EXPLICIT EXPRESSION FOR INVERSE OF AN AUTOMORPHISM OF A POWER SERIES RING

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In Memory of Professor Richard Hamilton

Abstract. Calculating the inverse of an automorphism of a formal power series ring presents a frequent challenge in a myriad of mathematical inquiries, especially in the realm of singularity theory. In instances involving non-linear and multivariable contexts, S. S. Abhyankar pioneered a methodology to tackle this problem. However, calculating the expressions up to a certain order using this method requires calculating higher-order terms and then carry out the selection, which leads to redundant computations in practice. This article introduces two novel approaches for determining the inverse of an automorphism of a formal power series ring over an arbitrary commutative ring with unit, grounded in the newly developed higher order Jacobian matrix theory. These approaches can be conceived as non-linear extensions of the inverse matrix method and the Gaussian elimination method respectively. They avoid redundant computations above. For the two new methods, we also give the application in calculating the explicit expression for the implicit function theorem.

Keywords. Implicit function theorem, automorphism of formal power series, higher order Jacobian matrix.

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Contents

1. Background

The implicit function theorem is fundamental in many branches of mathematics. In many cases, we want to get the explicit expression. More generally, for a certain automorphism of power series ring over an arbitrary commutative ring with unit, we want to calculate the expression of inverse. The linear terms are clear by the inverse of Jacobian matrix. However, calculations become difficult for the non-linear terms.

In this section, we will give a brief overview of the previous method. To present the ideas more clearly and concisely, we introduce the notations and conventions used in this article.

1.1. **Notations and Conventions.**

Definition 1.1. We define $\boldsymbol{x} := (x_1, x_2, \ldots, x_l)$, $\boldsymbol{x}' := (x_1, x_2, \ldots, x_{l'})$ and $\boldsymbol{x}'' := (x_1, x_2, \ldots, x_{l''})$ where $l, l', l'' \in \mathbb{N}_+$.

Definition 1.2. For $\alpha := (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$ and $\beta := (\beta_1, \ldots, \beta_l) \in \mathbb{N}^l$, we define

$$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdot \binom{\alpha_2}{\beta_2} \cdot \ldots \cdot \binom{\alpha_l}{\beta_l}. \tag{1}$$

Definition 1.3. For $\alpha := (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$ and $\beta := (\beta_1, \ldots, \beta_l) \in \mathbb{N}^l$, $\alpha \le \beta$ means $\alpha_i \le \beta_i$ for $i = 1, 2, \ldots, l$.

Definition 1.4. For $\alpha := (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$, we define $\alpha! := \alpha_1! \cdot \alpha_2! \cdot \ldots \cdot \alpha_l!$ and $|\alpha| := \alpha_1 + \alpha_2 + \ldots + \alpha_l$.

Definition 1.5. For $\alpha := (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$, we define $\boldsymbol{x}^\alpha := \prod_{i=1}^{l} x_i^{\alpha_i}$.

Definition 1.6. For an $r \times s$ matrix M and indexes $1 \le i_1, i_2, \ldots, i_t \le r$ and $1 \le j_1, j_2, \ldots, j_{t'} \le s$, we define $M_{(i_1, i_2, \ldots, i_t)}^{(j_1, j_2, \ldots, j_{t'})}$ to be the $t \times t'$ matrix whose (r', s')-th element is equal to $(i_{r'}, j_{s'})$-th element of M for any integer $1 \le r' \le t$ and $1 \le s' \le t'$.

Definition 1.7. In this article, $\mathcal{R}$ is defined to be an arbitrary commutative ring with unit.

Definition 1.8. We define $\Delta_i := (\delta_{1,i}, \delta_{2,i}, \ldots, \delta_{l,i})$ for $1 \le i \le l$ where $\delta_{j,i}$ $(1 \le j \le l)$ means Kronecker delta.

Definition 1.9. Assume that $k \in \mathbb{N}$. For $F = \sum_{I \in \mathbb{N}^l} a_I \boldsymbol{x}^I$ with a_I's in $\mathcal{R}$, $\mathrm{Jet}^{(k)}$ operates on the power series in terms of variables $\boldsymbol{x}$ means $\mathrm{Jet}^{(k)}(F) = \sum_{\substack{I \in \mathbb{N}^l; \\ |I| \le k}} a_I \boldsymbol{x}^I$. For the $\mathcal{R}$-algebra homomorphism $f : \mathcal{R}[[\boldsymbol{x}']] \to \mathcal{R}[[\boldsymbol{x}]]$, assume $Im(f)$ is in $\mathcal{R}[[\boldsymbol{y}]]$ where $\boldsymbol{y} := (y_1, y_2, \ldots, y_l)$. For such f, $\mathrm{Jet}^{(k)}$ operates on the power series in terms of variables $\boldsymbol{y}$ means $\mathrm{Jet}^{(k)}(f)$ is defined by $\left(\mathrm{Jet}^{(k)}(f)\right)(x_i) := \mathrm{Jet}^{(k)}(f(x_i))$ $(i = 1, 2, \ldots, l)$ in terms of variables $\boldsymbol{y}$. When there is no risk of ambiguity, we omit the notation for variables for simplicity.

