ON NON-EXISTENCE OF NEGATIVE WEIGHT DERIVATIONS ON MODULI ALGEBRAS: YAU CONJECTURE

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Abstract. Let $A = \mathbb{F}[x_1, \cdots, x_n]/(f_1, \cdots, f_n)$ be a graded complete intersection Artinian algebra where $\mathbb{F}$ is a field of characteristic zero. The grading on A induces a natural grading on $\mathrm{Der}_{\mathbb{F}}(A)$. Halperin proposed a famous conjecture: $\mathrm{Der}_{\mathbb{F}}(A)_{<0} = 0$, which implies the collapsing of the Serre spectral sequence for an orientable fibration with the fiber being an elliptic space and having no cohomology in odd degrees. In the context of singularity theory, the second author proposed the same conjecture in the special case that $f_i = \partial f/\partial x_i$ for a single polynomial f. H. Chen, the second author and Zuo ([CYZ]) proved Halperin conjecture assuming that the degrees of f_i are bounded below by a constant depending on the number n of variables and the degrees of variables. In this paper, in the special case that $f_i = \partial f/\partial x_i$ for a single polynomial f, we refine their result by giving a better bound which is independent of n.

1. Introduction

Throughout this article, we work over a field $\mathbb{F}$ of characteristic zero. Let $P_n = \mathbb{F}[x_1, \cdots, x_n]$ be the polynomial ring of n weighted variables $x_1, \cdots, x_n$ with positive integer weights $w_1, \cdots, w_n$. Suppose that $f_1, \cdots, f_n$ are weighted homogeneous polynomials such that $A = P_n/(f_1, \cdots, f_n)$ is a complete intersection Artinian algebra. The grading on A induces a natural grading on $\mathrm{Der}_{\mathbb{F}}(A)$, where $\mathrm{Der}_{\mathbb{F}}(A)$ is the A-module of derivations on A. In 1976, Halperin proposed a famous conjecture (see [FHT, p.516]):

Conjecture 1.1 (Halperin conjecture)**.** *There is no non-zero negative weight derivation on a graded complete intersection Artinian algebra.*

Halperin conjecture is one of the most important questions in rational homotopy theory. Indeed, a positive answer of this conjecture implies the collapsing of the Serre spectral sequence at E_2 level for an orientable fibration $F \hookrightarrow E \to B$ such that the fiber F is an elliptic space with cohomology vanishing in odd degrees (in this case the cohomology algebra of F is a complete intersection Artinian algebra). Recall that a 1-connected topological

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space X is called an elliptic space if its cohomology $H^*(X, \mathbb{F})$ and homotopy group $\pi_*(X) \otimes \mathbb{F}$ are finite dimensional vector spaces.

In the context of singularity theory, the second author proposed the same conjecture in the special case that $f_i = \partial f / \partial x_i$ for a single polynomial f.

Conjecture 1.2 (Yau conjecture)**.** *Suppose that f is a weighted homogeneous polynomial such that $A = P_n/(\partial f/\partial x_1, \cdots, \partial f/\partial x_n)$ is a complete intersection Artinian algebra. Then there is no non-zero negative weight derivation on A.*

In this case, A is the moduli algebra of the hypersurface singularity defined by $f = 0$. Assuming this conjecture is true, Xu and the second author ([XY, Theorem B]) gived a micro-local characterization of quasi-homogeneous hypersurface singularities, only using the Lie algebra of derivations on the moduli algebra (which is called Yau algebra).

Halperin conjecture has remained unresolved for a long time except in some special cases:

- $w_1 = \cdots = w_n$ ([AM, Proposition 4.1]);
- $n = 2$ ([AM, Theorem 4.3] and [Tho, Theorem 3]);
- $n = 3$ ([Che2, Theorem 3.1] and [Lup, Theorem 1]);
- A is the cohomology algebra of a homogeneous space G/U, where G is a connected compact Lie group and U is its closed subgroup of maximal rank ([ST, Theorem A′]);
- $P_n/(f_2, \cdots, f_n)$ is reduced and $\deg f_1 \geq \deg f_i$, $i = 1, \cdots, n$ ([PP]);
- all the polynomials $f_1, \cdots, f_n$ are homogeneous in the grading given by the length of monomials ([Mar, Theorem 3]);
- the formal dimension of A is at most 20 ([KW]);
- all the polynomials $f_1, \cdots, f_n$ have large enough degrees ([CYZ, Main Theorem A]).

Yau conjecture has been confirmed in the cases that:

- n=2,3 ([CXY, Theorem 2.2 and 2.3]);
- n=4 ([Che1, Theorem 2.1]);
- $w_1 \geq w_2 \cdots \geq w_n$ and $w_n \geq w_1/2$ ([YZ, Main Theorem]);

Let F be a 1-connected topological space such that the graded vector space $H^*(F, \mathbb{F})$ is finite dimensional and evenly graded (i.e. cohomology vanishes in odd degrees). Then $H^*(F, \mathbb{F})$ is a graded Artinian algebra. In [Mei, Lemma 2.5], it was shown that if the cohomology algebra $H^*(F, \mathbb{F})$ has no non-zero negative weight derivations, then the Serre spectral sequence of any orientable fibration with fiber F collapses at E_2 level. If the fiber F satisfies an additional condition that $\dim \pi_*(F) \otimes \mathbb{F} < \infty$, then F is an elliptic space. According to [Hal], in this case the cohomology algebra $H^*(F, \mathbb{F})$ is a complete intersection graded Artinian algebra and then the Halperin conjecture implies the collapsing of the Serre spectral sequences. However, if $\pi_*(F) \otimes \mathbb{F}$ is infinite dimensional, $H^*(F, \mathbb{F})$ may not be complete intersection and the Halperin conjecture is not applicable. This is the reason why the study of the nonexistence of negative weight derivations on non-complete intersection graded Artinian algebras is more important since we can allow the fiber to be even non-elliptic space.

H. Chen, the second author and Zuo ([CYZ]) gave a positive result to Halperin conjecture in the case that the degrees of f_j are larger than some lower bound, without the condition of being complete intersection, i.e. the number of f_j may be large than that of variables x_i.

Theorem 1.3 (Main Theorem A in [CYZ])**.** *Let $P_n = \mathbb{F}[x_1, \cdots, x_n]$ be the polynomial ring of n weighted variables $x_1, \cdots, x_n$ with positive integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$ ($n \geq 2$). Suppose that $f_1, \cdots, f_m$ are weighted homogeneous polynomials with degrees greater than $(m-1)(w_1 w_2)^{n-1}$ and $A = P_n/(f_1, \cdots, f_m)$ is an Artinian algebra. Then there are no non-zero negative weight derivations on A.*

Remark 1.4. Note that in the non-complete intersection case there are some counter examples if the degrees of f_j are small. Hence it is necessary to assume that the degrees of f_j are bounded below. However, the bound in Theorem 1.3 may not be sharp.

It is notable that the bound in Theorem 1.3 tends to infinity as n tends to infinity. In this paper, in the special case that $f_i = \partial f/\partial x_i$ for a single polynomial f, we refine Theorem 1.3 by giving a better bound which is independent of n (only depending on the degrees of variables).

Theorem 1.5. *Let $P_n = \mathbb{F}[x_1, \cdots, x_n]$ be the weighted polynomial ring of n weighted variables $x_1, \cdots, x_n$ with positive integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$. Let f be a weighted homogeneous polynomials with degrees greater than $w_1^2 - w_1 + 1$ such that*

$$A = P_n/(\partial f/\partial x_1, \cdots, \partial f/\partial x_n)$$

is an Artinian algebra. Then there is no non-zero negative weight derivation on A.

Remark 1.6. In the above theorem, A is a complete intersection algebra. In this case we expect that A has no non-zero negative weight derivations even when the degree of f is small. So the bound in the above theorem is not sharp.

In the general case, we also expect that the bound in Theorem 1.3 can be improved. Indeed, we have the following conjecture, inspired by Example 1.7.

Optimal Generalized Halperin Conjecture. Let $P_n = \mathbb{F}[x_1, \cdots, x_n]$ be the weighted polynomial ring of n weighted variables $x_1, \cdots, x_n$ with positive integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$. Suppose that $f_1, \cdots, f_m$ are weighted homogeneous polynomials with degrees greater than $(m-1)(w_1 - 1)$ and $A = P_n/(f_1, \cdots, f_m)$ is an Artinian algebra. Then there are no non-zero negative weight derivations on A.

Example 1.7. Let x, y be two weighted variables with positive integer weights w_1, w_2 such that $aw_2 = w_1 - 1$, where a is a positive integer. Consider m weighted homogeneous polynomials

$$(f_1, \cdots, f_m) = \left(x^{m-1}, x^{m-2}y^a, x^{m-3}y^{2a}, \cdots, xy^{(m-2)a}, y^{(m-1)a}\right).$$

Then $A = \mathbb{F}[x, y]/(f_1, \cdots, f_m)$ is an Artinian algebra. We have

$$\deg f_m = (m-1)aw_2 = (m-1)(w_1 - 1)$$

and

$$\begin{aligned} \deg f_j &= (m-j)w_1 + (j-1)aw_2 \\ &= (m-j)w_1 + (j-1)(w_1 - 1) \\ &> (m-1)(w_1 - 1) \end{aligned}$$

for any $j = 1, \cdots, m-1$. On A there exists a non-zero derivation

$$D = y^a \partial / \partial_x$$

of negative weight -1.

Sketch of the proof of Theorem 1.5. The main tool we use is the new weight type, which was first introduced by H. Chen [Che2] and further developed in [CYZ]. They associated any negative weight derivation D on P_n with a new weight type $(\ell_1, \cdots, \ell_n)$ controlled by parameters $\epsilon_1, \cdots, \epsilon_n$ (see Definition 3.1). Let f be a weighted homogeneous polynomial in P_n such that $A = P_n/(f_1, \cdots, f_n)$ is an Artinian algebra, where $f_i = \partial f / \partial x_i$. Suppose that there is a non-zero negative weight derivation on A. It induces a non-zero negative weight derivation D on P_n preserving the ideal $(f_1, \cdots, f_n)$. Let $(\ell_1, \cdots, \ell_n)$ be the new weight type associated to D controlled by parameters $\epsilon_1, \cdots, \epsilon_n$. In [CYZ] the author showed that, after changing the coordinate system and adjusting the parameters, the new weight type will satisfy the following: there exists $i_0 \in \{1, \cdots, n\}$ such that

(1) $\ell_{i_0}/w_{i_0} > \ell_i/w_i$ for any $i \neq i_0$, and
(2) $D_{\max} = p\partial/\partial x_{i_0}$, where $p \in P_n$ which is independent of the variable x_{i_0} and $D_{\max}$ is the part of D of highest degree with respect to the new weight type.

