

# ON HIGHER DERIVATIONS ASSOCIATED TO ISOLATED HYPERSURFACE SINGULARITIES

ZIDA XIAO, STEPHEN S.-T. YAU, AND HUAIQING ZUO

**ABSTRACT.** In this paper, we introduce many new invariants to singularities, i.e., higher order derivations of moduli algebras of isolated hypersurface singularities. We investigate their properties and propose several conjectures for these invariants. In particular, we verify an inequality conjecture on the dimension of  $Der^2/Der^1$  for the case of binomial singularities. In addition, we verify the Nakai Conjecture for the case of Brieskorn singularities.

**Keywords.** higher order derivations, isolated singularity, invariants.

MSC(2010). 14B05, 32S05.

## 1. INTRODUCTION

Let  $(V, 0) \subset (\mathbb{C}^n, 0)$  be an isolated hypersurface singularity defined by the holomorphic function  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ . Then the moduli algebra  $M(V) := \mathcal{O}_n / (f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$  is finite dimensional. We also denote it as  $M(f)$ . The well-known Mather-Yau Theorem [16] states that: Let  $(V_1, 0)$  and  $(V_2, 0)$  be two isolated hypersurface singularities,  $M(V_1)$  and  $M(V_2)$  be the moduli algebras, then  $(V_1, 0) \cong (V_2, 0) \iff M(V_1) \cong M(V_2)$ . In 1983, Yau introduced the Lie algebra of derivations of moduli algebra  $M(V)$ , i.e.,  $L(V) = \text{Der}(M(V), M(V))$ . The finite dimensional Lie algebra  $L(V)$  was called the Yau algebra ([14], [22]). This invariant  $L(V)$  plays an important role in singularity theory [21]. In this article, we will generalize the Yau algebra and search for new invariants of a singularity from the higher derivations of its moduli algebra.

For a  $k$ -algebra  $R$ , from the definition of its  $n$ -th order derivations  $Der_k^n(R) := Der_k^n(R, R)$ , (see Definition 2.6), there is a natural filtration  $0 = Der_k^0(R) \subset Der_k^1(R) \subset \dots \subset Der_k^n(R) \subset Der_k^{n+1}(R) \subset \dots$  on modules of higher derivations of  $R$ . Set  $Der_k^\infty(R) := \bigcup_{n \geq 0} Der_k^n(R)$ , the module of all higher derivations of  $R$ , and the associated graded ring is defined as  $GrDer(R) := R \oplus (\bigoplus_{n \geq 1} Der_k^n(R) / Der_k^{n-1}(R))$ . In fact,  $GrDer(R)$  is a commutative  $k$ -algebra (see section 2).

In section 3, we show that for an Artinian  $k$ -algebra  $R$ , its higher derivations form a finite dimensional  $k$ -vector space  $Der_k^\infty(R)$ , hence the associated graded  $k$ -algebra  $GrDer(R)$  is Artinian. We establish this fact as the following theorem.

**Theorem A.** *Let  $R$  be a local Artinian algebra, containing a subfield  $k$  isomorphic to its residue field, then the graded derivation algebra  $GrDer(R)$  of  $R$  is Artinian. Moreover, the  $R$ -module  $Der_k^\infty(R)$  is free of rank  $\dim_k R - 1$ , and  $\dim_k GrDer(R) = (\dim_k R)^2$ .*

Let  $A, B$  be two  $k$ -algebras, as demonstrated in [3], we have  $Der_k^1(A \otimes B) \simeq Der_k^1(A) \otimes B + A \otimes Der_k^1(B)$ , now for the case of higher derivations, we propose an analogous conjecture.

**Conjecture 1.1.** *Let  $A, B$  be two finitely generated  $k$ -algebras, then there is a canonical isomorphism of  $k$ -algebras*

$$GrDer(A \otimes B) \simeq GrDer(A) \otimes GrDer(B).$$

---

Zuo is supported by NSFC Grant 12271280 and BJNFS Grant 1252009. Yau is supported by Tsinghua University Education Foundation fund (042202008).

We find that when  $A, B$  are local Artinian  $k$ -algebras with  $k$  as their residue field, then this conjecture holds.

**Theorem B.** *Let  $A, B$  be two local Artinian  $k$ -algebras, each containing  $k$  as a subfield which is isomorphic to their respective residue fields, then there is a canonical isomorphism of  $k$ -algebras*

$$\mathrm{GrDer}(A \otimes B) \simeq \mathrm{GrDer}(A) \otimes \mathrm{GrDer}(B).$$

In section 4, we focus on calculating higher derivations of the moduli algebras for certain typical types of singularities, such as the simple hypersurface singularities (ADE singularities). We propose the following Conjecture 1.2, and demonstrate its validity for binomial singularities (see Theorem 4.14).

**Conjecture 1.2.** *Let  $(V(f), 0)$  be an isolated hypersurface singularity defined by a weighted homogeneous polynomial  $f$ , and let  $\{e_i : 1 \leq i \leq \mu(f)\}$  be a monomial basis of the moduli algebra  $M(f)$ , where  $e_1 = 1$  and  $\mathbb{C}e_{\mu(f)}$  is the socle (the definition can be seen in [8]) of  $M(f)$ . For each  $e_i, 2 \leq i \leq \mu(f)$ , let  $d_i$  be the monomial such that  $d_i e_i = e_{\mu(f)}$ . Then the following hold:*

(1) *There exists a higher derivation  $D_0$  on  $M(f)$  that sends  $e_{\mu(f)}$  to 1, and maps the other  $e_i$  to 0.*

(2) *The  $M(f)$ -module  $\mathrm{Der}^\infty(M(f))$  is free of rank  $\mu(f) - 1$ , generated by  $\langle D_0, d_i \rangle, 2 \leq i \leq \mu(f)$ . (The notation  $\langle D_0, d_i \rangle$  is defined in Proposition 2.7.)*

Another problem we are interested in is when the weight type of a weighted homogeneous hypersurface singularity  $(V(f), 0)$  is fixed, what is the upper bound for the  $\mathbb{C}$ -dimension of  $\mathrm{Der}^n(M(V))/\mathrm{Der}^{n-1}(M(V))$  (we omit the subscript  $\mathbb{C}$ ). For the case of  $n = 1$ , Hussain-Yau-Zuo had conjectured that  $\dim L(V)$  attains the maximum in the case of Brieskorn singularity (see the Conjecture 1.1 in [12]). In this paper, we propose the following new inequality conjecture.

**Conjecture 1.3.** *Let  $(V, 0)$  be an isolated hypersurface singularity defined by a weighted homogeneous polynomial  $f(x_1, \dots, x_n)$  of weight type  $(w_1, \dots, w_n; 1), 0 \leq w_i \leq 1/2, \forall i$ , then the following inequality holds,*

$$\dim \mathrm{Der}^k(M(V))/\mathrm{Der}^{k-1}(M(V)) \leq h_k\left(\frac{1}{w_1}, \frac{1}{w_2}, \dots, \frac{1}{w_n}\right),$$

where the function  $h_k(a_1, a_2, \dots, a_n)$  denotes the dimension of  $\mathrm{Der}^k(M(V'))/\mathrm{Der}^{k-1}(M(V'))$  for the Brieskorn singularity  $(V', 0) = (\{x_1^{a_1} + x_2^{a_2} + \dots + x_n^{a_n} = 0\}, 0)$ .

In section 5, we prove that the Conjecture 1.3 holds for the case of  $k = 2$  and  $(V, 0)$  is a binomial isolated singularity. The explicit expression for the function  $h_2(a_1, a_2, \dots, a_n)$  is provided in Definition 5.2.

**Theorem C.** *If  $(V, 0)$  is an isolated hypersurface singularity defined by a binomial  $f(x_1, x_2)$  of weight type  $(w_1, w_2; 1)$ , then the following inequality holds:*

$$\dim \mathrm{Der}^2(M(V))/\mathrm{Der}^1(M(V)) \leq h_2\left(\frac{1}{w_1}, \frac{1}{w_2}\right).$$

An important problem lies in the relationship between the higher derivations and geometry. Let  $R$  be the affine ring of an algebraic variety  $V$  over a field  $k$  of characteristic zero, it has been shown in [9] that if  $R$  is regular, then  $\mathrm{GrDer}(R)$  can be generated by  $\mathrm{Der}_k^1(R)$  as a  $k$ -algebra. And Y. Nakai (in [18]) proposed the following well-known conjecture.

**Nakai Conjecture:** the regularity of  $R$  is equivalent to the condition that  $GrDer(R)$  can be generated by  $Der_k^1(R)$ .

The Nakai Conjecture for a special case of the ring  $R = k[x_1, \dots, x_n]/(a_1x_1^m + a_2x_2^m + \dots + a_nx_n^m)$  has been proved in [4]. In section 6, we extended this result to the case of  $f = x_1^{a_1} + x_2^{a_2} + \dots + x_n^{a_n}$  as follows.

**Theorem D.** *Let  $k$  be a field of characteristic zero, then for  $R = k[x_1, x_2, \dots, x_n]/(x_1^{a_1} + x_2^{a_2} + \dots + x_n^{a_n})$ ,  $a_i \geq 2$ , its derivation algebra  $GrDer(R)$  cannot be generated by its first order derivations  $Der_k^1(R)$  as a  $k$ -algebra.*

**Remark 1.4.** We can replace the affine ring  $R$  in **Nakai Conjecture** by the moduli algebra  $M(V)$  of an isolated singularity  $(V, 0)$ , and propose a contrapositive of a special case of Nakai Conjecture. For the case of  $(V, 0)$  a weighted homogeneous hypersurface singularity, we find that this is obviously true if our Conjecture 1.2 and the Yau Conjecture (see section 2) hold, as the higher derivation  $D_0$  in Conjecture 1.2 is of negative weight (see Definition 2.1).

At the end of the text, we provide the magma codes for computing higher derivations of the moduli algebra associated with a weighted homogeneous isolated hypersurface singularity (see section 7). The example code is used to compute the third order derivations of the moduli algebra for the  $E_7$  singularity [1].

## 2. PRELIMINARIES

### 2.1. weighted homogeneous isolated hypersurface singularities.

**Definition 2.1.** Let  $k$  be a field, a polynomial  $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \in k[x_1, \dots, x_n]$  is called weighted homogeneous of weight type  $(w_1, \dots, w_n; d)$ , if  $w_1\alpha_1 + w_2\alpha_2 + \dots + w_n\alpha_n = d$  holds for each multi-index  $\alpha = (\alpha_1, \dots, \alpha_n)$  with  $a_\alpha \neq 0$ . We call  $w_i$  the weight of  $x_i$  and  $d$  the weighted degree of  $f$ , denoted as  $wt(x_i) = w_i$  and  $wt(f) = d$ .

Notice that when the isolated hypersurface singularity  $(V(f), 0)$  is defined by a weighted homogeneous polynomial  $f \in \mathbb{C}[x_1, \dots, x_n]$ , the ideal  $(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$  of  $\mathbb{C}\{x_1, \dots, x_n\}$  is generated by weighted homogeneous polynomials, hence  $M(V) = M(f) = \mathbb{C}\{x_1, \dots, x_n\}/(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$  has a graded structure, we call a higher derivation  $D$  (see Definition 2.6) of  $M(f)$  to have weight  $d$  if  $D : M(f)^s \rightarrow M(f)^{s+d}$ , where  $M(f)^s$  denotes the set of weight  $s$  elements of  $M(f)$ .

An important class of weighted homogeneous isolated hypersurface singularities is fewnomial singularities, defined as follows.

**Definition 2.2.** A weighted homogeneous polynomial  $f$  is called fewnomial if the number of variables coincides with the number of the monomials, and an isolated hypersurface singularity  $(V(f), 0)$  is called a fewnomial singularity if it is defined by a fewnomial. Especially, when  $f$  is in 2 variables,  $f$  (resp.  $(V(f), 0)$ ) is called a binomial (singularity); when  $f$  is in 3 variables,  $f$  (resp.  $(V(f), 0)$ ) is called a trinomial (singularity).

**Proposition 2.3.** *Let  $(V(f), 0)$  be a fewnomial singularity with  $\text{mult}(f) \geq 3$ , then  $f$  is analytically equivalent to a linear combination of the following three types of series:*

- (1)  $x_1^{a_1} + x_2^{a_2} + \dots + x_n^{a_n}, n \geq 1;$
- (2)  $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \dots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}, n \geq 2;$
- (3)  $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \dots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1, n \geq 2.$

**Proposition 2.4.** *Each binomial singularity  $(V(f), 0)$  is analytically equivalent to one of these three cases: (A)  $f = x^a + y^b$ ; (B)  $f = x^ay + y^b$ ; (C)  $f = x^ay + y^bx$ .*

**Proposition 2.5.** *Let  $(V(f), 0)$  be a trinomial singularity with  $\text{mult}(f) \geq 3$ , then  $f(x_1, x_2, x_3)$  is analytically equivalent to the following five types:*

- (1)  $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$ ;
- (2)  $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$ ;
- (3)  $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$ ;
- (4)  $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}x_1$ ;
- (5)  $x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$ .

## 2.2. higher derivations and higher differentials.

We review some basic concepts for  $n$ -th order derivations and  $n$ -th order *Kähler* differentials. Let  $k$  be a field,  $R$  be a  $k$ -algebra,  $A$  be an  $R$ -module, and  $\text{Hom}_k(R, A)$  represents the  $k$ -linear maps from  $R$  to  $A$ . There are two ways to define the derivations of order  $n$ , see [20].

The first way is to define inductively,  $\text{Diff}_k^n(R, A) = 0$  for  $n < 0$ , and for  $n > 0$ ,  $\text{Diff}_k^n(R, A) = \{D \in \text{Hom}_k(R, A) : [D, a] \in \text{Diff}_k^{n-1}(R, A), \forall a \in R\}$ , it is easy to see that  $\text{Diff}_k^0(R, A) = A$ . We call elements in  $\text{Diff}_k^n(R, A)$  the  $n$ -th order differential operators on  $A$ ,  $\text{Diff}_k^\infty(R, A) := \bigcup_{n \geq 0} \text{Diff}_k^n(R, A)$ . When  $R = A$ ,  $\text{Diff}_k^n(R) := \text{Diff}_k^n(R, R)$ , sometimes the subscript  $k$  is omitted.

The second way is more precise as following.

**Definition 2.6.** ([18])  $D \in \text{Hom}_k(R, A)$  is called an  $n$ -th order derivation of  $R$  into  $A$  over  $k$  if for any  $n+1$  elements of  $R$ , denoted as  $a_0, a_1, \dots, a_n$ , we have

$$D(a_0 a_1 \cdots a_n) = \sum_{s=1}^n (-1)^{s+1} \sum_{i_1 < \dots < i_s} a_{i_1} \cdots a_{i_s} D(a_0 \cdots \hat{a}_{i_1} \cdots \hat{a}_{i_s} \cdots a_n),$$

where the notation  $\hat{\phantom{a}}$  means the corresponding term is omitted. We use  $\text{Der}_k^n(R, A)$  to denote the set of  $n$ -th order derivations of  $R$  into  $A$  over  $k$ . And for  $n \leq 0$ ,  $\text{Der}_k^n(R, A) = 0$  by convention.

In other words, the definition means that  $[\dots[[D, a_1], a_2], \dots, a_n] = 0, \forall a_1, a_2, \dots, a_n \in R$  holds for  $D$ , see [18].

The relation between these two definitions is  $\text{Der}_k^n(R, A) = \{D \in \text{Diff}_k^n(R, A) | D(1) = 0\}$ , and there is an isomorphism  $\text{Diff}_k^n(R, A) \simeq A \oplus \text{Der}_k^n(R, A)$ , which maps  $D$  to  $(D(1), D - D(1)_R)$ , where  $D(1)_R$  is the  $R$ -linear map  $r \rightarrow r \cdot D(1)$  of  $R$  into  $A$ . More details can be found in [20].

When  $A = R$ , we simply denote  $\text{Der}_k^n(R, A)$  by  $\text{Der}_k^n(R)$ . Since for a higher derivation  $D \in \text{Der}_k^n(R)$  and a higher derivation  $D' \in \text{Der}_k^m(R)$ , their compositions  $DD', D'D \in \text{Der}_k^{m+n}(R)$ , and their Lie bracket  $[D, D'] \in \text{Der}_k^{m+n-1}(R)$ , thus we obtain a graded  $k$ -algebra structure. Denote  $\text{GrDer}_k(R) := R \oplus (\oplus_{n \geq 1} \text{Der}_k^n(R) / \text{Der}_k^{n-1}(R))$ , the multiplication on it is just the composition of higher derivations, moreover, it is commutative.

Now we consider the case of  $R = k[x_1, x_2, \dots, x_s]/I$ , where  $k$  is a field of characteristic 0, and  $I$  is an ideal (the case of  $R = k\{x_1, x_2, \dots, x_s\}/I$  for  $k$  a valued field of characteristic 0 is similar). Let  $\partial_x^{(\alpha)} = (1/\alpha!) \cdot \partial^\alpha / \partial x^\alpha$  be the derivation from  $k[x_1, \dots, x_s]$  to  $R$ , where  $\alpha = (\alpha_1, \dots, \alpha_s)$  is a multi-index,  $\alpha! = \alpha_1! \cdot \alpha_2! \cdots \alpha_s!$ . It is easily known that  $\text{Diff}_k^\infty(k[x_1, \dots, x_s], R)$  is a free  $R$ -module generated by all  $\partial_x^{(\alpha)}$  s. The Proposition 2.10 in [20] characterizes the algorithm of the higher derivations of  $R$ .

**Proposition 2.7.** ([20]) *With notations as above, let  $D = \sum_{|\alpha| \leq n} c_\alpha(D) \partial_x^{(\alpha)} \in \text{Diff}_k^n(k[x_1, \dots, x_s], R)$ , in which  $c_\alpha(D) \in R$ , then the following conditions are equivalent:*

(1)  $D(I) = 0$  in  $R$ , i.e.  $D$  can be viewed in  $\text{Diff}_k^n(R)$ .

(2)  $\sum_{|\alpha| \leq n} c_{\alpha+\beta}(D) \partial_x^{(\alpha)}(I) = 0$  in  $R$ , for all  $\beta \in \mathbb{N}^s$ .

(We use the notation  $\langle D, x^\beta \rangle$  to denote the higher differential operator  $\sum_{|\alpha| \leq n} c_{\alpha+\beta}(D) \partial_x^{(\alpha)}$ )

throughout the entire text, it has been shown in [20] that if we express  $x^\beta$  in the form  $x^\beta = x_{i_1}x_{i_2}\cdots x_{i_{|\beta|}}$ , then  $\langle D, x^\beta \rangle$  is just the higher differential operator  $[\dots[D, x_{i_1}], x_{i_2}], \dots, x_{i_{|\beta|}}]$ .

(3)  $\sum_{|\alpha| \leq n} c_{\alpha+\beta}(D)\partial_x^{(\alpha)}(I) = 0$  in  $R$ , for all  $\beta \in \mathbb{N}^s$  satisfying  $|\beta| \leq n-1$ .

(4) The assertion in (3) holds for a set of generators of  $I$ .

When  $R = k\{x_1, x_2, \dots, x_s\}/I$  for  $k$  a valuation field of characteristic 0, the equivalence of these conditions still hold after replacing  $\text{Diff}_k^n(k[x_1, \dots, x_s], R)$  by  $\text{Diff}_k^n(k\{x_1, \dots, x_s\}, R)$ .

An open problem for derivations was proposed by Halperin.

**Halperin Conjecture**([11]): For a complete intersection algebra  $R = k[x_1, \dots, x_n]/I$ , where  $I$  is generated by weighted homogeneous polynomials of the same weight type, then there is no first order derivations of  $R$  with negative weight.

One of the special cases of Halperin Conjecture is the Yau Conjecture.

**Yau Conjecture**([17]): Let  $(V(f), 0)$  be an isolated hypersurface singularity defined by a weighted homogeneous polynomial  $f(x_1, \dots, x_n)$  of weight type  $(w_1, \dots, w_n; d)$ , assume  $d \geq 2w_1 \geq 2w_2 \geq \dots \geq 2w_n$  without loss of generality, then  $L(V) := \text{Der}^1(M(V))$  is non-negatively graded.

The Yau Conjecture remains an open problem, with only low-dimensional cases having been proved, primarily through explicit calculations. Despite this, there have also been many progresses in these types of problems (see [5],[6],[17]). And from the examples we calculate in section 4, the Halperin Conjecture will fail if we enlarge the first order derivations to higher derivations.

Next we introduce another object which can be viewed as the dual of the module of  $n$ -th order derivations, that is the  $n$ -th order *Kähler* differentials.

**Definition 2.8.** Let  $k$  be a field,  $R$  be a  $k$ -algebra, an  $R$ -module  $\Omega_{R/k}^{(n)}$  together with a canonical  $k$ -linear map  $d_R^n : R \rightarrow \Omega_{R/k}^{(n)}$  is called the  $n$ -th order *Kähler* differentials of  $R$  over  $k$  if it satisfies the following universal property:

For an arbitrary  $R$ -module  $A$ , and an  $n$ -th order derivation  $D \in \text{Der}_k^n(R, A)$ , there exists a unique  $R$ -linear homomorphism  $h : \Omega_{R/k}^{(n)} \rightarrow A$ , such that  $D = h \circ d_R^n$ .

