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Research Paper

On (k, l) -th singular locus moduli algebras of singularities and their derivation Lie algebras [☆]



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ABSTRACT

In this paper, we introduce a series of new invariants to singularities. A new conjecture about the non-existence of negative weight derivations of these new (k, l) -th singular locus moduli algebras for weighted homogeneous isolated hypersurface singularities is proposed. We verify this conjecture in some cases.

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1. Introduction

This article has two purposes. On the one hand, we introduce a series of new local Artinian algebras associated with singularities in Section 2. These algebras and their dimensions are natural new invariants of singularities. On the other hand, motivated by the famous Halperin Conjecture, we investigate the derivation Lie algebras of these new local Artinian algebras. Again these Lie algebras and their dimensions are new invariants of singularities. We generalize the Halperin Conjecture and verify the conjecture in some cases.

We first recall the Halperin Conjecture. Suppose that A is a weighted zero-dimensional complete intersection algebra, i.e., a commutative algebra of the form

$$A = \mathbb{C}[z_1, z_2, \dots, z_n]/(g_1, g_2, \dots, g_n)$$

where the g_1, g_2, \dots, g_n is a regular sequence of length n . Here the variables z_i 's have strictly positive integral weights, denoted by $\text{wt}(z_i) = \alpha_i$ for $1 \leq i \leq n$, and the equations g_i 's are weighted homogeneous with respect to these weights α_i . Without loss of generality, we assume $d_1 \geq d_2 \geq \dots \geq d_n$, where $d_i := \text{wt}(g_i)$ for $i = 1, 2, \dots, n$. Consequently, the algebra A is naturally graded and one may speak about its weighted degree k derivations (k is an integer). A linear map $D : A \rightarrow A$ is a derivation if $D(ab) = D(a)b + aD(b)$, for any $a, b \in A$. We use $\text{Der}^k(A)$ (resp. $\text{Der}^{<k}(A)$) to denote the set of weighted degree k (resp. $< k$) derivations of A , i.e., D belongs to $\text{Der}^k(A)$ if $D : A^* \rightarrow A^{*+k}$. One of the most important open problems in rational homotopy theory (see [11]) is related to the vanishing of the above derivations in strictly negative degrees:

Halperin Conjecture. [14] *Let $A = \mathbb{C}[z_1, \dots, z_n]/(g_1, g_2, \dots, g_n)$ be a weighted homogeneous zero-dimensional complete intersection algebra. Here the variables z_i have strictly positive integral weights, denoted by $\text{wt}(z_i) = \alpha_i, 1 \leq i \leq n$, and g_i are weighted homogeneous polynomials with respect to these weights. Then $\text{Der}^{<0}(A) = 0$.*

Assuming that all the weights α_i are even, this has the following topological interpretation. If a space X has $H^*(X, \mathbb{C}) = A$ as graded algebras, then it is known that the vanishing of $\text{Der}^{<0}(A) = 0$ implies the collapsing at the E_2 -term of the Serre spectral sequence with \mathbb{C} -coefficients of any orientable fibration having X as fiber. The above collapsing properties also imply vanishing properties when \mathbb{C} is replaced by \mathbb{Q} and X a rational space, see e.g. [14]. The Halperin Conjecture has been verified in several particular cases:

- 1) equal weights ($\alpha_1 = \alpha_2 = \dots = \alpha_n$), see [31];
- 2) $n = 2, 3$, see [24], [2];
- 3 “fibered” algebras, see [13];
- 4) assuming $\mathbb{C}[z_1, z_2, \dots, z_n]/(g_1, g_2, \dots, g_{n-1})$ is reduced, see [19];
- 5) homogeneous spaces of equal rank compact connected Lie groups ($A = H^*(G/K)$), see [23].

For recent progress on the Halperin conjecture and its generalizations, interested readers are referred to [4–6].

Moreover, in the classification theory of isolated singularities, one always wants to find various invariants associated with isolated singularities. Hopefully with enough invariants found, one can distinguish between different isolated singularities up to a certain equivalence. However, not many effective invariants are known. Moreover, most known invariants, such as the cohomological invariant geometric genus, are difficult to compute in general. In this article, we shall introduce two series of new numerical invariants to isolated hypersurface singularities. One is the dimension of the new local Artinian algebra, i.e., $\delta_{\text{Sing}^k X}^l$ (see Definition 2.7), another one is the dimension of its derivation Lie algebra, i.e., $\rho_{\text{Sing}^k X}^l$ (see Definition 2.9). These new invariants can be calculated easily compared with other invariants of isolated singularities.

Let \mathcal{O}_n be the germs of holomorphic functions at the origin. The \mathcal{O}_n can be naturally identified with the algebra of convergent power series in n indeterminates with complex coefficients. As the ring \mathcal{O}_n has a unique maximal ideal \mathfrak{m} , the set of germs of holomorphic functions which vanish at the origin. Let $(X, 0)$ be an isolated hypersurface singularity defined by a holomorphic function $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$. The multiplicity $\text{mult}(f)$ of the singularity $(X, 0)$ is defined to be the order of the lowest nonvanishing term in the power series expansion of f at 0.

For any isolated hypersurface singularity $(X, 0) \subset (\mathbb{C}^n, 0)$ defined by f , the algebra $A(X) := \mathcal{O}_n / \left(f, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right)$ is called the moduli algebra of $(X, 0)$ [15]. The second author first introduced the Lie algebra of derivations of moduli algebra $A(X)$, i.e., $L(X) = \text{Der}(A(X), A(X))$. It is known that $L(X)$ is a finite-dimensional solvable Lie algebra [27]. $L(X)$ is called the Yau algebra of X in [12] and [28] to distinguish from Lie algebras of other types appearing in singularity theory [8]. The Yau algebra plays an important role in singularities [21]. In this paper, we will introduce a series of new derivation Lie algebras which are natural generalizations of the Yau algebra.

Theorem A and Theorem B in Section 2 are our main results. This paper is organized as follows. In Section 2, we introduce a series of new local Artinian algebras $M_{\text{Sing}^k X}^l$ (see Definition 2.7) associated with singularities. Motivated by Halperin Conjecture, a new conjecture (Conjecture 2.10) is proposed. Our main results are to verify this conjecture in several special cases. In Section 3, we give proofs of the main results.

2. Singular locus algebras

In this paper, we focus on weighted homogeneous isolated hypersurface singularities. However, our theory also is applicable to general isolated hypersurface singularities possessing more complicated symbols. In what follows, we restrict our consideration to weighted homogeneous singularities.