Since $P_n/(f_1, \cdots, f_n)$ is an Artinian algebra, there is f_e such that $x_{i_0}^a$ appears in f_e with non-zero coefficient. Let $(f_e)_{\max}$ be the part of f of highest degree with respect to the new weight type. It follows from (1) that $(f_e)_{\max} = x_{i_0}^a$ (omitting the coefficient) as f_e is weighted homogeneous with respect to the original weight type. Let $g = D^a f_e$ and $g_{\max}$ be the part of g of highest degree with respect to the new weight type. Then we have

$$g_{\max} = (D_{\max})^a (f_e)_{\max} = p^a \text{ (omitting the coefficient)},$$

which implies that $g \neq 0$. Note that $g \in (f_1, \cdots, f_n)$ as D preserves the ideal $(f_1, \cdots, f_n)$, there is f_d such that $\deg f_d \leq \deg g \leq \deg f_e - a$ (with respect to the original weight type), where the second inequality follows from the fact that D is of negative weight and $g = D^a f_e$. As $\deg f_i = \deg f - w_i$ for any i, we have $f_e - f_d = w_d - w_e$, which implies that $a \leq w_d - w_e$. Therefore,

$$\deg f = \deg f_e + w_e = a w_{i_0} + w_e \leq (w_d - w_e) w_{i_0} + w_e.$$

It is not hard to check that the above formula is no more than $w_1^2 - w_1 + 1$ (assuming that $w_1 \geq \cdots \geq w_n$). Therefore, if we assume that $\deg f > w_1^2 - w_1 + 1$, A has no no-zero negative weight derivations.

The paper is organized as follows. In Section 2 we recall some definitions and properties related to derivations. In Section 3 we introduce the main technical tool – the new weight type associated with a negative weight derivation and controlled by a group of parameters. In Section 4 we adjust parameters to obtain a new weight type which meets our needs. We shall give the proof of Theorem 1.5 in Section 5.

Notation. (1) Let $f \in \mathbb{F}[x_1, \cdots, x_n]$ be a polynomial and $g = \prod x_i^{a_i}$ be a monic monomial. We say $g \in f$ if $\prod x_i^{a_i}$ appears in the expansion of f with non-zero coefficient.

(2) Let S be a finite set. We denote the cardinality of S by $\#S$.

(3) Let a, b be two real numbers. We say a is divisible by b if a/b is an integer.

2. DERIVATIONS

Let A be a commutative algebra over $\mathbb{F}$. We say a linear endomorphism D of A is a derivation on A if it satisfies the Leibniz rule:

$$D(ab) = D(a)b + aD(b) \quad \forall a, b \in A. \tag{2.1}$$

We denote the A-module that consists of all the derivations on A by $\mathrm{Der}_{\mathbb{F}}(A)$.

We say a commutative algebra A is a graded algebra, if $A = \oplus_{i=0}^{\infty} A_i$, where A_i are linear subspaces of A and satisfy that $A_i \cdot A_j \subseteq A_{i+j}$ for any non-negative integers i, j. Let D be a derivation on a graded algebra $A = \oplus_{i=0}^{\infty} A_i$. We say D is of weight k if $D(A_i) \subseteq A_{i+k}$ for any non-negative integer i. In this case we denote the weight k of D by $\mathrm{wt}\ D$.

Let $P_n = \mathbb{F}[x_1, \cdots, x_n]$ be the polynomial ring of weighted variables $x_1, \cdots, x_n$ with weights $w_1, \cdots, w_n$, where each w_i is a positive integer. We call $(w_1, \cdots, w_n)$ a weight type of variables $x_1, \cdots, x_n$. For any monomial $\prod_{i=1}^n x_i^{a_i}$, we define its (weighted) degree with respect to the weight type $(w_1, \cdots, w_n)$ by $\sum_{i=1}^n a_i w_i$. Then $P_n = \oplus_{i=0}^{\infty} (P_n)_i$ is a graded algebra, where $(P_n)_i$ is the linear subspace spanned by monomials of degree i. For any non-zero polynomial $f \in P_n$, if $f \in (P_n)_i$ for some i, we say f is a weighted homogeneous polynomial. We call i the (weighted) degree of f and denote it by $\deg f$.

Every derivation on the weighted polynomial ring P_n can be written as

$$D = p_1 \partial/\partial_{x_1} + \cdots + p_n \partial/\partial_{x_n}, \tag{2.2}$$

where $p_i \in P_n$. Then D is of weight k if and only if p_i is either the zero polynomial or a weighted homogeneous polynomial with $\deg p_i - w_i = k$ for $i = 1, \cdots, n$.

Let $I \subseteq P_n$ be an ideal generated by weighted homogeneous polynomials $f_1, \cdots, f_m$. Then $P_n/I = \oplus_{i=0}^{\infty}(P_n/I)_i$ is a graded algebra, where the i-th graded pieces $(P_n/I)_i$ is the image of $(P_n)_i$ under the projection $P_n \to P_n/I$. The lemma below tells us that every derivation on P_n/I is induced by a derivation on P_n.

Lemma 2.1. *Let $\overline{D}$ be a derivation on P_n/I of weight d. Then there exists a derivation D on P_n of weight d such that*

(1) $\overline{D(g)} = \overline{D}(\overline{g})$ *for any $g \in P_n$;*

(2) $D(I) \subseteq I$.

Here we denote by $\overline{g}$ the image of g under the projection $P_n \to P_n/I$ for any $g \in P_n$.

Proof. Since $\overline{D}(\overline{x_i}) \in (P_n/I)_{w_i+d}$, there exists $g_i \in (P_n)_{w_i+d}$ such that $\overline{D}(\overline{x_i}) = \overline{g_i}$ for $i = 1, \cdots, n$. Define

$$D = g_1\partial/\partial_{x_1} + \cdots + g_n\partial/\partial_{x_n}. \tag{2.3}$$

Then D is a derivation on P_n of weight d and $\overline{D(x_i)} = \overline{D}(\overline{x_i})$ for any i. Hence the first assertion holds since both D and $\overline{D}$ satisfy the Leibniz rule.

For any $f \in I$, we have

$$\overline{Df} = \overline{D}\left(\overline{f}\right) = \overline{D}(0) = 0. \tag{2.4}$$

Hence $Df \in I$ and the second assertion holds. $\square$

3. New weight type

In this section, we will introduce our main technical tool – new weight type. It was first introduced in H. Chen's paper [Che2]. Then it was developed in [CYZ] and played a crucial role in the proof of Theorem 1.3. Though some definitions and properties in this section can been found in [CYZ], for the convenience of readers we will give complete proofs here.

Let P_n be the weighted polynomial ring of n weighted variables $x_1, \cdots, x_n$ with positive integer weights $w_1, \cdots, w_n$. Suppose that

$$w_1 \geq w_2 \geq \cdots \geq w_n. \tag{3.1}$$

Fix a negative weight derivation D on P_n and write

$$D = p_1\partial/\partial_{x_1} + \cdots + p_n\partial/\partial_{x_n}, \tag{3.2}$$

where p_i is either a weighted homogeneous polynomial of degree $w_i + \mathrm{wt}\ D$ or the zero polynomial. Since $\mathrm{wt}\ D < 0$, we have $\deg p_i < w_i$, which by (3.1) implies that p_i is a polynomial of only variables $x_{i+1}, \cdots, x_n$. In particular, p_n is a constant.

Definition 3.1 (New weight type)**.** Fix a non-zero negative weight derivation D on P_n as in (3.2). Given n real parameters $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$, we define a new weight type $(\ell_1, \cdots, \ell_n)$ of variables $x_1, \cdots, x_n$ as follows.

Set $\ell_n = \epsilon_n$. Suppose $\ell_n, \ell_{n-1}, \cdots, \ell_{q+1}$ have been defined. Note that the coefficient p_q of $\partial/\partial x_q$ in D is a polynomial of $x_{q+1}, \cdots, x_n$. Then

(1) if p_q is the zero polynomial, we set

$$\ell_q = \epsilon_q; \tag{3.3}$$

(2) if p_q is a non-zero polynomial, we set

(3.4)
$$\ell_q = \epsilon_q + \max\{\ell_{q+1} i_{q+1} + \ell_{q+2} i_{q+2} + \cdots + \ell_n i_n \mid x_{q+1}^{i_{q+1}} x_{q+2}^{i_{q+2}} \dots x_n^{i_n} \in p_q\}.$$

We call $(\ell_1, \ell_2, \dots, \ell_n)$ the new weight type associated with D and controlled by parameters $\epsilon_1, \epsilon_2, \dots, \epsilon_n$.

Remark 3.2. If we set

$$\epsilon_i = \begin{cases} -\text{wt } D, & \text{when } p_i \text{ is a non-zero polynomial,} \\ w_i, & \text{when } p_i \text{ is the zero polynomial,} \end{cases} \tag{3.5}$$

the new weight type $(\ell_1, \dots, \ell_n)$ coincides with the original weight type $(w_1, \cdots, w_n)$.

Definition 3.3. Given a new weight type $(\ell_1, \ell_2, \dots, \ell_n)$, to distinguish with the degree with respect to the original weight type $(w_1, \cdots, w_n)$, we denote the degree with respect to the new weight type by Q-deg. More precisely, for any polynomial $f \in P_n$, we denote

$$\text{Q-deg } f = \max\{-\infty, \ell_1 i_1 + \cdots + \ell_n i_n \mid x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} \in f\}. \tag{3.6}$$

If $f = 0$, then Q-deg $f = -\infty$.

We denote by $f_{\max}$ the sum of terms in expansion of f with maximum degrees with respect to the new weight type $(\ell_1, \ell_2, \dots, \ell_n)$. That is to say, if we write $f = \sum_\alpha g_\alpha$ where each g_α is a monomial, then

$$f_{\max} = \sum_{\text{Q-deg } g_\alpha = \text{Q-deg } f} g_\alpha. \tag{3.7}$$

If $f = 0$, then $f_{\max} = 0$.