**Remark 2.9.** The above definition is equivalent to say that the functor  $\text{Der}_k^n(R, -)$  from the category of  $R$ -modules to the category of sets is representable, and the canonical map  $d_R^n : R \rightarrow \Omega_{R/k}^{(n)}$  is the universal element.

We give more concrete descriptions for the  $n$ -th order *Kähler* differentials, they are similar to the case of the first order *Kähler* differentials.

**Theorem 2.10.** ([18]) Let  $R$  be a  $k$ -algebra, denote  $I_R$  as the kernel of the multiplication map from  $R \otimes_k R$  to  $R$ , giving the structure of  $R$ -module to  $R \otimes_k R$  by multiplying on the left. Then the  $n$ -th order *Kähler* differentials  $\Omega_{R/k}^{(n)}$  are isomorphic to  $I_R/I_R^{n+1}$ , and the canonical map is given by  $d_R^n(r) = (1 \otimes r - r \otimes 1) + I_R^{n+1}$ . Moreover,  $d_R^n(r)$  is a higher derivation of order  $n$ , we call it the canonical derivation of  $R$  in  $\Omega_{R/k}^{(n)}$ .

At the end of this subsection, we briefly introduce the relation between the derivations, differentials, and the smoothness.

**Theorem 2.11.** ([18]) Let  $k$  be a field,  $R$  be a  $k$ -algebra.  $I \subset R$  be an ideal, and  $S = R/I$ , define the  $k$ -linear map  $\rho : I/I^{n+1} \rightarrow \Omega_{R/k}^{(n)} \otimes_R S$  as  $\rho(\bar{r}) = d_R^n(r) \otimes 1$ , and denote  $N$  to be the  $S$ -module generated by the image of  $\rho$ . Then one has the following exact sequence:

$$0 \rightarrow N \rightarrow \Omega_{R/k}^{(n)} \otimes_R S \rightarrow \Omega_{S/k}^{(n)} \rightarrow 0.$$

When  $R = k[x_1, x_2, \dots, x_s]/I$  for  $k$  a field of characteristic 0 (resp.  $R = k\{x_1, x_2, \dots, x_s\}/I$  for  $k$  a valuation field of characteristic 0), this theorem provides an idea to give a free resolution for  $R$ . Remark  $B = k[x_1, x_2, \dots, x_s]$  (resp.  $B = k\{x_1, x_2, \dots, x_s\}$ ),  $I = (f_1, f_2, \dots, f_r)$ . As for any  $a, b \in B$ , one holds the identity in  $B \otimes_k B$ :

$$1 \otimes ab - ab \otimes 1 = (1 \otimes a - a \otimes 1)(1 \otimes b - b \otimes 1) + a(1 \otimes b - b \otimes 1) + b(1 \otimes a - a \otimes 1),$$

this implies that  $d_B^n(ab) = d_B^n(a)d_B^n(b) + ad_B^n(b) + bd_B^n(a)$ ,  $\forall a, b \in B$ .

Let  $h = \sum_{i=1}^r g_i f_i \in I$ , then  $\rho(\bar{h}) = \sum_i d_B^n(g_i f_i) \otimes 1 = \sum_i (d_B^n(g_i)d_B^n(f_i) \otimes 1 + g_i(d_B^n(f_i) \otimes 1) + f_i(d_B^n(g_i) \otimes 1)) = \sum_i ((d_B^n(g_i)d_B^n(f_i) \otimes 1 + g_i(d_B^n(f_i) \otimes 1)))$ . Remark  $F_\beta^i = (d_B^n(x))^\beta d_B^n(f_i)$ ,  $\beta \in \mathbb{N}^s, 1 \leq i \leq r$ . An easy calculation tells that the module  $N$  in the above Theorem 2.11 is generated by the set  $\{F_\beta^i \otimes 1 \mid |\beta| \leq n-1\}$  as an  $R$ -module.

**Lemma 2.12.** *With notations as above, for  $\beta \in \mathbb{N}^s, |\beta| \leq n-1, 1 \leq i \leq r$ , one has*

$$F_\beta^i = \sum_{\substack{\alpha \in \mathbb{N}^s \\ 1 \leq |\alpha| \leq n}} \frac{1}{(\alpha - \beta)!} \frac{\partial^{\alpha - \beta}(f_i)}{\partial x^{\alpha - \beta}} (d_B^n(x))^\alpha,$$

and we make a convention that  $\frac{1}{(\alpha - \beta)!} \frac{\partial^{\alpha - \beta}(f_i)}{\partial x^{\alpha - \beta}} = 0$ , whenever  $\begin{cases} |\alpha| < 1 + |\beta|, \\ \alpha_i < \beta_i \text{ for some } i, \\ \alpha = \beta. \end{cases}$

*Proof.* A direct calculation, omitted. □

**Definition 2.13.** The  $n$ -th order Jacobian matrix  $Jac_n(f_1, f_2, \dots, f_r)$  is a matrix of size  $r \binom{s+n-1}{s}$  times  $\binom{s+n}{s} - 1$ , whose rows are the vectors generated by the coefficients of  $F_\beta^i$ 's, as presented in above lemma.

**Remark 2.14.** When  $I = (f)$  is generated by a single polynomial, then the  $n$ -th order Jacobian matrix  $Jac_n(f)$  is related to the higher Nash blow up (ref. [7]) of the hypersurface  $V(f)$ . And Hussain-Ma-Yau-Zuo raised new invariants of singularities from the  $n$ -th Jacobian matrix (see [13]).

Generalized Jacobian criterion also holds as following.

**Theorem 2.15.** ([19]) *Let  $f \in \mathbb{C}[x_1, x_2, \dots, x_s]$  be a reduced non-constant polynomial,  $p \in V(f)$ . Then  $p$  is non-singular if and only if  $\text{rank}(Jac_n(f))|_p = \binom{n+s-1}{s}$ .*

**Theorem 2.16.** ([10] Chapter II-8) *Let  $R$  be a local ring containing a field  $k$  isomorphic to its residue field. Assume further more that  $k$  is perfect and  $R$  is a localization of a finitely generated  $k$ -algebra. Then  $\Omega_{R/k}^{(1)}$  is a free module of rank  $\dim R$  if and only if  $R$  is a regular local ring.*

The above Theorem 2.16 has also been generalized to the  $n$ -th order *Kähler* differentials for the hypersurface case.

**Theorem 2.17.** ([19]) *Let  $A = k[x_1, x_2, \dots, x_s]/(f)$ ,  $f$  is irreducible, and  $R = A_{\mathfrak{m}}$  be the localization of  $A$  at a maximal ideal  $\mathfrak{m}$  of  $A$ . Then  $\Omega_{R/k}^{(n)}$  is a free module of rank  $\binom{n+s-1}{s-1}$  if and only if  $R$  is a regular local ring.*

Geometrically speaking, the locally freeness of the *Kähler* differentials implies the smoothness. The Zariski-Lipman Conjecture makes the prediction for the module of derivations that it has the similar property [15].

**Zariski-Lipman Conjecture:** when  $R$  is the affine ring of an algebraic variety  $V$  over a characteristic zero field  $k$ , or  $R$  is a local analytic ring, and if  $\text{Der}_k^1(R)$  is free, then  $R$  is regular.

## 3. PROOF OF MAIN THEOREM A AND THEOREM B

The fact in Theorem 2.10 implies the main Theorem A, now we begin to prove it.

*Proof.* (of Theorem A) Denote  $\mathfrak{m}$  to be the unique maximal ideal of  $R$ ,  $R/\mathfrak{m} \simeq k$ . Then  $R \otimes_k R$  is also a local Artinian algebra with the unique maximal ideal  $\mathfrak{m} \otimes R + R \otimes \mathfrak{m}$ , so  $I_R \subset \mathfrak{m} \otimes R + R \otimes \mathfrak{m}$ . Since  $R$  is Artinian,  $\mathfrak{m}$  is nilpotent, write  $\mathfrak{m}^n = 0$ , then  $(\mathfrak{m} \otimes R + R \otimes \mathfrak{m})^{2n-1} = 0$ , this implies  $I_R^{2n-1} = 0$ .

Therefore, when  $m \geq 2n - 2$ ,

$$\mathrm{Der}_k^m(R) \simeq \mathrm{Hom}_R(\Omega_{R/k}^{(m)}, R) \simeq \mathrm{Hom}_R(I_R/I_R^{m+1}, R) \simeq \mathrm{Hom}_R(I_R, R),$$

and as this isomorphism is also compatible with the inclusion  $\mathrm{Der}_k^m(R) \subset \mathrm{Der}_k^{m+1}(R)$ , we have  $\mathrm{Der}_k^{m+1}(R)/\mathrm{Der}_k^m(R) = 0$  for  $m \geq 2n - 2$ . So  $\mathrm{GrDer}_k(R)$  is a finite dimensional  $k$ -vector space, it is Artinian as a  $k$ -algebra.

Next we calculate the  $k$ -dimension of  $\mathrm{GrDer}_k(R)$ . By the definition of  $I_R$ , there is an exact sequence of  $R$ -modules:

$$0 \rightarrow I_R \rightarrow R \otimes_k R \rightarrow R \rightarrow 0.$$

As  $R$  is a free  $R$ -module, hence projective, this exact sequence splits. So  $I_R$  is a free  $R$ -module of rank  $\dim_k R - 1$ , and  $\mathrm{Der}_k^\infty(R) \simeq \mathrm{Hom}_R(I_R, R)$  is also a free  $R$ -module of rank  $\dim_k R - 1$ . Thus  $\dim_k \mathrm{GrDer}_k(R) = \dim_k \mathrm{Diff}_k^\infty(R) = \dim_k R + \dim_k \mathrm{Der}_k^\infty(R) = \dim_k R + (\dim_k R) \cdot (\dim_k R - 1) = (\dim_k R)^2$ .  $\square$

To prove the main Theorem B, we propose the following two lemmas to give the inclusion on one side.

**Lemma 3.1.** *Let  $A, B$  be two finitely generated  $k$ -algebras,  $D_1 \in \mathrm{Diff}_k^i(A)$ ,  $D_2 \in \mathrm{Diff}_k^j(B)$ , then as a  $k$ -linear endomorphism of  $A \otimes B$ ,  $D_1 \otimes D_2$  induces an  $(i + j)$ -th order differential operator on  $A \otimes B$ .*

*Proof.* As  $A \otimes \mathrm{Der}_k^j(B) \subset \mathrm{Der}_k^j(A \otimes B) \subset \mathrm{Diff}_k^{i+j}(A \otimes B)$ ,  $\mathrm{Der}_k^i(A) \otimes B \subset \mathrm{Der}_k^i(A \otimes B) \subset \mathrm{Diff}_k^{i+j}(A \otimes B)$  from Definition 2.6, we may assume  $D_1 \in \mathrm{Der}_k^i(A)$  and  $D_2 \in \mathrm{Der}_k^j(B)$  without loss of generality.

Write  $A = k[x_1, \dots, x_r]/I$ ,  $B = k[y_1, \dots, y_s]/J$ ,  $D_1$  is induced from  $\tilde{D}_1 \in \mathrm{Der}_k^i(k[x_1, \dots, x_r], A)$ ,  $D_2$  is induced from  $\tilde{D}_2 \in \mathrm{Der}_k^j(k[y_1, \dots, y_s], B)$ , then  $A \otimes B \simeq k[x_1, \dots, x_r, y_1, \dots, y_s]/(I, J)$ , and  $D_1 \otimes D_2$  is induced from  $\tilde{D}_1 \otimes \tilde{D}_2 \in \mathrm{Der}_k^{i+j}(k[x_1, \dots, x_r, y_1, \dots, y_s], A \otimes B)$ . Now for each monomial  $x^\alpha y^\beta$ ,  $|(\alpha, \beta)| = |\alpha| + |\beta| \leq i + j - 1$ ,  $\langle \tilde{D}_1 \otimes \tilde{D}_2, x^\alpha y^\beta \rangle = \langle \tilde{D}_1, x^\alpha \rangle \langle \tilde{D}_2, y^\beta \rangle$ , and from Proposition 2.7,  $\langle \tilde{D}_1, x^\alpha \rangle(I) = 0$  in  $A$ ,  $\langle \tilde{D}_2, y^\beta \rangle(J) = 0$  in  $B$ , therefore  $\langle \tilde{D}_1 \otimes \tilde{D}_2, x^\alpha y^\beta \rangle(I \otimes k[y_1, \dots, y_s] + k[x_1, \dots, x_r] \otimes J) = 0$  in  $A \otimes B$ , and  $D_1 \otimes D_2$  is an  $(i + j)$ -th order derivation of  $A \otimes B$ .  $\square$

**Lemma 3.2.** *For two finitely generated  $k$ -algebras  $A, B$ , there is a canonical inclusion*

$$\bigoplus_{i+j=n} \frac{\mathrm{Diff}_k^i(A)}{\mathrm{Diff}_k^{(i-1)}(A)} \otimes \frac{\mathrm{Diff}_k^j(B)}{\mathrm{Diff}_k^{(j-1)}(B)} \hookrightarrow \frac{\mathrm{Diff}_k^n(A \otimes B)}{\mathrm{Diff}_k^{(n-1)}(A \otimes B)}, \forall n \geq 0.$$

*Proof.* We do induction on  $n$ , the cases of  $n = 0, 1$  have been already known.

Assume this inclusion holds for the case of  $n - 1$ , consider the case of  $n$ ,  $n \geq 2$ . Let  $(\bar{\psi}_0, \bar{\psi}_1, \dots, \bar{\psi}_n)$  maps to zero of the right hand side, where  $\bar{\psi}_i \in \frac{\mathrm{Diff}_k^i(A)}{\mathrm{Diff}_k^{(i-1)}(A)} \otimes \frac{\mathrm{Diff}_k^{(n-i)}(B)}{\mathrm{Diff}_k^{(n-i-1)}(B)}$  and  $\psi_i$  is a preimage of it in  $\mathrm{Diff}_k^i(A) \otimes \mathrm{Diff}_k^{n-i}(B)$ . Write  $A = k[x_1, \dots, x_r]/I$ ,  $B = k[y_1, \dots, y_s]/J$ , then for any  $a \in \{x_1, \dots, x_r\}$ ,  $\langle \sum \psi_i, a \otimes 1 \rangle \in \mathrm{Diff}_k^{(n-2)}(A \otimes B)$ , and  $\langle \psi_i, a \otimes 1 \rangle \in \mathrm{Diff}_k^{i-1}(A) \otimes \mathrm{Diff}_k^{n-i}(B)$

for each  $i$ . By the induction hypothesis, we have  $\langle \psi_i, a \otimes 1 \rangle$  is zero in  $\frac{\text{Diff}_k^{(i-1)}(A)}{\text{Diff}_k^{(i-2)}(A)} \otimes \frac{\text{Diff}_k^{(n-i)}(B)}{\text{Diff}_k^{(n-i-1)}(B)}$ . Similarly for each  $b \in \{y_1, y_2, \dots, y_s\}$ ,  $\langle \psi_i, 1 \otimes b \rangle$  is zero in  $\frac{\text{Diff}_k^i(A)}{\text{Diff}_k^{(i-1)}(A)} \otimes \frac{\text{Diff}_k^{(n-i-1)}(B)}{\text{Diff}_k^{(n-i-2)}(B)}$ . If there exists  $t$  such that  $\bar{\psi}_t \neq 0$ , express  $\psi_t = \sum C_{\alpha, \beta} \partial_x^{(\alpha)} \partial_y^{(\beta)}$ ,  $C_{\alpha, \beta} \in A \otimes B$ , then there exists multi-index  $\alpha, \beta$ , such that  $C_{\alpha, \beta} \neq 0$  in  $A \otimes B$ , and  $|\alpha| = i, |\beta| = n - i$ . Without loss of generality, let  $\alpha_1 \neq 0$ , then  $\langle \psi_t, \bar{x}_1 \otimes 1 \rangle$  is not equal to zero in  $\frac{\text{Diff}_k^{(i-1)}(A)}{\text{Diff}_k^{(i-2)}(A)} \otimes \frac{\text{Diff}_k^{(n-i)}(B)}{\text{Diff}_k^{(n-i-1)}(B)}$ , a contradiction!

Therefore the inclusion holds for the case of  $n$ , we have finished the proof.  $\square$

*Proof.* (of Theorem B) Lemma 3.1 and 3.2 tell that the inclusion  $\supseteq$  holds on each degree of the graded derivation rings. And Theorem A implies that  $\dim_k \text{GrDer}(A \otimes B) = (\dim_k A \otimes B)^2 = (\dim_k A)^2 \cdot (\dim_k B)^2 = \dim_k \text{GrDer}(A) \cdot \dim_k \text{GrDer}(B) = \dim_k (\text{GrDer}(A) \otimes \text{GrDer}(B))$ , they have the same dimension as  $k$ -vector spaces, hence must be isomorphic.  $\square$

**Remark 3.3.** Now we know that  $\text{Diff}_k^\infty(R)$  is free  $R$ -module of rank  $\dim_k R$  and has  $k$ -dimension  $(\dim_k R)^2$ . In other words, each  $k$ -linear endomorphism of  $R$  can be realized as a higher differential operator of  $R$ . However, we must notice that  $\text{GrDer}_k(R)$  is not a free  $R$ -module.

**Corollary 3.4.** *Let  $(V(f), 0)$  be an isolated hypersurface singularity defined by a weighted homogeneous polynomial  $f$ , then the graded derivation algebra  $\text{GrDer}(M(f))$  of its moduli algebra  $M(f) = \mathbb{C}\{x\}/(f, J(f))$  is an Artinian  $\mathbb{C}$ -algebra, whose dimension is  $\mu(f)^2$ , where  $\mu(f) = \dim_{\mathbb{C}} M(f)$  is the Milnor number (equal to the Tjurina number) of  $(V(f), 0)$ .*

#### 4. SOME EXAMPLES

In this section, we list the following two conjectures, and verify them for some concrete examples.

**Conjecture 4.1.** *Let  $(V, 0)$  be an isolated hypersurface singularity defined by a weighted homogeneous polynomial  $f$ , then the highest degree of  $\text{GrDer}(M(V))$  is of one dimensional, i.e. there exists an integral  $n$ , such that  $\dim \text{GrDer}(M(V))_n = 1$  and  $\text{GrDer}(M(V))_m = 0, \forall m > n$ , where  $\text{GrDer}(M(V))_i := \text{Der}^i(M(V))/\text{Der}^{i-1}(M(V))$  for  $i > 0$  and  $\text{GrDer}(M(V))_0 := M(V)$ .*

**Conjecture 4.2.** *Let  $(V, 0)$  be an isolated hypersurface singularity defined by a weighted homogeneous polynomial  $f$ , then elements in  $\text{Der}^\infty(M(V))$  can be discussed regarding their homogeneity (see the paragraph below Definition 2.1), we claim that the amount of higher derivations of weight  $k$  is the same as the amount of higher derivations of weight  $-k$ .*

##### 4.1. The simple hypersurface singularities case.

We know that simple hypersurface singularities have classifications with good form, which are called the ADE singularities (see [1]). Since ADE singularities are defined by weighted homogeneous polynomials, the weight type of the polynomial  $f$  induces a weighted graded structure on its moduli algebra  $M(f)$ , and further on the commutative ring  $\text{GrDer}(M(f))$ . Let  $\text{GrDer}(M(f))^s$  be the set of weight  $s$  elements in  $\text{GrDer}(M(f))$ , we will also compute the Hilbert-Poincare series  $P(t) := \sum_s \dim \text{GrDer}(M(f))^s \cdot t^s$  for them.

###### 4.1.1. The $A_k$ case.

First, as an example, we present the calculation of the higher derivations of the moduli algebra associated with the  $A_3$  singularity, the algorithm employed is based on Proposition 2.7.

**Example 4.3.** The  $A_3$  singularity is the isolated hypersurface singularity  $(V(f), 0)$  defined by  $f = x_1^4 + x_2^2 + \cdots + x_n^2$ , whose moduli algebra  $M(V) \simeq \mathbb{C}\{x_1\}/(x_1^3)$  is a three dimensional vector space, with a basis  $1, x_1, x_1^2$ . From Proposition 2.7, each  $D \in \text{Der}^m(M(V))$  must have the form  $D = \sum_{i=1}^m c_i \partial_{x_1}^{(i)}$ ,  $c_i \in M(V)$ , where  $\partial_{x_1}^{(i)} = 1/i! \partial_{x_1}^i$ , we will continue this notation in the whole text. Now we treat for each  $m$  separately.