Let $(X, 0) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$ be an isolated hypersurface singularity with f a weighted homogeneous polynomial. In [18], we introduced the concepts of higher singular locus and higher singular locus moduli algebra. (More precisely, if

we use the symbol in this paper, those higher singular locus algebras $M_{\text{Sing}^k X}$ is just $M_{\text{Sing}^k X}^n$ for $k \geq 2$ and $M_{\text{Sing}^1 X}$ is just $M_{\text{Sing}^1 X}^1$.) The ideal $I_{\text{Sing}^k X}$ is the n -th Fitting ideal of $\Omega_{M_{\text{Sing}^{k-1} X}/\mathbb{C}}$ for $k \geq 2$. Now, we generalize those definitions to any Fitting ideals.

Firstly, we recall some basic knowledge of the module of Kähler differential and Fitting ideal briefly. For more details, readers can refer to [7] and [22].

Definition 2.1. Let S be a \mathbb{C} -algebra, the module of Kähler differentials of S over \mathbb{C} , denoted by $\Omega_{S/\mathbb{C}}$, is the S -module generated by the set $\{df : f \in S\}$ satisfies the relations

$$\begin{aligned} d(ss') &= sd(s') + s'd(s) \\ d(rs + r's') &= rd(s) + r'd(s') \end{aligned}$$

for all $r, r' \in \mathbb{C}$ and $s, s' \in S$.

Let $S = \mathbb{C}[z_1, \dots, z_n]/I$ be a finitely generated \mathbb{C} -algebra. If $I = (g_1, \dots, g_s)$, then $\Omega_{S/\mathbb{C}} = \text{coker}(d : I/I^2 \rightarrow \bigoplus_i Sdz_i)$ where $\bigoplus_i Sdz_i = S \otimes_{\mathbb{C}} \Omega_{\mathbb{C}[z_1, \dots, z_n]/\mathbb{C}}$ is a free S -module on generators dz_i . We view I/I^2 as a homomorphic image of a free S -module with generators e_i mapping to the classes of the g_i , the composition

$$\mathcal{J} : \bigoplus S e_i \rightarrow I/I^2 \rightarrow \bigoplus_i S dz_i$$

is a map of free S -modules. Hence, $\Omega_{S/\mathbb{C}}$ is the cokernel of the Jacobian matrix $\mathcal{J} = [\frac{\partial g_i}{\partial z_j}]$. The readers can find more details and examples in Chapter 16.1 of [7].

Fitting ideals were first introduced by Fitting in [10]. The readers can also find this definition in Chapter 20 of [7] and Section 1 of the lecture notes [22].

Definition 2.2. Let M be a finite generated \mathbb{C} -module. We represent M as the cokernel of an \mathbb{C} -linear map between free \mathbb{C} -modules of finite rank:

$$F \xrightarrow{\varphi} G \rightarrow M \rightarrow 0$$

where F is a free \mathbb{C} -module of rank q and G is a free \mathbb{C} -module of rank p . For each j we associate to M the ideal $F_j(M)$, called j -th Fitting ideal of M , generated by the $(p - j) \times (p - j)$ minors of the matrix representing φ . If $j \geq p$, we set $F_j(M) = \mathbb{C}$. If $p - j > q$, we set $F_j(M) = 0$.

Remark 2.3. The Fitting ideal $F_j(M)$ depends solely on the \mathbb{C} -module M and is independent of the choice of presentation.

Proposition 2.4 ([7], Corollary 16.20). Let $R = \mathbb{C}[z_1, \dots, z_n]/I$ be an affine ring and suppose that I has pure codimension c . Suppose that $I = (g_1, \dots, g_s)$. If J is the ideal of R generated by $c \times c$ minors of the Jacobian matrix $[\frac{\partial g_i}{\partial z_j}]$, then J defines the singular

locus of R in the sense that a prime P of R contains J if and only if R_P is not a regular local ring.

J is the $(n - c)$ -th Fitting ideal of $\Omega_{R/\mathbb{C}}$. Hence, J depends only on R . We just recall the notation of k -th singular locus $\text{Sing}^k X$, the k -th singular locus ideal $I_{\text{Sing}^k X}$, and the k -th singular locus moduli algebra $M_{\text{Sing}^k X}$ introduced in [18].

Remark 2.5. Let $(X, 0) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$ be an isolated hypersurface singularity defined by a weighted homogeneous polynomial f . We introduce the following notations (see also [18]).

$$\begin{aligned} \text{Jac}^1 &:= [f_1 \quad f_2 \quad \cdots \quad f_n], \\ \text{Sing}^1 X &= \{\mathbf{z} \mid f_1(\mathbf{z}) = \cdots = f_n(\mathbf{z}) = 0\} = \{0\}, \\ \text{Jac}^2 &:= \begin{bmatrix} f_{11} & f_{21} & \cdots & f_{n1} \\ f_{12} & f_{22} & \cdots & f_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1n} & f_{2n} & \cdots & f_{nn} \end{bmatrix}, \\ \text{Sing}^2 X &= \{\mathbf{z} \mid f_1(\mathbf{z}) = \cdots = f_n(\mathbf{z}) = h(\mathbf{z}) = 0\}, \text{ where } h := \det[f_{ij}]_{1 \leq i, j \leq n}. \\ \text{Jac}^3 &:= \begin{bmatrix} f_{11} & f_{21} & \cdots & f_{n1} & h_1 \\ f_{12} & f_{22} & \cdots & f_{n2} & h_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{1n} & f_{2n} & \cdots & f_{nn} & h_n \end{bmatrix}, \\ \text{Sing}^3 X &= \{\mathbf{z} \mid f_1(\mathbf{z}) = \cdots = f_n(\mathbf{z}) = h(\mathbf{z}) = {}^3h^1(\mathbf{z}) = \cdots = {}^3h^n(\mathbf{z}) = 0\}, \end{aligned}$$

where ${}^3h^j$, $1 \leq j \leq n$ are defined as follows. Let ${}^3\bar{h}_i := [f_{i1} \quad f_{i2} \quad \cdots \quad f_{in}]^T$ be the i -th column of Jac^3 for $1 \leq i \leq n$ and ${}^3\bar{h}_{n+1} := [h_1 \quad h_2 \quad \cdots \quad h_n]^T$ be the $(n + 1)$ -th column of Jac^3 .