By the definition of the new weight type, if p_i is a non-zero polynomial, then

$$\ell_i = \epsilon_i + \text{Q-deg } p_i. \tag{3.8}$$

Recall that D is a non-zero negative weight derivation on P_n as in (3.2). We denote

$$\text{Q-deg } D = \max\{\text{Q-deg } p_i - \ell_i \mid i = 1, \cdots, n\} \tag{3.9}$$

and

$$D_{\max} = \sum_{j \in S} (p_j)_{\max} \partial/\partial x_j, \tag{3.10}$$

where $S = \{i = 1, \cdots, n \mid \text{Q-deg } p_i - \ell_i = \text{Q-deg } D\}$.

We denote

$$\epsilon_{\min} = \min\{\epsilon_i \mid i = 1, \cdots, n \mid p_i \text{ is a non-zero polynomial}\}. \tag{3.11}$$

Proposition 3.4. *With the above notations, we have*

$$\textit{Q-deg } D = -\epsilon_{\min} \tag{3.12}$$

and

$$D_{\max} = \sum_{\substack{p_i \neq 0 \\ \epsilon_i = \epsilon_{\min}}} (p_i)_{\max} \partial/\partial x_i. \tag{3.13}$$

Proof. It follows from (3.8), (3.9), (3.10) and (3.11) directly. □

Proposition 3.5. *With the above notations, for any non-zero polynomial $f \in P_n$ we have*

(1) *if $D_{\max} f_{\max} \neq 0$, then $D_{\max} f_{\max} = (Df)_{\max}$ and Q-deg Df = Q-deg $f - \epsilon_{min}$;*

(2) *if $D_{\max} f_{\max} = 0$, then Q-deg $Df <$ Q-deg $f - \epsilon_{min}$.*

Proof. First we prove that

$$\text{Q-deg } Dg \leq \text{Q-deg } D + \text{Q-deg } g \tag{3.14}$$

for any $g \in P_n$. Indeed, we have

$$Dg = p_1 \frac{\partial g}{\partial x_1} + p_2 \frac{\partial g}{\partial x_2} + \cdots + p_n \frac{\partial g}{\partial x_n} \tag{3.15}$$

and

$$\begin{aligned} \text{Q-deg } (p_i \frac{\partial g}{\partial x_i}) &\leq \text{Q-deg } p_i + \text{Q-deg } g - \ell_i \\ &\leq \text{Q-deg } D + \text{Q-deg } g \end{aligned} \tag{3.16}$$

for any $i = 1, \cdots, n$ (the second inequality follows from (3.9)). Hence the equality (3.14) holds.

For any $f \in P_n$, denote $f' = f - f_{\max}$ and $D' = D - D_{\max}$. Then Q-deg $f' <$ Q-deg f and Q-deg $D' <$ Q-deg D. We have

$$\begin{aligned} Df &= (D_{\max} + D')(f_{\max} + f') \\ &= D_{\max} f_{\max} + D_{\max} f' + D' f_{\max} + D' f'. \end{aligned} \tag{3.17}$$

If $D_{\max} f_{\max} \neq 0$, by (3.10), we have

$$D_{\max} f_{\max} = \sum_{i \in S} (p_i)_{\max} \frac{\partial f_{\max}}{\partial x_i}, \tag{3.18}$$

where $S = \{i = 1, \cdots, n \mid \text{Q-deg } p_i - \ell_i = \text{Q-deg } D\}$. Hence

$$\text{Q-deg } (D_{\max} f_{\max}) = \text{Q-deg } D + \text{Q-deg } f. \tag{3.19}$$

On the other hand, by (3.14) we have

$$\begin{aligned}\max\{\text{Q-deg }(D_{\max}f'), \text{Q-deg }(D'f_{\max}), \text{Q-deg }(D'f')\}\\ < \text{Q-deg } D + \text{Q-deg } f.\end{aligned} \tag{3.20}$$

Hence Q-deg Df = Q-deg D + Q-deg f. By Proposition 3.4, Q-deg $D = -\epsilon_{\min}$, which implies that Q-deg Df = Q-deg $f - \epsilon_{\min}$.

If $D_{\max}f_{\max} = 0$, then $Df = D_{\max}f' + D'f_{\max} + D'f'$. By the inequality (3.20), we have Q-deg $Df <$ Q-deg D + Q-deg f = Q-deg $f - \epsilon_{\min}$. □

Corollary 3.6. *If $Df = 0$, then $D_{\max}f_{\max} = 0$.*

Proof. It follows from Proposition 3.5(1) directly. □

Proposition 3.7. *Let $(\ell_1, \ell_2, \dots, \ell_n)$ be the new weight type associated with D and controlled by parameters $\epsilon_1, \epsilon_2, \dots, \epsilon_n$. Let ϵ be a real number such that all ϵ_i are divisible by ϵ. Then*

(1) *all ℓ_i are divisible by ϵ;*

(2) *for any $i, j \in \{1, \cdots, n\}$, if $\ell_i/w_i - \ell_j/w_j > 0$ then*

$$\ell_i/w_i - \ell_j/w_j \geq \epsilon/(w_iw_j). \tag{3.21}$$

Proof. (1) We prove the first assertion by induction on i. When $i = n$, by the definition of the new weight type, we have $\ell_n = \epsilon_n$, which implies that ℓ_n is divisible by ϵ.

Suppose that the first assertion holds for $i = k+1, \dots, n$. When $i = k$, if p_k is the zero polynomial, then $\ell_k = \epsilon_k$ is divisible by ϵ. If $p_k \neq 0$, since p_k is a polynomial of variables $x_{k+1}, \cdots, x_n$, by inductive assumption Q-deg p_k is divisible by ϵ, which implies that ℓ_k = Q-deg $p_k + \epsilon_k$ is divisible by ϵ.

(2) By the first assertion we can write $\ell_i = a\epsilon$ and $\ell_j = b\epsilon$ where a and b are integers. Then

$$\frac{\ell_i}{w_i} - \frac{\ell_j}{w_j} = \frac{\ell_iw_j - w_i\ell_j}{w_iw_j} = \frac{(aw_j - bw_i)\epsilon}{w_iw_j}. \tag{3.22}$$

Since $\ell_i/w_i - \ell_j/w_j > 0$, we have $aw_j - bw_i > 0$. It follows from $aw_j - bw_i$ is an integer that $aw_j - bw_i \geq 1$. Therefore $\ell_i/w_i - \ell_j/w_j \geq \epsilon/(w_iw_j)$. □

The next proposition gives a upper bound for the ratio of the new weight type and the original weight type.

Proposition 3.8. *If $\ell_{i_0}/w_{i_0} = \max\{\ell_i/w_i \mid i = 1, \cdots, n\}$ and p_{i_0} is a non-zero polynomial, then*

$$\ell_i/w_i \leq \epsilon_{i_0}/(-wt\ D) \tag{3.23}$$

for any $i = 1, \cdots, n$.

Proof. It suffices to prove that $\ell_{i_0}/w_{i_0} \leq \epsilon_{i_0}/(-\text{wt } D)$. Indeed, we have

$$\frac{\ell_{i_0}}{w_{i_0}} = \frac{\text{Q-deg } p_{i_0} + \epsilon_{i_0}}{\deg p_{i_0} - \text{wt } D}. \tag{3.24}$$

Since $\ell_i/w_i \leq \ell_{i_0}/w_{i_0}$ for any $i = 1, \cdots, n$ and p_{i_0} is weighted homogeneous with respect to the original weight type, we have

$$\frac{\text{Q-deg}\, p_{i_0}}{\deg p_{i_0}} \leq \frac{\ell_{i_0}}{w_{i_0}}. \tag{3.25}$$

It follows from (3.24) and (3.25) that

$$\frac{\epsilon_{i_0}}{-\text{wt } D} \geq \frac{\ell_{i_0}}{w_{i_0}}. \tag{3.26}$$

□

4. Adjusting parameters

The main result of this section (Theorem 4.5) can be found in the proof of [CYZ, Main Theorem A]. For the convenience of readers, we will give a complete proof below.

Let $P_n = \mathbb{F}[x_1, \cdots, x_n]$ be the polynomial ring of n weighted variables $x_1, \cdots, x_n$ with positive integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$. Fix a non-zero negative weight derivation D on P_n as in (3.2), the new weight type $(\ell_1, \cdots, \ell_n)$ is controlled by parameters $\epsilon_1, \cdots, \epsilon_n$. By adjusting parameters, we can make the new weight type suit our needs.

4.1. **Preparation.**

In this subsection, we will explain how the new weight type changes as the parameters change.

Proposition 4.1. *Take two groups of parameters $(\epsilon_1, \cdots, \epsilon_n)$ and $(\epsilon'_1, \cdots, \epsilon'_n)$. Suppose that there exist $i_0 \in \{1, \cdots, n\}$ and a real number $\Delta \geq 0$ such that*

$$\epsilon'_i = \begin{cases} \epsilon_i + \Delta, & \text{if } i = i_0, \\ \epsilon_i, & \text{if } i \neq i_0. \end{cases} \tag{4.1}$$

Let $(\ell_1, \cdots, \ell_n)$ and $(\ell'_1, \cdots, \ell'_n)$ be the new weight types controlled by the ϵ_i and the ϵ'_i respectively. We denote the degree with respect to the new weight types $(\ell_1, \cdots, \ell_n)$ and $(\ell'_1, \cdots, \ell'_n)$ by Q-deg and Q′-deg respectively (cf. Definition 3.3). Then we have

(1) *$\ell'_i = \ell_i$ for any $i \in \{i_0 + 1, i_0 + 2, \cdots, n\}$;*
(2) *$\ell'_{i_0} - \ell_{i_0} = \Delta$;*
(3) *$0 \leq \ell'_i - \ell_i \leq w_i \Delta$ for any $i \in \{1, \cdots, n\}$.*

Proof. (1) When $i_0 = n$, there is nothing further to prove, so we may suppose that $i_0 < n$. By (4.1), we have $\epsilon'_i = \epsilon_i$ for any $i = i_0 + 1, \cdots, n$. We will show the first assertion by decreasing induction on i. When $i = n > i_0$, we have $\ell_n = \epsilon_n$ and $\ell'_n = \epsilon'_n$, which implies that $\ell'_n = \ell_n$.