$D = c_1 \partial_{x_1}$  induces a first order derivation on  $M(V)$  if and only if  $D(x_1^3) = 3c_1 x_1^2$  equal to 0 in  $M(V)$ , thus  $c_1 = \lambda_1 x_1 + \lambda_2 x_1^2$ ,  $\lambda_1, \lambda_2 \in \mathbb{C}$ , and  $\text{Der}^1(M(V)) = \mathbb{C}\langle x_1 \partial_{x_1}, x_1^2 \partial_{x_1} \rangle$ ;  $D = c_1 \partial_{x_1} + c_2 \partial_{x_1}^{(2)}$  induces a second order derivation on  $M(V)$  if and only if  $D(x_1^3) = 3c_1 x_1^2 + 3c_2 x_1 = 0$  in  $M(V)$  and  $\langle D, x_1 \rangle(x_1^3) = c_1 x_1^3 + 3c_2 x_1^2 = 0$  in  $M(V)$ , thus  $(c_1, c_2) = \lambda_1(1, -x_1) + \lambda_2(x_1, 0) + \lambda_3(x_1^2, 0) + \lambda_4(0, x_1^2)$ ,  $\lambda_i \in \mathbb{C}$ , and  $\text{Der}^2(M(V)) = \mathbb{C}\langle \partial_{x_1} - x_1 \partial_{x_1}^{(2)}, x_1 \partial_{x_1}, x_1^2 \partial_{x_1}, x_1^2 \partial_{x_1}^{(2)} \rangle$ . Similarly,  $D = c_1 \partial_{x_1} + c_2 \partial_{x_1}^{(2)} + c_3 \partial_{x_1}^{(3)} \in \text{Der}^3(M(V))$  if and only if  $D(x_1^3) = 3c_1 x_1^2 + 3c_2 x_1 + c_3 = 0$  in  $M(V)$ ,  $\langle D, x_1 \rangle(x_1^3) = c_1 x_1^3 + 3c_2 x_1^2 + 3c_3 x_1 = 0$  in  $M(V)$  and  $\langle D, x_1^2 \rangle(x_1^3) = c_2 x_1^3 + 3c_3 x_1^2 = 0$  in  $M(V)$ , thus  $(c_1, c_2, c_3) = \lambda_1(1, 0, -3x_1^2) + \lambda_2(1, -x_1, 0) + \lambda_3(x_1, 0, 0) + \lambda_4(x_1^2, 0, 0) + \lambda_5(0, x_1^2, 0)$ ,  $\lambda_i \in \mathbb{C}$ , and  $\text{Der}^3(M(V)) = \mathbb{C}\langle \partial_{x_1} - 3x_1^2 \partial_{x_1}^{(3)}, \partial_{x_1} - x_1 \partial_{x_1}^{(2)}, x_1 \partial_{x_1}, x_1^2 \partial_{x_1}, x_1^2 \partial_{x_1}^{(2)} \rangle$ .  $D = c_1 \partial_{x_1} + c_2 \partial_{x_1}^{(2)} + c_3 \partial_{x_1}^{(3)} + c_4 \partial_{x_1}^{(4)} \in \text{Der}^4(M(V))$  if and only if  $D(x_1^3) = 3c_1 x_1^2 + 3c_2 x_1 + c_3 = 0$  in  $M(V)$ ,  $\langle D, x_1 \rangle(x_1^3) = c_1 x_1^3 + 3c_2 x_1^2 + 3c_3 x_1 + c_4 = 0$  in  $M(V)$ ,  $\langle D, x_1^2 \rangle(x_1^3) = c_2 x_1^3 + 3c_3 x_1^2 + 3c_4 x_1 = 0$  in  $M(V)$  and  $\langle D, x_1^3 \rangle(x_1^3) = c_3 x_1^3 + 3c_4 x_1^2 = 0$  in  $M(V)$ , solving these four equations in  $M(V)$ , we obtain  $(c_1, c_2, c_3, c_4) = \lambda_1(1, 0, -3x_1^2, 0) + \lambda_2(1, -x_1, 0, 0) + \lambda_3(x_1, 0, 0, 0) + \lambda_4(x_1^2, 0, 0, 0) + \lambda_5(0, 1, -3x_1, 6x_1^2) + \lambda_6(0, x_1^2, 0, 0)$ ,  $\lambda_i \in \mathbb{C}$ , and  $\text{Der}^4(M(V)) = \mathbb{C}\langle \partial_{x_1}^{(2)} - 3x_1 \partial_{x_1}^{(3)} + 6x_1^2 \partial_{x_1}^{(4)}, \partial_{x_1} - 3x_1^2 \partial_{x_1}^{(3)}, \partial_{x_1} - x_1 \partial_{x_1}^{(2)}, x_1 \partial_{x_1}, x_1^2 \partial_{x_1}, x_1^2 \partial_{x_1}^{(2)} \rangle$ .

Now we have calculated from the first to the 4-th order derivations of  $M(V)$ , especially, we obtain  $\dim \text{Der}^4(M(V)) = 6 = 3 \times 2 = \dim M(V) \cdot (\dim M(V) - 1)$ . Then from Theorem A,  $\dim \text{Der}^4(M(V)) = \dim \text{Der}^\infty(M(V))$ , we have  $\text{Der}^m(M(V)) = \text{Der}^4(M(V))$ ,  $\forall m \geq 4$ . Therefore, we obtain the  $\mathbb{C}$ -basis for each degree of  $\text{GrDer}(M(V))$ .

At last, we choose the weights by  $wt(x_1) = 1$ ,  $wt(x_i) = 2, \forall 2 \leq i \leq n$ ,  $(V(f), 0)$  is weighted homogeneous, then the Hilbert-Poincare series for  $\text{GrDer}(M(V))$  is  $P(t) = \frac{1}{t^2} + \frac{2}{t} + 3 + 2t + t^2$ .

From the above concrete example, we find that to calculate higher derivations of the moduli algebra  $M(V)$  of an isolated singularity  $(V, 0)$ , just by applying Proposition 2.7 to translate the conditions to several equations in  $M(V)$ , and reduced to solving linear equations as  $M(V)$  is finite dimensional  $\mathbb{C}$ -vector space. However this workload is too heavy when the singularity is more complex, so we use the magma programming to help us calculate some examples. Here we list some calculation results for  $A_k$  singularity in the following Table 1.

Table 1: higher derivations of moduli algebras of  $A_k$  singularities

| symbol | moduli algebra $M(V)$              | each positive degree parts of $\text{GrDer}(M(V))$                                                                                                                                                                                                                                                                                                                                                                                                                      |
|--------|------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $A_1$  | $M(V) = \mathbb{C}$                | $\text{GrDer}(M(V))_m = 0$ for $m > 0$ .                                                                                                                                                                                                                                                                                                                                                                                                                                |
| $A_2$  | $M(V) = \mathbb{C}\{x_1\}/(x_1^2)$ | $\text{GrDer}(M(V))_1 = \mathbb{C}\langle x_1 \partial_{x_1} \rangle$ ,<br>$\text{GrDer}(M(V))_2 = \mathbb{C}\langle x_1 \partial_{x_1} - 2x_1 \partial_{x_1}^2 \rangle$ ,<br>$\text{GrDer}(M(V))_m = 0$ for $m > 2$ .                                                                                                                                                                                                                                                  |
| $A_3$  | $M(V) = \mathbb{C}\{x_1\}/(x_1^3)$ | $\text{GrDer}(M(V))_1 = \mathbb{C}\langle x_1 \partial_{x_1}, x_1^2 \partial_{x_1} \rangle$ ,<br>$\text{GrDer}(M(V))_2 = \mathbb{C}\langle \partial_{x_1} - x_1 \partial_{x_1}^{(2)}, x_1^2 \partial_{x_1}^{(2)} \rangle$ ,<br>$\text{GrDer}(M(V))_3 = \mathbb{C}\langle \partial_{x_1} - 3x_1^2 \partial_{x_1}^{(3)} \rangle$ ,<br>$\text{GrDer}(M(V))_4 = \mathbb{C}\langle \partial_{x_1}^{(2)} - 3x_1 \partial_{x_1}^{(3)} + 6x_1^2 \partial_{x_1}^{(4)} \rangle$ , |

|       |                                    |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |
|-------|------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|       |                                    | $GrDer(M(V))_m = 0$ for $m > 4$ .                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |
| $A_4$ | $M(V) = \mathbb{C}\{x_1\}/(x_1^4)$ | $GrDer(M(V))_1 = \mathbb{C}\langle x_1\partial_{x_1}, x_1^2\partial_{x_1}, x_1^3\partial_{x_1} \rangle,$<br>$GrDer(M(V))_2 = \mathbb{C}\langle \partial_{x_1} - \frac{2}{3}x_1\partial_{x_1}^{(2)}, x_1^2\partial_{x_1}^{(2)}, x_1^3\partial_{x_1}^{(2)} \rangle,$<br>$GrDer(M(V))_3 = \mathbb{C}\langle \partial_{x_1} - x_1^2\partial_{x_1}^{(3)}, x_1^3\partial_{x_1}^{(3)} \rangle,$<br>$GrDer(M(V))_4 = \mathbb{C}\langle \partial_{x_1} - 4x_1^3\partial_{x_1}^{(4)}, \partial_{x_1}^{(2)} - 2x_1\partial_{x_1}^{(3)} + 2x_1^2\partial_{x_1}^{(4)} \rangle,$<br>$GrDer(M(V))_5 = \mathbb{C}\langle \partial_{x_1}^{(2)} - 6x_1^2\partial_{x_1}^{(4)} + 20x_1^3\partial_{x_1}^{(5)} \rangle,$<br>$GrDer(M(V))_6 = \mathbb{C}\langle \partial_{x_1}^{(3)} - 4x_1\partial_{x_1}^{(4)} + 10x_1^2\partial_{x_1}^{(5)} - 20x_1^3\partial_{x_1}^{(6)} \rangle,$<br>$GrDer(M(V))_m = 0$ for $m > 6$ . |

**Proposition 4.4.** *For general  $k \geq 2$ , the  $A_k$  singularity defined by  $f = x_1^{k+1} + x_2^2 + \cdots + x_n^2$ , whose moduli algebra  $M(f) = \mathbb{C}\{x\}/(x^k)$  has the following properties:*

- (1)  $\dim_{\mathbb{C}} GrDer(M(f)) = k^2$ .
- (2)  $Der^m(M(f))/Der^{m-1}(M(f)) = 0$  for all  $m > 2k - 2$ .
- (3)  $\dim Der^{2k-2}(M(f))/Der^{2k-3}(M(f)) = 1$ , moreover,  $Der^{2k-2}(M(f))/Der^{2k-3}(M(f))$  is spanned by  $\partial_x^{(k-1)} - \binom{k}{1}x\partial_x^{(k)} + \binom{k+1}{2}x^2\partial_x^{(k+1)} - \cdots + (-1)^{k-1}\binom{2k-2}{k-1}x^{k-1}\partial_x^{(2k-2)}$ .

Before proving this proposition, we give the following lemma about a combinatorial identity.

**Lemma 4.5.** *Let  $n \geq 2$  be an integer, then for any  $0 \leq s \leq n - 1$ , we have:*

$$\binom{n}{s}\binom{n-1}{0} - \binom{n}{s+1}\binom{n}{1} + \binom{n}{s+2}\binom{n+1}{2} + \cdots + (-1)^{n-s}\binom{n}{n}\binom{2n-1-s}{n-s} = 0.$$

*Proof.* This identity has such a combinatorial explanation.

The left hand side can be viewed as the coefficient of the term  $x^{n-1}$  in the polynomial  $\sum_{i=0}^n \binom{n}{i}(x-1)^{(n-1-s+i)}$ . On the other hand  $\sum_{i=0}^n \binom{n}{i}(x-1)^{(n-1-s+i)} = ((x-1)+1)^n(x-1)^{n-1-s} = x^n(x-1)^{n-1-s}$ , the coefficient of the term  $x^{n-1}$  is zero. Hence the identity has been proved.  $\square$

Now we begin to prove the proposition.

*Proof.* (of Proposition 4.4)

We know that  $M(f) \otimes_{\mathbb{C}} M(f) \simeq \mathbb{C}\{x, y\}/(x^k, y^k)$ , and the multiplication from  $M(f) \otimes_{\mathbb{C}} M(f)$  to  $M(f)$  can be viewed as the  $\mathbb{C}$ -algebra homomorphism  $\phi : \mathbb{C}\{x, y\}/(x^k, y^k) \rightarrow \mathbb{C}\{t\}/(t^k)$ , in which  $\phi(x) = t, \phi(y) = t$ . Then we can compute the kernel of  $\phi$ , which is denoted as  $I_{M(f)}$ . It is the  $\mathbb{C}$ -vector space spanned by  $\{x^i y^{j-1} - x^{i-1} y^j : 1 \leq i, j \leq k-1, i+j \leq k\}$  and  $\{x^i y^j : 1 \leq i, j \leq k-1, i+j \geq k\}$ . Moreover, it is a free  $M(f)$ -module of rank  $k-1$ , which is generated by  $\{xy^{j-1} - y^j : 1 \leq j \leq k-1\}$ .

(1) and (2) are implied from the proof of Theorem A. More precisely, as the unique maximal ideal of  $M(f)$  is  $(x)$ , and  $(x)^k = 0$  in  $M(f)$ . Then  $Der^{\infty}(M(f)) = Der^{2k-2}(M(f))$  from Theorem 2.10, and  $Der^{2k-2}(M(f)) = Hom_{M(f)}(I_{M(f)}, M(f)) \simeq M(f)^{k-1}$  as  $M(f)$ -module. Hence  $\dim Der^{2k-2}(M(f)) = (k-1)\dim M(f) = (k-1)k$ , and  $\dim_{\mathbb{C}} GrDer(M(f)) = \dim M(f) + \dim Der^{2k-2}(M(f)) = k^2$ .

For (3), define  $D = \partial_x^{(k-1)} - \binom{k}{1}x\partial_x^{(k)} + \binom{k+1}{2}x^2\partial_x^{(k+1)} - \cdots + (-1)^{k-1}\binom{2k-2}{k-1}x^{k-1}\partial_x^{(2k-2)}$ , from Proposition 2.7, we know that  $D \in Der^{2k-2}(M(f))$  if and only if  $\langle D, x^s \rangle(x^k)$  is equal to 0 in  $M(f)$  for  $0 \leq s \leq 2k-3$ .

When  $0 \leq s \leq k-2$ ,  $\langle D, x^s \rangle = \sum_{i=0}^{k-1} (-1)^i x^i \binom{k-1+i}{i} \partial_x^{(k-1-s+i)}$ , so  $\langle D, x^s \rangle (x^k) = \sum_{i=0}^{k-1} (-1)^i \binom{k-1+i}{i} \binom{k}{k-1-s+i} x^{s+1}$ . As  $1 \leq k-1-s \leq k-1$ ,  $\sum_{i=0}^{k-1} (-1)^i \binom{k-1+i}{i} \binom{k}{k-1-s+i} = 0$  from the above lemma,  $\langle D, x^s \rangle (x^k) = 0$ .

When  $k-1 \leq s \leq 2k-3$ ,  $\langle D, x^s \rangle = \sum_{i=s+1-k}^{k-1} (-1)^i x^i \binom{k-1+i}{i} \partial_x^{(k-1-s+i)}$ , and we can see that  $\langle D, x^s \rangle (x^k) \in (x^k)$ , it must be 0 in  $M(f)$ .

Now we have checked that  $D$  is a derivation of order  $2k-2$ .

Since  $D(x^{k-1}) = 1$  and  $D(x^i) = 0, \forall 0 \leq i \leq k-2$ . Then for each  $0 \leq j \leq k-2$ ,  $\langle D, x^j \rangle (x^{k-1-j}) = 1$  and  $\langle D, x^j \rangle (x^i) = 0, \forall 0 \leq i \leq k-2-j$ . Thus  $D, \langle D, x \rangle, \langle D, x^2 \rangle, \dots, \langle D, x^{k-2} \rangle$  are  $M(f)$ -linearly independent in  $\text{Der}^{2k-2}(M(f))$ , combining with  $\dim \text{Der}^{2k-2}(M(f)) = (k-1)\dim M(f) = (k-1)k$  we have proved,  $\text{Der}^{2k-2}(M(f))$  is just the free  $M(f)$ -module generated by  $D, \langle D, x \rangle, \langle D, x^2 \rangle, \dots, \langle D, x^{k-2} \rangle$ , (3) is proved.  $\square$

**Proposition 4.6.** *For general  $k \geq 2$ , if we set  $\text{wt}(x_1) = 1$ , and  $\text{wt}(x_i) = \frac{k+1}{2}, 2 \leq i \leq n$  in the polynomial  $f = x_1^{k+1} + x_2^2 + \dots + x_n^2$ . Then the Hilbert series for  $\text{GrDer}(M(f))$  is  $P(t) = \frac{1}{t^{k-1}} + \frac{2}{t^{k-2}} + \dots + k + (k-1)t + \dots + t^{k-1}$ .*

*Proof.* Denote  $D = \partial_x^{(k-1)} - \binom{k}{1} x \partial_x^{(k)} + \binom{k+1}{2} x^2 \partial_x^{(k+1)} - \dots + (-1)^{k-1} \binom{2k-2}{k-1} x^{k-1} \partial_x^{(2k-2)}$  as a higher derivation of  $M(f) = \mathbb{C}\{x\}/(x^k)$  from the proof of Proposition 4.4, we know that  $\text{Der}^\infty(M(f))$  is the free  $M(f)$ -module of rank  $k-1$  generated by  $D, \langle D, x \rangle, \langle D, x^2 \rangle, \dots, \langle D, x^{k-2} \rangle$ , where  $\langle D, x^s \rangle = \sum_{i=0}^{k-1} (-1)^i x^i \binom{k-1+i}{i} \partial_x^{(k-1-s+i)}, 0 \leq s \leq k-2$ .

Since  $\langle D, x^s \rangle$  is a derivation of weight  $-(k-1) + s$  and  $x^j \langle D, x^s \rangle$  is a derivation of weight  $-(k-1) + s + j$  for  $0 \leq j \leq k-1$ . We know that the dimension of weight  $i$  derivations is the cardinality of the set  $\{(j, s) \mid 0 \leq s \leq k-2, 0 \leq j \leq k-1, -(k-1) + s + j = i\}$ , which equals to  $k+i$  when  $-(k-1) \leq i \leq 0$ , and equals to  $k-1-i$  when  $1 \leq i \leq (k-1)$ . Together with the basis  $\{1, x, x^2, \dots, x^{k-1}\}$  of  $M(f)$  has weights  $0, 1, 2, \dots, (k-1)$ , the Hilbert series for  $\text{GrDer}(M(f))$  is  $P(t) = \frac{1}{t^{k-1}} + \frac{2}{t^{k-2}} + \dots + k + (k-1)t + \dots + t^{k-1}$ .  $\square$

#### 4.1.2. The $D_k$ case.

For  $D_k$  singularities, we still present some calculation result by magma programming in the following Table 2 first.

