{12}] = 4e_{10}, \quad [e_1, e_{13}] = 5e_{12}, \quad [e_1, e_{15}] = -e_2, \quad [e_1, e_{16}] = -e_4, \quad [e_1, e_{17}] = -e_7,$
$[e_1, e_{18}] = -e_{10}, \quad [e_1, e_{19}] = -e_{12} + 4e_{18}, \quad [e_1, e_{20}] = -e_{14}, \quad [e_2, e_5] = 2e_4, \quad [e_2, e_8] = 3e_7,$
$[e_2, e_{11}] = 4e_{10}, \quad [e_3, e_5] = e_6, \quad [e_3, e_6] = e_7, \quad [e_3, e_8] = 2e_9, \quad [e_3, e_9] = 2e_{10}, \quad [e_3, e_{11}] = 3e_{12},$
$[e_3, e_{13}] = 4e_{14}, \quad [e_3, e_{15}] = -e_4, \quad [e_3, e_{16}] = -e_7, \quad [e_3, e_{17}] = -e_{10}, \quad [e_3, e_{19}] = -e_{14},$
$[e_4, e_5] = e_7, \quad [e_4, e_8] = 2e_{10}, \quad [e_5, e_8] = e_{11}, \quad [e_5, e_9] = e_{12}, \quad [e_5, e_{11}] = 2e_{13}, \quad [e_5, e_{12}] = 2e_{14},$
$[e_5, e_{13}] = -3e_2, \quad [e_5, e_{16}] = e_{17}, \quad [e_5, e_{17}] = 2e_{18}, \quad [e_5, e_{19}] = 4e_{20}, \quad [e_6, e_8] = e_{12}, \quad [e_6, e_{11}]$
$= 2e_{14}, \quad [e_6, e_{15}] = -e_7, \quad [e_6, e_{16}] = -e_{10}, \quad [e_8, e_{11}] = -e_2, \quad [e_8, e_{13}] = -2e_4, \quad [e_8, e_{16}] = e_{18},$
$[e_9, e_{15}] = -e_{10}, \quad [e_{11}, e_{13}] = -e_7.$

For A_k $(k \geq 6)$ singularity, the nilradical $g(V) =< e_1, e_2, e_3, \cdots, e_{4k} >$ have following multiplication table:

$[e_1, e_3] = e_2, \quad [e_1, e_5] = 2e_3, \quad [e_1, e_6] = 2e_4, \quad [e_1, e_8] = 3e_6, \quad [e_1, e_9] = 3e_7, \quad [e_1, e_{11}] = 4e_9,$
$[e_1, e_{12}] = 4e_{10}, \cdots, [e_1, e_{3k-4}] = (k-1)e_{3k-6}, \quad [e_1, e_{3k-3}] = (k-1)e_{3k-5}, \quad [e_1, e_{3k-2}] = ke_{3k-3},$
$[e_1, e_{3k}] = -e_2, \quad [e_1, e_{3k+1}] = -e_4, \quad [e_1, e_{3k+2}] = -e_7, \quad [e_1, e_{3k+3}] = -e_{10}, \cdots, [e_1, e_{4k-2}]$
$= -e_{3k-5}, \quad [e_1, e_{4k-1}] = -e_{3k-3} + (k-1)e_{4k-2}, \quad [e_1, e_{4k}] = -e_{3k-1}, \quad [e_2, e_5] = 2e_4, \quad [e_2, e_8] =$
$3e_7, \quad [e_2, e_{11}] = 4e_{10}, \cdots, [e_2, e_{3k-4}] = (k-1)e_{3k-5}, \quad [e_3, e_5] = e_6, \quad [e_3, e_6] = e_7, \quad [e_3, e_8] = 2e_9,$
$[e_3, e_9] = 2e_{10}, \quad [e_3, e_{11}] = 3e_{12}, \quad [e_3, e_{12}] = 3e_{13}, \cdots, [e_3, e_{3k-4}] = (k-2)e_{3k-3}, \quad [e_3, e_{3k-2}] =$
$(k-1)e_{3k-1}, \quad [e_3, e_{3k}] = -e_4, \quad [e_3, e_{3k+1}] = -e_7, \quad [e_3, e_{3k+2}] = -e_{10}, \cdots, [e_3, e_{4k-3}] = -e_{3k-5},$
$[e_3, e_{4k-1}] = -e_{3k-1}, \quad [e_4, e_5] = e_7, \quad [e_4, e_8] = 2e_{10}, \quad [e_4, e_{11}] = 3e_{13}, \cdots, [e_4, e_{3k-7}] = (k-3)e_{3k-5},$
$[e_5, e_8] = e_{11}, \quad [e_5, e_9] = e_{12}, \quad [e_5, e_{10}] = e_{13}, \quad [e_5, e_{11}] = 2e_{14}, \quad [e_5, e_{12}] = 2e_{15}, \quad [e_5, e_{13}] = 2e_{16},$
$[e_5, e_{14}] = 3e_{17}, \quad [e_5, e_{15}] = 3e_{18}, \quad [e_5, e_{16}] = 3e_{19}, \cdots, [e_5, e_{3k-7}] = (k-4)e_{3k-4}, \quad [e_5, e_{3k-6}] =$
$(k-4)e_{3k-3}, \quad [e_5, e_{3k-4}] = (k-3)e_{3k-2}, \quad [e_5, e_{3k-3}] = (k-3)e_{3k-1}, \quad [e_5, e_{3k-2}] = -(k-2)e_2,$
$[e_5, e_{3k+1}] = e_{3k+2}, \quad [e_5, e_{3k+2}] = 2e_{3k+3}, \quad [e_5, e_{3k+3}] = 3e_{3k+4}, \cdots, [e_5, e_{4k-3}] = (k-3)e_{4k-2},$
$[e_5, e_{4k-1}] = (k-1)e_{4k}, \quad [e_6, e_8] = e_{12}, \quad [e_6, e_9] = e_{13}, \quad [e_6, e_{11}] = 2e_{15}, \quad [e_6, e_{12}] = 2e_{16},$
$[e_6, e_{14}] = 3e_{18}, \quad [e_6, e_{15}] = 3e_{19}, \cdots, [e_6, e_{3k-7}] = (k-4)e_{3k-3}, \quad [e_6, e_{3k-4}] = (k-3)e_{3k-1},$
$[e_6, e_{3k}] = -e_7, \quad [e_6, e_{3k+1}] = -e_{10}, \quad [e_6, e_{3k+2}] = -e_{13}, \cdots, [e_6, e_{4k-4}] = -e_{3k-5}, \quad [e_7, e_8] = e_{13},$
$[e_7, e_{11}] = 2e_{16}, \quad [e_7, e_{14}] = 3e_{19}, \cdots, [e_7, e_{3k-10}] = (k-5)e_{3k-5}, \quad [e_8, e_{11}] = e_{17}, \quad [e_8, e_{12}] = e_{18},$
$[e_8, e_{13}] = e_{19}, \quad [e_8, e_{14}] = 2e_{20}, \quad [e_8, e_{15}] = 2e_{21}, \quad [e_8, e_{16}] = 2e_{22}, \quad [e_8, e_{17}] = 3e_{23}, \quad [e_8, e_{18}] =$
$3e_{24}, \quad [e_8, e_{19}] = 3e_{25}, \cdots, [e_8, e_{3k-10}] = (k-6)e_{3k-4}, \quad [e_8, e_{3k-9}] = (k-6)e_{3k-3}, \quad [e_8, e_{3k-7}] =$
$(k-5)e_{3k-2}, \quad [e_8, e_{3k-6}] = (k-5)e_{3k-1}, \quad [e_8, e_{3k-4}] = -(k-4)e_2 \quad [e_8, e_{3k-2}] = -(k-3)e_4,$
$[e_8, e_{3k+1}] = e_{3k+3}, \quad [e_8, e_{3k+2}] = 2e_{3k+4}, \quad [e_8, e_{3k+3}] = 3e_{3k+5}, \cdots, [e_8, e_{4k-4}] = (k-4)e_{4k-2},$
$[e_9, e_{11}] = e_{18}, \quad [e_9, e_{12}] = e_{19} \quad [e_9, e_{14}] = 2e_{21}, \quad [e_9, e_{15}] = 2e_{22}, \quad [e_9, e_{17}] = 3e_{24}, \quad [e_9, e_{18}] =$
$3e_{25}, \cdots, [e_9, e_{3k-10}] = (k-6)e_{3k-3}, \quad [e_9, e_{3k-7}] = (k-5)e_{3k-1}, \quad [e_9, e_{3k}] = -e_{10}, \quad [e_9, e_{3k+1}] =$
$-e_{13}, \quad [e_9, e_{3k+2}] = -e_{16}, \cdots, [e_9, e_{4k-5}] = -e_{3k-5}, \quad [e_{10}, e_{11}] = e_{19}, \quad [e_{10}, e_{14}] = 2e_{22}, \quad [e_{10}, e_{17}]$
$= 3e_{25}, \cdots, [e_{10}, e_{3k-13}] = (k-7)e_{3k-5}, \quad [e_{11}, e_{14}] = e_{22}, \quad [e_{11}, e_{15}] = e_{23}, \quad [e_{11}, e_{16}] = e_{24},$
$[e_{11}, e_{17}] = 2e_{25}, \quad [e_{11}, e_{18}] = 2e_{26}, \quad [e_{11}, e_{19}] = 2e_{27}, \quad [e_{11}, e_{20}] = 3e_{28}, \quad [e_{11}, e_{21}] = 3e_{29},$
$[e_{11}, e_{22}] = 3e_{30}, \cdots, [e_{11}, e_{3k-13}] = (k-8)e_{3k-4}, \quad [e_{11}, e_{3k-12}] = (k-8)e_{3k-3}, \quad [e_{11}, e_{3k-10}] =$
$(k-7)e_{3k-2}, \quad [e_{11}, e_{3k-9}] = (k-7)e_{3k-1}, \quad [e_{11}, e_{3k-7}] = -(k-6)e_2, \quad [e_{11}, e_{3k-4}] = -(k-5)e_4,$
$[e_{11}, e_{3k-2}] = -(k-4)e_7, \quad [e_{11}, e_{3k+1}] = e_{3k+4}, \quad [e_{11}, e_{3k+2}] = 2e_{3k+5}, \quad [e_{11}, e_{3k+3}] = 3e_{3k+6}, \cdots,$