Suppose that the assertion (1) holds for $i = k + 1, \ldots, n$ where $k > i_0$. When $i = k$, since $k > i_0$, we have $\epsilon_k = \epsilon'_k$. If p_k is the zero polynomial, then $\ell_k = \epsilon_k$ and $\ell'_k = \epsilon'_k$, which implies that $\ell'_k = \ell_k$. If $p_k \neq 0$, since p_k

is a polynomial of variables $x_{k+1}, \cdots, x_n$, by inductive assumption we have Q-deg $p_k =$ Q$'$-deg p_k. Therefore

$$\ell'_k - \ell_k = (\text{Q}'\text{-deg } p_k + \epsilon'_k) - (\text{Q-deg } p_k + \epsilon_k) = \epsilon'_k - \epsilon_k = 0. \tag{4.2}$$

(2) If p_{i_0} is the zero polynomial, then $\ell_{i_0} = \epsilon_{i_0}$ and $\ell'_{i_0} = \epsilon'_{i_0} = \epsilon_{i_0} + \Delta$, which implies that $\ell'_{i_0} = \ell_{i_0} + \Delta$. If $p_{i_0} \neq 0$, since p_{i_0} is a polynomial of variables $x_{i_0+1}, \cdots, x_n$, it follows from the assertion (1) that Q-deg $p_{i_0} =$ Q$'$-deg p_{i_0}. Therefore

$$\ell'_{i_0} - \ell_{i_0} = (\text{Q}'\text{-deg } p_{i_0} + \epsilon'_{i_0}) - (\text{Q-deg } p_{i_0} + \epsilon_{i_0}) = \epsilon'_{i_0} - \epsilon_{i_0} = \Delta. \tag{4.3}$$

(3) We will show the assertion (3) by decreasing induction on i. When $i = n$, we have $\ell_n = \epsilon_n$ and $\ell'_n = \epsilon'_n$, which implies that $0 \leq \ell'_n - \ell_n \leq \Delta \leq w_n\Delta$.

Suppose the assertion (3) holds for $i = k+1, \ldots, n$. When $i = k$, if $p_k = 0$ then we have $\ell_k = \epsilon_k$ and $\ell'_k = \epsilon'_k$, which implies that $0 \leq \ell'_k - \ell_k \leq \Delta \leq w_k\Delta$. So we may suppose that $p_k \neq 0$. Note that p_k is a polynomial of variables $x_{k+1}, x_{k+2}, \cdots, x_n$. By inductive assumption, for any monomial $g = x_{k+1}^{a_{k+1}} \cdots x_n^{a_n} \in p_k$ we have

$$\begin{aligned} 0 \leq \text{Q}'\text{-deg } g - \text{Q-deg } g &= \sum_{i=k+1}^{n} a_i(\ell'_i - \ell_i) \\ &\leq \sum_{i=k+1}^{n} a_i w_i \Delta \\ &= \Delta \deg g = \Delta \deg p_k. \end{aligned} \tag{4.4}$$

Hence $0 \leq$ Q$'$-deg $p_k -$ Q-deg $p_k \leq \Delta \deg p_k$. Therefore

$$\begin{aligned} 0 \leq \ell'_k - \ell_k &= (\text{Q}'\text{-deg } p_k + \epsilon'_k) - (\text{Q-deg } p_k + \epsilon_k) \\ &= (\text{Q}'\text{-deg } p_k - \text{Q-deg } p_k) + (\epsilon'_k - \epsilon_k) \\ &\leq \Delta \deg p_k + \Delta \\ &\leq (\deg p_k - \text{wt } D)\Delta \\ &= w_k\Delta. \end{aligned} \tag{4.5}$$

□

Corollary 4.2. *Under the assumptions in Proposition 4.1, we suppose additionally that there exists $\epsilon > 0$ such that each ϵ_i is divisible by ϵ and that Δ in* (4.1) *satisfies $w_1 w_2 \Delta < \epsilon$. Then for any $i, j \in \{1, \cdots, n\}$, we have*

$$\ell_i/w_i > \ell_j/w_j \Longrightarrow \ell'_i/w_i > \ell'_j/w_j. \tag{4.6}$$

Proof. Take $i, j \in \{1, \cdots, n\}$ with $\ell_i/w_i > \ell_j/w_j$. It follows from Proposition 3.7(2) and $w_1 \geq \cdots \geq w_n$ that

$$\ell_i/w_i - \ell_j/w_j \geq \epsilon/(w_i w_j) \geq \epsilon/(w_1 w_2). \tag{4.7}$$

On the other hand, by Proposition 4.1(3) we have

$$\ell'_j/w_j - \ell_j/w_j \leq (w_j\Delta)/w_j = \Delta < \epsilon/(w_1 w_2). \tag{4.8}$$

Therefore

$$\ell_j'/w_j < \ell_i/w_i \leq \ell_i'/w_i, \tag{4.9}$$

where the second inequality follows from $\ell_i' \geq \ell_i$ (the third assertion in Proposition 4.1). □

Corollary 4.3. *Under the assumptions in Proposition 4.1, we suppose additionally that there exists $\epsilon > 0$ such that each ϵ_i is divisible by ϵ and that Δ in* (4.1) *satisfies $M\Delta \leq \epsilon$, where M is a positive real number. Let g_1, g_2 be two monomials such that* $\deg g_1 < M$. *Then*

$$Q\text{-}deg\ g_1 < Q\text{-}deg\ g_2 \Longrightarrow Q'\text{-}deg\ g_1 < Q'\text{-}deg\ g_2. \tag{4.10}$$

Proof. We claim that for any monomial g we have

$$0 \leq \mathrm{Q}'\text{-deg } g - \mathrm{Q}\text{-deg } g \leq \Delta \deg g. \tag{4.11}$$

Indeed, if we write $g = x_{k+1}^{a_{k+1}} \cdots x_n^{a_n}$ (omitting the coefficient), by Proposition 4.1(3) we have

$$\begin{aligned} 0 \leq \mathrm{Q}'\text{-deg } g - \mathrm{Q}\text{-deg } g &= \sum_{i=1}^{n} a_i(\ell_i' - \ell_i) \\ &\leq \sum_{i=1}^{n} a_i w_i \Delta \\ &= \Delta \deg g. \end{aligned} \tag{4.12}$$

Hence the inequality (4.11) holds.

Let g_1, g_2 be two monomials with $\deg g_1 < M$ and Q-deg $g_1 <$ Q-deg g_2. By Proposition 3.7(1) we have each ℓ_i is divisible by ϵ, which implies that Q-deg g_1 and Q-deg g_2 are divisible by ϵ. Therefore

$$\mathrm{Q}\text{-deg } g_2 - \mathrm{Q}\text{-deg } g_1 \geq \epsilon. \tag{4.13}$$

It follows from the inequality (4.11) that

$$\mathrm{Q}'\text{-deg } g_1 - \mathrm{Q}\text{-deg } g_1 \leq \Delta \deg g_1 < M\Delta \leq \epsilon. \tag{4.14}$$

Therefore

$$\mathrm{Q}'\text{-deg } g_1 < \mathrm{Q}\text{-deg } g_2 \leq \mathrm{Q}'\text{-deg } g_2. \tag{4.15}$$

□

Proposition 4.4. *Fix a positive real number M, an integer $i_0 \in \{1, \cdots, n\}$ and n real numbers $(e_1, \cdots, e_n)$. Let $b : i \mapsto b_i$ be a one to one mapping from $\{1, 2, \ldots, n\}$ to itself. We set $\epsilon_i = e_i + 1/M^{b_i}$ for any $i = 1, \cdots, n$ and let $(\ell_1, \cdots, \ell_n)$ be the new weight type controlled by the ϵ_i. If g, h are two monic monomials such that* $\deg g, \deg h < M$, *then*

$$Q\text{-}deg\ g = Q\text{-}deg\ h \Longleftrightarrow g = h. \tag{4.16}$$

Proof. Let d be the inverse mapping of $b : i \mapsto b_i$. That is to say, $b_{d(i)} = i$ for any $i = 1, 2, \ldots, n$. We will define by induction a sequences of parameters $\epsilon_i^{(0)}, \epsilon_i^{(1)}, \ldots, \epsilon_i^{(n)}$. First we set

$$(\epsilon_1^{(0)}, \cdots, \epsilon_n^{(0)}) = (e_1, \cdots, e_n). \tag{4.17}$$

Suppose the $(j-1)$-th group of parameters $(\epsilon_1^{(j-1)}, \ldots, \epsilon_n^{(j-1)})$ has been defined. We define the j-th group of parameters as follows ($j > 0$):

$$\epsilon_i^{(j)} = \begin{cases} \epsilon_i^{(j-1)} + 1/M^j & i = d(j), \\ \epsilon_i^{(j-1)} & i \neq d(j). \end{cases} \tag{4.18}$$

It is easy to see that the n-th group of parameters $(\epsilon_1^{(n)}, \cdots, \epsilon_n^{(n)}) = (\epsilon_1, \cdots, \epsilon_n)$. Let $(\ell_1^{(j)}, \cdots, \ell_n^{(j)})$ be the new weight type controlled by the j-th group of parameters $(\epsilon_1^{(j)}, \cdots, \epsilon_n^{(j)})$ and we denote the degree with respect to this new weight type by $\mathrm{Q}^{(j)}$-deg (cf. Definition 3.3). Note that $\mathrm{Q}^{(n)}$-deg $f =$ Q-deg f for any polynomial f.

Let g, h be two monic monomials with $\deg g, \deg h < M$ and Q-deg $g =$ Q-deg h. Then $\mathrm{Q}^{(n)}$-deg $g = \mathrm{Q}^{(n)}$-deg h. We claim that

$$\mathrm{Q}^{(j)}\text{-deg } g = \mathrm{Q}^{(j)}\text{-deg } h \quad \forall j = 0, \cdots, n. \tag{4.19}$$

Indeed, if this is not the case, there exists j such that $\mathrm{Q}^{(j)}$-deg $g \neq \mathrm{Q}^{(j)}$-deg h. Without loss of generality, we may suppose that $\mathrm{Q}^{(j)}$-deg $g < \mathrm{Q}^{(j)}$-deg h. It follows from Corollary 4.3 (take $\epsilon = 1/M^j$ and $\Delta = 1/M^{j+1}$) that $\mathrm{Q}^{(j+1)}$-deg $g < \mathrm{Q}^{(j+1)}$-deg h. Applying Corollary 4.3 again (taking $\epsilon = 1/M^{j+1}$ and $\Delta = 1/M^{j+2}$), we obtain $\mathrm{Q}^{(j+2)}$-deg $g < \mathrm{Q}^{(j+2)}$-deg h. Repeat the above process and finally we obtain $\mathrm{Q}^{(n)}$-deg $g < \mathrm{Q}^{(n)}$-deg h. This is a contradiction and hence (4.19) holds.