Table 2: higher derivations of moduli algebras of  $D_k$  singularities

| symbol | moduli algebra $M(V)$                                    | each positive degree parts of $\text{GrDer}(M(V))$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        |
|--------|----------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $D_4$  | $M(V) = \mathbb{C}\{x_1, x_2\}/(3x_1^2 + x_2^2, x_1x_2)$ | $\text{GrDer}(M(V))_1 = \mathbb{C}\langle x_2\partial_{x_2} + x_1\partial_{x_1}, x_2^2\partial_{x_2}, x_1\partial_{x_2} + \frac{1}{3}x_2\partial_{x_1}, x_2^2\partial_{x_1} \rangle,$<br>$\text{GrDer}(M(V))_2 = \mathbb{C}\langle \partial_{x_2} - x_2\partial_{x_2}^{(2)} - x_1\partial_{x_1}\partial_{x_2} - \frac{1}{3}x_2\partial_{x_1}^{(2)}, x_2\partial_{x_2} - \frac{2}{3}x_2^2\partial_{x_1}^{(2)}, x_2^2\partial_{x_2}^{(2)} - \frac{1}{3}x_2^2\partial_{x_1}^{(2)}, x_1\partial_{x_2}^{(2)} - \frac{1}{3}\partial_{x_1} + \frac{1}{3}x_2\partial_{x_1}\partial_{x_2} + \frac{1}{3}x_1\partial_{x_1}^{(2)}, x_1\partial_{x_2} + \frac{1}{3}x_2^2\partial_{x_1}\partial_{x_2} \rangle,$<br>$\text{GrDer}(M(V))_3 = \mathbb{C}\langle x_2\partial_{x_2}^{(2)} - 3x_2^2\partial_{x_2}^{(3)} - \frac{1}{3}x_2\partial_{x_1}^{(2)} + \frac{1}{3}x_2^2\partial_{x_1}^{(2)}\partial_{x_2}, \frac{1}{3}x_2^2\partial_{x_1}\partial_{x_2}^{(2)} - \frac{2}{3}x_1\partial_{x_1}^{(2)} - \frac{1}{3}x_2^2\partial_{x_1}^{(3)} + \frac{1}{3}\partial_{x_1} - \frac{1}{3}x_2\partial_{x_1}\partial_{x_2} \rangle,$<br>$\text{GrDer}(M(V))_4 = \mathbb{C}\langle \partial_{x_2}^{(2)} - 3x_2\partial_{x_2}^{(3)} + 6x_2^2\partial_{x_2}^{(4)} - x_1\partial_{x_1}\partial_{x_2}^{(2)} - \frac{1}{3}\partial_{x_1}^{(2)} + \frac{1}{3}x_2\partial_{x_1}^{(2)}\partial_{x_2} - \frac{2}{3}x_2^2\partial_{x_1}^{(2)}\partial_{x_2}^{(2)} + x_1\partial_{x_1}^{(3)} + \frac{2}{3}x_2^2\partial_{x_1}^{(4)} \rangle,$<br>$\text{GrDer}(M(V))_m = 0$ for $m > 4$ . |
|        |                                                          | $\text{GrDer}(M(V))_1 = \mathbb{C}\langle x_2\partial_{x_2} + \frac{2}{3}x_1\partial_{x_1}, x_2^2\partial_{x_2}, x_1^2\partial_{x_1} + \frac{1}{4}x_2\partial_{x_1}, x_2^2\partial_{x_1}, x_1^2\partial_{x_1} \rangle,$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |

|       |                                                          |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |
|-------|----------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $D_5$ | $M(V) = \mathbb{C}\{x_1, x_2\}/(4x_1^3 + x_2^2, x_1x_2)$ | $\begin{aligned} GrDer(M(V))_2 &= \mathbb{C}\langle x_2\partial_{x_2} + \frac{2}{3}x_1^2\partial_{x_1}^{(2)}, x_1\partial_{x_2} - x_1^2\partial_{x_1}\partial_{x_2} - \frac{1}{4}x_2\partial_{x_1}^{(2)}, x_2^2\partial_{x_1}^{(2)}, x_2\partial_{x_1} - x_2^2\partial_{x_1}\partial_{x_2}, x_2^2\partial_{x_2}^{(2)} + \frac{1}{3}x_1^2\partial_{x_1}^{(2)}, x_1^2\partial_{x_2}^{(2)} - \frac{1}{4}\partial_{x_1} + \frac{1}{4}x_2\partial_{x_1}\partial_{x_2} + \frac{1}{6}x_1\partial_{x_1}^{(2)} \rangle, \\ GrDer(M(V))_3 &= \mathbb{C}\langle x_1^2\partial_{x_1}^{(2)} + \frac{3}{4}x_2^2\partial_{x_1}^{(3)}, x_2\partial_{x_1}^{(2)} - x_2^2\partial_{x_1}^{(2)}\partial_{x_2}, x_1^2\partial_{x_2}^{(2)} + \frac{1}{4}x_2^2\partial_{x_1}\partial_{x_2}^{(2)} - \frac{1}{6}x_1\partial_{x_1}^{(2)} + \frac{1}{4}x_1^2\partial_{x_1}^{(3)}, \partial_{x_2} - 3x_2\partial_{x_2}^{(2)} + 6x_2^2\partial_{x_2}^{(3)} - x_1\partial_{x_1}\partial_{x_2} + x_1^2\partial_{x_1}^{(2)}\partial_{x_2} + \frac{1}{4}x_2\partial_{x_1}^{(3)} \rangle, \\ GrDer(M(V))_4 &= \mathbb{C}\langle x_1\partial_{x_1}^{(2)} - 3x_1^2\partial_{x_1}^{(3)} - \frac{3}{2}x_2^2\partial_{x_1}^{(4)}, x_2\partial_{x_2}^{(2)} - 3x_2^2\partial_{x_2}^{(3)} - \frac{1}{4}x_2\partial_{x_1}^{(3)} + \frac{1}{4}x_2^2\partial_{x_1}^{(3)}\partial_{x_2}, x_1\partial_{x_2}^{(2)} - x_1^2\partial_{x_1}\partial_{x_2}^{(2)} - \frac{1}{4}\partial_{x_1}^{(2)} + \frac{1}{4}x_2\partial_{x_1}^{(2)}\partial_{x_2} - \frac{1}{2}x_2^2\partial_{x_1}^{(2)}\partial_{x_2}^{(2)} + \frac{1}{2}x_1\partial_{x_1}^{(3)} - \frac{1}{2}x_1^2\partial_{x_1}^{(4)} \rangle, \\ GrDer(M(V))_5 &= \mathbb{C}\langle \partial_{x_1}^{(2)} - x_2\partial_{x_1}^{(2)}\partial_{x_2} + x_2^2\partial_{x_1}^{(2)}\partial_{x_2}^{(2)} - 3x_1\partial_{x_1}^{(3)} + 6x_1^2\partial_{x_1}^{(4)} + \frac{5}{2}x_2^2\partial_{x_1}^{(5)} \rangle, \\ GrDer(M(V))_6 &= \mathbb{C}\langle \partial_{x_2}^{(2)} - 3x_2\partial_{x_2}^{(3)} + 6x_2^2\partial_{x_2}^{(4)} - x_1\partial_{x_1}\partial_{x_2}^{(2)} + x_1^2\partial_{x_1}^{(2)}\partial_{x_2}^{(2)} - \frac{1}{4}\partial_{x_1}^{(3)} + \frac{1}{4}x_2\partial_{x_1}^{(3)}\partial_{x_2} + x_1\partial_{x_1}^{(4)} - \frac{5}{2}x_1^2\partial_{x_1}^{(5)} - \frac{5}{4}x_2^2\partial_{x_1}^{(6)} \rangle, \\ GrDer(M(V))_m &= 0 \text{ for } m > 6. \end{aligned}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |
| $D_6$ | $M(V) = \mathbb{C}\{x_1, x_2\}/(5x_1^4 + x_2^2, x_1x_2)$ | $\begin{aligned} GrDer(M(V))_1 &= \mathbb{C}\langle x_2\partial_{x_2} + \frac{1}{2}x_1\partial_{x_1}, x_2^2\partial_{x_2}, x_1^3\partial_{x_2} + \frac{1}{5}x_2\partial_{x_1}, x_2^2\partial_{x_1}, x_1^3\partial_{x_1} \rangle, \\ GrDer(M(V))_2 &= \mathbb{C}\langle x_1^2\partial_{x_2} - x_1^3\partial_{x_2}\partial_y - \frac{1}{5}x_2\partial_{x_1}^{(2)}, x_2^2\partial_{x_2}^{(2)} + \frac{1}{6}x_1^2\partial_{x_1}^{(2)}, x_1^3\partial_{x_2}^{(2)} - \frac{1}{5}\partial_{x_1} + \frac{1}{5}x_2\partial_{x_1}\partial_{x_2} + \frac{1}{10}x_1\partial_{x_1}^{(2)}, x_2\partial_{x_1} - x_2^2\partial_{x_1}\partial_{x_2}, x_1\partial_{x_1} - \frac{2}{3}x_1^2\partial_{x_1}^{(2)}, x_2^2\partial_{x_1}^{(2)}, x_1^3\partial_{x_1}^{(2)} \rangle, \\ GrDer(M(V))_3 &= \mathbb{C}\langle x_1\partial_{x_2} - x_1^2\partial_{x_1}\partial_{x_2} + x_1^3\partial_{x_1}^{(2)}\partial_{x_2} + \frac{1}{5}x_2\partial_{x_1}^{(3)}, \partial_{x_1} - x_2\partial_{x_1}\partial_{x_2} + x_2^2\partial_{x_1}\partial_{x_2}^{(2)} - x_1\partial_{x_1}^{(2)} + \frac{1}{2}x_1^2\partial_{x_1}^{(3)}, x_2\partial_{x_1}^{(2)} - x_2^2\partial_{x_1}^{(2)}\partial_{x_2}, x_1^2\partial_{x_1}^{(2)} - \frac{3}{2}x_1^3\partial_{x_1}^{(3)}, x_2^2\partial_{x_1}^{(3)} \rangle, \\ GrDer(M(V))_4 &= \mathbb{C}\langle \partial_{x_2} - x_2\partial_{x_2}^{(2)} - x_1\partial_{x_1}\partial_{x_2} + x_1^2\partial_{x_1}^{(2)}\partial_{x_2} - x_1^3\partial_{x_1}^{(3)}\partial_{x_2} - \frac{1}{5}x_2\partial_{x_1}^{(4)}, x_1^2\partial_{x_2}^{(2)} - x_1^3\partial_{x_1}\partial_{x_2}^{(2)} - \frac{1}{5}\partial_{x_1}^{(2)} + \frac{1}{5}x_2\partial_{x_1}^{(2)}\partial_{x_2} - \frac{2}{5}x_2^2\partial_{x_1}^{(2)}\partial_{x_2}^{(2)} + \frac{3}{10}x_1\partial_{x_1}^{(3)} - \frac{1}{5}x_1^2\partial_{x_1}^{(4)}, x_1\partial_{x_1}^{(2)} - 2x_1^2\partial_{x_1}^{(3)} + 2x_1^3\partial_{x_1}^{(4)}, x_2\partial_{x_1}^{(3)} - x_2^2\partial_{x_1}^{(3)}\partial_{x_2}, x_1^3\partial_{x_1}^{(3)} + \frac{4}{5}x_2^2\partial_{x_1}^{(4)} \rangle, \\ GrDer(M(V))_5 &= \mathbb{C}\langle x_2\partial_{x_2}^{(2)} - 3x_2^2\partial_{x_2}^{(3)} - \frac{1}{5}x_2\partial_{x_1}^{(4)} + \frac{1}{5}x_2^2\partial_{x_1}^{(4)}\partial_{x_2}, \partial_{x_1}^{(2)} - x_2\partial_{x_1}^{(2)}\partial_{x_2} + x_2^2\partial_{x_1}^{(2)}\partial_{x_2}^{(2)} - \frac{9}{4}x_1\partial_{x_1}^{(3)} + 3x_1^2\partial_{x_1}^{(4)} - \frac{5}{2}x_1^3\partial_{x_1}^{(5)}, x_1^2\partial_{x_1}^{(3)} - 4x_1^3\partial_{x_1}^{(4)} - 2x_2^2\partial_{x_1}^{(5)} \rangle, \\ GrDer(M(V))_6 &= \mathbb{C}\langle x_1\partial_{x_2}^{(2)} - x_1^2\partial_{x_1}\partial_{x_2}^{(2)} + x_1^3\partial_{x_1}^{(2)}\partial_{x_2}^{(2)} - \frac{1}{5}\partial_{x_1}^{(3)} + \frac{1}{5}x_2\partial_{x_1}^{(3)}\partial_{x_2} + \frac{3}{5}x_1\partial_{x_1}^{(4)} - x_1^2\partial_{x_1}^{(5)} + x_1^3\partial_{x_1}^{(6)}, x_1\partial_{x_1}^{(3)} - 4x_1^2\partial_{x_1}^{(4)} + 10x_1^3\partial_{x_1}^{(5)} + 4x_2^2\partial_{x_1}^{(6)} \rangle, \\ GrDer(M(V))_7 &= \mathbb{C}\langle \partial_{x_1}^{(3)} - x_2\partial_{x_1}^{(3)}\partial_{x_2} + x_2^2\partial_{x_1}^{(3)}\partial_{x_2}^{(2)} - 4x_1\partial_{x_1}^{(4)} + 10x_1^2\partial_{x_1}^{(5)} - 20x_1^3\partial_{x_1}^{(6)} - 7x_2^2\partial_{x_1}^{(7)} \rangle, \\ GrDer(M(V))_8 &= \mathbb{C}\langle \partial_{x_2}^{(2)} - 3x_2\partial_{x_2}^{(3)} + 6x_2^2\partial_{x_2}^{(4)} - x_1\partial_{x_1}\partial_{x_2}^{(2)} + x_1^2\partial_{x_1}^{(2)}\partial_{x_2}^{(2)} - x_1^3\partial_{x_1}^{(3)}\partial_{x_2}^{(2)} - \frac{1}{5}\partial_{x_1}^{(4)} + \frac{1}{5}x_2\partial_{x_1}^{(4)}\partial_{x_2} - \frac{2}{5}x_2^2\partial_{x_1}^{(4)}\partial_{x_2}^{(2)} + x_1\partial_{x_1}^{(5)} - 3x_1^2\partial_{x_1}^{(6)} + 7x_1^3\partial_{x_1}^{(7)} + \frac{14}{5}x_2^2\partial_{x_1}^{(8)} \rangle, \\ GrDer(M(V))_m &= 0 \text{ for } m > 8. \end{aligned}$ |

**Lemma 4.7.** For  $k \geq 4$ , denote  $D = \partial_y^{(2)} - 3y\partial_y^{(3)} + 6y^2\partial_y^{(4)} - \frac{1}{k-1}(\sum_{i=0}^{k-2}(-1)^i \binom{k-2+i}{i} x^i \partial_x^{(k-2+i)}) + \sum_{i=1}^{k-3}(-1)^i x^i \partial_x^{(i)} \partial_y^{(2)} + \frac{1}{k-1}y\partial_x^{(k-2)}\partial_y + ((-1)^{k-1} - 1)\frac{1}{k-1}y^2\partial_x^{(k-2)}\partial_y^{(2)}$ , we claim that  $D$  induces a  $(2k-4)$ -th order derivation on the  $\mathbb{C}$ -algebra  $\mathbb{C}\{x, y\}/((k-1)x^{k-2} + y^2, xy)$ .

*Proof.* We directly check it by using Proposition 2.7, saying that  $\langle D, x^i y^j \rangle((k-1)x^{k-2} + y^2)$ ,  $\langle D, x^i y^j \rangle(xy)$  both equal to 0 in  $\mathbb{C}\{x, y\}/((k-1)x^{k-2} + y^2, xy)$ , for any  $i, j \geq 0$ . It is easy to see that  $\langle D, x^i y^j \rangle = 0$  when  $j > 4$  or  $i > 2k-4$ , we only need to treat the case for  $i \leq 2k-4, j \leq 3$ . To simplify the notations, remark  $f_1 = (k-1)x^{k-2} + y^2$ ,  $f_2 = xy$ , and the following calculations are all in the algebra  $\mathbb{C}\{x, y\}/((k-1)x^{k-2} + y^2, xy)$ .

- (1) When  $i = j = 0$ ,  $D(f_1) = \partial_y^{(2)}(f_1) - \frac{1}{k-1}\partial_x^{(k-2)}(f_1) = 1 - 1 = 0$ ;  $D(f_2) = 0$ .
- (2) When  $1 \leq i \leq k-3, j = 0$ ,  $\langle D, x^i \rangle(f_1) = -\frac{1}{k-1}(\sum_{r=0}^{k-2}(-1)^r \binom{k-2+r}{r} x^r \partial_x^{(k-2+r-i)}(f_1)) + (-1)^i x^i \partial_y^{(2)}(f_1) = -\sum_{r=0}^i (-1)^r \binom{k-2+r}{r} \binom{k-2}{k-2-i+r} x^i + (-1)^i x^i$ , similar to the trick in Lemma 3.1, we know that  $\sum_{r=0}^i (-1)^r \binom{k-2+r}{r} \binom{k-2}{k-2-i+r}$  is the coefficient of the term  $x^{k-2}$  in  $((x-1)+1)^{k-2}(x-1)^i$ , which is  $(-1)^i$ , thus  $\langle D, x^i \rangle(f_1) = 0$ . And  $\langle D, x^i \rangle(f_2) = 0$  if  $i < k-3$ ,  $\langle D, x^{k-3} \rangle(f_2) = -\frac{1}{k-1}\partial_x(f_2) + \frac{1}{k-1}y\partial_x\partial_y(f_2) = 0$ .
- (3) When  $i = k-2, j = 0$ ,  $\langle D, x^{k-2} \rangle = -\frac{1}{k-1}\sum_{r=0}^{k-2}(-1)^r \binom{k-2+r}{r} x^r \partial_x^{(r)} + \frac{1}{k-1}y\partial_y + ((-1)^{k-1} - 1)\frac{1}{k-1}y^2\partial_y^{(2)}$ , then  $\langle D, x^{k-2} \rangle(f_1) = -\frac{1}{k-1}f_1 - \sum_{r=1}^{k-2}(-1)^r \binom{k-2+r}{r} \binom{k-2}{r} x^{k-2} + \frac{1}{k-1} \cdot 2y^2 + ((-1)^{k-1} - 1)\frac{1}{k-1}y^2 = (-1)^{k-1}(x^{k-2} + \frac{1}{k-1}y^2) = 0$ , and  $\langle D, x^{k-2} \rangle(f_2) = 0$ .
- (4) When  $k-1 \leq i \leq 2k-4, j = 0$ ,  $\langle D, x^i \rangle = -\frac{1}{k-1}\sum_{r=i-k+2}^{k-2}(-1)^r \binom{k-2+r}{r} x^r \partial_x^{(k-2+r-i)}$ , then  $\langle D, x^i \rangle(f_1) = -\sum_{r=i-k+2}^{k-2}(-1)^r \binom{k-2+r}{r} \binom{k-2}{k-2+r-i} x^i = 0$ , as  $x^{k-1} = 0$  in  $\mathbb{C}\{x, y\}/((k-1)x^{k-2} + y^2, xy)$ , and  $\langle D, x^i \rangle(f_2) = 0$ .
- (5) When  $i = 0, j = 1$ ,  $\langle D, y \rangle = \partial_y - 3y\partial_y^{(2)} + 6y^2\partial_y^{(3)} + \sum_{r=1}^{k-3}(-1)^r x^r \partial_x^{(r)} \partial_y + \frac{1}{k-1}y\partial_x^{(k-2)} + ((-1)^{k-1} - 1)\frac{1}{k-1}y^2\partial_x^{(k-2)}\partial_y$ , then  $\langle D, y \rangle(f_1) = 2y - 3y + y = 0$ ,  $\langle D, y \rangle(f_2) = x - x = 0$ .
- (6) When  $1 \leq i \leq k-3, j = 1$ ,  $\langle D, x^i y \rangle = \sum_{r=i}^{k-3}(-1)^r x^r \partial_x^{(r-i)} \partial_y + \frac{1}{k-1}y\partial_x^{(k-2-i)} + ((-1)^{k-1} - 1)\frac{1}{k-1}\partial_x^{(k-2-i)}\partial_y$ , then  $\langle D, x^i y \rangle(f_1) = 0$ . And  $\langle D, x^i y \rangle(f_2) = (-1)^i x^{i+1} + (-1)^{i+1} x^{i+1} = 0$  for  $i < k-3$ ,  $\langle D, x^{k-3} y \rangle(f_2) = (-1)^{k-3} x^{k-2} + \frac{1}{k-1}y^2 + ((-1)^{k-1} - 1)\frac{1}{k-1}y^2 = 0$ .
- (7) When  $k-2 \leq i \leq 2k-4, j = 1$ , then  $\langle D, x^i y \rangle = \frac{1}{k-1}y + ((-1)^{k-1} - 1)\frac{1}{k-1}y^2\partial_y$  if  $i = k-2$ , and vanishes if  $i > k-2$ .  $\langle D, x^{k-2} y \rangle(f_1) = 0$ ,  $\langle D, x^{k-2} y \rangle(f_2) = 0$ .
- (8) When  $i = 0, j = 2$ ,  $\langle D, y^2 \rangle = 1 - 3y\partial_y + 6y^2\partial_y^{(2)} + \sum_{r=1}^{k-3}(-1)^r x^r \partial_x^{(r)} + ((-1)^{k-1} - 1)\frac{1}{k-1}y^2\partial_x^{(k-2)}$ , then  $\langle D, y^2 \rangle(f_1) = f_1 - 6y^2 + 6y^2 + (k-1)\sum_{r=1}^{k-3}(-1)^r \binom{k-2}{r} x^{k-2} + ((-1)^{k-1} - 1)y^2 = -(k-1)(1 + (-1)^{k-2})x^{k-2} + ((-1)^{k-1} - 1)y^2 = 0$ , and  $\langle D, y^2 \rangle(f_2) = 0$ .
- (9) When  $i \geq 1, j = 2$ ,  $\langle D, x^i y^2 \rangle(f_1) = 0$ ,  $\langle D, x^i y^2 \rangle(f_2) = 0$ .

The rest case of  $j = 3, 4$  is trivial, we omit them.

Therefore, we have checked that  $D$  induces a  $(2k-4)$ -th order derivation on  $\mathbb{C}\{x, y\}/((k-1)x^{k-2} + y^2, xy)$ .  $\square$

**Proposition 4.8.** *For general  $k \geq 4$ , the  $D_k$  singularity defined by  $f = x_1^{k-1} + x_1 x_2^2 + x_3^2 + \cdots + x_n^2$ , whose moduli algebra  $M(f) \simeq \mathbb{C}\{x, y\}/((k-1)x^{k-2} + y^2, xy)$  has the following properties:*

- (1)  $\dim_{\mathbb{C}} \text{GrDer}(M(f)) = k^2$ .
- (2)  $\text{Der}^m(M(f))/\text{Der}^{m-1}(M(f)) = 0$  for all  $m > 2k-4$ .
- (3)  $\dim \text{Der}^{2k-4}(M(f))/\text{Der}^{2k-5}(M(f)) = 1$ , moreover,  $\text{Der}^{2k-4}(M(f))/\text{Der}^{2k-5}(M(f))$  is spanned by  $\partial_y^{(2)} - 3y\partial_y^{(3)} + 6y^2\partial_y^{(4)} - \frac{1}{k-1}(\sum_{i=0}^{k-2}(-1)^i \binom{k-2+i}{i} x^i \partial_x^{(k-2+i)}) + \sum_{i=1}^{k-3}(-1)^i x^i \partial_x^{(i)} \partial_y^{(2)} + \frac{1}{k-1}y\partial_x^{(k-2)}\partial_y + ((-1)^{k-1} - 1)\frac{1}{k-1}y^2\partial_x^{(k-2)}\partial_y^{(2)}$ .