$[e_{11}, e_{4k-5}] = (k-5)e_{4k-2}, \quad [e_{12}, e_{14}] = e_{23}, \quad [e_{12}, e_{15}] = e_{24}, \quad [e_{12}, e_{17}] = 2e_{26}, \quad [e_{12}, e_{18}] = 2e_{27},$
$[e_{12}, e_{20}] = 3e_{29}, \quad [e_{12}, e_{21}] = 3e_{30}, \cdots, [e_{12}, e_{3k-13}] = (k-8)e_{3k-3}, \quad [e_{12}, e_{3k-10}] = (k-7)e_{3k-1},$
$[e_{12}, e_{3k}] = -e_{13} \quad [e_{12}, e_{3k+1}] = -e_{16}, \quad [e_{12}, e_{3k+2}] = -e_{19}, \cdots, [e_{12}, e_{4k-6}] = -e_{3k-5},$

$$\vdots$$

$[e_{3k-7}, e_{3k-4}] = -e_{3k-14}, \quad [e_{3k-7}, e_{3k-2}] = -2e_{3k-11}, \quad [e_{3k-7}, e_{3k+1}] = e_{4k-2},$
$[e_{3k-6}, e_{3k}] = -e_{3k-5}, \quad [e_{3k-4}, e_{3k-2}] = -e_{3k-8}.$

□

Proposition 3.14. *Let $V = \{(x_1, x_2) \in \mathbb{C}^2 : x_1^2 x_2 + x_2^{k-1} = 0\}$ be the D_k singularity, $k \geq 4$ and $\mathcal{L}_2^3(V)$ be a derivation Lie algebra. Then*

$$\rho_2^3(V) = \begin{cases} 19; & k = 4 \\ 4k+2; & k \geq 5. \end{cases}$$

Proof. It follows that the local algebra

$$\mathcal{M}_2^3(V) = \mathcal{O}^2 \Big/ \Big\langle f, \left(\frac{\partial f}{\partial x_1}\right)^3, \left(\frac{\partial f}{\partial x_2}\right)^3, \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right)^2, \left(\frac{\partial f}{\partial x_2}\right)^2 \frac{\partial^2 f}{\partial x_1^2} + \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial^2 f}{\partial x_2^2} - 2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_1 \partial x_2}\Big\rangle.$$

has a monomial basis of the form:

(1) if $k = 4$, $\{x_1^{i_1} x_2^{i_2}, 0 \leq i_1 \leq 4, 0 \leq i_2 \leq 1; x_2^5; x_1^2; x_1^3; x_1^4; x_1^5\}$,

(2) if $k \geq 5$, $\{x_1^{i_1} x_2^{i_2}, 0 \leq i_1 \leq k, 0 \leq i_2 \leq 1; x_2^{i_2}, k+1 \leq i_2 \leq 2k-4; x_1^2; x_1^3; x_1^4; x_1^5\}$.