Write $g = x_1^{a_1} \cdots x_n^{a_n}$ and $h = x_1^{a_1'} \cdots x_n^{a_n'}$. We will prove by induction that $a_i = a_i'$ for any $i = 1, \cdots, n$. Suppose that $a_i = a_i'$ for $i = 1, \cdots k-1$. When $i = k$, let $j = b_k$, then we have $d(j) = k$. It follows from Proposition 4.1 that

$$\begin{cases} \ell_k^{(j)} - \ell_k^{(j-1)} = 1/M^j > 0, \\ \ell_i^{(j)} - \ell_i^{(j)} = 0 \quad \forall i = k+1, \cdots, n. \end{cases} \tag{4.20}$$

Therefore

$$\begin{cases} \mathrm{Q}^{(j)}\text{-deg } g = \mathrm{Q}^{(j-1)}\text{-deg } g + a_1(\ell_1^{(j)} - \ell_1^{(j-1)}) + \cdots + a_k(\ell_k^{(j)} - \ell_k^{(j-1)}), \\ \mathrm{Q}^{(j)}\text{-deg } h = \mathrm{Q}^{(j-1)}\text{-deg } h + a_1'(\ell_1^{(j)} - \ell_1^{(j-1)}) + \cdots + a_k'(\ell_k^{(j)} - \ell_k^{(j-1)}). \end{cases} \tag{4.21}$$

By (4.19) we have $\mathrm{Q}^{(j)}$-deg $g = \mathrm{Q}^{(j)}$-deg h and $\mathrm{Q}^{(j-1)}$-deg $g = \mathrm{Q}^{(j-1)}$-deg h. Note that $a_i = a_i'$ for any $i < k$ by inductive assumption. Therefore

$$a_k(\ell_k^{(j)} - \ell_k^{(j-1)}) = a_k'(\ell_k^{(j)} - \ell_k^{(j-1)}). \tag{4.22}$$

Since $\ell_k^{(j)} - \ell_k^{(j-1)} > 0$, we have $a_k = a_k'$ and the proof is completed. $\square$

4.2. **Perfect parameters.** In this subsection, we will adjust parameters and take a coordinate transformation to obtain a new weight type meeting our needs, which is crucial in the proof of Main Theorem 1.5.

We say a coordinate transformation

$$x_i' = \phi_i(x_1, \cdots, x_n), \quad i = 1, \cdots, n, \tag{4.23}$$

preserves the weight type $(w_1, \cdots, w_n)$ if each ϕ_i is a weighted homogeneous polynomial of degree w_i with respect to the weight type $(w_1, \cdots, w_n)$. The weight of the new variable x_i' is still w_i.

Theorem 4.5. *Let P_n be the polynomial ring of n variables $x_1, \cdots, x_n$ with positive integer weights $w_1 \geq w_2 \geq \cdots w_n$ $(n \geq 2)$. Fix a positive real number $M > w_1 w_2$. Let D be a non-zero negative weight derivation on P_n as in* (3.2). *Then after a coordinate transformation which preserves the weight type $(w_1, \cdots, w_n)$, in the new coordinate system there exist parameters $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ such that the new weight type $(\ell_1, \ell_2, \ldots, \ell_n)$ associated with D and controlled by these parameters has the following properties:*

(1) *there exists $i_0 \in \{1, \cdots, n\}$ such that*

(1a) *$\epsilon_{i_0} < \epsilon_i$ for any $i = 1, \cdots, n$ such that $p_i \neq 0$ and $i \neq i_0$;*

(1b) *$\ell_{i_0}/w_{i_0} > \ell_i/w_i$ for any $i = 1, \cdots, n$ such that $i \neq i_0$;*

(1c) *p_{i_0} is a non-zero polynomial;*

(2) *for any two monic monomials g, h such that $\deg g, \deg h < M$, we have*

$$Q\text{-}deg\ g = Q\text{-}deg\ h \Longleftrightarrow g = h. \tag{4.24}$$

To prove this theorem, we rewrite it into a more complicated form (Proposition 4.6) that can be proved by induction. We will prove Proposition 4.6 first and use it to prove Theorem 4.5.

Proposition 4.6. *Fix a positive real number $M > w_1 w_2$ and a non-zero negative weight derivation D on P_n as in* (3.2). *Let I be non-empty subset of $\{1, 2, \ldots, n\}$ with k elements and let $b : i \mapsto b_i$ be a one to one mapping from $\{1, 2, \ldots, n\} \setminus I$ to $\{1, 2, \ldots, n-k\}$. Set parameters $\epsilon_1, \cdots, \epsilon_n$ as follows:*

$$\epsilon_i = \begin{cases} 1 & i \in I \text{ and } p_i \neq 0, \\ 0 & i \in I \text{ and } p_i = 0, \\ 1 + 1/M^{b_i} & i \notin I \text{ and } p_i \neq 0, \\ 1/M^{b_i} & i \notin I \text{ and } p_i = 0. \end{cases} \tag{4.25}$$

Let $(\ell_1, \cdots, \ell_n)$ be the new weight type associated with D and controlled by parameters $\epsilon_1, \cdots, \epsilon_n$. Denote $I_{\max} = \{e \mid \ell_e/w_e \geq \ell_i/w_i, i = 1, \ldots, n\}$ and $J = \{e \mid p_e \neq 0\}$. Suppose $I_{\max} \subseteq I$ and $I_{\max} \subseteq J$. Then after a coordinate transformation which preserves the original weight type $(w_1, \cdots, w_n)$ (we denote the new coordinate system by $x_1', \cdots, x_n'$), one of the following two assertions holds.

(a) *In the new coordinate system, there exists another group of parameter* $(\epsilon'_1, \cdots, \epsilon'_n)$ *such that the new weight type* $(\ell'_1, \cdots, \ell'_n)$ *controlled by these parameters satisfies Theorem 4.5(1) and (2).*

(b) *In the new coordinate system, if we write*

$$D = p'_1 \frac{\partial}{\partial x'_1} + p'_2 \frac{\partial}{\partial x'_2} + \cdots + p'_n \frac{\partial}{\partial x'_n} \tag{4.26}$$

and denote $J' = \{e \mid p'_e \neq 0\}$, *then* $\#J' < \#J$.

Remark 4.7. When $\#J = 1$, the assertion (b) can not hold because J' can not be the empty set (since $D \neq 0$).

Proof. We prove the proposition by induction on the cardinality of I. When $\#I = 1$, denote the unique element in I by i_0. Since $I_{\max} \subseteq I$, $I_{\max} = \{i_0\}$. Hence

$$\ell_{i_0}/w_{i_0} > \ell_i/w_i \tag{4.27}$$

for any $i = 1, \cdots, n$ with $i \neq i_0$. Since $i_0 \in I_{\max} \subseteq J$, we have p_{i_0} is a non-zero polynomial, which implies that $\epsilon_{i_0} = 1$ (see (4.25)). For any i such that $p_i \neq 0$ and $i \neq i_0$ (so $i \notin I$), we have $\epsilon_i \geq 1 + M^{n-1}$ (see (4.25)). Define a new group of parameters $(\epsilon'_1, \cdots, \epsilon'_n)$ as follows:

$$\epsilon'_i = \begin{cases} \epsilon_i & i \neq i_0, \\ \epsilon_i + 1/M^n & i = i_0. \end{cases} \tag{4.28}$$

Then $\epsilon'_{i_0} = 1 + 1/M^n$. For any i such that $p_i \neq 0$ and $i \neq i_0$, we have

$$\epsilon'_i = \epsilon_i \geq 1 + 1/M^{n-1} > \epsilon'_{i_0}. \tag{4.29}$$

Let $(\ell'_1, \cdots, \ell'_n)$ be the new weight type controlled by parameters $\epsilon'_1, \cdots, \epsilon'_n$. By (4.27) and Corollary 4.2 (take $\epsilon = 1/M^{n-1}$,$\Delta = 1/M^n$, note that $M > w_1 w_2$), we have

$$\ell'_{i_0}/w_{i_0} > \ell'_i/w_i \tag{4.30}$$

for any $i = 1, \cdots, n$ with $i \neq i_0$. Hence the new weight type $(\ell'_1, \cdots, \ell'_n)$ satisfies Theorem 4.5(1). By Proposition 4.4 it also satisfies Theorem 4.5(2). Therefore the assertion (a) holds in the case that $\#I = 1$.

Suppose that Proposition 4.6 holds when $\#I = 1, \cdots, k-1$. Now consider that case that $\#I = k$ $(k \geq 2)$. There are two sub-cases: $I_{\max} = I$ or $I_{\max}$ is a proper subset of I.

(1) Suppose that $I_{\max}$ is a proper subset of I. Take $j_0 \in I \setminus I_{\max}$. Define another group of parameters as follows:

$$\epsilon'_i = \begin{cases} \epsilon_i + 1/(w_1 w_2)^{n-k+1} & i = j_0, \\ \epsilon_i & i \neq j_0. \end{cases} \tag{4.31}$$

Let $(\ell'_1, \ldots, \ell'_n)$ be the new weight type controlled by parameters $\epsilon'_1, \cdots, \epsilon'_n$. Denote $I'_{\max} = \{e \mid \ell'_e/w_e \geq \ell'_i/w_i, i = 1, \ldots, n\}$.

We claim that $I'_{\max} \subseteq I_{\max}$. Indeed, take $k_0 \in I_{\max}$, then $\ell_i/w_i < \ell_{k_0}/w_{k_0}$ for any $i \notin I_{\max}$. By Corollary 4.2 (take $\epsilon = 1/M^{n-k}$ and $\Delta = 1/M^{n-k+1}$, note that $M > w_1w_2$), we have $\ell'_i/w_i < \ell'_{k_0}/w_{k_0}$, which implies that $i \notin I'_{\max}$. Therefore the claim that $I'_{\max} \subseteq I_{\max}$ holds, which implies that $I'_{\max} \subseteq I \setminus \{j_0\}$.

For any $i \in I'_{\max}$, we have $i \in I_{\max} \subseteq J$, so p_i is a non-zero polynomial. Let $I' = I \setminus \{j_0\}$, then $\#I' = k-1$ and $I'_{\max} \subseteq I'$. By inductive assumption, Proposition 4.6 holds.

(2) Suppose that $I_{\max} = I$. Write $I = I_{\max} = \{i_1, \ldots, i_k\}$ and suppose that $i_1 < i_2 < \cdots < i_k$. For any $j \in I = I_{\max} \subseteq J$, since p_j is a non-zero polynomial, by (4.25) we have $\epsilon_j = 1$. That is to say, $\epsilon_{i_e} = 1$ for any $e = 1, \cdots, k$.

We claim that

$$(p_{i_{k-1}})_{\max} = c x_{i_k}^a (p_{i_k})_{\max}, \tag{4.32}$$

where c is a non-zero coefficient and a is a non-negative integer. We will give the proof of this claim in Lemma 4.8 below since it is too long.