*Proof.* We know that  $M(f) \simeq \mathbb{C}\{x, y\}/((k-1)x^{k-2} + y^2, xy)$  is a  $k$ -dimensional vector space with  $1, x, x^2, \dots, x^{k-2}, y$  as a basis, so  $M(f) \otimes_{\mathbb{C}} M(f) \simeq \mathbb{C}\{x_1, x_2, y_1, y_2\}/((k-1)x_1^{k-2} + y_1^2, x_1 y_1, (k-1)x_2^{k-2} + y_2^2, x_2 y_2)$ . And the multiplication map from  $M(f) \otimes_{\mathbb{C}} M(f)$  to  $M(f)$  can be viewed as the

$\mathbb{C}$ -algebra homomorphism  $\phi : \mathbb{C}\{x_1, x_2, y_1, y_2\}/((k-1)x_1^{k-2} + y_1^2, x_1y_1, (k-1)x_2^{k-2} + y_2^2, x_2y_2) \rightarrow \mathbb{C}\{x, y\}/((k-1)x^{k-2} + y^2, xy)$ , in which  $\phi(x_1) = \phi(x_2) = x, \phi(y_1) = \phi(y_2) = y$ .

Now we can compute  $I_{M(f)}$ , which is denoted as the kernel of  $\phi$ . It is the  $\mathbb{C}$ -vector space spanned by  $y_1 - y_2, x_1^i y_2$  ( $1 \leq i \leq k-2$ ),  $x_2^j y_1$  ( $1 \leq j \leq k-2$ ),  $x_1^i x_2^{j-1} - x_1^{i-1} x_2^j$  ( $1 \leq i, j \leq k-2, i+j \leq k-1$ ),  $(k-1)x_1^{k-2} + y_1 y_2$  and  $x_1^i x_2^j$  ( $0 \leq i, j \leq k-2, i+j \geq k-1$ ). Moreover, it is a free  $M(f)$ -module of rank  $k-1$ , which is generated by  $y_1 - y_2, x_1 x_2^{j-1} - x_2^j$  ( $1 \leq j \leq k-2$ ). Then (1) and (2) can be directly implied from the proof of Theorem A, we omit them.

For (3), denote  $D := \partial_y^{(2)} - 3y\partial_y^{(3)} + 6y^2\partial_y^{(4)} - \frac{1}{k-1}(\sum_{i=0}^{k-2}(-1)^i \binom{k-2+i}{i} x^i \partial_x^{(k-2+i)}) + \sum_{i=1}^{k-3}(-1)^i x^i \partial_x^{(i)} \partial_y^{(2)} + \frac{1}{k-1}y\partial_x^{(k-2)} \partial_y + ((-1)^{k-1} - 1)\frac{1}{k-1}y^2\partial_x^{(k-2)} \partial_y^{(2)}$ , from the above Lemma 4.7,  $D$  is a  $(2k-4)$ -th order derivation on  $M(f)$ . Moreover, we see that the higher derivations  $D, \langle D, x \rangle, \langle D, x^2 \rangle, \dots, \langle D, x^{k-3} \rangle, \langle D, y \rangle$  are  $M(f)$ -linearly independent in  $Der^\infty(M(f)) = Der^{2k-4}(M(f))$ . Together with the fact in Theorem A,  $Der^\infty(M(f))$  is a free  $M(f)$ -module of rank  $k-1$ , generated by these  $k-1$  higher derivations. Therefore,  $Der^{2k-4}(M(f))/Der^{2k-5}(M(f))$  is one dimensional, spanned by the higher derivation  $D$ .  $\square$

**Proposition 4.9.** *For the  $D_k$  singularity defined by  $f = x_1^{k-1} + x_1 x_2^2 + x_3^2 + \dots + x_n^2$ , the following holds for the Hilbert series  $P(t)$  of  $GrDer(M(f))$ :*

(1) *If  $k$  is odd and we set  $wt(x_1) = 2, wt(x_2) = k-2, wt(x_i) = k-1, \forall 3 \leq i \leq n$ , then the Hilbert series is  $P(t) = k + \sum_{r=1}^{k-2} (\frac{k-1-r}{t^{2r}} + \frac{1-(-1)^r}{t^r} + (1 - (-1)^r)t^r + (k-1-r)t^{2r}$ ;*

(2) *If  $k$  is even and we set  $wt(x_1) = 1, wt(x_2) = \frac{k-2}{2}, wt(x_i) = \frac{k-1}{2}, \forall 3 \leq i \leq n$ , then the Hilbert series is  $P(t) = k + 2 + \sum_{r=1}^{(k-2)/2} (\frac{k-1-2r}{t^{2r}} + \frac{k-2r}{t^{2r-1}} + \frac{2}{t^r} + 2t^r + (k-2r)t^{2r-1} + (k-1-2r)t^{2r}$ ;*

*Proof.* Just deduce from the fact that  $Der^\infty(M(f))$  is the free  $M(f)$ -module generated by  $D, \langle D, x_1 \rangle, \langle D, x_1^2 \rangle, \dots, \langle D, x_1^{k-3} \rangle$  and  $\langle D, x_2 \rangle$ ,  $D$  is defined in Lemma 4.7 (identify  $x_1, x_2$  with  $x, y$ ), we omit the details.  $\square$

#### 4.1.3. The $E_6, E_7, E_8$ case.

The  $E_6$  singularity is the isolated hypersurface singularity defined by  $f = x_1^3 + x_2^4 + x_3^2 + \dots + x_n^2$ , we see that its moduli algebra is the tensor product of those of  $A_2$  and  $A_3$ , then  $GrDer(M(f)) \simeq GrDer(\mathbb{C}\{x\}/(x^2)) \otimes GrDer(\mathbb{C}\{x\}/(x^3))$  from Theorem B, and the dimensions of each degrees of  $GrDer(M(f))$  are 6, 7, 9, 6, 5, 2, 1. If we choose  $wt(x_1) = 4, wt(x_2) = 3$ , and  $wt(x_i) = 6, \forall 3 \leq i \leq n$ , then the Hilbert series for  $GrDer(M(f))$  is  $P(t) = \frac{1}{t^{10}} + \frac{2}{t^7} + \frac{2}{t^6} + \frac{3}{t^4} + \frac{4}{t^3} + \frac{1}{t^2} + \frac{2}{t} + 6 + 2t + t^2 + 4t^3 + 3t^4 + 2t^6 + 2t^7 + t^{10}$ .

The  $E_7$  singularity is the isolated hypersurface singularity defined by  $f = x_1^3 + x_1 x_2^3 + x_3^2 + \dots + x_n^2$ , by means of programming, we calculate the dimensions of each degrees of  $GrDer(M(f))$  are 7, 8, 10, 8, 7, 4, 3, 1, 1. If we choose  $wt(x_1) = 3, wt(x_2) = 2$ , and  $wt(x_i) = 9/2, \forall 3 \leq i \leq n$ , then the Hilbert series for  $GrDer(M(f))$  is  $P(t) = \frac{1}{t^8} + \frac{2}{t^6} + \frac{2}{t^5} + \frac{3}{t^4} + \frac{4}{t^3} + \frac{5}{t^2} + \frac{4}{t} + 7 + 4t + 5t^2 + 4t^3 + 3t^4 + 2t^5 + 2t^6 + t^8$ .

The  $E_8$  singularity is the isolated hypersurface singularity defined by  $f = x_1^3 + x_2^5 + x_3^2 + \dots + x_n^2$ , we see that its moduli algebra is the tensor product of those of  $A_2$  and  $A_4$ , then  $GrDer(M(f)) \simeq GrDer(\mathbb{C}\{x\}/(x^2)) \otimes GrDer(\mathbb{C}\{x\}/(x^4))$ , and the dimensions of each degrees of  $GrDer(E_8)$  are 8, 10, 13, 10, 9, 6, 5, 2, 1. If we choose  $wt(x_1) = 5, wt(x_2) = 3$ , and  $wt(x_i) = 15/2, \forall 3 \leq i \leq n$ , then its Hilbert series is  $P(t) = \frac{1}{t^{14}} + \frac{2}{t^{11}} + \frac{2}{t^9} + \frac{3}{t^8} + \frac{4}{t^6} + \frac{4}{t^5} + \frac{1}{t^4} + \frac{6}{t^3} + \frac{3}{t^2} + \frac{2}{t} + 8 + 2t + 3t^2 + 6t^3 + t^4 + 4t^5 + 4t^6 + 3t^8 + 2t^9 + 2t^{11} + t^{14}$ .

Now we give some conclusions.

**Theorem 4.10.** *The Conjecture 1.2, 4.1 and 4.2 hold for the simple hypersurface singularity case.*

*Proof.* It follows from proofs of Proposition 4.4, 4.6, 4.8, 4.9 and our explicit calculations immediately.  $\square$

**Theorem 4.11.** *The Conjecture 1.2, 4.1 and 4.2 hold for the case of Brieskorn singularity.*

*Proof.* Let the isolated hypersurface singularity  $(V(f), 0)$  be defined by  $f = x_1^{a_1} + x_2^{a_2} + \dots + x_n^{a_n}$ , then its moduli algebra  $M(f) \simeq \mathbb{C}\{x_1, x_2, \dots, x_n\}/(x_1^{a_1-1}, x_2^{a_2-1}, \dots, x_n^{a_n-1}) \simeq \bigotimes_{i=1}^n \mathbb{C}\{x_i\}/(x_i^{a_i-1})$ . So  $\text{GrDer}(M(f)) \simeq \bigotimes_{i=1}^n \text{GrDer}(\mathbb{C}\{x_i\}/(x_i^{a_i-1}))$  from Theorem B. We have verified that the Conjecture 4.1 and 4.2 hold for  $A_k$  singularities, hence hold for  $M(f)$  directly from this isomorphism. Now let  $\text{Der}^{2a_i-4}(\mathbb{C}\{x_i\}/(x_i^{a_i-1}))/\text{Der}^{2a_i-5}(\mathbb{C}\{x_i\}/(x_i^{a_i-1})) = \mathbb{C}\langle D_i \rangle$  for each  $i$ , then  $D_0 := \bigotimes_{i=1}^n D_i$  satisfies the conditions in Conjecture 1.2.  $\square$

#### 4.2. The binomial isolated singularities case.

As the binomial singularity only has three types up to analytical equivalence, we will concentrate on the binomial singularity  $(V(f), 0)$  of types  $f = x^a y + y^b$  and  $f = x^a y + y^b x$ .

For the isolated hypersurface singularity  $(V(f), 0)$  defined by  $f = x^a y + y^b$ , we construct higher derivations of  $M(f)$  directly as following.

**Proposition 4.12.** *Remark  $D_1 = \sum_{i=0}^{2a-2} \sum_{j=0}^{b-1} (-1)^{i+j} \binom{a-2+i}{i} \binom{b-1+j}{j} x^i y^j \partial_x^{(a-2+i)} \partial_y^{(b-1+j)}$ , and  $D_2 = \sum_{i=0}^{2a-2} \sum_{j=0}^{b-1} (-1)^{i+j} \binom{2a-2+i}{i} x^i y^j \partial_x^{(2a-2+i)} \partial_y^{(j)}$ , then*

(1)  $D_0 := D_1 - bD_2$  induces a higher derivation on  $M(f) = \mathbb{C}\{x, y\}/(x^{a-1}y, x^a + by^{b-1})$ .

(2)  $\text{Der}^\infty(M(f))$  is a free  $M(f)$ -module of rank  $b(a-1)$ , moreover,  $\langle D_0, x^i \rangle, 0 \leq i \leq a-1$ , and  $\langle D_0, x^i y^j \rangle, 0 \leq i \leq a-2, 1 \leq j \leq b-1, i+j < a+b-3$ , form a basis of it.

For the isolated hypersurface singularity  $(V(f), 0)$  defined by  $f = x^a y + y^b x$ , we construct higher derivations of  $M(f)$  directly as following.

**Proposition 4.13.** *Remark  $D_1 = \sum_{i=0}^{2a-2} \sum_{j=0}^{2b-2} (-1)^{i+j} \binom{a-1+i}{i} \binom{b-1+j}{j} x^i y^j \partial_x^{(a-1+i)} \partial_y^{(b-1+j)}$ ,  $D_2 = \sum_{i=0}^{2a-2} \sum_{j=0}^{2b-2} (-1)^{i+j} \binom{2a-2+i}{i} x^i y^j \partial_x^{(2a-2+i)} \partial_y^{(j)}$ ,  $D_3 = \sum_{i=0}^{2a-2} \sum_{j=0}^{2b-2} (-1)^{i+j} \binom{2b-2+j}{j} x^i y^j \partial_x^{(i)} \partial_y^{(2b-2+j)}$ , then*

(1)  $D_0 := D_1 - bD_2 - aD_3$  induces a higher derivation on  $M(f) = \mathbb{C}\{x, y\}/(ax^{a-1}y + y^b, x^a + bxy^{b-1})$ .

(2)  $\text{Der}^\infty(M(f))$  is a free  $M(f)$ -module of rank  $ab-1$ , moreover,  $\langle D_0, x^i y^j \rangle, 0 \leq i \leq a-1, 0 \leq j \leq b-1, i+j < a+b-2$ , form a basis of it.

We omit the proofs of the above two propositions, they are just obtained from the calculations of combinatorial numbers similar as before. And the following theorem is immediately obtained from Proposition 4.12 and 4.13.

**Theorem 4.14.** *The Conjecture 1.2, 4.1 and 4.2 hold for the binomial singularity case.*

#### 4.3. The simple elliptic singularities case.

We check the Conjecture 4.1 and 4.2 for simple elliptic singularities of types  $\tilde{E}_7$  and  $\tilde{E}_8$ .

The simple elliptic singularity  $\tilde{E}_7$  is an isolated hypersurface singularity defined by  $f_t = x^4 + y^4 + tx^2y^2 + z^2$ , with  $t^2 \neq 4$ . By means of programming, we calculate the dimensions of each degrees of  $\text{GrDer}(M(f_t))$  are 9, 11, 14, 13, 13, 10, 8, 2, 1. If we choose  $wt(x) = wt(y) = 1, wt(z) = 2$ , then the Hilbert series for  $\text{GrDer}(M(f_t))$  is  $P(s) = \frac{1}{s^4} + \frac{4}{s^3} + \frac{10}{s^2} + \frac{16}{s} + 19 + 16s + 10s^2 + 4s^3 + s^4$ .

The simple elliptic singularities  $\tilde{E}_8$  is an isolated hypersurface singularity defined by  $f_t = x^6 + y^3 + z^2 + tx^4y$ , with  $4t^3 + 27 \neq 0$ . By means of programming, we calculate the dimensions of each degrees of  $GrDer(M(f_t))$  are 10,12,15,13,13,10,9,6,5,3,2,1,1. If we choose  $wt(x) = 1, wt(y) = 2, wt(z) = 3$ , then the Hilbert series for  $GrDer(M(f_t))$  is  $P(s) = \frac{1}{s^6} + \frac{2}{s^5} + \frac{5}{s^4} + \frac{8}{s^3} + \frac{12}{s^2} + \frac{14}{s} + 16 + 14s + 12s^2 + 8s^3 + 5s^4 + 2s^5 + s^6$ .

**Theorem 4.15.** *The Conjecture 4.1 and 4.2 hold for  $\tilde{E}_7$  and  $\tilde{E}_8$  types of simple elliptic singularity case.*

## 5. THE INEQUALITY CONJECTURE FOR THE CASE OF BINOMIAL SINGULARITIES

In this section, we prove the inequality Conjecture 1.3 for the binomial singularities case. We have known that each binomial singularity  $(V(f), 0)$  is analytically equivalent to one of these three cases (2.4): (A)  $f = x^a + y^b$ ; (B)  $f = x^a y + y^b$ ; (C)  $f = x^a y + y^b x$ . First, we give the explicit expression of the function  $h_2(w_1, w_2, \dots, w_n)$  as stated in Conjecture 1.3.

**Proposition 5.1.** *Let  $(V, 0)$  be an isolated hypersurface singularity defined by  $f = x_1^{a_1} + x_2^{a_2} + \dots + x_n^{a_n}$ , where  $a_1, a_2, \dots, a_n \geq 2$ , then*

$$\dim Der^2(M(V))/Der^1(M(V)) = \frac{n(n+1)}{2} \prod_{i=1}^n (a_i - 1) - n \sum_{i=1}^n \prod_{k \neq i} (a_k - 1) + \sum_{1 \leq i < j \leq n} \prod_{k \neq i, j} (a_k - 1).$$

*Proof.* Same as the proof of Theorem 4.10, we know that  $M(V) \simeq \bigotimes_{i=1}^n \mathbb{C}\{x_i\}/(x_i^{a_i-1})$ .

For each  $\mathbb{C}\{x_i\}/(x_i^{a_i-1})$ , we have seen in section 4 that  $Der^1(\mathbb{C}\{x_i\}/(x_i^{a_i-1}))$  is of dimension  $(a_i - 2)$ , with a basis  $x_i \partial_{x_i}, x_i^2 \partial_{x_i}, \dots, x_i^{a_i-2} \partial_{x_i}$ . And  $Der^2(\mathbb{C}\{x_i\}/(x_i^{a_i-1}))/Der^1(\mathbb{C}\{x_i\}/(x_i^{a_i-1}))$  is of dimension  $(a_i - 2)$ , with a basis  $\partial_{x_i} - \frac{2}{a_i-2} x_i \partial_{x_i}^{(2)}, x_i^2 \partial_{x_i}^{(2)}, x_i^3 \partial_{x_i}^{(2)}, \dots, x_i^{a_i-2} \partial_{x_i}^{(2)}$ .

From Theorem B, we have  $GrDer(M(V)) \simeq \bigotimes_{i=1}^n GrDer(\mathbb{C}\{x_i\}/(x_i^{a_i-1}))$ , therefore  $\dim Der^2(M(V))/Der^1(M(V)) = \sum_{1 \leq i < j \leq n} (a_i - 2)(a_j - 2) \prod_{k \neq i, j} (a_k - 1) + \sum_{i=1}^n (a_i - 2) \prod_{k \neq i} (a_k - 1) = \frac{n(n+1)}{2} \prod_{i=1}^n (a_i - 1) - n \sum_{i=1}^n \prod_{k \neq i} (a_k - 1) + \sum_{1 \leq i < j \leq n} \prod_{k \neq i, j} (a_k - 1)$ .  $\square$

**Definition 5.2.** For  $a_1, a_2, \dots, a_n \geq 1$ , define the function  $h_2(a_1, a_2, \dots, a_n)$  to be

$$h_2(a_1, \dots, a_n) = \frac{n(n+1)}{2} \prod_{i=1}^n (a_i - 1) - n \sum_{i=1}^n \prod_{k \neq i} (a_k - 1) + \sum_{1 \leq i < j \leq n} \prod_{k \neq i, j} (a_k - 1).$$

Now we begin to calculate  $\dim Der^2(M(V))/Der^1(M(V))$  for the binomial singularity  $(V, 0)$  of type (B) and type (C).

**Proposition 5.3.** *Let  $(V(f), 0)$  be an isolated hypersurface singularity defined by  $f = x^a y + y^b$ ,  $b \geq 2$ , then*

$$\dim Der^2(M(f))/Der^1(M(f)) = \begin{cases} 3ab - 4a - 5b + 10; & \text{if } a \geq 2, b \geq 3, \\ 2a - 2; & \text{if } a \geq 2, b = 2, \\ 0; & \text{if } a = 1. \end{cases}$$

*Proof.* Let  $\Delta := A\partial_x^2 + B\partial_x\partial_y + C\partial_y^2 + D\partial_x + E\partial_y$  be a derivation in  $Der^2(M(f))$ , where the coefficients  $A, B, C, D, E \in M(f)$ , since  $M(f) = \mathbb{C}\{x, y\}/(f, f_x, f_y) \simeq \mathbb{C}\{x, y\}/(x^{a-1}y, x^a + by^{b-1})$ ,  $\Delta$  is nonzero in  $Der^2(M(f))/Der^1(M(f))$  if and only if  $A, B, C$  are not all zero and  $\Delta(x^{a-1}y), \Delta(x^a + by^{b-1}), \Delta(x^a y), \Delta(x^{a-1}y^2), \Delta(x^{a+1} + bxy^{b-1}), \Delta(x^a y + by^b)$  all equal to zero in  $M(f)$ .