Case 1: When $k = 4$, then we obtain following bases of Lie algebra $\mathcal{L}_2^3(V)$:

$e_0 = x_1\partial_1 + x_2\partial_2, \quad e_1 = x_2^2\partial_1, \quad e_2 = x_2^3\partial_1, \quad e_3 = x_2^4\partial_1, \quad e_4 = x_2^5\partial_1, e_5 = x_1x_2\partial_1,$
$e_6 = x_1x_2^2\partial_1, \quad e_7 = x_1x_2^3\partial_1, \quad e_8 = x_1x_2^4\partial_1, \quad e_9 = x_1^2\partial_1, \quad e_{10} = x_1^3\partial_1 e_{11} = x_1^4\partial_1, \quad e_{12} = x_1^5\partial_1,$
$e_{13} = x_2^4\partial_2, \quad e_{14} = x_2^5\partial_2, \quad e_{15} = x_1x_2^3\partial_2, \quad e_{16} = x_1x_2^4\partial_2, \quad e_{17} = x_1^4\partial_2, \quad e_{18} = x_1^5\partial_2.$

For D_4 singularity, the nilradical $g(V) =< e_1, e_2, e_3, \cdots, e_{18} >$ have following multiplication table:

$[e_1, e_5] = e_2, \quad [e_1, e_6] = e_3, \quad [e_1, e_7] = e_4, \quad [e_1, e_9] = 2e_6, \quad [e_1, e_{10}] = -3e_3, \quad [e_1, e_{11}] = -4e_8,$
$[e_1, e_{13}] = -2e_4, \quad [e_1, e_{15}] = -2e_8 + e_{14}, \quad [e_1, e_{17}] = -4e_{16}, \quad [e_2, e_5] = e_3, \quad [e_2, e_6] = e_4,$
$[e_2, e_9] = 2e_7, \quad [e_2, e_{10}] = -3e_4, \quad [e_3, e_5] = e_4, \quad [e_3, e_9] = 2e_8, \quad [e_5, e_9] = -e_2, \quad [e_5, e_{10}] = -2e_7,$
$[e_5, e_{13}] = -e_8, \quad [e_5, e_{15}] = e_4 + e_{16}, \quad [e_5, e_{17}] = -e_{12}, \quad [e_6, e_9] = -e_3, \quad [e_6, e_{10}] = -2e_8,$
$[e_7, e_9] = -e_4, \quad [e_9, e_{10}] = e_{11}, \quad [e_9, e_{11}] = 2e_{12}, \quad [e_9, e_{15}] = -e_{14}, \quad [e_9, e_{17}] = 4e_{18}.$

Case 2: When $k \geq 5$, then we obtain following bases of Lie algebra $\mathcal{L}_2^3(V)$:

$e_0 = x_1\partial_1 + \frac{2x_2}{k-2}\partial_2, \quad e_1 = x_2^{k-2}\partial_1, \quad e_2 = x_2^{k-1}\partial_1, \quad e_3 = x_2^k\partial_1, \cdots, e_{k-1} = x_2^{2k-4}\partial_1,$
$e_k = x_1x_2\partial_1, \quad e_{k+1} = x_1x_2^2\partial_1, \quad e_{k+2} = x_1x_2^3\partial_1, \cdots, e_{2k-1} = x_1x_2^k\partial_1, \quad e_{2k} = x_1^2\partial_1, \quad e_{2k+1} = x_1^3\partial_1,$
$e_{2k+2} = x_1^4\partial_1, \quad e_{2k+3} = x_1^5\partial_1, \quad e_{2k+4} = x_2^{k-1}\partial_2, \quad e_{2k+5} = x_2^k\partial_2, \quad e_{2k+6} = x_2^{k+1}\partial_2, \cdots,$
$e_{3k+1} = x_2^{2k-4}\partial_2, \quad e_{3k+2} = x_1x_2^3\partial_2, \quad e_{3k+3} = x_1x_2^4\partial_2, \quad e_{3k+4} = x_1x_2^5\partial_2, \cdots, e_{4k-1} = x_1x_2^k\partial_2,$

$e_{4k} = x_1^4\partial_2, \quad e_{4k+1} = x_1^5\partial_2.$

For D_5 singularity, the nilradical $g(V) =< e_1, e_2, e_3, \cdots, e_{21} >$ have following multiplication table:

$[e_1, e_5] = e_2, \quad [e_1, e_6] = e_3, \quad [e_1, e_7] = e_4, \quad [e_1, e_{10}] = 2e_7, \quad [e_1, e_{11}] = -3e_4, \quad [e_1, e_{14}] = -3e_4,$
$[e_1, e_{17}] = -3e_9 + e_{16}, \quad [e_2, e_5] = e_3, \quad [e_2, e_6] = e_4, \quad [e_2, e_{10}] = 2e_8, \quad [e_3, e_5] = e_4,$
$[e_3, e_{10}] = 2e_9, \quad [e_5, e_{10}] = -e_2, \quad [e_5, e_{11}] = -2e_8, \quad [e_5, e_{14}] = -e_8, \quad [e_5, e_{15}] = -e_9,$
$[e_5, e_{17}] = e_4 + e_{18}, \quad [e_5, e_{18}] = e_{19}, \quad [e_5, e_{20}] = -e_{13}, \quad [e_6, e_{10}] = -e_3, \quad [e_6, e_{11}] = -2e_9,$
$[e_6, e_{14}] = -2e_9, \quad [e_6, e_{17}] = e_{19}, \quad [e_7, e_{10}] = -e_4, \quad [e_{10}, e_{11}] = e_{12}, \quad [e_{10}, e_{12}] = 2e_{13},$
$[e_{10}, e_{17}] = -e_{16}, \quad [e_{10}, e_{20}] = 4e_{21}.$

For D_6 singularity, the nilradical $g(V) =< e_1, e_2, e_3, \cdots, e_{25} >$ have following multiplication table:

$[e_1, e_6] = e_2, \quad [e_1, e_7] = e_3, \quad [e_1, e_8] = e_4, \quad [e_1, e_9] = e_5, \quad [e_1, e_{12}] = 2e_9, \quad [e_1, e_{13}] = -3e_5,$
$[e_1, e_{16}] = -4e_5, \quad [e_1, e_{20}] = -4e_{11} + e_{18}, \quad [e_1, e_{21}] = e_{19}, \quad [e_2, e_6] = e_3, \quad [e_2, e_7] = e_4,$
$[e_2, e_8] = e_5, \quad [e_2, e_{12}] = 2e_{10}, \quad [e_2, e_{20}] = e_{19}, \quad [e_3, e_6] = e_4, \quad [e_3, e_7] = e_5,$
$[e_3, e_{12}] = 2e_{11}, \quad [e_4, e_6] = e_5, \quad [e_6, e_{12}] = -e_2, \quad [e_6, e_{13}] = -2e_{10}, \quad [e_6, e_{16}] = -e_{10},$
$[e_6, e_{17}] = -e_{11}, \quad [e_6, e_{20}] = e_4 + e_{21}, \quad [e_6, e_{21}] = e_5 + e_{22}, \quad [e_6, e_{22}] = e_{23}, \quad [e_6, e_{24}] = -e_{15},$
$[e_7, e_{12}] = -e_3, \quad [e_7, e_{13}] = -2e_{11} \quad [e_7, e_{16}] = -2e_{11}, \quad [e_7, e_{20}] = 2e_5 + e_{22}, \quad [e_7, e_{21}] = e_{23},$
$[e_8, e_{12}] = -e_4, \quad [e_8, e_{20}] = e_{23}, \quad [e_9, e_{12}] = -e_5, \quad [e_{12}, e_{13}] = e_{14}, \quad [e_{12}, e_{14}] = 2e_{15},$
$[e_{12}, e_{20}] = -e_{18}, \quad [e_{12}, e_{21}] = -e_{19}, \quad [e_{12}, e_{24}] = 4e_{25}.$