Set ${}^3H^i := [{}^3\bar{h}_1 \quad {}^3\bar{h}_2 \quad \cdots \quad \hat{{}^3\bar{h}}_i \quad \cdots \quad {}^3\bar{h}_{n+1}]$ for $1 \leq i \leq n$ by deleting the column \bar{h}_i . We define ${}^3h^i := \det({}^3H^i)$.

If the k -th singular locus is defined by

$$\text{Sing}^k X = \{\mathbf{z} \mid f_1(\mathbf{z}) = \cdots = f_n(\mathbf{z}) = h(\mathbf{z}) = {}^k h^1(\mathbf{z}) = \cdots = {}^k h^{l^k}(\mathbf{z}) = 0\},$$

then Jac^{k+1} is the Jacobian matrix of the defined polynomials $f_1, \dots, f_n, h, {}^k h^1, \dots, {}^k h^{l^k}$ of $\text{Sing}^k X$.

Hence,

$$\text{Jac}^{k+1} := \begin{bmatrix} f_{11} & f_{21} & \cdots & f_{n1} & h_1 & ({}^k h^1)_1 & \cdots & ({}^k h^{l^k})_1 \\ f_{12} & f_{22} & \cdots & f_{n2} & h_2 & ({}^k h^1)_2 & \cdots & ({}^k h^{l^k})_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{1n} & f_{2n} & \cdots & f_{nn} & h_n & ({}^k h^1)_n & \cdots & ({}^k h^{l^k})_n \end{bmatrix}.$$

Hence, $\Omega_{M_{\text{Sing}^k}/\mathbb{C}}$ can be viewed as the cokernel of the Jacobian matrix $\text{Jac}^{k+1} X$.

Definition 2.6. We define $\bar{I}^l_{\text{Sing}^{k+1} X}$ to be the ideal generated by all the $l \times l$ minors of $\text{Jac}^{k+1} X$. Hence, $\bar{I}^l_{\text{Sing}^{k+1} X}$ is the Fitting ideal of $\Omega_{M_{\text{Sing}^{k+1}}/\mathbb{C}}$. Let $I^l_{\text{Sing}^{k+1} X} = (I_{\text{Sing}^k X}, \bar{I}^l_{\text{Sing}^{k+1} X})$ be the ideal generated by all the $l \times l$ minors of $\text{Jac}^{k+1} X$ and $I_{\text{Sing}^k X}$.

Definition 2.7. Let $(X, 0)$ be an isolated hypersurface singularity. We define the (k, l) -th singular locus moduli algebra as follows

$$M^l_{\text{Sing}^k X} := \mathcal{O}_n / I^l_{\text{Sing}^k X}.$$

The \mathbb{C} -dimension of $M^l_{\text{Sing}^k X}$ as a vector space is denoted as $\delta^l_{\text{Sing}^k X}$.

Remark 2.8. For $k = 1$, l can only equal 1. Hence, we still use $M_{\text{Sing}^1 X}$ (resp. $I_{\text{Sing}^1 X}$) instead of $M^1_{\text{Sing}^1 X}$ (resp. $I^1_{\text{Sing}^1 X}$) in the following. For $k \geq 2$, the range of l is $1 \leq l \leq n$. $M^n_{\text{Sing}^k X}$ is the k -th singular locus moduli algebra $M_{\text{Sing}^k X}$ introduced firstly in our previous paper [18], thus $M^l_{\text{Sing}^k X}$ are natural generalizations of higher singular locus moduli algebra. All (k, l) -th singular locus moduli algebras are invariants of isolated hypersurface singularities whose dimensions are numerical invariants of isolated hypersurface singularities. Those algebras are Artinian local algebras.

For $k = 2, 2 \leq l \leq n$, the ideal $I^l_{\text{Sing}^2 X} = (I_{\text{Sing}^1 X}, \bar{I}^l_{\text{Sing}^2 X})$ as we defined above. For $k \geq 3$, $\text{Jac}^k X$ can be viewed as a matrix that added extra columns to $\text{Jac}^{k-1} X$. In particular, Jac^k can be viewed as a matrix that added extra columns to $\text{Jac}^2 X$ inductively. Hence, we denote the ideal $I^l_{\text{Sing}^k X}$ to be $(I_{\text{Sing}^k X}, \bar{I}^l_{\text{Sing}^2 X}, \bar{I}^l_{\text{Sing}^k X})$ where $\bar{I}^l_{\text{Sing}^k X} = \bar{I}^l_{\text{Sing}^k X} - \bar{I}^l_{\text{Sing}^2 X}$. Inspired by Yau algebra, we give the following definition.

Definition 2.9. We define $L^l_{\text{Sing}^k X}$ is the derivation Lie algebra of $M^l_{\text{Sing}^k X}$ for $k \geq 2$ and $1 \leq l \leq n$, i.e., $L^l_{\text{Sing}^k X} = \text{Der}(M^l_{\text{Sing}^k X}, M^l_{\text{Sing}^k X})$. The dimension of $L^l_{\text{Sing}^k X}$ as a \mathbb{C} -vector space is denoted as $\rho^l_{\text{Sing}^k X}$.

By Definition 2.9, the $L_{\text{Sing}^1 X}$ is the same as the Yau algebra $L(X)$. Thus the $L^l_{\text{Sing}^k X}$ is a natural generalization of the Yau algebra.

Motivated by the Halperin Conjecture, we proposed the following conjecture called generalized Yau conjecture.

Conjecture 2.10. Let $(X, 0) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1, \dots, \alpha_n; d)$. Assume that $d \geq 2\alpha_1 \geq \dots \geq 2\alpha_n > 0$ without loss of generality [20]. Furthermore, we assume that $\text{mult}(f) \geq 4$. Then, for any $k \geq 2$ and $2 \leq l \leq n$, there is no non-zero negative weight derivation on the $M^l_{\text{Sing}^k X}$, i.e., $L^l_{\text{Sing}^k X}$ is non-negatively graded.

This Conjecture 2.10 is a generalization of Conjecture 1.5 in [16] and Conjecture 2.8 in [18]. For $k = 1$, this is just the original Yau conjecture. It was investigated only for small k before. This Conjecture was verified in low dimensions.

Remark 2.11. The condition $\text{mult}(f) \geq 4$ is necessary, the following three examples show that when $\text{mult}(f) = 3$, there exists a negative weight derivation.