(1) When  $a \geq 3, b \geq 4$ , the following equations hold in  $M(f)$ ,

$$\begin{aligned}\Delta(x^{a-1}y) &= (a-1)(a-2)x^{a-3}yA + (a-1)x^{a-2}B + (a-1)x^{a-2}yD + x^{a-1}E = 0; \\ \Delta(x^a + by^{b-1}) &= a(a-1)x^{a-2}A + b(b-1)(b-2)y^{b-3}C + ax^{a-1}D + b(b-1)y^{b-2}E = 0; \\ \Delta(x^ay) &= a(a-1)x^{a-2}yA + ax^{a-1}B + ax^{a-1}yD + x^aE = 0; \\ \Delta(x^{a-1}y^2) &= (a-1)(a-2)x^{a-3}y^2A + 2(a-1)x^{a-2}yB + 2x^{a-1}C + (a-1)x^{a-2}y^2D + 2x^{a-1}yE = 0; \\ \Delta(x^{a+1} + bxy^{b-1}) &= (a+1)ax^{a-1}A + b(b-1)y^{b-2}B + b(b-1)(b-2)xy^{b-3}C + ((a+1)x^a + by^{b-1})D + \\ &+ b(b-1)xy^{b-2}E = (a+1)ax^{a-1}A + b(b-1)y^{b-2}B + b(b-1)(b-2)xy^{b-3}C + ax^aD + b(b-1)xy^{b-2}E = 0; \\ \Delta(x^ay + by^b) &= a(a-1)x^{a-2}yA + ax^{a-1}B + b^2(b-1)y^{b-2}C + ax^{a-1}yD + (x^a + b^2y^{b-1})E = \\ &= a(a-1)x^{a-2}yA + ax^{a-1}B + b^2(b-1)y^{b-2}C + ax^{a-1}yD + b(b-1)y^{b-1}E = 0.\end{aligned}$$

We know that  $M(f)$  is Artinian, and has the monomial basis:  $x^iy^j, 0 \leq i \leq a-2, 0 \leq j \leq b-1, x^{a-1}$ , its dimension  $\mu(f) = ab - b + 1$ . Write  $A := \sum_{0 \leq i \leq a-2, 0 \leq j \leq b-1} A_{i,j}x^iy^j + A_{a-1,0}x^{a-1}$ , and express  $B, C, D, E$  in the same way, then the above six equalities in  $M(f)$  will be turned into a huge system of linear equations with  $5\mu(f)$  variables and  $6\mu(f)$  equations. It seems like the system of linear equations is too huge to start with, however, we can only concentrate on a certain group of variables at one time, and find all equations containing them. Equivalently speaking, we decompose  $Der^2(M(f))/Der^1(M(f))$  into smaller subspaces, each of which is the nullspace of a smaller system of linear equations.

After reordering the equations following the above idea, the conditions to make  $\Delta$  a second order derivation in  $Der^2(M(f))$  are as follows:

$$\begin{aligned}a(a-1)A_{1,0} + aD_{0,0} &= 0, \quad -(a+1)abA_{1,0} + b(b-1)B_{0,1} - abD_{0,0} = 0; \\ (a-1)(a-2)A_{1,k} + (a-1)B_{0,k+1} + (a-1)D_{0,k} &= 0, \quad (a-1)(a-2)A_{1,k} + 2(a-1)B_{0,k+1} + (a-1)D_{0,k} = \\ &= 0, \quad 0 \leq k \leq b-4; \\ (a-1)(a-2)A_{1,b-3} + (a-1)B_{0,b-2} + (a-1)D_{0,b-3} &= 0, \quad (a-1)(a-2)A_{1,b-3} + 2(a-1)B_{0,b-2} - \\ &- 2bC_{a-1,0} + (a-1)D_{0,b-3} = 0; \\ (a-1)(a-2)A_{1,b-2} + (a-1)B_{0,b-1} + (a-1)D_{0,b-2} - bE_{a-1,0} &= 0; \\ (a-1)(a-2)A_{0,k} &= 0, 0 \leq k \leq b-4; \\ a(a-1)A_{0,b-3} + b(b-1)(b-2)C_{a-2,0} &= 0, \quad a(a-1)A_{0,b-3} + b^2(b-1)C_{a-2,0} = 0; \\ (a-1)(a-2)A_{0,b-2} - (a-1)bB_{a-1,0} - bE_{a-2,0} &= 0, \quad a(a-1)A_{0,b-2} + b(b-1)(b-2)C_{a-2,1} + \\ &+ b(b-1)E_{a-2,0} = 0, \quad a(a-1)A_{0,b-2} - abB_{a-1,0} - bE_{a-2,0} = 0, \quad a(a-1)A_{0,b-2} - abB_{a-1,0} + \\ &+ b^2(b-1)C_{a-2,1} + b(b-1)E_{a-2,0} = 0; \\ a(a-1)A_{0,b-1} + b(b-1)(b-2)C_{a-2,2} - abD_{a-1,0} + b(b-1)E_{a-2,1} &= 0; \\ -a(a-1)bA_{k,0} + b(b-1)(b-2)C_{k-2,2} - abD_{k-1,0} + b(b-1)E_{k-2,1} &= 0, \quad -(a+1)abA_{k,0} + b(b-1) \\ &+ b(b-1)B_{k-1,1} + b(b-1)(b-2)C_{k-2,2} - abD_{k-1,0} + b(b-1)E_{k-2,1} = 0, 2 \leq k \leq a-1; \\ B_{0,0} &= 0; \\ (a-1)B_{k,0} + E_{k-1,0} &= 0, \quad b(b-1)(b-2)C_{k-1,1} + b(b-1)E_{k-1,0} = 0, \quad -abB_{k,0} - bE_{k-1,0} = 0, \\ b(b-1)B_{k,0} + b(b-1)(b-2)C_{k-1,1} + b(b-1)E_{k-1,0} &= 0, \quad -abB_{k,0} + b^2(b-1)C_{k-1,1} + b(b-1)E_{k-1,0} = \\ &= 0, 1 \leq k \leq a-2; \\ C_{k,0} &= 0, 0 \leq k \leq a-3.\end{aligned}$$

Since  $\Delta$  belongs to  $Der^1(M(f))$  if and only if  $A = B = C = 0$  and the above linear equations hold, we obtain the following basis for  $Der^2(M(f))/Der^1(M(f))$ :

$$\begin{aligned}&x^iy^j\partial_x^2, 2 \leq i \leq a-2, 1 \leq j \leq b-1; \quad xy^{b-1}\partial_x^2; \\ &x^iy^j\partial_x\partial_y, 1 \leq i \leq a-2, 2 \leq j \leq b-1; \quad x^iy^j\partial_y^2, 0 \leq i \leq a-2, 3 \leq j \leq b-1; \\ &xy^k\partial_x^2 - (a-2)y^k\partial_x, 1 \leq k \leq b-2; \\ &xy^{b-3}\partial_x^2 - (a-2)y^{b-2}\partial_x\partial_y - \frac{(a-1)(a-2)}{2b}x^{a-1}\partial_y^2; \quad y^{b-1}\partial_x\partial_y - y^{b-2}\partial_y;\end{aligned}$$

$$\begin{aligned} & \frac{1}{a(a-1)}y^{b-2}\partial_x^2 + \frac{2}{ab}x^{a-1}\partial_x\partial_y + \frac{1}{b(b-1)}x^{a-2}y\partial_y^2 - \frac{1}{b}x^{a-2}\partial_y; \\ & y^{b-1}\partial_x^2 + \frac{a-1}{b}x^{a-1}\partial_x; \quad x^{k-2}y^2\partial_y^2 + (2-b)x^{k-2}y\partial_y, 2 \leq k \leq a; \\ & (b-1)x^k\partial_x^2 + 2ax^{k-1}y\partial_x\partial_y + a(a-1)x^{k-2}y\partial_y, 2 \leq k \leq a-1. \end{aligned}$$

Therefore,  $\dim Der^2(M(f))/Der^1(M(f)) = (a-3)(b-1) + 1 + (a-2)(b-2) + (a-1)(b-3) + (b-2) + 4 + (a-1) + (a-2) = 3ab - 4a - 5b + 10$ .

(2) When  $a \geq 3, b = 3$ , we just need to replace the equation  $a(a-1)A_{1,0} + aD_{0,0} = 0$  by  $a(a-1)A_{1,0} + b(b-1)(b-2)C_{a-1,0} + aD_{0,0} = 0$  and replace  $-(a+1)abA_{1,0} + b(b-1)B_{0,1} - abD_{0,0} = 0$  by  $-(a+1)abA_{1,0} + b(b-1)B_{0,1} - b^2(b-1)(b-2)C_{a-1,0} - abD_{0,0} = 0$ . A basis for  $Der^2(M(f))/Der^1(M(f))$  can be chosen as:

$$\begin{aligned} & x^i y^j \partial_x^2, 2 \leq i \leq a-2, 1 \leq j \leq 2; \quad xy^2 \partial_x^2; \\ & x^i y^j \partial_x \partial_y, 1 \leq i \leq a-2, j = 2; \quad x \partial_x^2 + ay \partial_x \partial_y + \frac{a(a-1)}{6}x^{a-1} \partial_y^2 + 2(1-a)\partial_x; \\ & xy \partial_x^2 - (a-2)y \partial_x; \quad y^2 \partial_x \partial_y - y^1 \partial_y; \quad \frac{1}{a(a-1)}y^{b-2}\partial_x^2 + \frac{2}{3a}x^{a-1}\partial_x\partial_y + \frac{1}{6}x^{a-2}y\partial_y^2 - \frac{1}{3}x^{a-2}\partial_y; \\ & y^2 \partial_x^2 + \frac{a-1}{3}x^{a-1}\partial_x; \quad x^{k-2}y^2 \partial_y^2 - x^{k-2}y \partial_y, 2 \leq k \leq a; \\ & 2x^k \partial_x^2 + 2ax^{k-1}y \partial_x \partial_y + a(a-1)x^{k-2}y \partial_y, 2 \leq k \leq a-1. \end{aligned}$$

Therefore,  $\dim Der^2(M(f))/Der^1(M(f)) = 2(a-3) + 1 + (a-2) + 5 + (a-1) + (a-2) = 5a - 5 = 3ab - 4a - 5b + 10$ .

(3) When  $a = 2, b \geq 3$ , then  $f = x^2y + y^b$ , which is the case of  $D_{b+1}$  singularity, then from the case of simple hypersurface singularity we have calculated before, we know that  $\dim Der^2(M(f))/Der^1(M(f)) = b + 2 = 3ab - 4a - 5b + 10$ .

(4) When  $b = 2, M(f) \simeq \mathbb{C}\{x, y\}/(x^{a-1}y, x^a + 2y) \simeq \mathbb{C}\{x\}/(x^{2a-1})$ , which reduces to the case of  $A_{2a-1}$  singularity, so  $\dim Der^2(M(f))/Der^1(M(f)) = 2a - 2$ .

(5) When  $a = 1, M(f) \simeq \mathbb{C}\{x, y\}/(y, x + by^{b-1}) \simeq \mathbb{C}$ ,  $\dim Der^2(M(f))/Der^1(M(f)) = 0$ .

Now, we have proved the proposition completely.  $\square$

**Proposition 5.4.** *Let  $(V(f), 0)$  be an isolated hypersurface singularity defined by  $f = x^a y + y^b x$ , where  $a, b \geq 2$ , then*

$$\dim Der^2(M(f))/Der^1(M(f)) = \begin{cases} 3ab - 4a - 4b + 13; & \text{if } a \geq 3, b \geq 3, \\ 2b + 1; & \text{if } a = 2, \\ 2a + 1; & \text{if } b = 2. \end{cases}$$

*Proof.* Let  $\Delta := A\partial_x^2 + B\partial_x\partial_y + C\partial_y^2 + D\partial_x + E\partial_y$  be a derivation in  $Der^2(M(f))$ , where the coefficients  $A, B, C, D, E \in M(f)$ , since  $M(f) = \mathbb{C}\{x, y\}/(f, f_x, f_y) \simeq \mathbb{C}\{x, y\}/(ax^{a-1}y + y^b, x^a + by^{b-1}x)$ ,  $\Delta$  is nonzero in  $Der^2(M(f))/Der^1(M(f))$  if and only if  $A, B, C$  are not all zero and  $\Delta(ax^{a-1}y + y^b), \Delta(x^a + by^{b-1}x), \Delta(ax^a y + y^b x), \Delta(ax^{a-1}y^2 + y^{b+1}), \Delta(x^{a+1} + by^{b-1}x^2), \Delta(x^a y + by^b x)$  all equal to zero in  $M(f)$ .

(1) When  $a, b \geq 3$ , the following equations hold in  $M(f)$ ,  
 $\Delta(ax^{a-1}y + y^b) = a(a-1)(a-2)x^{a-3}yA + a(a-1)x^{a-2}B + b(b-1)y^{b-2}C + a(a-1)x^{a-2}yD + (ax^{a-1} + by^{b-1})E = 0;$   
 $\Delta(x^a + by^{b-1}x) = a(a-1)x^{a-2}A + b(b-1)y^{b-2}B + b(b-1)(b-2)y^{b-3}xC + (ax^{a-1} + by^{b-1})D + b(b-1)y^{b-2}xE = 0;$   
 $\Delta(ax^a y + y^b x) = a^2(a-1)x^{a-2}yA + (a^2x^{a-1} + by^{b-1})B + b(b-1)y^{b-2}xC + a(a-1)x^{a-1}yD +$

$$\begin{aligned}
b(1-a)y^{b-1}xE &= 0; \\
\Delta(ax^{a-1}y^2 + y^{b+1}) &= a(a-1)(a-2)x^{a-3}y^2A + 2a(a-1)x^{a-2}yB + (2ax^{a-1} + (b+1)by^{b-1})C + \\
&+ a(a-1)x^{a-2}y^2D + a(1-b)x^{a-1}yE = 0; \\
\Delta(x^{a+1} + by^{b-1}x^2) &= ((a+1)ax^{a-1} + 2by^{b-1})A + 2b(b-1)y^{b-2}xB + b(b-1)(b-2)y^{b-3}x^2C + \\
&+ b(1-a)y^{b-1}xD + b(b-1)y^{b-2}x^2E = 0; \\
\Delta(x^ay + by^bx) &= a(a-1)x^{a-2}yA + (ax^{a-1} + b^2y^{b-1})B + b^2(b-1)y^{b-2}xC + a(1-b)x^{a-1}yD + \\
&+ b(b-1)y^{b-1}xE = 0.
\end{aligned}$$

(1.1) We do the cases of  $a, b \geq 4$  first.

Since  $M(f)$  is an Artinian algebra with the monomial basis:  $x^iy^j, 0 \leq i \leq a-1, 0 \leq j \leq b-1$ , we express  $A := \sum_{0 \leq i \leq a-1, 0 \leq j \leq b-1} A_{i,j}x^iy^j$ , similar for  $B, C, D$  and  $E$ . Then the conditions to make  $\Delta$  a second order derivation in  $Der^2(M(f))$  is as the following:

$$\begin{aligned}
A_{0,0} &= 0; \\
a(a-1)(a-2)A_{1,0} + a(a-1)B_{0,1} + a(a-1)D_{0,0} &= 0, \quad a(a-1)A_{1,0} + aD_{0,0} = 0, \quad b(b-1)B_{0,1} + bD_{0,0} = 0, \\
a^2(a-1)A_{1,0} + a(a-b)B_{0,1} + a(a-1)D_{0,0} &= 0, \quad a(a-1)(a-2)A_{1,0} + 2a(a-1)B_{0,1} + a(a-1)D_{0,0} = 0, \\
(2-(a+1)a)bA_{1,0} + 2b(b-1)B_{0,1} + (1-a)bD_{0,0} &= 0, \\
a(a-1)A_{1,0} + a(1-b^2)B_{0,1} + (1-b)aD_{0,0} &= 0; \\
-a(a-1)bA_{k,0} + b(b-1)B_{k-1,1} + b(b-1)(b-2)C_{k-2,2} + b(1-a)D_{k-1,0} + b(b-1)E_{k-2,1} &= 0, \\
(2-(a+1)a)bA_{k,0} + 2b(b-1)B_{k-1,1} + b(b-1)(b-2)C_{k-2,2} + b(1-a)D_{k-1,0} + b(b-1)E_{k-2,1} &= 0, \quad 2 \leq k \leq a-2; \\
-a(a-1)bA_{a-1,0} + a(a-1)A_{0,b-1} + b(b-1)B_{a-2,1} + b(b-1)(b-2)C_{a-3,2} + b(1-a)D_{a-2,0} + b(b-1)E_{a-3,1} &= 0, \\
(2-(a+1)a)bA_{a-1,0} + (a+1-2b)aA_{0,b-1} + 2b(b-1)B_{a-2,1} + b(b-1)(b-2)C_{a-3,2} + (1-a)bD_{a-2,0} + b(b-1)E_{a-3,1} &= 0; \\
a(a-1)(a-2)A_{2,k} + a(a-1)B_{1,k+1} - ab(b-1)C_{0,k+2} + a(a-1)D_{1,k} + a(1-b)E_{0,k+1} &= 0, \\
a(a-1)(a-2)A_{2,k} + 2a(a-1)B_{1,k+1} + (2-(b+1)b)aC_{0,k+2} + a(a-1)D_{1,k} + a(1-b)E_{0,k+1} &= 0, \quad 0 \leq k \leq b-2; \\
a(a-1)(a-2)A_{2,b-3} + a(a-1)B_{1,b-2} - ab(b-1)C_{0,b-1} + b(b-1)C_{a-1,0} + a(a-1)D_{1,b-3} + a(1-b)E_{0,b-2} &= 0, \\
a(a-1)(a-2)A_{2,b-3} + 2a(a-1)B_{1,b-2} + (2-(b+1)b)aC_{0,b-1} + (b+1-2a)bC_{a-1,0} + a(a-1)D_{1,b-3} + a(1-b)E_{0,b-2} &= 0; \\
a(a-1)(a-2)A_{2,b-2} + a(a-1)B_{1,b-1} + b(b-1)C_{a-1,1} + a(a-1)D_{1,b-2} + a(1-b)E_{0,b-1} + b(1-a)E_{a-1,0} &= 0; \\
a(a-1)(a-2)A_{1,k} + a(a-1)B_{0,k+1} + a(a-1)D_{0,k} &= 0, \quad a(a-1)A_{1,k} - ab(b-1)B_{0,k+1} + a(1-b)D_{0,k} = 0, \\
a^2(a-1)A_{1,k} + a(a-b)B_{0,k+1} + a(a-1)D_{0,k} &= 0, \quad a(a-1)(a-2)A_{1,k} + 2a(a-1)B_{0,k+1} + a(a-1)D_{0,k} = 0, \\
a(a-1)A_{1,k} + a(1-b^2)B_{0,k+1} + a(1-b)D_{0,k} &= 0, \quad 1 \leq k \leq b-4; \\
a(a-1)(a-2)A_{1,b-3} + a(a-1)B_{0,b-2} + b(b-1)C_{a-2,0} + a(a-1)D_{0,b-3} &= 0, \quad a(a-1)A_{1,b-3} - ab(b-1)B_{0,b-2} + b(b-1)(b-2)C_{a-2,0} + a(1-b)D_{0,b-3} = 0, \\
a^2(a-1)A_{1,b-3} + a(a-b)B_{0,b-2} + b(b-1)C_{a-2,0} + a(a-1)D_{0,b-3} &= 0, \quad a(a-1)(a-2)A_{1,b-3} + 2a(a-1)B_{0,b-2} + (b+1-2a)bC_{a-2,0} + a(a-1)D_{0,b-3} = 0, \\
a(a-1)A_{1,b-3} + a(1-b^2)B_{0,b-2} + b^2(b-1)C_{a-2,0} + a(1-b)D_{0,b-3} &= 0; \\
a(a-1)(a-2)A_{1,b-2} + a(a-1)B_{0,b-1} - a(a-1)bB_{a-1,0} + b(b-1)C_{a-2,1} + a(a-1)D_{0,b-2} + b(1-a)E_{a-2,0} &= 0, \\
a(a-1)A_{1,b-2} - ab(b-1)B_{0,b-1} + b(b-1)B_{a-1,0} + b(b-1)(b-2)C_{a-2,1} + a(1-b)D_{0,b-2} + b(b-1)E_{a-2,0} &= 0, \\
a^2(a-1)A_{1,b-2} + a(a-b)B_{0,b-1} + b(1-a^2)B_{a-1,0} + b(b-1)C_{a-2,1} + a(a-1)D_{0,b-2} + b(1-a)E_{a-2,0} &= 0, \\
a(a-1)A_{1,b-2} + a(1-b^2)B_{0,b-1} + b(b-1)B_{a-1,0} + b^2(b-1)C_{a-2,1} + a(1-b)D_{0,b-2} + b(b-1)E_{a-2,0} &= 0; \\
a(a-1)A_{1,b-1} + b(b-1)B_{a-1,1} + b(b-1)(b-2)C_{a-2,2} + a(1-b)D_{0,b-1} + b(1-a)D_{a-1,0} + b(b-1)E_{a-2,1} &= 0; \\
A_{0,k} &= 0, \quad 1 \leq k \leq b-4; \\
a(a-1)(a-2)A_{0,b-3} + b(b-1)C_{a-3,0} &= 0, \quad a(a-1)A_{0,b-3} + b(b-1)(b-2)C_{a-3,0} = 0, \\
a^2(a-1)A_{0,b-3} + b(b-1)C_{a-3,0} &= 0, \quad a(a-1)(a-2)A_{0,b-3} + (b+1-2a)bC_{a-3,0} = 0,
\end{aligned}$$

$$\begin{aligned}
& a(a+1-2b)A_{0,b-3} + b(b-1)(b-2)C_{a-3,0} = 0, \quad a(a-1)A_{0,b-3} + b^2(b-1)C_{a-3,0} = 0; \\
& a(a-1)(a-2)A_{0,b-2} - a(a-1)bB_{a-2,0} + b(b-1)C_{a-3,1} + b(1-a)E_{a-3,0} = 0, \quad a(a-1)A_{0,b-2} + \\
& b(b-1)B_{a-2,0} + b(b-1)(b-2)C_{a-3,1} + b(b-1)E_{a-3,0} = 0, \quad a^2(a-1)A_{0,b-2} + b(1-a^2)B_{a-2,0} + \\
& b(b-1)C_{a-3,1} + b(1-a)E_{a-3,0} = 0, \quad a(a+1-2b)A_{0,b-2} + 2b(b-1)B_{a-2,0} + b(b-1)(b-2)C_{a-3,1} + \\
& b(b-1)E_{a-3,0} = 0, \quad a(a-1)A_{0,b-2} + b(b-a)B_{a-2,0} + b^2(b-1)C_{a-3,1} + b(b-1)E_{a-3,0} = 0; \\
& \dots\dots
\end{aligned}$$