For D_k $(k \geq 7)$ singularity, the nilradical $g(V) =< e_1, e_2, e_3, \cdots, e_{4k+1} >$ have following multiplication table:

$[e_1, e_k] = e_2, \quad [e_1, e_{k+1}] = e_3, \quad [e_1, e_{k+2}] = e_4, \cdots, [e_1, e_{2k-3}] = e_{k-1}, \quad [e_1, e_{2k}] = 2e_{2k-3},$
$[e_1, e_{2k+1}] = -3e_{k-1}, \quad [e_1, e_{2k+4}] = -(k-2)e_{k-1}, \quad [e_1, e_{3k+2}] = -(k-2)e_{2k-1} + e_{2k+6},$
$[e_1, e_{3k+3}] = e_{2k+7}, \quad [e_1, e_{3k+4}] = e_{2k+8}, \quad [e_1, e_{3k+5}] = e_{2k+9}, \cdots, [e_1, e_{4k-3}] = e_{3k+1}, \quad [e_2, e_k]$
$= e_3, [e_2, e_{k+1}] = e_4, \quad [e_2, e_{k+2}] = e_5, \cdots, [e_2, e_{2k-4}] = e_{k-1}, \quad [e_2, e_{2k}] = 2e_{2k-2}, \quad [e_2, e_{3k+2}]$
$= e_{2k+7}, \quad [e_2, e_{3k+3}] = e_{2k+8}, \quad [e_2, e_{3k+4}] = e_{2k+9}, \cdots, [e_2, e_{4k-4}] = e_{3k+1}, \quad [e_3, e_k] = e_4,$
$[e_3, e_{k+1}] = e_5, \quad [e_3, e_{k+2}] = e_6, \cdots, [e_3, e_{2k-5}] = e_{k-1}, \quad [e_3, e_{2k}] = 2e_{2k-1}, \quad [e_3, e_{3k+2}] = e_{2k+8},$
$[e_3, e_{3k+3}] = e_{2k+9}, \quad [e_3, e_{3k+4}] = e_{2k+10}, \cdots, [e_3, e_{4k-5}] = e_{3k+1}, \quad [e_4, e_k] = e_5, \quad [e_4, e_{k+1}] = e_6,$
$[e_4, e_{k+2}] = e_7, \cdots, [e_4, e_{2k-6}] = e_{k-1}, \quad [e_4, e_{3k+2}] = e_{2k+9}, \quad [e_4, e_{3k+3}] = e_{2k+10}, \quad [e_4, e_{3k+4}]$
$= e_{2k+11}, \cdots, [e_4, e_{4k-6}] = e_{3k+1} \quad [e_5, e_k] = e_6, \quad [e_5, e_{k+1}] = e_7, \quad [e_5, e_{k+2}] = e_8, \cdots, [e_5, e_{2k-7}]$
$= e_{k-1}, [e_5, e_{3k+2}] = e_{2k+10}, \quad [e_5, e_{3k+3}] = e_{2k+11}, \quad [e_5, e_{3k+4}] = e_{2k+12}, \cdots, [e_5, e_{4k-7}] = e_{3k+1}$
$[e_6, e_k] = e_7, \quad [e_6, e_{k+1}] = e_8, \quad [e_6, e_{k+2}] = e_9, \cdots, [e_6, e_{2k-8}] = e_{k-1},$
$[e_6, e_{3k+2}] = e_{2k+11}, \quad [e_6, e_{3k+3}] = e_{2k+12}, \quad [e_6, e_{3k+4}] = e_{2k+13}, \cdots, [e_6, e_{4k-8}] = e_{3k+1}$

$\vdots$

$[e_{k-3}, e_k] = e_{k-2}, \quad [e_{k-3}, e_{k+1}] = e_{k-1}, \quad [e_{k-2}, e_k] = e_{k-1},$
$[e_k, e_{2k}] = -e_2, \quad [e_k, e_{2k+1}] = -2e_{2k-2}, \quad [e_k, e_{2k+4}] = -e_{2k-2}, \quad [e_k, e_{2k+5}] = -e_{2k-1},$

$[e_k, e_{3k+2}] = e_4 + e_{3k+3}, \quad [e_k, e_{3k+3}] = e_5 + e_{3k+4}, \quad [e_k, e_{3k+4}] = e_6 + e_{3k+5}, \cdots, [e_k, e_{4k-3}] = e_{k-1} + e_{4k-2}, \quad [e_k, e_{4k-2}] = e_{4k-1}, \quad [e_k, e_{4k}] = -e_{2k+3}, \quad [e_{k+1}, e_{2k}] = -e_3, \quad [e_{k+1}, e_{2k+1}] = -2e_{2k-1}, [e_{k+1}, e_{2k+4}] = -2e_{2k-1}, \quad [e_{k+1}, e_{3k+2}] = 2e_5 + e_{3k+4}, \quad [e_{k+1}, e_{3k+3}] = 2e_6 + e_{3k+5}, [e_{k+1}, e_{3k+4}] = 2e_7 + e_{3k+6}, \cdots, [e_{k+1}, e_{4k-4}] = 2e_{k-1} + e_{4k-2}, \quad [e_{k+1}, e_{4k-3}] = e_{4k-1}, \quad [e_{k+2}, e_{2k}] = -e_4, [e_{k+2}, e_{3k+2}] = 3e_6 + e_{3k+5}, \quad [e_{k+2}, e_{3k+3}] = 3e_7 + e_{3k+6}, \quad [e_{k+2}, e_{3k+4}] = 3e_8 + e_{3k+7}, \cdots, [e_{k+2}, e_{4k-4}] = e_{4k-1}, \quad [e_{k+3}, e_{2k}] = -e_5, \quad [e_{k+3}, e_{3k+2}] = 4e_7 + e_{3k+6}, \quad [e_{k+3}, e_{3k+3}] = 4e_8 + e_{3k+7}, [e_{k+3}, e_{3k+4}] = 4e_9 + e_{3k+8}, \cdots, [e_{k+3}, e_{4k-5}] = e_{4k-1}, \quad [e_{k+4}, e_{2k}] = -e_6, \quad [e_{k+4}, e_{3k+2}] = 5e_8 + e_{3k+7}, [e_{k+4}, e_{3k+3}] = 5e_9 + e_{3k+8}, \quad [e_{k+4}, e_{3k+4}] = 5e_{10} + e_{3k+9}, \cdots, [e_{k+4}, e_{4k-6}] = e_{4k-1},$