Note that $\deg(p_{i_{k-1}})_{\max} = w_{i_{k-1}} + \text{wt } D$ and $\deg(p_{i_k})_{\max} = w_{i_k} + \text{wt } D$, we have $w_{i_{k-1}} + \text{wt } D = a w_{i_k} + w_{i_k} + \text{wt } D$, which implies that

$$w_{i_{k-1}} = (a+1) w_{i_k}. \tag{4.33}$$

Since $i_{k-1}, i_k \in I_{\max}$, we have $\ell_{i_{k-1}}/w_{i_{k-1}} = \ell_{i_k}/w_{i_k}$. Hence

$$\ell_{i_{k-1}} = (a+1)\ell_{i_k}. \tag{4.34}$$

Next we will apply a coordinate transformation which preserves the original weight type $(w_1, \cdots, w_n)$. The coordinate change is of the following form:

$$\begin{aligned} x_1 &= x'_1 \\ &\cdots\cdots \\ x_{i_{k-1}} &= x'_{i_{k-1}} + c(x'_{i_k})^{a+1}/(a+1) \\ &\cdots\cdots \\ x_n &= x'_n. \end{aligned} \tag{4.35}$$

Then we have

$$\frac{\partial}{\partial x'_1} = \frac{\partial}{\partial x_1}$$
$$\cdots\cdots$$
$$\frac{\partial}{\partial x'_{i_{k-1}}} = \frac{\partial}{\partial x_{i_{k-1}}} \tag{4.36}$$
$$\frac{\partial}{\partial x'_{i_k}} = \frac{\partial}{\partial x_{i_k}} + c(x'_{i_k})^a \frac{\partial}{\partial x_{i_{k-1}}}$$
$$\cdots\cdots$$
$$\frac{\partial}{\partial x'_n} = \frac{\partial}{\partial x_n}.$$

In the new coordinate system, we write

$$D = p'_1 \frac{\partial}{\partial x'_1} + p'_2 \frac{\partial}{\partial x'_2} + \cdots + p'_n \frac{\partial}{\partial x'_n}. \tag{4.37}$$

It follows from (4.36) that

$$\begin{cases} p'_j = p_j \quad \text{if } j \neq i_{k-1}, \\ p'_{i_{k-1}} = p_{i_{k-1}} - c(x'_{i_k})^a p_{i_k} = p_{i_{k-1}} - c x^a_{i_k} p_{i_k}. \end{cases} \tag{4.38}$$

Denote $J' = \{e \mid p'_e \neq 0\}$. If $p'_{i_{k-1}}$ is the zero polynomial, then $\#J' = \#J - 1$, which implies that assertion (b) holds. So we may suppose that $p'_{i_{k-1}} \neq 0$. In this case we have

$$J' = J. \tag{4.39}$$

Let $(\ell'_1, \ldots, \ell'_n)$ be the new weight type of the new coordinate system $(x'_1, \cdots, x'_n)$ associated with D and controlled by the same group of parameters $(\epsilon_1, \ldots, \epsilon_n)$. We denote by Q$'$-deg the degree with respect to the new weight type $(\ell'_1, \ldots, \ell'_n)$ (cf. Definition 3.3).

We claim that $\ell'_j = \ell_j$ if $j > i_{k-1}$. Indeed, for any $j > i_{k-1}$ we have $p'_j = p_j$. Since p_j is independent of the variable $x_{i_{k-1}}$, the expansion of p_j in the original coordinate system is the same of that of p'_j in the original coordinate system (because $x_l = x'_l$ for any $l \neq i_{k-1}$). Therefore $\ell'_j = \ell_j$ for any $j > i_{k-1}$.

We claim that $\ell'_{i_{k-1}} < \ell_{i_{k-1}}$. Indeed, by (4.32) and the second equality in (4.38), we have Q-deg $p'_{i_{k-1}} <$ Q-deg $p_{i_{k-1}}$. Since $p'_{i_{k-1}}$ is a polynomial of only variables x'_j $(j > i_{k-1})$, it has the same expansion in the original and new coordinate system (because $x'_j = x_j$ for any $j > i_{k-1}$). Note that $\ell'_j = \ell_j$ for any $j > i_{k-1}$, we have

$$\text{Q}'\text{-deg } p'_{i_{k-1}} = \text{Q-deg } p'_{i_{k-1}} < \text{Q-deg } p_{i_{k-1}}, \tag{4.40}$$

which implies that $\ell'_{i_{k-1}} < \ell_{i_{k-1}}$.

We claim that $\ell'_j \le \ell_j$ for any $j = 1, \cdots, n$. Indeed, by the above argument we know that this claim holds for $j \ge i_{k-1}$. Suppose that the claim holds for $j+1, j+2, \ldots, n$, we will prove it holds for j (here $j < i_{k-1}$). Take a monomial $g = x_{j+1}^{a_{j+1}} \ldots x_n^{a_n}$ in the expansion of p_j in the original coordinate system. Then in the new coordinate system

$$g = (x'_{j+1})^{a_{j+1}} \ldots \left(x'_{i_{k-1}} + \frac{c}{a+1}(x'_{i_k})^{a+1}\right)^{a_{i_{k-1}}} \ldots (x'_n)^{a_n}. \tag{4.41}$$

Note that

$$\mathrm{Q}'\text{-deg } (x'_{i_k})^{a+1} = (a+1)\ell'_{i_k} = (a+1)\ell_{i_k} = \ell_{i_{k-1}} \tag{4.42}$$

(The last equality follows from (4.34)). By inductive assumption and (4.42) we have Q′-deg $g \le$ Q-deg g. Hence Q′-deg $p_j \le$ Q-deg p_j. Note that $p'_j = p_j$ (see (4.38), here $j < i_{k-1}$), we have Q′-deg $p'_j \le$ Q-deg p_j, which implies that $\ell'_j \le \ell_j$ and the claim holds.

Denote $I'_{\max} = \{e \mid \ell'_e/w_e \ge \ell'_i/w_i, i = 1, \ldots, n\}$. For any $i \notin I_{\max}$, we have

$$\ell'_i/w_i \le \ell_i/w_i < \ell_{i_k}/w_{i_k} = \ell'_{i_k}/w_{i_k}, \tag{4.43}$$

which implies that $i \notin I'_{\max}$. Therefore $I'_{\max} \subseteq I_{\max}$.

Note that

$$\ell'_{i_{k-1}}/w_{i_{k-1}} < \ell_{i_{k-1}}/w_{i_{k-1}} = \ell_{i_k}/w_{i_k} = \ell'_{i_k}/w_{i_k}, \tag{4.44}$$

we have $I'_{\max} \subseteq I_{\max} \setminus \{i_{k-1}\}$. Hence $I'_{\max}$ is a proper subset of $I_{\max} = I$.

Since $I'_{\max} \subseteq I_{\max} \subseteq J$ and $J = J'$ (see (4.39)), the condition that $I'_{\max} \subseteq J'$ in Proposition 4.6 is still satisfied. Thus we successfully reduce the case that $I_{\max} = I$ to case that $I'_{\max}$ is a proper subset of I, which has been solved.

□

Lemma 4.8. *Under the assumptions of Proposition 4.6, we additionally suppose that $k = \#I \ge 2$ and $I = I_{\max}$. Write $I = \{i_1, i_2, \cdots, i_k\}$, where $i_1 < i_2 < \cdots < i_k$. Then*

$$(p_{i_{k-1}})_{\max} = c x_{i_k}^a (p_{i_k})_{\max}, \tag{4.45}$$

where c is a non-zero coefficient and a is a non-negative integer.

Proof. Let d be the inverse mapping of $b : i \mapsto b_i$. That is to say, $b_{d(i)} = i$ for any $i = 1, 2, \ldots, n-k$. We define by induction a sequence of parameters $\epsilon_i^{(0)}, \epsilon_i^{(1)}, \ldots, \epsilon_i^{(n-k)}$. First we set $(\epsilon_1^{(0)}, \ldots, \epsilon_n^{(0)})$ as follow:

$$\epsilon_i^{(0)} = \begin{cases} 1 & p_i \ne 0, \\ 0 & p_i = 0. \end{cases} \tag{4.46}$$

Suppose that the $(j-1)$-th group of parameters $(\epsilon_1^{(j-1)}, \ldots, \epsilon_n^{(j-1)})$ has been defined. We define the j-th group of parameters ($j > 0$) as follows:

$$\epsilon_i^{(j)} = \begin{cases} \epsilon_i^{(j-1)} + 1/M^j & i = d(j), \\ \epsilon_i^{(j-1)} & i \ne d(j). \end{cases} \tag{4.47}$$

It is easy to see that the $(n-k)$-th group of parameters $(\epsilon_1^{(n-k)}, \cdots, \epsilon_n^{(n-k)}) = (\epsilon_1, \cdots, \epsilon_n)$. Let $(\ell_1^{(j)}, \cdots, \ell_n^{(j)})$ be the new weight type controlled by the j-th group parameters $\epsilon_i^{(j)}$. Then $(\ell_1^{(n-k)}, \cdots, \ell_n^{(n-k)}) = (\ell_1, \cdots, \ell_n)$. We denote the degree with respect to the new weight type $(\ell_1^{(j)}, \cdots, \ell_n^{(j)})$ by $\mathrm{Q}^{(j)}$-deg (cf. Definition 3.3). Note that $\mathrm{Q}^{(n-k)}$-deg $h =$ Q-deg h for any polynomial h.

For convenience we denote $i_{k-1} = s$ and $i_k = t$. Then $s < t$. Since $s, t \in I_{\max}$, we have $\ell_s/w_s = \ell_t/w_t$, which implies that $\ell_s^{(n-k)}/w_s = \ell_t^{(n-k)}/w_t$. We claim

$$\ell_s^{(j)}/w_s = \ell_t^{(j)}/w_t \quad \forall j = 0, \cdots, n-k. \tag{4.48}$$

Indeed, if this is not the case, then there exists j such that $\ell_s^{(j)}/w_s \neq \ell_t^{(j)}/w_t$. Without loss of generality we may suppose that $\ell_s^{(j)}/w_s < \ell_t^{(j)}/w_t$. By Corollary 4.2 (take $\epsilon = 1/M^j$and $\Delta = 1/M^{j+1}$, note that $M > w_1 w_2$), we have $\ell_s^{(j+1)}/w_s < \ell_t^{(j+1)}/w_t$. Apply Corollary 4.2 again (Take $\epsilon = 1/M^{j+1}$ and $\Delta = 1/M^{j+2}$) we obtain $\ell_s^{(j+2)}/w_s < \ell_t^{(j+2)}/w_t$. Repeat the above process and finally we obtain $\ell_s^{(n-k)}/w_s < \ell_t^{(n-k)}/w_t$. This is a contradiction and hence (4.48) holds.