(We omit the remaining equations, as they can be derived from the symmetry of  $(x; a; A)$  and  $(y; b; C)$ . Then we can obtain the following basis for  $Der^2(M(f))/Der^1(M(f))$ :

$$\begin{aligned}
& x^i y^j \partial_x^2, 3 \leq i \leq a-1, 1 \leq j \leq b-1; \quad x^2 y^{b-1} \partial_x^2; \\
& x^i y^j \partial_x \partial_y, 2 \leq i \leq a-1, 2 \leq j \leq b-1; \quad x^i y^j \partial_y^2, 1 \leq i \leq a-1, 3 \leq j \leq b-1; \\
& x^{a-1} y^2 \partial_y^2; \quad \frac{b-1}{a-1} x^2 \partial_x^2 + 2xy \partial_x \partial_y + \frac{a-1}{b-1} y^2 \partial_y^2 - x \partial_x - y \partial_y; \\
& (b-1)x^k \partial_x^2 + 2(a-1)x^{k-1} y \partial_x \partial_y + (a-1)(a-2)x^{k-2} y^2 \partial_y^2, 3 \leq k \leq a-2; \\
& x^{k-2} y^2 \partial_y^2 + (2-b)x^{k-2} y \partial_y, 3 \leq k \leq a-2; \\
& (b-1)x^{a-1} \partial_x^2 + (2a-2)x^{a-2} y \partial_x \partial_y + (a-1)(a-2)x^{a-3} y^2 \partial_y^2; \\
& by^{b-1} \partial_x^2 + 2ax^{a-2} y \partial_x \partial_y + ax^{a-2} \partial_x - 2ax^{a-3} y \partial_y; \quad x^{a-3} y^2 \partial_y^2 + (2-b)x^{a-3} y \partial_y; \\
& x^2 y^k \partial_x^2 + (2-a)xy^k \partial_x, 1 \leq k \leq b-4; \\
& 2(b-1)xy^{k+1} \partial_x \partial_y + (a-1)y^{k+2} \partial_y^2 + (b-1)(b-2)xy^k \partial_x, 1 \leq k \leq b-4; \\
& x^2 y^{b-3} \partial_x^2 + (2-a)xy^{b-3} \partial_x; \quad 2bxy^{b-2} \partial_x \partial_y + ax^{a-1} \partial_y^2 - 2bxy^{b-3} \partial_x + by^{b-2} \partial_y; \\
& 2(b-1)xy^{b-2} \partial_x \partial_y + (a-1)y^{b-1} \partial_y^2 + (b-1)(b-2)xy^{b-3} \partial_x; \\
& x^2 y^{b-2} \partial_x^2 + (2-a)xy^{b-2} \partial_x; \quad xy^{b-1} \partial_x \partial_y - xy^{b-2} \partial_x; \quad x^{a-1} y \partial_y^2 + \frac{b-1}{a-1} x^{a-1} \partial_y; \\
& \frac{b(b-1)}{a-1} xy^{b-3} \partial_x^2 + 2by^{b-2} \partial_x \partial_y + ax^{a-2} \partial_y^2 - b(b+1)y^{b-3} \partial_x; \\
& \frac{1}{2a(a-1)} xy^{b-2} \partial_x^2 + \frac{1}{a(b-1)} y^{b-1} \partial_x \partial_y + \frac{1}{2b(b-1)} x^{a-2} y \partial_y^2 + \frac{b+1}{2b(b-1)} x^{a-2} \partial_y; \\
& \frac{1}{2a(a-1)} xy^{b-2} \partial_x^2 + \frac{1}{(a-1)b} x^{a-1} \partial_x \partial_y + \frac{1}{2b(b-1)} x^{a-2} y \partial_y^2 + \frac{a+1}{2a(a-1)} y^{b-2} \partial_x; \\
& xy^{b-1} \partial_x^2 + \frac{a-1}{b-1} y^{b-1} \partial_x; \quad x^{a-1} y \partial_x \partial_y - x^{a-2} y \partial_y; \quad x^{a-2} y^2 \partial_y^2 + (2-b)x^{a-2} y \partial_y; \\
& by^{b-2} \partial_x^2 + 2ax^{a-2} \partial_x \partial_y + \frac{a(a-1)}{b-1} x^{a-3} y \partial_y^2 - a(a+1)x^{a-3} \partial_y.
\end{aligned}$$

Therefore,  $\dim Der^2(M(f))/Der^1(M(f)) = (a-3)(b-1) + 1 + (a-2)(b-2) + (a-1)(b-3) + 2 + 2(a-4) + 3 + 2(b-4) + 13 = 3ab - 4a - 4b + 13$ .

(1.2) Next, we consider the cases of  $a \geq 4, b = 3$ . From the above restrictions, only the equations containing variables  $A_{1,0}$  and  $A_{2,0}$  will be changed.

More explicitly, the equations relating to  $A_{1,0}$  should be enlarged from the above equations containing  $A_{1,b-3}$  by adding  $a(a-1)A_{1,0} + 6C_{a-2,0} + aD_{0,0} = 0$  and  $3(2-a(a+1))A_{1,0} + 12B_{0,1} - 18C_{a-2,0} + 3(1-a)D_{0,0} = 0$ . These lead to  $A_{1,0} = B_{0,1} = C_{a-2,0} = D_{0,0} = 0$ . And the equations relating to  $A_{2,0}$  should be enlarged from the above equations containing  $A_{2,b-3}$  by adding equations  $-3a(a-1)A_{2,0} + 6B_{1,1} + 6C_{0,2} - 18C_{a-1,0} + 3(1-a)D_{1,0} + 6E_{0,1} = 0$  and  $3(2-a(a+1))A_{2,0} + 12B_{1,1} + 6C_{0,2} - 18C_{a-1,0} + 3(1-a)D_{1,0} + 6E_{0,1} = 0$ . We will obtain a 2-dimensional subspace of  $Der^2((M(f)))/Der^1((M(f)))$ , which has a basis  $x^2 \partial_x^2 + (a-1)xy \partial_x \partial_y +$

$\frac{a(a-1)}{6}x^{a-1}\partial_y^2 + 2(1-a)x\partial_x, (3(1-a)y^2 + 2ax^{a-1})\partial_y^2 + (9a-3)\partial_y$ . Then we obtain the following basis for  $Der^2(M(f))/Der^1(M(f))$ :

$$\begin{aligned}
& x^i y^j \partial_x^2, 3 \leq i \leq a-1, 1 \leq j \leq 2; \quad x^2 y^2 \partial_x^2; \\
& x^i y^2 \partial_x \partial_y, 2 \leq i \leq a-1; \quad x^{a-1} y^2 \partial_y^2; \\
& 2x^k \partial_x^2 + 2(a-1)x^{k-1}y\partial_x \partial_y + (a-1)(a-2)x^{k-2}y\partial_y, 3 \leq k \leq a-2; \\
& x^{k-2}y^2 \partial_y^2 - x^{k-2}y\partial_y, 3 \leq k \leq a-2; \\
& 2x^{a-1}\partial_x^2 + (2a-2)x^{a-2}y\partial_x \partial_y + (a-1)(a-2)x^{a-3}y\partial_y; \\
& 3y^2 \partial_x^2 + 2ax^{a-2}y\partial_x \partial_y + ax^{a-2}\partial_x - 2ax^{a-3}y\partial_y; \quad x^{a-3}y^2 \partial_y^2 - x^{a-3}y\partial_y; \\
& 4xy\partial_x \partial_y + (a-1)y^2 \partial_y^2 + 2x\partial_x; \\
& x^2 y \partial_x^2 + (2-a)xy\partial_x; \quad xy^2 \partial_x \partial_y - xy\partial_x; \quad x^{a-1}y\partial_y^2 + \frac{2}{a-1}x^{a-1}\partial_y; \\
& \frac{1}{2a(a-1)}xy\partial_x^2 + \frac{1}{2a}y^2 \partial_x \partial_y + \frac{1}{12}x^{a-2}y\partial_y^2 + \frac{1}{3}x^{a-2}\partial_y; \\
& \frac{1}{2a(a-1)}xy\partial_x^2 + \frac{1}{3(a-1)}x^{a-1}\partial_x \partial_y + \frac{1}{12}x^{a-2}y\partial_y^2 + \frac{a+1}{2a(a-1)}y\partial_x; \\
& xy^2 \partial_x^2 + \frac{a-1}{2}y^2 \partial_x; \quad x^{a-1}y\partial_x \partial_y - x^{a-2}y\partial_y; \quad x^{a-2}y^2 \partial_y^2 - x^{a-2}y\partial_y; \\
& 3y\partial_x^2 + 2ax^{a-2}\partial_x \partial_y + \frac{a(a-1)}{2}x^{a-3}y\partial_y^2 - a(a+1)x^{a-3}\partial_y; \\
& x^2 \partial_x^2 + (a-1)xy\partial_x \partial_y + \frac{a(a-1)}{6}x^{a-1}\partial_y^2 + 2(1-a)x\partial_x; \quad (3(1-a)y^2 + 2ax^{a-1})\partial_y^2 + (9a-3)\partial_y.
\end{aligned}$$

Therefore,  $\dim Der^2(M(f))/Der^1(M(f)) = 2(a-3) + 1 + (a-2) + 1 + 2(a-4) + 15 = 5a + 1 = 3ab - 4a - 4b + 13$ .

(1.3) For the case of  $a = b = 3$ , we directly using magma calculated that:

$Der^2(M(f))/Der^1(M(f)) = \mathbb{C}\langle \partial_y - \frac{1}{4}y\partial_y^2 - \frac{1}{2}x\partial_x \partial_y - \frac{1}{4}y\partial_x^2, \frac{1}{2}y^2 \partial_y^2 + xy\partial_x \partial_y + \frac{1}{2}x^2 \partial_x^2, y^3 \partial_y^2 - y^3 \partial_x^2, y^4 \partial_y^2, \frac{1}{2}x\partial_y^2 - 2\partial_x + y\partial_x \partial_y + \frac{1}{2}x\partial_x^2, xy\partial_y^2 + (x^2 + y^2)\partial_x \partial_y + xy\partial_x^2, xy^2 \partial_y^2 - xy^2 \partial_x^2, x^2 \partial_y^2 + 2xy\partial_x \partial_y + y^2 \partial_x^2, y\partial_x + \frac{1}{4}(x^2 - y^2)\partial_x \partial_y, x\partial_x - \frac{1}{4}(x^2 - y^2)\partial_x^2, xy\partial_x + \frac{1}{3}y^3 \partial_x^2, x^2 \partial_x + xy^2 \partial_x^2, y^3 \partial_x \partial_y + 3xy^2 \partial_x^2, y^4 \partial_x \partial_y, xy^2 \partial_x \partial_y + \frac{1}{3}y^3 \partial_x^2, y^4 \partial_x^2 \rangle$ , it is 16-dimensional, satisfies the proposition.

(2) When  $a = 2$ , the following equations holds in  $M(f)$ ,

$$\begin{aligned}
& \Delta(f_x) = 2B + b(b-1)y^{b-2}C + 2yD + (2x + by^{b-1})E = 0; \\
& \Delta(f_y) = 2A + b(b-1)y^{b-2}B + b(b-1)(b-2)y^{b-3}xC + (2x + by^{b-1})D + b(b-1)y^{b-2}xE = 0; \\
& \Delta(xf_x) = 4yA + (4x + by^{b-1})B + b(b-1)y^{b-2}xC + 2xyD - by^{b-1}xE = 0; \\
& \Delta(yf_x) = 4yB + (4x + (b+1)by^{b-1})C + 2y^2D + 2(1-b)xyE = 0; \\
& \Delta(xf_y) = (6x + 2by^{b-1})A + 2b(b-1)y^{b-2}xB + b(b-1)(b-2)y^{b-3}x^2C - by^{b-1}xD + b(b-1)y^{b-2}x^2E = 0; \\
& \Delta(yf_y) = 2yA + (2x + b^2y^{b-1})B + b^2(b-1)y^{b-2}xC + 2(1-b)xyD + b(b-1)y^{b-1}xE = 0.
\end{aligned}$$

(2.1) For the case of  $b \geq 3$ , after some cumbersome calculations similar as above, we obtain the following basis for  $Der^2(M(f))/Der^1(M(f))$ :

$$\begin{aligned}
& xy^k \partial_y^2, 2 \leq k \leq b-1; \quad bxy^{b-2} \partial_x^2 + 4x\partial_x \partial_y + \frac{2}{b-1}y\partial_y^2 + by^{b-2}\partial_x - 4\partial_y; \\
& (2b-1)xy^{b-1} \partial_x^2 + (2x + y^{b-1})\partial_x; \quad b(b-1)^2xy^{b-1} \partial_x^2 - 2(b-1)xy\partial_x \partial_y + y^2 \partial_y^2 + (b-2)y\partial_y; \\
& 2(b-1)xy^k \partial_x \partial_y + y^{k+1} \partial_y^2 + (b-2)y^k \partial_y, 2 \leq k \leq b-3; \\
& 2(b-1)xy^{b-2} \partial_x \partial_y + y^{b-1} \partial_y^2 + (2-b)y^{b-2} \partial_y; \quad bxy^{b-2} \partial_x \partial_y + x\partial_y^2 - \frac{b(b+1)}{2}xy^{b-3} \partial_x;
\end{aligned}$$

$$\frac{b(b-1)}{2}xy^{b-1}\partial_x\partial_y - xy\partial_y^2; \quad xy^{b-1}\partial_x\partial_y - xy^{b-2}\partial_x.$$

So  $\dim Der^2(M(f))/Der^1(M(f)) = (b-2) + 3 + (b-4) + 4 = 2b + 1$ .

(2.2) For the case of  $a = b = 2$ , we directly using magma calculated that:

$Der^2(M(f))/Der^1(M(f)) = \mathbb{C}\langle \partial_y + x\partial_y^2 + \partial_x - 2(x+y)\partial_x\partial_y + y\partial_x^2, (2x+y)\partial_y^2 + 3\partial_x - (2x+4y)\partial_x\partial_y + (y-x)\partial_x^2, y^2\partial_y^2 - y^2\partial_x\partial_y + y^2\partial_x^2, 2y\partial_x - 2y^2\partial_x\partial_y + y^2\partial_x^2, 2x\partial_x + y^2\partial_x\partial_y - 2y^2\partial_x^2 \rangle$ , it is 5-dimensional, satisfies the proposition.

Now we have finished the whole proof of the proposition.  $\square$

After these tedious calculations, we begin to prove the main Theorem C.

*Proof.* (of Theorem C) As the binomial singularity  $(V(f), 0)$  has only three types up to analytical equivalence, we only need to treat these three typical cases.

(1) When  $f = x^a + y^b$ , the equality holds just from the definition of  $h_2(a, b)$ , and in this case,  $\dim Der^2(M(V))/Der^1(M(V)) = h_2(a, b) = h_2(\frac{1}{w_1}, \frac{1}{w_2}) = 3ab - 5a - 5b + 8$ .

(2) When  $f = x^a y + y^b$ ,  $w_1 := wt(x) = \frac{b-1}{ab}$ ,  $w_2 := wt(y) = \frac{1}{b}$ , and  $h_2(\frac{1}{w_1}, \frac{1}{w_2}) = 3(\frac{1}{w_1} - 2)(\frac{1}{w_2} - 2) + (\frac{1}{w_1} - 2) + (\frac{1}{w_2} - 2) = 3ab - 5b - 2\frac{ab}{b-1} + 8$ .

If  $a = 1$ , then  $h_2(\frac{1}{w_1}, \frac{1}{w_2}) = -2b - 2\frac{b}{b-1} + 8$ , and since  $w_1, w_2 \leq 1/2$ , we must have  $b = 2$ , and  $h_2(\frac{1}{w_1}, \frac{1}{w_2}) = 0$ .

If  $a \geq 2, b = 2$ , then  $h_2(\frac{1}{w_1}, \frac{1}{w_2}) = 2a - 2 = \dim Der^2(M(V))/Der^1(M(V))$ .

If  $a \geq 2, b \geq 3$ , from Proposition 5.3,  $\dim Der^2(M(V))/Der^1(M(V)) = 3ab - 4a - 5b + 10 \leq 3ab - 5b - 3a + 8 \leq 3ab - 5b - \frac{2b}{b-1}a + 8 = h_2(\frac{1}{w_1}, \frac{1}{w_2})$ .

(3) When  $f = x^a y + y^b x$ ,  $w_1 := wt(x) = \frac{b-1}{ab-1}$ ,  $w_2 := wt(y) = \frac{a-1}{ab-1}$ , and  $h_2(\frac{1}{w_1}, \frac{1}{w_2}) = 3ab - 2a - 2b + 5 - \frac{2(a-1)}{b-1} - \frac{2(b-1)}{a-1}$ .

If  $a = 2$ , then  $h_2(\frac{1}{w_1}, \frac{1}{w_2}) = 2b + 3 - \frac{2}{b-1}$ , and from Proposition 5.4,  $\dim Der^2(M(f))/Der^1(M(f)) = 2b + 1 \leq h_2(\frac{1}{w_1}, \frac{1}{w_2})$ , the case of  $b = 2$  is symmetrical.

If  $a, b \geq 3$ , from Proposition 5.4,  $\dim Der^2(M(f))/Der^1(M(f)) = 3ab - 4a - 4b + 13$ , since  $\frac{2(a-1)}{b-1} + \frac{2(b-1)}{a-1} \leq (a-1) + (b-1) \leq 2a + 2b - 8$ , then  $\dim Der^2(M(f))/Der^1(M(f)) \leq h_2(\frac{1}{w_1}, \frac{1}{w_2})$ .  $\square$

## 6. AN EXAMPLE FOR THE NAKAI CONJECTURE

In section 2, we give a brief statement for Zariski-Lipman Conjecture. It has been shown that the Nakai Conjecture implies the Zariski-Lipman Conjecture, and both of these two conjectures can be reduced to the case of isolated singularities [2].

For the case of hypersurface singularities, Singh gave a stronger conjecture [20], which states that for a  $k$ -algebra  $R = k[x_1, \dots, x_n]/(F)$ , if  $Der_k^2(R)$  is generated by  $Der_k^1(R)$ , then  $R$  is regular. Singh's conjecture for the ring  $k[x_1, x_2, \dots, x_n]/(a_1 x_1^m + \dots + a_n x_n^m)$  has been checked in [4]. In this section, we will imitate their method to prove Singh's conjecture for the case of Brieskorn singularities.

We fix some notations in this section first. Let  $k$  be a field of characteristic zero, remark  $S = k[x_1, x_2, \dots, x_n]$  the polynomial ring,  $F = x_1^{a_1} + x_2^{a_2} + \dots + x_n^{a_n}$ , and  $R = S/(F)$  the affine algebra. Our goal is to show that  $Der_k^2(R)$  can not be generated by  $Der_k^1(R)$ .

**Lemma 6.1.** ([4]) *For a weighted homogeneous polynomial  $f \in S$  of weight type  $(w_1, w_2, \dots, w_n; 1)$ , denote  $D_{ij} = f_{x_i} \partial_{x_j} - f_{x_j} \partial_{x_i}$  the Hamiltonian derivations, and  $E = \sum_{i=1}^n w_i x_i \partial_{x_i}$  the Euler derivation, then  $Der_k^1(S/(f))$  is generated by  $D_{ij}, 1 \leq i < j \leq n$  and  $E$  as  $S/(f)$ -module.*

*Proof.* Let  $D \in \text{Der}_k^1(S)$  such that  $D(f) = hf$ ,  $h \in S$ , as  $E(f) = f$ , then  $D - hE \in \text{Der}_k^1(S)$ . So it is no matter to assume  $D(f) = 0$ , and we prove that  $D$  can be generated by  $D_{ij}$ 's.