$\vdots$

$[e_{2k-3}, e_{2k}] = -e_{k-1}, \quad [e_{2k}, e_{2k+1}] = e_{2k+2}, \quad [e_{2k}, e_{2k+2}] = 2e_{2k+3}, \quad [e_{2k}, e_{3k+2}] = -e_{2k+6}, [e_{2k}, e_{3k+3}] = -e_{2k+7}, \quad [e_{2k}, e_{3k+4}] = -e_{2k+8}, \cdots, [e_{2k}, e_{4k-3}] = -e_{3k+1}, \quad [e_{2k}, e_{4k}] = 4e_{4k+1} [e_{3k+2}, e_{3k+3}] = -e_{2k+9}, \quad [e_{3k+2}, e_{3k+4}] = -2e_{2k+10}, \quad [e_{3k+2}, e_{3k+5}] = -3e_{2k+11}, \cdots, [e_{3k+2}, e_{4k-5}] = -(k-7)e_{3k+1}.$

□

Proposition 3.15. *Let $V = \{(x_1, x_2) \in \mathbb{C}^2 : x_1^3 + x_2^4 = 0\}$ be the E_6 singularity and $\mathcal{L}_2^3(V)$ be a derivation Lie algebra. Then*

$$\rho_2^3(V) = 26.$$

Proof. It follows that the local algebra

$$\mathcal{M}_2^3(V) = \mathcal{O}^2 \Big/ \Big\langle f, \left(\frac{\partial f}{\partial x_1}\right)^3, \left(\frac{\partial f}{\partial x_2}\right)^3, \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right)^2, \left(\frac{\partial f}{\partial x_2}\right)^2 \frac{\partial^2 f}{\partial x_1^2} + \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial^2 f}{\partial x_2^2} - 2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_1 \partial x_2}\Big\rangle.$$

has a monomial basis of the form:

$$\{1, x_2, x_2^2, x_2^3, \cdots, x_2^7, x_1, x_1x_2, x_1x_2^2, \cdots, x_1x_2^5, x_1^2, x_1^2x_2, x_1^2x_2^2, \cdots, x_1^2x_2^5\}.$$

We obtain following bases of Lie algebra $\mathcal{L}_2^3(V)$:

$e_0 = x_1\partial_1 + \frac{3x_2}{4}\partial_2, \quad e_1 = x_2^3\partial_1, \quad e_2 = x_2^4\partial_1, \quad e_3 = x_2^5\partial_1, \quad e_4 = x_2^6\partial_1, \quad e_5 = x_2^7\partial_1,$
$e_6 = x_1x_2\partial_1, \quad e_7 = x_1x_2^2\partial_1, \quad e_8 = x_1x_2^3\partial_1, \quad e_9 = x_1x_2^4\partial_1, \quad e_{10} = x_1x_2^5\partial_1, \quad e_{11} = x_1^2\partial_1,$
$e_{12} = x_1^2x_2\partial_1, \quad e_{13} = x_1^2x_2^2\partial_1, \quad e_{14} = x_1^2x_2^3\partial_1, \quad e_{15} = x_1^2x_2^4\partial_1, \quad e_{16} = x_1^2x_2^5\partial_1, \quad e_{17} = x_2^5\partial_2,$
$e_{18} = x_2^6\partial_2, \quad e_{19} = x_2^7\partial_2, \quad e_{20} = x_1x_2^3\partial_2, \quad e_{21} = x_1x_2^4\partial_2, \quad e_{22} = x_1x_2^5\partial_2, \quad e_{23} = x_1^2x_2^3\partial_2,$
$e_{24} = x_1^2x_2^4\partial_2, \quad e_{25} = x_1^2x_2^5\partial_2.$

The nilradical $g(V) =< e_1, e_2, e_3, \cdots, e_{25} >$ have following multiplication table:

$[e_1, e_6] = e_2, \quad [e_1, e_7] = e_3, \quad [e_1, e_8] = e_4, \quad [e_1, e_9] = e_5, \quad [e_1, e_{11}] = 2e_8, \quad [e_1, e_{12}] = 2e_9,$
$[e_1, e_{13}] = 2e_{10}, \quad [e_1, e_{17}] = -3e_5, \quad [e_1, e_{20}] = -3e_{10} + e_{18}, \quad [e_1, e_{21}] = e_{19}, \quad [e_1, e_{23}] = -3e_{16},$
$[e_2, e_6] = e_3, \quad [e_2, e_7] = e_4, \quad [e_2, e_8] = e_5, \quad [e_2, e_{11}] = 2e_9, \quad [e_2, e_{12}] = 2e_{10}, \quad [e_2, e_{20}] = e_{19},$
$[e_3, e_6] = e_4, \quad [e_3, e_7] = e_5, \quad [e_3, e_{11}] = 2e_{10}, \quad [e_4, e_6] = e_5, \quad [e_6, e_{11}] = e_{12}, \quad [e_6, e_{12}] = e_{13},$

$[e_6, e_{13}] = e_{14}, \quad [e_6, e_{14}] = e_{15}, \quad [e_6, e_{15}] = e_{16}, \quad [e_6, e_{17}] = -e_{10}, \quad [e_6, e_{20}] = -e_{14} + e_{21},$
$[e_6, e_{21}] = -e_{15} + e_{22}, \quad [e_6, e_{22}] = -e_{16}, \quad [e_6, e_{23}] = e_5 + 2e_{24}, \quad [e_6, e_{24}] = 2e_{25},$
$[e_7, e_{11}] = e_{13}, \quad [e_7, e_{12}] = e_{14}, \quad [e_7, e_{13}] = e_{15}, \quad [e_7, e_{14}] = e_{16}, \quad [e_7, e_{20}] = -2e_{15} + e_{22},$
$[e_7, e_{21}] = -2e_{16}, \quad [e_7, e_{23}] = 2e_{25}, \quad [e_8, e_{11}] = e_{14}, \quad [e_8, e_{12}] = e_{15}, \quad [e_8, e_{13}] = e_{16},$
$[e_8, e_{13}] = e_{16}, \quad [e_8, e_{20}] = -3e_{16}, \quad [e_9, e_{11}] = e_{15}, \quad [e_9, e_{12}] = e_{16}, \quad [e_{10}, e_{11}] = e_{16},$
$[e_{11}, e_{20}] = e_{23}, \quad [e_{11}, e_{21}] = e_{24}, \quad [e_{11}, e_{22}] = e_{25}, \quad [e_{11}, e_{23}] = -2e_{19}, \quad [e_{12}, e_{17}] = -e_{16},$
$[e_{12}, e_{20}] = e_5 + e_{24}, \quad [e_{12}, e_{21}] = e_{25}, \quad [e_{13}, e_{20}] = e_{25}.$