Example 2.12. Let $f = z_1^3 + z_1 z_2 z_3^2 + z_2^3 z_3 + z_3^5$ with the weighted type $(5, 4, 3; 15)$, $D_2^1 = z_3 \frac{\partial}{\partial z_2}$ is a negative weight derivation (weighted degree of D_2^1 is -1 , i.e., $\text{wt}(D_2^1) = -1$) of $\mathbb{C}[z_1, z_2, z_3]/(\frac{\partial^2 f}{\partial z_1^2}, \frac{\partial^2 f}{\partial z_2^2}, \frac{\partial^2 f}{\partial z_3^2}, \frac{\partial^2 f}{\partial z_1 \partial z_2}, \frac{\partial^2 f}{\partial z_1 \partial z_3}, \frac{\partial^2 f}{\partial z_2 \partial z_3})$, i.e., $D_2^1 \in L_{\text{Sing}^2 X}^1$.

Example 2.13. Let $f = z_1^2 z_3 + z_1 z_3^5 + z_2^3$ with the weighted type $(4, 3, 1; 9)$, $D_2^2 = z_2 \frac{\partial}{\partial z_1} \in L_{\text{Sing}^2 X}^2$ is a negative weight derivation ($\text{wt}(D_2^2) = -1$).

Example 2.14. Let $f = z_1^2 z_3 + z_3^3 z_1 + z_2^3 + z_4^5$ with the weighted type $(6, 5, 3, 3; 15)$, $D_2^2 := z_2 \frac{\partial}{\partial z_1} \in L_{\text{Sing}^2 X}^2$ is a negative weight derivation ($\text{wt}(D_2^2) = -1$).

Remark 2.15. Conjecture 2.10 is not true for $k = 1$, even for $l = 2$. There is a counterexample in four variables, see Example 2.16. For two and three variables, Conjecture 2.10 is true, see Proposition 3.6 and Proposition 3.7.

Example 2.16. Let $f = z_1^3 z_4 + z_2^3 z_3 + z_3^3 z_1 + z_4^5$ with the weighted type $(36, 34, 33, 27; 135)$, $D_2^1 = z_3 \frac{\partial}{\partial z_2}$ in $L_{\text{Sing}^2 X}^1$ is a negative weight derivation ($\text{wt}(D_2^1) = -1$).

In this paper, we obtain the following two main results which verify the Conjecture 2.10 for arbitrary k in low dimensions.

Theorem A. Let $(X, 0) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1, \dots, \alpha_n; d)$. Assume that $n \leq 4$ and $d \geq 2\alpha_1 \geq \dots \geq 2\alpha_n > 0$ without loss of generality. We also need to assume that $\text{mult}(f) \geq 4$. Then for any $k \geq 3$, $2 \leq l \leq n$, there is no non-zero negative weight derivation on the $M_{\text{Sing}^k X}^l$, i.e., $L_{\text{Sing}^k X}^l$ is non-negatively graded.

For $k = 1$ or 2 , Theorem A has already been proved in [1], [3], [16]. The following Theorem 2.17 verified the Conjecture 2.10 for $k = 1$ and $n \leq 4$.

Theorem 2.17 ([1], [3]). Let $(X, 0) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1, \dots, \alpha_n; d)$, $n \leq 4$. Assume that $d \geq 2\alpha_1 \geq \dots \geq 2\alpha_n > 0$ without loss of generality. Then there is no non-zero negative weight derivation on the $M_{\text{Sing}^1 X}$, i.e., $L_{\text{Sing}^1 X}$ is non-negatively graded.

The following Theorem 2.18 verified the Conjecture 2.10 for $k = 2$, $2 \leq l \leq n$, and $n \leq 4$.

Theorem 2.18 ([16], [17]). *Let $(X, 0) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1, \dots, \alpha_n; d)$, $n \leq 4$. Assume that $d \geq 2\alpha_1 \geq \dots \geq 2\alpha_n > 0$ without loss of generality. We also need to assume that $\text{mult}(f) \geq 4$. For $k = 2$ and $2 \leq l \leq n$, there is no non-zero negative weight derivation on the $M_{\text{Sing}^2 X}^l$, i.e., $L_{\text{Sing}^2 X}^l$ is non-negatively graded.*

The following Theorem 2.19 verified the Conjecture 2.10 for $k = 2$, $l = 1$, and $n \leq 3$.

Theorem 2.19 ([17]). *Let $(X, 0) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1, \dots, \alpha_n; d)$, $n \leq 3$. Assume that $d \geq 2\alpha_1 \geq \dots \geq 2\alpha_n > 0$ without loss of generality. We also need to assume that $\text{mult}(f) \geq 4$. For $k = 2$ and $l = 1$, there is no non-zero negative weight derivation on the $M_{\text{Sing}^2 X}^l$, i.e., $L_{\text{Sing}^2 X}^l$ is non-negatively graded.*

The following Theorem 2.20 verified the Conjecture 2.10 for $k = 1$ under the condition $\alpha_1 \leq 2\alpha_n$.

Theorem 2.20 ([29]). *Let $(X, 0) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1, \dots, \alpha_n; d)$. Assume that $d \geq 2\alpha_1 \geq \dots \geq 2\alpha_n > 0$ without loss of generality. If $\alpha_1 \leq 2\alpha_n$, then there is no non-zero negative weight derivation on the $M_{\text{Sing}^1 X}$, i.e., $L_{\text{Sing}^1 X}$ is non-negatively graded.*

The following Theorem 2.21 verified the Conjecture 2.10 for $k \geq 2$ and $l = n$ under the condition $\alpha_1 \leq 2\alpha_n$.

Theorem 2.21 ([18]). *Let $(X, 0) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1, \dots, \alpha_n; d)$. Assume that $d \geq 2\alpha_1 \geq \dots \geq 2\alpha_n > 0$ without loss of generality. If $\alpha_1 \leq 2\alpha_n$, then there is no non-zero negative weight derivation on the $M_{\text{Sing}^k X}^n$, i.e., $L_{\text{Sing}^k X}^n$ is non-negatively graded.*

Theorem B. *Let $(X, 0) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1, \dots, \alpha_n; d)$. Assume that $d \geq 2\alpha_1 \geq \dots \geq 2\alpha_n > 0$ without loss of generality. We also need to assume that $\text{mult}(f) \geq 4$. For $k \geq 3$ and $2 \leq l \leq n$, if $\alpha_1 \leq 2\alpha_n$, then there is no non-zero negative weight derivation on the $M_{\text{Sing}^k X}^l$, i.e., $L_{\text{Sing}^k X}^l$ is non-negatively graded.*

3. Proof of theorems

First, we recall some known results that will be frequently used in proving our main theorems.