Set  $D = \sum_{i=1}^n h_i \partial_{x_i}$ , we have  $D(f) = \sum_{i=1}^n h_i f_{x_i} = 0$ ,  $h_1 f_{x_1} = -\sum_{i=2}^n h_i f_{x_i}$ . Because  $\{f_{x_1}, f_{x_2}, \dots, f_{x_n}\}$  is a regular sequence, then  $h_1 \in (f_{x_2}, f_{x_3}, \dots, f_{x_n})$ , let  $h_1 = \sum_{i=2}^n g_i f_{x_i}$ , then  $D = (\sum_{j=2}^n g_j f_{x_j}) \partial_{x_1} + \sum_{i=2}^n h_i \partial_{x_i} = \sum_{j=2}^n (-g_j D_{1j}) + \sum_{i=2}^n (h_i + g_i f_1) \partial_{x_i}$ . We continue the same operator for  $\sum_{i=2}^n (h_i + g_i f_1) \partial_{x_i}$ . Finally, we will obtain  $D = D' + p \partial_{x_n}$ , where  $D'$  is generated by  $D_{ij}$ 's.  $D(f) = p f_{x_n} = 0$ , and since  $f_{x_n}$  is regular in  $S$ ,  $p = 0$ , thus  $D$  is generated by  $D_{ij}$ 's.  $\square$

**Lemma 6.2.** ([4]) *With notations as above, let  $d \in \text{Der}_k^1(S)$ ,  $D \in \text{Der}_k^2(S)$  satisfying  $D(F) \subset (F)$ ,  $d(F) \subset (F)$ , the following facts hold:*

- 1)  $d(x_i) \in J_i = (x_1^{a_1-1}, \dots, x_{i-1}^{a_{i-1}-1}, x_i, x_{i+1}^{a_{i+1}-1}, \dots, x_n^{a_n-1})$  and  $d(J_i) \subset J_i$ ;
- 2)  $d(x_i F) \in J_i \cdot (F)$  and  $d(J_i \cdot (F)) \subset J_i \cdot (F)$ ;
- 3) If  $2 \leq l \in \mathbb{N}$ , then for each  $j \in \{1, \dots, n\}$ ,  $D(x_1 x_j^l) \in (x_1, x_j^{l-1})$  and  $D(x_1 F) \in J_1$ .

*Proof.* 1) and 2) are just direct computations, we omit them.

For 3), we do induction on  $l$ . For  $l = 2$ , by the definition of second order derivations,  $D(x_1 x_j^2) = x_1 D(x_j^2) + 2x_j D(x_1) - 2x_1 x_j D(x_j) - x_j^2 D(x_1) \in (x_1, x_j)$ .

Assume 3) holds for  $l-1$ , the for the case of  $l$ , we have  $D(x_1 x_j^l) = D(x_1 x_j x_j^{l-1}) = x_1 D(x_j^l) + x_j D(x_1 x_j^{l-1}) + x_j^{l-1} D(x_1 x_j) - x_1 x_j D(x_j^{l-1}) - x_1 x_j^{l-1} D(x_j) - x_j^l D(x_1)$ . From the induction hypothesis,  $D(x_1 x_j^{l-1}) \in (x_1, x_j^{l-2})$ ,  $D(x_j^{l-1}) \in (x_j^{l-2})$ , thus  $D(x_1 x_j^l) \in (x_1, x_j^{l-1})$ , and  $D(x_1 F) = \sum_{i=1}^n D(x_1 x_i^{a_i}) \in J_1$ .  $\square$

Now we define a special second order derivation of  $S/(F)$ , and we prove that it does not belong to  $\text{Der}_k^1(S/(F)) + \text{Der}_k^1(S/(F))\text{Der}_k^1(S/(F))$ .

**Definition 6.3.** Remark  $G = \prod_{i=1}^n x_i^{a_i-2}$ , we define a second order derivation in  $\text{Der}^2(S)$  as following,

$$D_0 = \left( \frac{a_1 - 1}{a_1} - \sum_{j=2}^n \frac{a_j - 1}{a_j} \right) G \partial_{x_1} - \frac{1}{a_1} x_1 G \partial_{x_1}^2 - 2G \sum_{j=2}^n \frac{x_j}{a_j} \partial_{x_1} \partial_{x_j} + x_1^{a_1-1} \sum_{j=2}^n \frac{a_1 G}{a_j^2 x_j^{a_j-2}} \partial_{x_j}^2.$$

**Proposition 6.4.** *The derivation  $D_0$  defined above induces a second order derivation on  $S/(F)$ .*

*Proof.* From the Proposition 2.7, we just need to check that  $D_0(F) \in (F)$ ,  $[D_0, x_i](F) \in (F)$ ,  $\forall 1 \leq i \leq n$ .

$$D_0(F) = ((a_1 - 1)x_1^{a_1-1} - \sum_{j=2}^n \frac{a_1(a_j-1)}{a_j} x_1^{a_1-1})G - \frac{1}{a_1} x_1 G a_1(a_1-1)x_1^{a_1-2} + x_1^{a_1-1} \sum_{j=2}^n \frac{a_1(a_j-1)}{a_j} G = 0.$$

$$[D_0, x_j] = -2G \sum_{j=2}^n \frac{x_j}{a_j} \partial_{x_1} + 2x_1^{a_1-1} \sum_{j=2}^n \frac{a_1 G}{a_j^2 x_j^{a_j-2}} \partial_{x_j} \text{ for } j \geq 2, \text{ then } [D_0, x_j](F) = -2G \sum_{j=2}^n \frac{a_1}{a_j} x_1^{a_1-1} x_j + 2x_1^{a_1-1} \sum_{j=2}^n \frac{a_1}{a_j} x_j G = 0.$$

$$[D_0, x_1] = \left( \frac{a_1-1}{a_1} - \sum_{j=2}^n \frac{a_j-1}{a_j} \right) G - \frac{2}{a_1} x_1 G \partial_{x_1} - 2G \sum_{j=2}^n \frac{x_j}{a_j} \partial_{x_j} = \left( \frac{a_1-1}{a_1} - \sum_{j=2}^n \frac{a_j-1}{a_j} \right) G - 2G \cdot E, \text{ where } E = \sum_{i=1}^n \frac{1}{a_i} x_i \partial_{x_i} \text{ is the Euler derivation, then } [D_0, x_1](F) = \left( \frac{a_1-1}{a_1} - \sum_{j=2}^n \frac{a_j-1}{a_j} - 2 \right) GF \in (F).$$

Therefore  $D_0$  induces a second order derivation on the affine algebra  $S/(F)$ .  $\square$

**Theorem 6.5.** *For the  $k$ -algebra  $S/(F)$  defined above, we have  $\text{Der}_k^2(S/(F)) \neq \text{der}_k^2(S/(F))$ , where  $\text{der}_k^2(S/(F)) = \text{Der}_k^1(S/(F)) + \text{Der}_k^1(S/(F))\text{Der}_k^1(S/(F))$ . In other words, the Nakai Conjecture holds for the case of Brieskorn singularity.*

*Proof.* Consider the derivation  $D_0$  we have constructed above, from the previous proposition,  $D_0$  induces a second order derivation on  $S/(F)$ , we still denote it by  $D_0$ . Assume  $D_0 \in \text{der}_k^2(S/(F))$ , then  $D_0 = d + \sum d_i \circ d'_i + FD$  in  $\text{Der}_k^2(S)$ , where  $d, d_i, d'_i \in \text{Der}_k^1(S)$  and  $D \in \text{Der}_k^2(S)$ .

From Lemma 6.2, we know that  $D_0(x_1F) \in F \cdot J_1$ , where  $J_1 = (x_1, x_2^{a_2-1}, \dots, x_n^{a_n-1})$ . However, from the calculation in the proof of Proposition 6.4,  $D_0(x_1F) = [D_0, x_1](F) + x_1D_0(F) = (\frac{a_1-1}{a_1} - \sum_{j=2}^n \frac{a_j-1}{a_j} - 2)GF$ , and  $(\frac{a_1-1}{a_1} - \sum_{j=2}^n \frac{a_j-1}{a_j} - 2)G$  does not belong to  $J_1$ . As  $F$  is regular in  $S$ ,  $D_0(x_1F) \notin F \cdot J_1$ , which leads to a contradiction.  $\square$

Now we see that Theorem D immediately follows from it.

## 7. APPENDIX: MAGMA PROGRAM FOR CALCULATING HIGHER DERIVATIONS

Here we list the magma code for calculating the third order derivations of  $M(V)$  for  $E_7$  singularity  $(V, 0) = (\{x^3 + xy^3 = 0\}, 0)$ . One need to run the program twice: for the second running, input the result below the dashed line of the first running.

```
>QQ:=RationalField();
>R < x, y >:= PolynomialRing(QQ,2);
>% change FF
>FF:= x^3+x*y^3;
>f1:=Derivative(FF,1,1);
>f2:=Derivative(FF,1,2);
>printf "f1:= %o;\n" , f1;
>printf "f2:= %o;\n" , f2;
>printf "FF:= %o;\n" , FF;
>% change TM: a basis of the moduli algebra
> TM:=[ 1, y, y^2, y^3, y^4, x, x*y];
>rk:=#TM;
>% change m: the order of the higher derivations
>m:=3;
>l:=rk*(Binomial(m+2,2)-1);
>p:=Binomial(m+2,2)-1;

>b:=Matrix(R, m+1,m+1,[<i,j,0>:i,j in [1..m+1]]);
>for i:=0 to m do
> for j:=0 to m do
> if (i+j ne 0 and i+j le m) then
> b[i+1][j+1]:=f1;
> b[i+1][j+1]:=Derivative(b[i+1][j+1],i,1)/Factorial(i);
> b[i+1][j+1]:=Derivative(b[i+1][j+1],j,2)/Factorial(j);
> end if;
> end for;
>end for;

>c:=Matrix(R, m+1,m+1,[<i,j,0>:i,j in [1..m+1]]);
>for i:=0 to m do
> for j:=0 to m do
> if (i+j ne 0 and i+j le m) then
> c[i+1][j+1]:=f2;
```

```

> c[i+1][j+1]:=Derivative(c[i+1][j+1],i,1)/Factorial(i);
> c[i+1][j+1]:=Derivative(c[i+1][j+1],j,2)/Factorial(j);
> end if;
> end for;
>end for;

>"-----";
>printf "QQ:=RationalField();\n";
>printf "dimA:=%o;\n", rk;
>printf "m:=%o;\n",m;
>printf "l:=%o;\n",l;
>printf "p:=%o;\n",p;
>printf "R:= PolynomialRing(QQ,l);\n";
>printf "F<";
>for i:=0 to m do
> for j:=0 to m do
> if (i+j ne 0 and i+j le m) then
> for k:=1 to rk do
> printf "a_%o_%o_%o",i,j,k;
> if(i ne m or j ne 0 or k ne rk) then
> printf ",";
> else printf ">";
> end if;
> end for;
> end for;
> printf "\n";
> end if;
> end for;
>end for;
>printf":= FieldOfFractions(R); \n";
>printf "A<x, y> := AffineAlgebra<F, x, y|\n";
>printf "%o,\n%o,\n%o>; \n ",FF,f1,f2;
>printf "FF:=%o; \n",FF;
>printf "TM:=[ ";
>for i:=1 to rk-1 do
>printf "%o, ", TM[i];
>end for;
>printf "%o ];\n", TM[rk];

>printf "a:=Matrix(A, %o,%o,<i,j,0>:i,j in [1..%o]);\n",p,p,p;
>printf "s:=1;\n";
>printf "for i:=0 to m do\n";
>printf " for j:=0 to m do\n";
>printf " if (i+j ne 0 and i+j le m) then\n";
>printf " P:=0;\n";
>printf " for k:=1 to dimA do\n";
>printf " P:=P+F.s*TM[k];\n";
>printf " s:=s+1;\n";
>printf " end for;\n";

```

```

>printf" a[i+1][j+1]:=P;\n";
>printf" end if;\n";
>printf" end for; \n";
>printf" end for;\n";

>printf"b:=Matrix(A, m+1,m+1,[<i,j,0>:i,j in [1..m+1]]);\n";
>printf"c:=Matrix(A, m+1,m+1,[<i,j,0>:i,j in [1..m+1]]);\n";
>printf"DF1:=Matrix(A, m,m,[<i,j,0>:i,j in [1..m]]);\n";
>printf"DF2:=Matrix(A, m,m,[<i,j,0>:i,j in [1..m]]);\n";

>for i:=0 to m do
> for j:=0 to m do
> if (i+j ne 0 and i+j le m) then
> printf "b[%o][%o]:=%o;\n", i+1,j+1,b[i+1][j+1];
> end if;
> end for;
>end for;

>for i:=0 to m do
> for j:=0 to m do
> if (i+j ne 0 and i+j le m) then
> printf "c[%o][%o]:=%o;\n", i+1,j+1,c[i+1][j+1];
> end if;
> end for;
>end for;

>printf"for u:=0 to m-1 do \n";
>printf" for v:=0 to m-1 do \n";
>printf" if (u+v le m-1) then \n";
>printf" DF1[u+1][v+1]:=0; \n";
>printf" for i:=0 to m do \n";
>printf" for j:=0 to m do \n";
>printf" if (i+j ne 0 and i+j le m) then \n";
>printf" DF1[u+1][v+1]:=DF1[u+1][v+1]+a[u+i+1][v+j+1]*b[i+1][j+1];\n";
>printf" end if; \n";
>printf" end for; \n";
>printf" end for; \n";
>printf" end if; \n";
>printf" end for; \n";
>printf"end for; \n";

>printf"for u:=0 to m-1 do \n";
>printf" for v:=0 to m-1 do \n";
>printf" if (u+v le m-1) then \n";
>printf" DF2[u+1][v+1]:=0; \n";
>printf" for i:=0 to m do \n";
>printf" for j:=0 to m do \n";
>printf" if (i+j ne 0 and i+j le m) then \n";
>printf" DF2[u+1][v+1]:=DF2[u+1][v+1]+a[u+i+1][v+j+1]*c[i+1][j+1];\n";

```

```

>printf" end if; \n";
>printf" end for; \n";
>printf" end for; \n";
>printf" end if; \n";
>printf" end for; \n";
>printf"end for; \n";

>printf " V,h:=VectorSpace(A); \n";

>printf "T:=[";
>for i:=1 to l-1 do
>printf "R.%o,", i;
>end for;
>printf "R.%o];\n", l;

>printf " LC:=function(a)\n";
>printf " return [ MonomialCoefficient(a, TE): TE in T ];\n";
>printf " end function;\n";

>printf " M:=[];\n";
>printf " for u:=0 to m-1 do\n";
>printf " for v:=0 to m-1 do\n";
>printf " if (u+v le m-1) then \n";
>printf " for i in [1 .. dimA] do\n";
>printf " L:=(h(DF1[u+1][v+1])[i]); a:=Numerator(L);\n";
>printf " Append( M,LC(a));\n";
>printf " L:=(h(DF2[u+1][v+1])[i]); a:=Numerator(L);\n";
>printf " Append( M,LC(a));\n";
>printf " end for;\n";
>printf " end if;\n";
>printf " end for;\n";
>printf " end for;\n";

>printf "MM:=Matrix(M);\n";
>printf "l-Rank(MM);\n";
>printf "N:=NullSpace(Transpose(MM)); \n";
>printf "B:=Basis(N);\n";
>printf "Rank(N);\n";
>printf "B:=Basis(N); \n";
>printf "rk:=#B; rk;\n";

>printf "RP<x,y>:= PolynomialRing(QQ,2);\n";

>printf "TM:=[";
>for i:=1 to rk-1 do
>printf "%o, ", TM[i];
>end for;
>printf "%o ];\n", TM[rk];

>printf "check:=function(a)\n";

```

```

>printf "Z:=Matrix(RP, %o,%o,[<i,j,0>:i,j in [1..%o]]);\n",m+1,m+1,m+1;
>r:=1;
>for i:=0 to m do
> for j:=0 to m do
> if (i+j ne 0 and i+j le m) then
>printf "alpha:= ";
> for s:=1 to (rk-1) do
>printf "a[%o]*TM[%o]+", r, s;
> r:=r+1;
> end for;
>printf "a[%o]*TM[%o];\n", r, rk ;
>r:=r+1;
>printf "Z[%o][%o]:=alpha;\n",i+1,j+1;
> end if;
> end for;
>end for;
>printf "return Z;\n";
>printf "end function;\n ";

>printf " for s:=1 to rk do\n ";
>printf " printf \"e_%o &=\", s;\n";
>printf "for i:=0 to (m-1) do\n";
>printf " for j:=0 to m do\n";
>printf " if (i+j ne 0 and i+j le m) then\n";
>printf " if (check(B[s])[i+1][j+1] ne 0) then";
>printf " printf \"_%o\\\\\\\\partial_x ^(_%o) partial_y ^(_%o) + \",check(B[s])[i+1][j+1],i,j;\n";
>printf " end if;\n";
>printf " end if;\n";
>printf " end for;\n";
>printf "end for;\n";
>printf " V:=check(B[s])[m+1][1];\n";

>printf " printf \"_%o\\\\\\\\partial_x ^(_%o), \\\n \",check(B[s])[m+1][1],m;\n";
>printf " end for;\n";

```

## REFERENCES

- [1] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of differentiable maps. Volume 1.*, Mod. Birkhäuser Class. Birkhäuser/Springer, New York, 2012. xii+382 pp.
- [2] J. Becker, *Higher derivations and integral closure*, Amer. J. Math. 100.3 (1978), 495-521.
- [3] R. E. Block, *Determination of the differentiably simple rings with a minimal ideal*, Ann. of Math. (1969), 433-459.
- [4] P. R. Brumatti and M. O. Veloso, *A note on Nakai's conjecture for the ring  $K[X_1, \dots, X_n]/(a_1X_1^m + \dots + a_nX_n^m)$* , Colloq. Math. 123 (2011), 277-283.
- [5] H. Chen, S. S.-T. Yau, and H. Zuo, *Non-existence of negative weight derivations on positively graded Artinian algebras*, Trans. Amer. Math. Soc. 372 (2019), no.4, 2493-2535.
- [6] B. Chen, H. Chen, S. S.-T. Yau, and H. Zuo, *The nonexistence of negative weight derivations on positive dimensional isolated singularities: generalized Wahl conjecture*, J. Differential Geom. 115(2020), no.2, 195-224.
- [7] D. Duarte, *Computational aspects of the higher Nash blowup of hypersurfaces*, J. Algebra 477 (2017), 211-230.
- [8] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Grad. Texts in Math. 150, 1995.

- [9] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas*, Inst. Hautes Études Sci. Publ. Math. 32 (1967).
- [10] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, Vol. 52, 1977.
- [11] V. Hauschild, *Discriminants, resultants and a conjecture of S. Halperin*, Jahresber. Deutsch. Math.-Verein. 104 (2002), no.1, 26–47.
- [12] N. Hussain, S. S.-T. Yau, and H. Zuo, *On the derivation Lie algebras of fewnomial singularities*, Bull. Aust. Math. Soc. 98: 1 (2018), 77–88.
- [13] N. Hussain, G. Ma, S. S.-T. Yau, and H. Zuo, *Higher Nash blow-up local algebras of singularities and its derivation Lie algebras*, J. Algebra 618 (2023), 165–194.
- [14] G. Khimshiashvili, *Yau algebras of fewnomial singularities*, Universiteit Utrecht Preprints (1352), 2006, <http://www.math.uu.nl/publications/preprints/1352.pdf>.
- [15] J. Lipman, *Free derivation modules on algebraic varieties*, Amer. J. Math. 87 (1965), 874–898.
- [16] J. Mather and S. S.-T. Yau, *Classification of isolated hypersurface singularities by their moduli algebras*, Invent. Math. 69 (1982), 243–251.
- [17] G. Ma, S. S.-T. Yau, and H. Zuo, *On the non-existence of negative weight derivations of the new moduli algebras of singularities*, J. Algebra 564 (2020), 199–246.
- [18] Y. Nakai, *High order derivations I*, Osaka J. Math. 7 (1970), 1–27.
- [19] P. Barajas and D. Duarte, *On the module of differentials of order  $n$  of hypersurfaces*, J. Pure Appl. Algebra 224 (2020), no.2, 536–550.
- [20] B. Singh, *Differential operators on a hypersurface*, Nagoya Math. J. 103 (1986), 67–84.
- [21] C. Seeley and S. S.-T. Yau, *Variation of complex structure and variation of Lie algebras*, Invent. Math. 99 (1990), 545–565.
- [22] Y. Yu, *On Jacobian ideals invariant by reducible  $sl(2;C)$  action*, Trans. Amer. Math. Soc. 348 (1996), 2759–2791.

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING, 100084, P. R. CHINA.

*Email address:* xiaozd21@mails.tsinghua.edu.cn

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING, 100084, P. R. CHINA.; YANQI LAKE BEIJING INSTITUTE OF MATHEMATICAL SCIENCES AND APPLICATIONS, HUIAIROU 101400, P. R. CHINA

*Email address:* yau@uic.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING, 100084, P. R. CHINA.

*Email address:* hqzuo@mail.tsinghua.edu.cn