□

Proposition 3.16. *Let $V = \{(x_1, x_2) \in \mathbb{C}^2 : x_1^3 + x_2^3 x_1 = 0\}$ be the E_7 singularity and $\mathcal{L}_2^3(V)$ be a derivation Lie algebra. Then*

$$\rho_2^3(V) = 30.$$

Proof. It follows that the local algebra

$$\mathcal{M}_2^3(V) = \mathcal{O}^2 \Big/ \Big\langle f, \left(\frac{\partial f}{\partial x_1}\right)^3, \left(\frac{\partial f}{\partial x_2}\right)^3, \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right)^2, \left(\frac{\partial f}{\partial x_2}\right)^2 \frac{\partial^2 f}{\partial x_1^2} + \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial^2 f}{\partial x_2^2} - 2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_1 \partial x_2}\Big\rangle.$$

has a monomial basis of the form:

$$\{1, x_2, x_2^2, x_2^3, \cdots, x_2^9, x_1, x_1x_2, x_1x_2^2, \cdots, x_1x_2^6, x_1^2, x_1^2x_2, x_1^2x_2^2, \cdots, x_1^2x_2^5\}.$$

We obtain following bases of Lie algebra $\mathcal{L}_2^3(V)$:

$e_0 = x_1\partial_1 + \frac{2x_2}{3}\partial_2, \quad e_1 = x_2^6\partial_1, \quad e_2 = x_2^7\partial_1, \quad e_3 = x_2^8\partial_1, \quad e_4 = x_2^9\partial_1, \quad e_5 = x_1x_2\partial_1,$
$e_6 = x_1x_2^2\partial_1, \quad e_7 = x_1x_2^3\partial_1, \quad e_8 = x_1x_2^4\partial_1, \quad e_9 = x_1x_2^5\partial_1, \quad e_{10} = x_1x_2^6\partial_1, \quad e_{11} = x_1^2\partial_1,$
$e_{12} = x_1^2x_2\partial_1, \quad e_{13} = x_1^2x_2^2\partial_1, \quad e_{14} = x_1^2x_2^3\partial_1, \quad e_{15} = x_1^2x_2^4\partial_1, \quad e_{16} = x_1^2x_2^5\partial_1, \quad e_{17} = x_2^3\partial_2,$
$e_{18} = x_2^4\partial_2, \quad e_{19} = x_2^5\partial_2, \quad e_{20} = x_2^6\partial_2, \quad e_{21} = x_2^7\partial_2, \quad e_{22} = x_2^8\partial_2, \quad e_{23} = x_2^9\partial_2,$
$e_{24} = x_1x_2^5\partial_2, \quad e_{25} = x_1x_2^6\partial_2, \quad e_{26} = x_1^2x_2^2\partial_2, \quad e_{27} = x_1^2x_2^3\partial_2, \quad e_{28} = x_1^2x_2^4\partial_2, \quad e_{29} = x_1^2x_2^5\partial_2.$

The nilradical $g(V) =< e_1, e_2, e_3, \cdots, e_{29} >$ have following multiplication table:

$[e_1, e_5] = e_2, \quad [e_1, e_6] = e_3, \quad [e_1, e_7] = e_4, \quad [e_1, e_{11}] = 2e_{10}, [e_1, e_{17}] = -6e_3, \quad [e_1, e_{18}] = -6e_4$
$[e_2, e_5] = e_3, \quad [e_2, e_6] = e_4, \quad [e_2, e_{17}] = -7e_4, \quad [e_3, e_5] = e_4, \quad [e_5, e_{11}] = e_{12}, \quad [e_5, e_{12}] = e_{13},$
$[e_5, e_{13}] = e_{14}, \quad [e_5, e_{14}] = e_{15}, \quad [e_5, e_{15}] = e_{16}, \quad [e_5, e_{16}] = -e_4, \quad [e_5, e_{17}] = -e_7, \quad [e_5, e_{18}]$
$= -e_8, [e_5, e_{19}] = -e_9, \quad [e_5, e_{20}] = -e_{10}, [e_5, e_{24}] = -e_{16} + e_{25}, [e_5, e_{25}] = e_4, [e_5, e_{26}] = e_9 + 2e_{27},$
$[e_5, e_{27}] = e_{10} + 2e_{28}, \quad [e_5, e_{28}] = 2e_{29}, \quad [e_5, e_{29}] = 2e_{23}, \quad [e_6, e_{11}] = e_{13}, \quad [e_6, e_{12}] = e_{14},$
$[e_6, e_{13}] = e_{15}, \quad [e_6, e_{14}] = e_{16}, \quad [e_6, e_{15}] = -e_4, \quad [e_6, e_{17}] = -2e_8, \quad [e_6, e_{18}] = -2e_9,$
$[e_6, e_{19}] = -2e_{10}, \quad [e_6, e_{24}] = 2e_4, \quad [e_6, e_{26}] = 2e_{10} + 2e_{28}, [e_6, e_{27}] = 2e_{29}, \quad [e_6, e_{28}] = -2e_{23},$
$[e_7, e_{11}] = e_{14}, \quad [e_7, e_{12}] = e_{15}, \quad [e_7, e_{13}] = e_{16}, \quad [e_7, e_{14}] = -e_4, \quad [e_7, e_{17}] = -3e_9,$
$[e_7, e_{18}] = -3e_{10}, \quad [e_7, e_{26}] = 2e_{29}, \quad [e_7, e_{27}] = -2e_{23}, \quad [e_8, e_{11}] = e_{15}, \quad [e_8, e_{12}] = e_{16},$
$[e_8, e_{13}] = -e_4, \quad [e_8, e_{17}] = -4e_{10}, \quad [e_8, e_{26}] = -2e_{23}, \quad [e_9, e_{11}] = e_{16}, \quad [e_9, e_{12}] = -e_4,$
$[e_{10}, e_{11}] = -e_4, \quad [e_{11}, e_{24}] = e_{29}, \quad [e_{11}, e_{25}] = -e_{23}, \quad [e_{11}, e_{26}] = -2e_{24}, \quad [e_{11}, e_{27}] = -2e_{25},$

$[e_{12},e_{17}]=-e_{14},\quad [e_{12},e_{18}]=-e_{15},\quad [e_{12},e_{19}]=-e_{16},\quad [e_{12},e_{20}]=e_4,\quad [e_{12},e_{24}]=-e_{23},$
$[e_{12},e_{26}]=e_{16}-2e_{25},\quad [e_{13},e_{17}]=-2e_{15},\quad [e_{13},e_{18}]=-2e_{16},\quad [e_{13},e_{19}]=2e_4,\quad [e_{14},e_{17}]$
$=-3e_{16},[e_{14},e_{18}]=3e_4,\quad [e_{15},e_{17}]=4e_4,\quad [e_{17},e_{18}]=e_{20},\quad [e_{17},e_{19}]=2e_{21},[e_{17},e_{20}]=3e_{22},$
$[e_{17},e_{21}]=4e_{23},\quad [e_{17},e_{26}]=-e_{28},\quad [e_{17},e_{28}]=-e_{23},\quad [e_{18},e_{19}]=e_{22},\quad [e_{18},e_{20}]=2e_{23},$
$[e_{18},e_{26}]=-2e_{29},\quad [e_{18},e_{27}]=e_{23},\quad [e_{19},e_{26}]=3e_{23}.$