Proposition 3.1 ([25], Prop.2.6). *Let $A = \bigoplus_{i=0}^k A_i$ be a graded commutative Artinian local algebra with $A_0 = \mathbb{C}$. Suppose the maximal ideal of A is generated by A_j for some $j > 0$. Then $L(A)$ is a graded Lie algebra without negative weight.*

Lemma 3.2 ([26]). *Let (A, \mathfrak{m}) be a commutative local Artinian algebra (\mathfrak{m} is the unique maximal ideal of A and $D \in L(A)$ be the derivation of A). Then D preserves the \mathfrak{m} -adic filtration of A , i.e., $D(\mathfrak{m}) \subset \mathfrak{m}$.*

Lemma 3.3 ([3] lemma 2.1). *Let f be a weighted homogeneous polynomial with isolated singularity in z_1, \dots, z_n variables of type $(\alpha_1, \dots, \alpha_n; d)$. Assume $\text{wt}(z_1) = \alpha_1 \geq \text{wt}(z_2) = \alpha_2 \geq \dots \geq \text{wt}(z_n) = \alpha_n$. Then f must be as in one of the following two cases: Case (1).*

$$f = z_1^m + a_1(z_2, \dots, z_n)z_1^{m-1} + \dots + a_{m-1}(z_2, \dots, z_n)z_1 + a_m(z_2, \dots, z_n).$$

Case (2).

$$f = z_1^m z_i + a_1(z_2, \dots, z_n)z_1^{m-1} + \dots + a_{m-1}(z_2, \dots, z_n)z_1 + a_m(z_2, \dots, z_n)$$

with $2 \leq i \leq n$.

Lemma 3.4 ([1] lemma 1.2). *Let f be a weighted homogeneous polynomial in z_1, \dots, z_n which defines an isolated singularity at the origin. Then there is a term of the form $z_i^{a_i}$ or $z_i^{a_i} z_j$ in f for any i ($a_i \geq 2$ in the case $z_i^{a_i}$ and $a_i \geq 1$ otherwise).*

Lemma 3.5. ([30]) *Let I be an ideal in $R = \mathbb{C}[z_1, \dots, z_n]$. Then there is a natural isomorphism of Lie algebras*

$$(Der_I R)/(I \cdot Der_{\mathbb{C}} R) \cong Der_{\mathbb{C}}(R/I).$$

If $\begin{vmatrix} f_{11} & f_{21} & \cdots & f_{l1} \\ f_{12} & f_{22} & \cdots & f_{l2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1l} & f_{2l} & \cdots & f_{ll} \end{vmatrix}$ is non-zero, then

$$\text{wt} \left(\begin{vmatrix} f_{11} & f_{21} & \cdots & f_{l1} \\ f_{12} & f_{22} & \cdots & f_{l2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1l} & f_{2l} & \cdots & f_{ll} \end{vmatrix} \right) = ld - 2 \sum_{i=1}^l \alpha_i.$$

$$\text{If } \begin{vmatrix} f_{(n-l+1)(n-l+1)} & f_{(n-l+2)(n-l+1)} & \cdots & f_{n(n-l+1)} \\ f_{(n-l+1)(n-l+2)} & f_{(n-l+2)(n-l+2)} & \cdots & f_{n(n-l+2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{(n-l+1)n} & f_{(n-l+2)n} & \cdots & f_{nn} \end{vmatrix} \text{ is non-zero, then}$$

$$\text{wt} \left(\begin{pmatrix} f_{(n-l+1)(n-l+1)} & f_{(n-l+2)(n-l+1)} & \cdots & f_{n(n-l+1)} \\ f_{(n-l+1)(n-l+2)} & f_{(n-l+2)(n-l+2)} & \cdots & f_{n(n-l+2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{(n-l+1)n} & f_{(n-l+2)n} & \cdots & f_{nn} \end{pmatrix} \right) = ld - 2 \sum_{i=n-l+1}^n \alpha_i.$$

Proposition 3.6. *Let $(X, 0) = \{(z_1, z_2) \in \mathbb{C}^2 : f(z_1, z_2) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1, \alpha_2; d)$ with $\text{mult}(f) \geq 4$. Assume that $d \geq 2\alpha_1 \geq 2\alpha_2 > 0$ without loss of generality. Then for any $k \geq 1$, there is no non-zero negative weight derivation on the $M_{\text{Sing}^k X}^1$, i.e., $L_{\text{Sing}^k X}^1$ is non-negatively graded.*

Proof. For $k = 1$, this has already been proven by Proposition 2.6 in [17]. For $k \geq 2$, we obtain that $\text{wt}(h_i) \geq 2d - 3\alpha_1 - 2\alpha_2 \geq (d - 2\alpha_2) + (3\alpha_1 + \alpha_2 - 3\alpha_1) \geq f_{22}$. If D_2^1 is a non-zero negative weight derivation of $M_{\text{Sing}^2 X}^1$, then D_2^1 is also a non-zero negative weight derivation of $M_{\text{Sing}^1 X}$ by Lemma 3.5. By Proposition 2.7 in [17], we obtain that such D_2^1 does not exist. For $k > 2$, if D_k^1 is a non-zero negative weight derivation of $M_{\text{Sing}^k X}^1$, by Lemma 3.9 in [18], we obtain that D_k^1 is also a non-zero negative weight derivation of $M_{\text{Sing}^2 X}^1$. By discussion above, we obtain that such D_k^1 does not exist. \square

Proposition 3.7. *Let $(X, 0) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : f(z_1, z_2, z_3) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1, \alpha_2, \alpha_3; d)$ with $\text{mult}(f) \geq 4$. Assume that $d \geq 2\alpha_1 \geq 2\alpha_2 \geq 2\alpha_3 > 0$ without loss of generality. Then for any $k \geq 1$, there is no non-zero negative weight derivation on the $M_{\text{Sing}^k X}^1$, i.e., $L_{\text{Sing}^k X}^1$ is non-negatively graded.*