Since $s, t \in I = I_{\max}$, by the assumption that $I_{\max} \subseteq J$ in Proposition 4.6, we have p_s and p_t are non-zero polynomials. We claim that

$(*)$ For any monic monomial $g_s \in (p_s)_{\max}$ and $g_t \in (p_t)_{\max}$, there exists a non-negative integer a such that $g_s = x_t^a g_t$.

Here $(p_s)_{\max}$ (resp. $(p_t)_{\max}$) is the sum of terms in the expansion of p_s (resp. p_t) with maximum degrees with respect to the new weight type $(\ell_1, \cdots, \ell_n)$. We will prove Claim $(*)$ in five steps.

Step 1. First we prove that

$$\begin{cases} \ell_s^{(j)} = \mathrm{Q}^{(j)}\text{-deg } g_s + 1, \\ \ell_t^{(j)} = \mathrm{Q}^{(j)}\text{-deg } g_t + 1, \end{cases} \tag{4.49}$$

for any $j = 0, \cdots, n-k$. We will only show the first equality since the same argument can be used to prove the second equality. For any monomial $h \in p_s$, since $g_s \in (p_s)_{\max}$, we have Q-deg $h \leq$ Q-deg g_s. So $\mathrm{Q}^{(n-k)}$-deg $h \leq \mathrm{Q}^{(n-k)}$-deg g_s. We claim that

$$\mathrm{Q}^{(j)}\text{-deg } h \leq \mathrm{Q}^{(j)}\text{-deg } g_s \quad \forall j = 0, \cdots, n-k. \tag{4.50}$$

Indeed, if this is not the case, then $\mathrm{Q}^{(j)}$-deg $h > \mathrm{Q}^{(j)}$-deg g_s for some j. Since $\deg p_s < w_s \leq w_1 w_2 < M$, by Corollary 4.3 (take $\epsilon = 1/M^j$ and $\Delta = 1/M^{j+1}$), we have $\mathrm{Q}^{(j+1)}$-deg $h > \mathrm{Q}^{(j+1)}$-deg g_s. Apply Corollary 4.3 again (Take $\epsilon = 1/M^{j+1}$ and $\Delta = 1/M^{j+2}$) we obtain $\mathrm{Q}^{(j+2)}$-deg $h > \mathrm{Q}^{(j+2)}$-deg g_s. Repeat this process and finally we obtain $\mathrm{Q}^{(n-k)}$-deg $h > \mathrm{Q}^{(n-k)}$-deg g_s. This is a contradiction and hence the inequality (4.50) holds.

Therefore

$$\mathrm{Q}^{(j)}\text{-deg } p_s = \mathrm{Q}^{(j)}\text{-deg } g_s \quad \forall j = 0, \cdots, n-k. \tag{4.51}$$

Since $s \in I = I_{\max} \subseteq J$, we have $p_s \neq 0$ and $\epsilon_s^{(j)} = 1$ for any $j = 0, \cdots, n-k$. It follows from (3.8) that $\ell_s^{(j)} = \mathrm{Q}^{(j)}\text{-deg } g_s + 1$.

Step 2. Since $g_s \in (p_s)_{\max}$, g_s is a monomial of variables $x_{s+1}, \cdots, x_n$. We will show that g_s is independent of variables $x_{s+1}, \cdots, x_{t-1}$, that is to say, g_s is a monomial of variables $x_t, x_{t+1}, \cdots, x_n$. Indeed, if this is not the case, there exists $e \in \{s+1, \ldots, t-1\}$ such that the exponent of x_e in g_s is positive. Denote $j = b_e$, then $d(j) = e$. By Proposition 4.1 we have

$$\begin{cases} \ell_i^{(j-1)} = \ell_i^{(j)} & \forall i > e, \\ \ell_e^{(j-1)} < \ell_e^{(j)}, \\ \ell_i^{(j-1)} \leq \ell_i^{(j)} & \forall i < e. \end{cases} \tag{4.52}$$

Note that g_t is a monomial of variables $x_{t+1}, \ldots, x_n$ and $t+1 > e$, we have

$$\mathrm{Q}^{(j-1)}\text{-deg } g_t = \mathrm{Q}^{(j)}\text{-deg } g_t. \tag{4.53}$$

Since the exponent of x_e in g_s is positive, we have

$$\mathrm{Q}^{(j-1)}\text{-deg } g_s < \mathrm{Q}^{(j)}\text{-deg } g_s. \tag{4.54}$$

By (4.48) and (4.49) we have

$$\begin{cases} \mathrm{Q}^{(j-1)}\text{-deg } g_s + 1)/w_s = (\mathrm{Q}^{(j-1)}\text{-deg } g_t + 1)/w_t, \\ \mathrm{Q}^{(j)}\text{-deg } g_s + 1)/w_s = (\mathrm{Q}^{(j)}\text{-deg } g_t + 1)/w_t. \end{cases} \tag{4.55}$$

The equations (4.53) and (4.54) are in contradiction with (4.55). Therefore g_s is a monomial of only variables $x_t, x_{t+1}, \cdots, x_n$.

Step 3. Write $g_s = x_t^a g_s'$, here g_s' is a monomial of variables $x_{t+1}, \cdots, x_n$ and a is a non-negative integer. By (4.48),(4.49), the fact that $w_s = \deg g_s -$ wt D and $w_t = \deg g_t -$ wt D, we have

$$\begin{aligned} \frac{\ell_t^{(j)}}{w_t} = \frac{\ell_s^{(j)}}{w_s} &= \frac{\mathrm{Q}^{(j)}\text{-deg } g_t + 1}{\deg g_t - \text{wt } D} \\ &= \frac{\mathrm{Q}^{(j)}\text{-deg } g_s + 1}{\deg g_s - \text{wt } D} \\ &= \frac{a\ell_t^{(j)} + \mathrm{Q}^{(j)}\text{-deg } g_s' + 1}{aw_t + \deg g_s' - \text{wt } D} \end{aligned} \tag{4.56}$$

for any $j = 0, 1, \ldots, n-k$. Hence

$$\begin{aligned} \frac{\ell_t^{(j)}}{w_t} = \frac{\ell_s^{(j)}}{w_s} &= \frac{\mathrm{Q}^{(j)}\text{-deg } g_t + 1}{\deg g_t - \text{wt } D} \\ &= \frac{\mathrm{Q}^{(j)}\text{-deg } g_s' + 1}{\deg g_s' - \text{wt } D} \end{aligned} \tag{4.57}$$

for any $j = 0, 1, \ldots, n-k$. Therefore

$$\frac{\mathrm{Q}^{(j)}\text{-deg } g_t - \mathrm{Q}^{(j-1)}\text{-deg } g_t}{\deg g_t - \mathrm{wt}\ D} = \frac{\mathrm{Q}^{(j)}\text{-deg } g'_s - \mathrm{Q}^{(j-1)}\text{-deg } g'_s}{\deg g'_s - \mathrm{wt}\ D} \tag{4.58}$$

for any $j = 1, \ldots, n-k$.

Step 4. Write $g'_s = x_{t+1}^{\alpha_{t+1}} \ldots x_n^{\alpha_n}$ and $g_t = x_{t+1}^{\beta_{t+1}} \ldots x_n^{\beta_n}$. In this step we will prove

$$\frac{\alpha_i}{\deg g'_s - \mathrm{wt}\ D} = \frac{\beta_i}{\deg g_t - \mathrm{wt}\ D} \qquad \forall i = t+1, \ldots, n \tag{4.59}$$

by induction on i.

Suppose (4.59) holds for $t+1, t+2, \ldots, i-1$ and we will show it holds for i. Denote $j = b_i$, then $d(j) = i$. By Proposition 4.1(1) and (2), we have $\ell_i^{(j)} - \ell_i^{(j-1)} > 0$ and $\ell_{i+1}^{(j)} - \ell_{i+1}^{(j-1)} = \cdots = \ell_n^{(j)} - \ell_n^{(j-1)} = 0$. Hence

$$\begin{cases} \mathrm{Q}^{(j)}\text{-deg } g'_s - \mathrm{Q}^{(j-1)}\text{-deg } g'_s = \alpha_{t+1}(\ell_{t+1}^{(j)} - \ell_{t+1}^{(j-1)}) + \cdots + \alpha_i(\ell_i^{(j)} - \ell_i^{(j-1)}), \\ \mathrm{Q}^{(j)}\text{-deg } g_t - \mathrm{Q}^{(j-1)}\text{-deg } g_t = \beta_{t+1}(\ell_{t+1}^{(j)} - \ell_{t+1}^{(j-1)}) + \cdots + \beta_i(\ell_i^{(j)} - \ell_i^{(j-1)}). \end{cases} \tag{4.60}$$

By (4.58), (4.60) and the inductive assumption for $t+1, t+2, \cdots, i-1$, we have $\alpha_i/(\deg g'_s - \mathrm{wt}\ D) = \beta_i/(\deg g_t - \mathrm{wt}\ D)$. Therefore (4.59) holds.