□

Proposition 3.17. *Let $V=\{(x_1,x_2)\in\mathbb{C}^2 : x_1^3+x_2^5=0\}$ be the E_8 singularity and $\mathcal{L}_2^3(V)$ be a derivation Lie algebra. Then*

$$\rho_2^3(V)=34.$$

Proof. It follows that the local algebra

$$\mathcal{M}_2^3(V)=\mathcal{O}^2\Big/\Big\langle f,\left(\frac{\partial f}{\partial x_1}\right)^3,\left(\frac{\partial f}{\partial x_2}\right)^3,\left(\frac{\partial f}{\partial x_1}\right)^2\frac{\partial f}{\partial x_2},\frac{\partial f}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right)^2, \left(\frac{\partial f}{\partial x_2}\right)^2\frac{\partial^2 f}{\partial x_1^2}+\left(\frac{\partial f}{\partial x_1}\right)^2\frac{\partial^2 f}{\partial x_2^2}-2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_1\partial x_2}\Big\rangle.$$

has a monomial basis of the form:

$$\{1,x_2,x_2^2,x_2^3,\cdots,x_2^9,x_1,x_1x_2,x_1x_2^2,\cdots,x_1x_2^7,x_1^2,x_1^2x_2,x_1^2x_2^2,\cdots,x_1^2x_2^7\}.$$

We obtain following bases of Lie algebra $\mathcal{L}_2^3(V)$:

$$\begin{aligned}
&e_0=x_1\partial_1+\frac{3x_2}{5}\partial_2,\quad e_1=x_2^4\partial_1,\quad e_2=x_2^5\partial_1,\quad e_3=x_2^6\partial_1,\quad e_4=x_2^7\partial_1,\quad e_5=x_2^8\partial_1,\\
&e_6=x_2^9\partial_1,\quad e_7=x_1x_2\partial_1,\quad e_8=x_1x_2^2\partial_1,\quad e_9=x_1x_2^3\partial_1,\quad e_{10}=x_1x_2^4\partial_1,\quad e_{11}=x_1x_2^5\partial_1,\\
&e_{12}=x_1x_2^6\partial_1,\quad e_{13}=x_1x_2^7\partial_1,\quad e_{14}=x_1^2\partial_1,\quad e_{15}=x_1^2x_2\partial_1,\quad e_{16}=x_1^2x_2^2\partial_1,\quad e_{17}=x_1^2x_2^3\partial_1,\\
&e_{18}=x_1^2x_2^4\partial_1,\quad e_{19}=x_1^2x_2^5\partial_1,\quad e_{20}=x_1^2x_2^6\partial_1,\quad e_{21}=x_1^2x_2^7\partial_1,\quad e_{22}=x_2^6\partial_2,\quad e_{23}=x_2^7\partial_2,\\
&e_{24}=x_2^8\partial_2,\quad e_{25}=x_2^9\partial_2,\quad e_{26}=x_1x_2^4\partial_2,\quad e_{27}=x_1x_2^5\partial_2,\quad e_{28}=x_1x_2^6\partial_2,\quad e_{29}=x_1x_2^7\partial_2,\\
&e_{30}=x_1^2x_2^4\partial_2,\quad e_{31}=x_1^2x_2^5\partial_2,\quad e_{32}=x_1^2x_2^6\partial_2,\quad e_{33}=x_1^2x_2^7\partial_2.
\end{aligned}$$

The nilradical $g(V)=<e_1,e_2,e_3,\cdots,e_{33}>$ have following multiplication table:

$[e_1,e_7]=e_2,\quad [e_1,e_8]=e_3,[e_1,e_9]=e_4,\quad [e_1,e_{10}]=e_5,\quad [e_1,e_{11}]=e_6,[e_1,e_{14}]=2e_{10},$
$[e_1,e_{15}]=2e_{11},[e_1,e_{16}]=2e_{12},[e_1,e_{17}]=2e_{13},[e_1,e_{22}]=-4e_6,[e_1,e_{26}]=-4e_{13}+e_{24},$
$[e_1,e_{27}]=e_{25},\quad [e_2,e_7]=e_3,\quad [e_2,e_8]=e_4,\quad [e_2,e_9]=e_5,\quad [e_2,e_{10}]=e_6,$
$[e_2,e_{14}]=2e_{11},\quad [e_2,e_{16}]=2e_{13},\quad [e_2,e_{26}]=e_{25},\quad [e_3,e_7]=e_4,\quad [e_3,e_8]=e_5,$
$[e_3,e_9]=e_6,\quad [e_3,e_{14}]=2e_{12},\quad [e_3,e_{15}]=2e_{13},\quad [e_4,e_7]=e_5,\quad [e_4,e_8]=e_6,$
$[e_4,e_{14}]=2e_{13},\quad [e_5,e_7]=e_6,\quad [e_7,e_{14}]=e_{15},\quad [e_7,e_{15}]=e_{16},\quad [e_7,e_{16}]=e_{17},$
$[e_7,e_{17}]=e_{18},\quad [e_7,e_{18}]=e_{19},\quad [e_7,e_{19}]=e_{20},\quad [e_7,e_{20}]=e_{21},\quad [e_7,e_{22}]=-e_{12},$
$[e_7,e_{23}]=-e_{13},\quad [e_7,e_{26}]=-e_{18}+e_{27},\quad [e_7,e_{27}]=-e_{19}+e_{28},\quad [e_7,e_{28}]=-e_{20}+e_{29},$
$[e_7,e_{29}]=-e_{21},\quad [e_7,e_{30}]=e_6+2e_{31},\quad [e_7,e_{31}]=2e_{32},\quad [e_7,e_{32}]=2e_{33},\quad [e_8,e_{14}]=e_{16},$
$[e_8,e_{15}]=e_{17},\quad [e_8,e_{16}]=e_{18},\quad [e_8,e_{17}]=e_{19},\quad [e_8,e_{18}]=e_{20},\quad [e_8,e_{19}]=e_{21},[e_8,e_{22}]$
$=-2e_{13},\quad [e_8,e_{26}]=-2e_{19}+e_{28},[e_8,e_{27}]=-2e_{20}+e_{29},[e_8,e_{28}]=-2e_{21},[e_8,e_{30}]=2e_{32},$