Proof. For $k = 1$, this has already been proven by Proposition 2.7 in [17]. For $k \geq 2$, we obtain that $\text{wt}(h_i) \geq 3d - 3\alpha_1 - 2\alpha_2 - 2\alpha_3 \geq (d - 2\alpha_3) + [2(3\alpha_1 + \alpha_3) - 3\alpha_1 - 2\alpha_2] \geq f_{33}$. If D_2^1 is a non-zero negative weight derivation of $M_{\text{Sing}^2 X}^1$, then D_2^1 is also a non-zero negative weight derivation of $M_{\text{Sing}^1 X}$ by Lemma 3.5. By Proposition 2.7 in [17], we obtain that such D_2^1 does not exist. For $k > 2$, if D_k^1 is a non-zero negative weight derivation of $M_{\text{Sing}^k X}^1$, by Lemma 3.9 in [18], we obtain that D_k^1 is also a non-zero negative weight derivation of $M_{\text{Sing}^2 X}^1$. By discussion above, we obtain that such D_k^1 does not exist. \square

For four variables, there are only partial results. For $k = 2, l = 1$, f is a fewnomial, see [9]. For $k \geq 3, l = 1$, we can also get a similar result when f is a fewnomial. It is a tedious statement, so we are not going to describe this inelegant statement. Now, we begin to prove our main results.

Proof of Theorem A. Since $\text{mult}(f) \geq 4$ and $d \geq 2\alpha_1 \geq \dots \geq 2\alpha_n > 0$ by assumption, so it follows from Lemma 3.3 that the weighted degree of f satisfied $d \geq 3\alpha_1 + \alpha_n$. Thus, $ld - 2 \sum_{i=1}^l \alpha_i = (d - \alpha_n) + (l - 1)d - 2 \sum_{i=1}^l \alpha_i + \alpha_n \geq (d - \alpha_n) + (l - 1)(3\alpha_1 + \alpha_n) - 2 \sum_{i=1}^l \alpha_i + \alpha_n$. When $l \geq 3$, we obtain that $(l - 3)\alpha_1 + l\alpha_n + 2 \sum_{i=2}^l (\alpha_1 - \alpha_i) > 0$. We set $I_{\text{Sing}^2 X}^l = (I_{\text{Sing}^1 X}, \bar{I}_{\text{Sing}^2 X}^l)$. Let F_2^l be an element with the smallest weight in $\bar{I}_{\text{Sing}^2 M}^l$, we obtain that $\text{wt}(F_2^l) \geq ld - 2 \sum_{i=1}^l \alpha_i$. Hence, $\text{wt}(F_2^l) > \text{wt}(\frac{\partial f}{\partial z_n})$.

To prove Theorem A, we need to prove the following lemma first.

Lemma 3.8. *With above notations, let f be a weighted homogeneous polynomial with isolated singularity at the origin with weight type $(\alpha_1, \dots, \alpha_n; d)$. Assume $\text{wt}(z_1) = \alpha_1 \geq \text{wt}(z_2) = \alpha_2 \geq \dots \geq \text{wt}(z_n) = \alpha_n$ and $d \geq 3\alpha_1 + \alpha_n$. For any $k \geq 2$ and $2 \leq l \leq n$, let F_k^l be a smallest weight element of the ideal $\bar{I}_{\text{Sing}^k X}^l$, G_2^l be a largest weight element of the ideal $I_{\text{Sing}^2 X}^2 = (I_{\text{Sing}^1 X}, \bar{I}_{\text{Sing}^2 M}^2)$. Then*

$$\text{wt}(F_k^l) \geq \text{wt}(G_2^l).$$

Proof. First, we consider $l = 2$. Let G_2^2 be a largest weight element of the ideal $I_{\text{Sing}^2 X}^2 = (I_{\text{Sing}^1 X}, \bar{I}_{\text{Sing}^2 M}^2)$, then $\text{wt}(G_2^2) \leq \max\{2d - 2\alpha_{n-1} - 2\alpha_n, d - \alpha_n\}$. Now, we consider $I_{\text{Sing}^{k+1} X}^2$.

$$Jac^{k+1} := \begin{bmatrix} f_{11} & f_{21} & \cdots & f_{n1} & h_1 & ({}^k h^1)_1 & \cdots & ({}^k h^{l_k})_1 \\ f_{12} & f_{22} & \cdots & f_{n2} & h_2 & ({}^k h^1)_2 & \cdots & ({}^k h^{l_k})_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{1n} & f_{2n} & \cdots & f_{nn} & h_n & ({}^k h^1)_n & \cdots & ({}^k h^{l_k})_n \end{bmatrix},$$

where $({}^k h^j)_i := \frac{\partial {}^k h^j}{\partial z_i}$ for $1 \leq i \leq n, 1 \leq j \leq l_k$. Without loss of generality, we assume that $\text{wt}({}^k h^1) \geq \text{wt}({}^k h^2) \geq \dots \geq \text{wt}({}^k h^{l_k})$.

If $\begin{vmatrix} f_{11} & h_1 \\ f_{12} & h_2 \end{vmatrix}$ is non-zero, then

$$\begin{aligned} & \text{wt} \left(\begin{vmatrix} f_{11} & h_1 \\ f_{12} & h_2 \end{vmatrix} \right) \\ &= d + \text{wt}(h) - 2\alpha_1 - \alpha_2 \\ &= d + \sum_{i=1}^n (d - 2\alpha_i) - 2\alpha_1 - \alpha_2 \\ &\geq (2d - 2\alpha_{n-1} - 2\alpha_n) + (d - 2\alpha_1 - \alpha_2) + \sum_{i=1}^{n-2} (d - 2\alpha_i) \\ &> 2d - 2\alpha_{n-1} - 2\alpha_n \end{aligned}$$

and $d + \text{wt}(h) - 2\alpha_1 - \alpha_2 \geq d - \alpha_n$, we obtain that $\text{wt} \left(\begin{vmatrix} f_{11} & h_1 \\ f_{12} & h_2 \end{vmatrix} \right) \geq \text{wt}(G_2^2)$.