Step 5. In this step we will prove $\deg g'_s - \mathrm{wt}\ D = \deg g_t - \mathrm{wt}\ D$. Suppose $\deg g'_s - \mathrm{wt}\ D > \deg g_t - \mathrm{wt}\ D$, then by (4.59) we have $\alpha_i \geq \beta_i$ for any $i = t+1, \ldots, n$ (it is possible that $\alpha_i = \beta_i = 0$). Let $h = x_{t+1}^{\alpha_{t+1} - \beta_{t+1}} \ldots x_n^{\alpha_n - \beta_n}$, then $g'_s = h g_t$. If $h = 1$, then $g'_s = g_t$ and the claim $\deg g'_s - \mathrm{wt}\ D = \deg g_t - \mathrm{wt}\ D$ holds. So we may suppose that h is not a constant, i.e. there exists $i \in \{t+1, \cdots, n\}$ such that $\alpha_i > \beta_i$. Note that $(\ell_1^{(n-k)}, \ldots, \ell_n^{(n-k)}) = (\ell_1, \ldots, \ell_n)$, by (4.57) (take $j = n-k$) we have

$$\begin{aligned} \frac{\ell_t}{w_t} = \frac{\ell_s}{w_s} &= \frac{\mathrm{Q}\text{-deg } g_t + 1}{\deg g_t - \mathrm{wt}\ D} \\ &= \frac{\mathrm{Q}\text{-deg } g'_s + 1}{\deg g'_s - \mathrm{wt}\ D} \\ &= \frac{\mathrm{Q}\text{-deg } h + \mathrm{Q}\text{-deg } g_t + 1}{\deg h + \deg g_t - \mathrm{wt}\ D}. \end{aligned} \tag{4.61}$$

It follows that

$$\frac{\ell_t}{w_t} = \frac{\mathrm{Q}\text{-deg } h}{\deg h}. \tag{4.62}$$

Since $t \in I_{\max}$ and $t+1, \cdots, n \notin I_{\max}$, we have

$$\max\left\{\frac{\ell_{t+1}}{w_{t+1}}, \frac{\ell_{t+2}}{w_{t+2}}, \cdots, \frac{\ell_n}{w_n}\right\} < \frac{\ell_t}{w_t}. \tag{4.63}$$

Since h is a monomial of variables $x_{t+1}, \ldots, x_n$, we have $\mathrm{Q}\text{-deg } h/\deg h < \ell_t/w_t$, which contradicts (4.62). Hence the assumption that $\deg g'_s - \mathrm{wt}\ D > \deg g_t - \mathrm{wt}\ D$ does not hold. By the same argument, it is impossible that $\deg g'_s - \mathrm{wt}\ D < \deg g_t - \mathrm{wt}\ D$. Hence $\deg g'_s - \mathrm{wt}\ D = \deg g_t - \mathrm{wt}\ D$. By

(4.59), we have $\alpha_i = \beta_i$ for any $i = t+1, \dots, n$, which implies that $g'_s = g_t$. Therefore $g_s = x_t^a g_t$ and Claim $(*)$ holds.

Fix a monic monomial $h \in (p_t)_{\max}$. For any two monic monomials $g_1, g_2 \in (p_s)_{\max}$, by Claim $(*)$, there exist non-negative integers a_1 and a_2 such that $g_1 = x_t^{a_1} h$ and $g_2 = x_t^{a_2} h$. Since p_s is weighted homogeneous with respect to $(w_1, \cdots, w_n)$, we have $\deg g_1 = \deg g_2$. So

$$a_1 w_t + \deg h = a_2 w_t + \deg h, \tag{4.64}$$

Since $w_t > 0$, we have $a_1 = a_2$. Hence $g_1 = g_2$, which implies that $(p_s)_{\max}$ is a monomial.

Fix a monic monomial $g \in (p_s)_{\max}$. For any two monic monomials $h_1, h_2 \in (p_t)_{\max}$, by Claim $(*)$, there exist non-negative integers a_1 and a_2 such that $g = x_t^{a_1} h_1$ and $g = x_t^{a_2} h_2$. Hence $x_t^{a_1} h_1 = x_t^{a_2} h_2$. Since p_t is a polynomial of variables $x_{t+1}, \cdots, x_n$, we have h_1, h_2 are independent of the variable x_t, which implies that $h_1 = h_2$. Therefore $(p_t)_{\max}$ is a monomial.

Since $(p_s)_{\max}$ and $(p_t)_{\max}$ are monomials, by Claim $(*)$, we have $(p_s)_{\max} = c x_t^a (p_t)_{\max}$, where c is a non-zero coefficient and a is a non-negative integer. Therefore Lemma 4.8 holds. □

Now we return to the proof of Theorem 4.5.

Proof of Theorem 4.5. Denote $J = \{e \mid p_e \neq 0\}$. We prove the theorem by induction on the cardinality of J. Suppose the theorem holds for $\#J = 1, \cdots, r-1$. Consider the case that $\#J = r$. Set parameters $\epsilon_1, \cdots, \epsilon_n$ as follows:

$$\epsilon_i = \begin{cases} 1 & \text{if } p_i \neq 0, \\ 0 & \text{if } p_i = 0. \end{cases} \tag{4.65}$$

Let $(\ell_1, \dots, \ell_n)$ be the new weight type associated with D and controlled by the ϵ_i. Then

$$\begin{cases} \ell_i > 0 & \text{if } p_i \neq 0, \\ \ell_i = 0 & \text{if } p_i = 0. \end{cases} \tag{4.66}$$

Hence p_i is a non-zero polynomial for any $i \in I_{\max}$. Let $I = \{1, 2, \cdots, n\}$, then $I_{\max} \subseteq I$. All conditions in Proposition 4.6 are satisfied. Hence either the assertion (a) or the assertion (b) in Proposition 4.6 holds. If the assertion (a) in Proposition 4.6 holds, then Theorem 4.5 is proved. If the assertion (b) holds, then by inductive assumption the proof of Theorem 4.5 is completed (note that the assertion (b) can not hold when $\#J = 1$, see Remark 4.7). □

5. Proof of Theorem 1.5

Lemma 5.1. *Let $\mathfrak{m}$ be the maximal ideal of P_n generated by $x_1, \cdots, x_n$. Let $f_1, f_2, \dots, f_m \in \mathfrak{m}$ such that $P_n/(f_1, f_2, \dots, f_m)$ is an Artinian algebra. Then for any $i \in \{1, 2, \dots, n\}$, there exists $j \in \{1, 2, \dots, m\}$ and a positive integer a such that $x_i^a \in f_j$.*

Proof. Assume there exists $i \in \{1, 2, \ldots, n\}$ such that $x_i^a \notin f_j$ for any positive integer a and any $j \in \{1, 2, \ldots, m\}$. Then

$$(f_1, \cdots, f_m) \subseteq (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n). \tag{5.1}$$

Since $P_n/(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ is a vector space of infinite dimension, we have $P_n/(f_1, \cdots, f_m)$ is also of infinite dimension as a vector space, which is in contradiction with that it is an Artinian algebra. $\square$

Proof of Theorem 1.5. When $n = 1$, the theorem holds obviously. So we may suppose that $n \geq 2$. For convenience, we denote $f_i = \partial f/\partial x_i$ for any $i = 1, \cdots, n$. Suppose that $P_n/(f_1, \cdots, f_n)$ has a non-zero negative weight derivation. By Lemma 2.1, there exists a non-zero negative weight derivation D on P_n as in (3.2) such that $Df_j \in (f_1, f_2, \ldots, f_n)$ for any $j = 1, \cdots, n$.

By Theorem 4.5, after a coordinate transformation which preserves the weight type $(w_1, \cdots, w_n)$, there exists a group of parameters $(\epsilon_1, \cdots, \epsilon_n)$ such that the assertions in Theorem 4.5 hold. Let $(\ell_1, \cdots, \ell_n)$ be the new weight type associated with D and controlled by the ϵ_i. By Theorem 4.5(1b), there exists a unique $i_0 \in \{1, \cdots, n\}$ such that $\ell_{i_0}/w_{i_0} > \ell_i/w_i$ for any $i \neq i_0$. By Lemma 5.1, there exist $e \in \{1, \cdots, n\}$ and a positive integer a such that $x_{i_0}^a \in f_e$. Note that f_e is weighted homogeneous with respect to the original weight type, we have

$$(f_e)_{\max} = c x_{i_0}^a, \tag{5.2}$$

where c is a non-zero coefficient. Hence

$$\deg f_e = a w_{i_0}. \tag{5.3}$$

By Proposition 3.4 and Theorem 4.5(1a)(1c), we have

$$D_{\max} = (p_{i_0})_{\max} \partial/\partial_{x_{i_0}}. \tag{5.4}$$

We define polynomials $g_0, g_1, \cdots, g_a$ as follows:

$$\begin{cases} g_0 = f_e & i = 0, \\ g_i = D g_{i-1} & 1 \leq i \leq a. \end{cases} \tag{5.5}$$

Then $g_i \in (f_1, f_2, \cdots, f_n)$ for any $i = 0, \cdots, a$. We claim that

$$(g_i)_{\max} = \frac{c(a!)}{(a-i)!} (p_{i_0})_{\max}^i x_{i_0}^{a-i} \quad \forall i = 0, 1, \cdots, a. \tag{5.6}$$

Indeed, when $i = 0$, $(g_0)_{\max} = (f_e)_{\max} = c x_{i_0}^a$and the claim holds. Suppose the claim holds for $i = k - 1$ $(1 \leq k \leq a)$, then

$$\begin{aligned} D_{\max}(g_{k-1})_{\max} &= (p_{i_0})_{\max} \frac{c(a!)}{(a-k+1)!} \frac{\partial \big((p_{i_0})_{\max}^{k-1} x_{i_0}^{a-k+1} \big)}{\partial x_{i_0}} \\ &= \frac{c(a!)}{(a-k)!} (p_{i_0})_{\max}^k x_{i_0}^{a-k} \neq 0. \end{aligned} \tag{5.7}$$

Here the second equality follows from the fact that p_{i_0} is independent of the variable x_{i_0}. By Proposition 3.5(1),

$$(g_k)_{\max} = (Dg_{k-1})_{\max} = D_{\max}(g_{k-1})_{\max}$$

and hence the claim (5.6) holds. In particular, when $i = a$ we have

$$(g_a)_{\max} = c(a!) \cdot (p_{i_0})^a_{\max} \neq 0. \tag{5.8}$$

Hence $g_a \neq 0$. Since $g_a \in (f_1, \cdots, f_n)$, there exists $d \in \{1, \cdots, n\}$ such that

$$\begin{aligned} \deg f_d &\leq \deg g_a = a \deg p_{i_0} \\ &\leq a(w_{i_0} - 1). \end{aligned} \tag{5.9}$$

By (5.3) and (5.9) we have

$$\deg f_e - \deg f_d \geq a. \tag{5.10}$$

On the other hand, we have

$$\begin{aligned} \deg f_e - \deg f_d &= (\deg f - w_e) - (\deg f - w_d) \\ &= w_d - w_e. \end{aligned}$$

Hence $a \leq w_d - w_e$. It follows from (5.3) that

$$\deg f_e \leq (w_d - w_e)w_{i_0}, \tag{5.11}$$

which implies that

$$\begin{aligned} \deg f = \deg f_e + w_e &\leq (w_d - w_e)w_{i_0} + w_e \\ &\leq (w_d - w_e)w_1 + w_e \\ &= w_d w_1 - (w_1 - 1)w_e \\ &\leq w_1^2 - w_1 + 1. \end{aligned} \tag{5.12}$$

This is a contradiction. Therefore $P_n/(f_1, \cdots, f_n)$ has no non-zero negative weight derivations. □

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