$[e_8, e_{31}] = 2e_{33}$, $[e_9, e_{14}] = e_{17}$, $[e_9, e_{15}] = e_{18}$, $[e_9, e_{16}] = e_{19}, [e_9, e_{17}] = e_{20}, [e_9, e_{18}] = e_{21}$, $[e_9, e_{26}] = -3e_{20} + e_{29}$, $[e_9, e_{27}] = -3e_{21}$, $[e_9, e_{30}] = 2e_{33}, [e_{10}, e_{14}] = e_{18}$, $[e_{10}, e_{15}] = e_{19}$, $[e_{10}, e_{16}] = e_{20}$, $[e_{10}, e_{17}] = e_{21}$, $[e_{11}, e_{14}] = e_{19}$, $[e_{11}, e_{15}] = e_{20}$, $[e_{11}, e_{16}] = e_{21}$, $[e_{12}, e_{14}] = e_{20}$, $[e_{12}, e_{15}] = e_{21}$, $[e_{13}, e_{14}] = e_{21}$, $[e_{14}, e_{26}] = e_{30}$, $[e_{14}, e_{27}] = e_{31}$, $[e_{14}, e_{28}] = e_{32}$, $[e_{14}, e_{29}] = e_{33}$, $[e_{14}, e_{30}] = -2e_{25}$, $[e_{15}, e_{22}] = -e_{20}$, $[e_{15}, e_{23}] = -e_{21}$, $[e_{15}, e_{26}] = e_6 + e_{31}$, $[e_{15}, e_{27}] = e_{32}$, $[e_{15}, e_{28}] = e_{33}$, $[e_{16}, e_{22}] = -2e_{21}$, $[e_{16}, e_{26}] = e_{32}$, $[e_{16}, e_{27}] = e_{33}$, $[e_{17}, e_{26}] = e_{33}$.

□

Proof of Theorem A.

Proof. From Propositions (3.8-3.12) we get following table:

Table 1.

Type	$\rho_2^2(V)$	dim $g(V)$	dim $\frac{g(V)}{[g(V),g(V)]}$
A_1	0	not *exist*	not *exist*
A_2	1	0	0
A_3	2	1	1
A_4	3	2	2
$A_k, k \geq 5$	$k-1$	$k-2$	2
D_4	10	7	3
D_5	11	9	3
D_6	12	10	4
$D_k, k \geq 7$	$k+6$	$k+4$	5
E_6	15	13	4
E_7	17	15	3
E_8	20	18	4

Since dimension of Lie algebra is invariant of singularity. From above table, we get following pairs which have same dimension of Lie algebra.
$(1)\mathcal{L}_2^2(D_4) \ncong \mathcal{L}_2^2(A_{11})$, $(2)\mathcal{L}_2^2(D_5) \ncong \mathcal{L}_2^2(A_{12})$, $(3)\mathcal{L}_2^2(D_6) \ncong \mathcal{L}_2^2(A_{13})$
$(4)\mathcal{L}_2^2(E_6) \ncong \mathcal{L}_2^2(A_{16})$, $(5)\mathcal{L}_2^2(E_6) \ncong \mathcal{L}_2^2(D_9)$, $(6)\mathcal{L}_2^2(E_7) \ncong \mathcal{L}_2^2(A_{18})$
$(7)\mathcal{L}_2^2(E_7) \ncong \mathcal{L}_2^2(D_{11})$, $(8)\mathcal{L}_2^2(E_8) \ncong \mathcal{L}_2^2(A_{21})$, $(9)\mathcal{L}_2^2(E_8) \ncong \mathcal{L}_2^2(D_{14})$. It is follows from above table the above 9 pairs have different dim $\frac{g(V)}{[g(V),g(V)]}$.

It is follows from propositions 3.13 - 3.17, the singularities $A_k(k \geq 1)$ and $D_k(k \geq 4)$ have different number of dimensions of Lie algebra $\mathcal{L}_2^3(V)$. Next we need to distinguish the remaining pairs which have same dimensions of Lie algebra $\mathcal{L}_2^3(V)$. It is noted from propositions 3.13 to 3.17, we only need to treat three cases:
$(1)\mathcal{L}_2^3(E_6) \ncong \mathcal{L}_2^3(D_6)$
$(2)\mathcal{L}_2^3(E_7) \ncong \mathcal{L}_2^3(D_7)$
$(3)\mathcal{L}_2^3(E_8) \ncong \mathcal{L}_2^3(D_8)$.
Case 1: It is follows from proposition 3.14, the minimal number of generators of nilradical of Lie algebra $\mathcal{L}_2^3(D_6)$ are $g(V)/[g(V), g(V)] =< e_1, e_6, e_7, e_8, e_{12}, e_{13}, e_{16}, e_{17}, e_{20}, e_{24} >$. It is follows from proposition 3.15, the minimal number of generators of nilradical of Lie algebra $\mathcal{L}_2^3(E_6)$ are

$g(V)/[g(V), g(V)] =< e_1, e_6, e_7, e_{11}, e_{17}, e_{20} >$. Therefore singularities E_6 and D_6 have different minimal number of generators of nilradical of Lie algebra $\mathcal{L}_2^3(V)$.
Similarly, we can prove cases 2 and 3. □

Proof of Theorem B.

Proof. Let $f \in \mathbb{C}\{x_1, x_2\}$ be a weighted homogeneous fewnomial isolated singularity. Then f can be divided into the following three types:

Type A. $x_1^{a_1} + x_2^{a_2}$,

Type B. $x_1^{a_1}x_2 + x_2^{a_2}$,

Type C. $x_1^{a_1}x_2 + x_2^{a_2}x_1$.

The Theorem B is an immediate corollary of Propositions 3.2, 3.3 and 3.4. □

Proof of Theorem C.

Proof. From Propositions 3.2, 3.3 and 3.4, it is easy to check that the inequality

$$d_2^2(V) \le \ell_2^2(\frac{1}{w_1}, \frac{1}{w_2}) = \begin{cases} \frac{2}{w_1w_2} - 3(\frac{1}{w_1} + \frac{1}{w_2}) + 6; & w_1 \le \frac{1}{3},\ w_2 \le \frac{1}{3}\ , \\ \frac{1}{w_1} - 1; & w_1 \le \frac{1}{2}, w_2 = \frac{1}{2}. \end{cases}$$

hold true.

It is also note from Propositions 3.5, 3.6 and 3.7, it is easy to check that the inequality

$$d_2^3(V) \le \ell_2^3(\frac{1}{w_1}, \frac{1}{w_2}) = \begin{cases} \frac{3}{w_1w_2} - 3(\frac{1}{w_1} + \frac{1}{w_2}) + 6; & w_1 \le \frac{1}{3},\ w_2 \le \frac{1}{3} \text{ and } w_1 = w_2, \\ 5; & w_1 = 2, w_2 = 2 \\ \frac{3}{w_2} - 2; & w_1 = 2, w_2 \le \frac{1}{3} \\ \frac{3}{w_1w_2} - 3(\frac{1}{w_1} + \frac{1}{w_2}) + 5; & w_1 \le \frac{1}{3},\ w_2 \le \frac{1}{3} \text{ and } w_1 > w_2 \text{ or } w_1 < w_2. \end{cases}$$

hold true. □

Proof of Theorem D.

Proof. From Propositions 2.7, 2.8, 2.9 and 2.10, 2.11, 2.12 and 3.2, 3.3, 3.4 it is easy to check that the inequality

$$\rho_2^3(V) > \rho_2^2(V) > \rho_2^1(V).$$

hold true. □

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