If $\begin{vmatrix} f_{11} & \binom{k}{h} h^{l_k}_1 \\ f_{12} & \binom{k}{h} h^{l_k}_2 \end{vmatrix}$ is non-zero, then

$$\text{wt} \left(\begin{vmatrix} f_{11} & \binom{k}{h} h^{l_k}_1 \\ f_{12} & \binom{k}{h} h^{l_k}_2 \end{vmatrix} \right) \geq d + \text{wt}(\binom{k}{h} h^{l_k}) - 2\alpha_1 - \alpha_2.$$

Since

$$\begin{aligned} & d + \text{wt}(\binom{k}{h} h^{l_k}) - 2\alpha_1 - \alpha_2 \\ & > d + \sum_{i=1}^n (d - 2\alpha_i) - 2\alpha_1 - \alpha_2 \\ & \geq (2d - 2\alpha_{n-1} - 2\alpha_n) + (d - 2\alpha_1 - \alpha_2) + \sum_{i=1}^{n-2} (d - 2\alpha_i) \\ & > 2d - 2\alpha_{n-1} - 2\alpha_n \end{aligned}$$

and $d + \text{wt}(\binom{k}{h} h^{l_k}) - 2\alpha_1 - \alpha_2 \geq d - \alpha_n$, we obtain that $\text{wt} \left(\begin{vmatrix} f_{11} & \binom{k}{h} h^{l_k}_1 \\ f_{12} & \binom{k}{h} h^{l_k}_2 \end{vmatrix} \right) \geq \text{wt}(G_2^2)$.

If $\begin{vmatrix} h_1 & \binom{k}{h} h^{l_k}_1 \\ h_2 & \binom{k}{h} h^{l_k}_2 \end{vmatrix}$ is non-zero,

$$\text{wt} \left(\begin{vmatrix} h_1 & \binom{k}{h} h^{l_k}_1 \\ h_2 & \binom{k}{h} h^{l_k}_2 \end{vmatrix} \right) \geq 2 \text{wt}(h) - \alpha_1 - \alpha_2.$$

Since

$$\begin{aligned} & 2 \text{wt}(h) - \alpha_1 - \alpha_2 \\ & = 2 \sum_{i=1}^n (d - 2\alpha_i) - \alpha_1 - \alpha_2 \\ & \geq (2d - 2\alpha_{n-1} - 2\alpha_n) + (2d - \alpha_1 - \alpha_2 - 2\alpha_{n-1} - 2\alpha_n) + 2 \sum_{i=1}^{n-2} (d - 2\alpha_i) \\ & > 2d - 2\alpha_{n-1} - 2\alpha_n \end{aligned}$$

and $2 \text{wt}(h) - \alpha_1 - \alpha_2 \geq d - \alpha_n$, we obtain that $\text{wt} \left(\begin{vmatrix} h_1 & \binom{k}{h} h^{l_k}_1 \\ h_2 & \binom{k}{h} h^{l_k}_2 \end{vmatrix} \right) \geq \text{wt}(G_2^2)$.

Let F_k^2 be a smallest weight element of the ideal $\tilde{I}_{\text{Sing}^k(X)}^2$ with $k \geq 3$. We obtain that

$$\text{wt}(F_k^2) \geq \min \left\{ \text{wt} \left(\begin{vmatrix} f_{11} & h_1 \\ f_{12} & h_2 \end{vmatrix} \right), \text{wt} \left(\begin{vmatrix} f_{11} & \binom{k}{h} h^{l_k}_1 \\ f_{12} & \binom{k}{h} h^{l_k}_2 \end{vmatrix} \right), \text{wt} \left(\begin{vmatrix} h_1 & \binom{k}{h} h^{l_k}_1 \\ h_2 & \binom{k}{h} h^{l_k}_2 \end{vmatrix} \right) \right\}.$$

Hence, $\text{wt}(F_k^2) \geq \text{wt}(G_2^2)$. For $3 \leq l \leq n$, by Lemma 3.9 in [17], we obtain that $\text{wt}({}^k h^j) > \text{wt}(h)$ for $k \geq 3, 1 \leq j \leq l_k$. Moreover, under the assumption $d \geq 3\alpha_1 + \alpha_n$, we obtain that $\text{wt}(h_r) = nd - 2 \sum_{i=1}^n \alpha_i - \alpha_r > \text{wt}(f_{st})$ for any $1 \leq r, s, t \leq n$. Thus $\text{wt}(({}^k h^j)_r) > \text{wt}(f_{st})$. F_k^l is a smallest weight element of the ideal $\tilde{I}_{\text{Sing}^k X}^l$. F_k^l can be obtained by adding more rows and columns to F_2^l . Hence, we obtain that $\text{wt}(F_k^l) \geq \text{wt}(G_2^l)$. \square

By the assumption of $\text{mult}(f) \geq 4$, we obtain that $d \geq 3\alpha_1 + \alpha_n$. By Lemma 3.8, we obtain that $\text{wt}(F_k^l) \geq \text{wt}(F_2^l)$. Let $I_{\text{Sing}^2 X}^l := (I_{\text{Sing}^1 X}, \tilde{I}_{\text{Sing}^2 X}^l)$ be the ideal corresponding to $\text{Sing}^2 M$. For any $k > 2, 2 \leq l \leq n$, we set $I_{\text{Sing}^k X}^l = (I_{\text{Sing}^1 X}, \tilde{I}_{\text{Sing}^2 X}^l, \tilde{I}_{\text{Sing}^k X}^l)$ as we describe above. For $k > 2, 2 \leq l \leq n$, if D_k^l is a non-zero negative weight derivation of $M_{\text{Sing}^k X}^l$, by Lemma 3.8, we obtain that $D_k^l(I_{\text{Sing}^k X}^l) \subset I_{\text{Sing}^k X}^l$, i.e., D_k^l is also a non-zero negative weight derivation of $M_{\text{Sing}^2 X}^l$ by Lemma 3.5. By Theorem 2.19, we obtain that such D_k^l does not exist. Hence, we finish the proof of Theorem A. \square

Proof of Theorem B. Since $\text{mult}(f) \geq 4$. First, we consider the following lemma.

Lemma 3.9. *Let $(X, 0) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f with $\text{mult}(f) \geq 4$ of weight type $(\alpha_1, \dots, \alpha_n; d)$. Assume that $d \geq 2\alpha_1 \geq \dots \geq 2\alpha_n > 0$ without loss of generality. For $2 \leq l \leq n$, if $\alpha_1 \leq 2\alpha_n$, then there is no non-zero negative weight derivation on the $M_{\text{Sing}^2 X}^l$, i.e., $L_{\text{Sing}^2 X}^l$ is non-negatively graded.*

Proof. Let D_2^l be a negative weight derivation of $M_{\text{Sing}^2 X}^l$. By Lemma 3.2, we can assume that D_2^l has to be the following form

$$D_2 = p_1(z_2, \dots, z_n) \frac{\partial}{\partial z_1} + p_2(z_3, \dots, z_n) \frac{\partial}{\partial z_2} + \dots + cz_n^r \frac{\partial}{\partial z_{n-1}}.$$

We compute the commutator $[\frac{\partial}{\partial z_i}, D]$ for $1 \leq i \leq n$ as follows.

$$\begin{aligned} [\frac{\partial}{\partial z_1}, D] &= 0, \\ [\frac{\partial}{\partial z_2}, D] &= \frac{\partial p_1}{\partial z_2} \frac{\partial}{\partial z_1}, \\ &\dots \quad \dots \quad \dots \\ [\frac{\partial}{\partial z_i}, D] &= \frac{\partial p_1}{\partial z_i} \frac{\partial}{\partial z_1} + \dots + \frac{\partial p_{i-1}}{\partial z_i} \frac{\partial}{\partial z_{i-1}}, \\ &\dots \quad \dots \quad \dots \\ [\frac{\partial}{\partial z_n}, D] &= \frac{\partial p_1}{\partial z_n} \frac{\partial}{\partial z_1} + \dots + \frac{\partial p_{n-2}}{\partial z_n} \frac{\partial}{\partial z_{n-2}} + \frac{\partial cz_n^r}{\partial z_n} \frac{\partial}{\partial z_{n-1}}. \end{aligned}$$

If $\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1l} \\ f_{12} & f_{22} & \cdots & f_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1l} & f_{2l} & \cdots & f_{ll} \end{bmatrix}$ is non-zero, then

$$\begin{aligned} & \text{wt} \left(\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1l} \\ f_{12} & f_{22} & \cdots & f_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1l} & f_{2l} & \cdots & f_{ll} \end{bmatrix} \right) \\ &= ld - 2 \sum_{i=1}^l \alpha_i \\ &= (d - \alpha_n) + (l - 1)d + \alpha_n - 2 \sum_{i=1}^l \alpha_i \\ &\geq (d - \alpha_n) + (3l - 3)\alpha_1 + l\alpha_n - 2 \sum_{i=1}^l \alpha_i \\ &\geq \text{wt} \left(\frac{\partial f}{\partial z_n} \right). \end{aligned}$$

Thus,

$$\begin{aligned} D \left(\frac{\partial f}{\partial z_1} \right) &= 0, \\ D \left(\frac{\partial f}{\partial z_2} \right) &\in \left(\frac{\partial f}{\partial z_1} \right), \\ &\dots \quad \dots \quad \dots \\ D \left(\frac{\partial f}{\partial z_n} \right) &\in \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_{n-1}} \right). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\partial(Df)}{\partial z_1} &= 0, \\ \frac{\partial(Df)}{\partial z_2} &\in \left(\frac{\partial f}{\partial z_1} \right), \\ &\dots \quad \dots \quad \dots \\ \frac{\partial(Df)}{\partial z_n} &\in \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_{n-1}} \right). \end{aligned}$$

We obtain that $\frac{\partial(Df)}{\partial z_2} = g(z_2, \dots, z_n) \frac{\partial f}{\partial z_1}$. If $g(z_2, \dots, z_n) \neq 0$, then $\text{wt}(g) \geq \alpha_n$. This implies that $\text{wt} \left(\frac{\partial(Df)}{\partial z_2} \right) = \text{wt}(D) + d - \alpha_2 \leq \text{wt}(D) + d - \alpha_n < d - \alpha_n \leq d - \alpha_1 + \alpha_n \leq$

$\text{wt} \left(g(z_2, \dots, z_n) \frac{\partial f}{\partial z_1} \right)$. This is a contradiction. Hence, $g \equiv 0$, i.e., $\frac{\partial(Df)}{\partial z_2} = 0$. For any $3 \leq i \leq n$, we assume that

$$\frac{\partial(Df)}{\partial z_i} = q_1(z_2, \dots, z_n) \frac{\partial f}{\partial z_1} + \dots + q_{i-1}(z_i, \dots, z_n) \frac{\partial f}{\partial z_{i-1}}.$$

If there exists a $1 \leq j \leq i - 1$ such that $q_j \neq 0$, then $\text{wt} \left(\frac{\partial(Df)}{\partial z_i} \right) = \text{wt}(D) + d - \alpha_i \leq \text{wt}(D) + d - \alpha_n < d - \alpha_n \leq d - \alpha_1 + \alpha_n \leq \text{wt} \left(q_1(z_2, \dots, z_n) \frac{\partial f}{\partial z_1} + \dots + q_{i-1}(z_i, \dots, z_n) \frac{\partial f}{\partial z_{i-1}} \right)$. This is a contradiction. Hence, $q_j \equiv 0$, for all $1 \leq j \leq i - 1$, i.e., $\frac{\partial(Df)}{\partial z_i} = 0$. Since Df does not depend on z_1, \dots, z_n , we conclude that Df is a constant which means that $\text{wt}(D) = -d$. Moreover, $\text{wt}(D) \geq -\alpha_1$. This is a contradiction. Hence, there is no non-zero negative weight derivation on the $M^l_{\text{Sing}^2 X}$. \square

By the assumption of $\text{mult}(f) \geq 4$, we obtain that $d \geq 3\alpha_1 + \alpha_n$. By Lemma 3.8, we obtain that $\text{wt}(G^l_k) \geq \text{wt}(F^l_2)$. Let $I^l_{\text{Sing}^2 X} := (I_{\text{Sing}^1 X}, \bar{I}^l_{\text{Sing}^2 X})$ be the ideal corresponding to $\text{Sing}^2 M$. For any $k > 2, 2 \leq l \leq n$, we set $I^l_{\text{Sing}^k X} = (I_{\text{Sing}^1 X}, \bar{I}^l_{\text{Sing}^2 X}, \tilde{I}^l_{\text{Sing}^k X})$ as we describe above. For $k > 2, 2 \leq l \leq n$, if D^l_k is a non-zero negative weight derivation of $M^l_{\text{Sing}^k X}$, by Lemma 3.8, we obtain that $D^l_k(I^l_{\text{Sing}^k X}) \subset I^l_{\text{Sing}^k X}$, i.e., D^l_k is also a non-zero negative weight derivation of $M^l_{\text{Sing}^2 X}$ by Lemma 3.5. By Theorem 2.19, we conclude that such D^l_k does not exist. Hence, we finish the proof of Theorem B. \square

Data availability

No data was used for the research described in the article.

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