1.2. **Previous Researches.**

In univariate case, one can apply Newton's Lemma to get the explicit expression for the implicit function (or more generally, the inverse of an automorphism of formal power series).

Lemma 1.10 (Newton's lemma [4]). *Let $F \in \mathbb{C}\{\boldsymbol{x}, y\}$ and $k \in \mathbb{N}_+$. Let $\overline{Y}(\boldsymbol{x}) \in \mathbb{C}\{\boldsymbol{x}\}$ be such that, for $D := \frac{\partial F}{\partial y}(\boldsymbol{x}, \overline{Y}(\boldsymbol{x}))$, we have*

$$F(\boldsymbol{x}, \overline{Y}(\boldsymbol{x})) \in \langle \boldsymbol{x} \rangle^k \cdot \langle D \rangle^2 \subset \mathbb{C}\{\boldsymbol{x}\}.$$

Then there exists a $Y(\boldsymbol{x}) \in \mathbb{C}\{\boldsymbol{x}\}$ with $Y(\boldsymbol{x}) - \overline{Y}(\boldsymbol{x}) \in \langle \boldsymbol{x} \rangle^k \cdot \langle D \rangle$ such that $F(\boldsymbol{x}, Y(\boldsymbol{x})) = 0$.

To compute the explicit expression of the solution $Y(\boldsymbol{x}) \in \mathfrak{m}_{K\langle \boldsymbol{x} \rangle}$ of the equation $F(\boldsymbol{x}, y) = 0$ (with $F \in \mathbb{C}\{\boldsymbol{x}, y\}$ satisfying $F|_{(\boldsymbol{x}, y)=0} = 0$ and $\frac{\partial F}{\partial y}|_{(\boldsymbol{x}, y)=0} \neq 0$, we may use Lemma 1.10. For instance, starting with the initial solution $Y^{(0)}(\boldsymbol{x}) = 0$, we may set

$$Y^{(j+1)}(\boldsymbol{x}) := Y^{(j)}(\boldsymbol{x}) - \frac{F\left(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\right)}{\frac{\partial F}{\partial y}\left(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\right)}$$

where $j \in \mathbb{N}$. Note that the denominator $\frac{\partial F}{\partial y}\left(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\right)$ is a unit in $\mathbb{C}\{\boldsymbol{x}\}$ as $\frac{\partial F}{\partial y}$ has a non-zero constant term, and as $Y^{(j)}(\boldsymbol{x}) \in \langle \boldsymbol{x} \rangle$. Moreover, by Taylor's expansion, we get

$$\begin{aligned} & F\left(\boldsymbol{x}, Y^{(j+1)}(\boldsymbol{x})\right) \\ = & F\left(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\right) - \frac{\partial F}{\partial y}\left(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\right) \cdot \frac{F\left(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\right)}{\frac{\partial F}{\partial y}\left(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\right)} + h(\boldsymbol{x}) \cdot \left(\frac{F\left(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\right)}{\frac{\partial F}{\partial y}\left(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\right)}\right)^2 \\ = & h(\boldsymbol{x}) \cdot \left(\frac{F\left(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\right)}{\frac{\partial F}{\partial y}\left(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\right)}\right)^2 \end{aligned} \tag{2}$$

for some $h(\boldsymbol{x}) \in \mathbb{C}\{\boldsymbol{x}\}$. Thus,

$$F\left(\boldsymbol{x}, Y^{(j+1)}(\boldsymbol{x})\right) \in \left\langle F\left(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\right)\right\rangle^2,$$

and for $j \in \mathbb{N}$,

$$F\left(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x})\right) \in \left\langle F\left(\boldsymbol{x}, Y^{(0)}(\boldsymbol{x})\right)\right\rangle^{2^j} \subset \langle \boldsymbol{x} \rangle^{2^j} = \langle \boldsymbol{x} \rangle^{2^j} \cdot \langle \frac{\partial F}{\partial y}(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x}))\rangle^2.$$

By Newton's lemma, there exists a $Y(\boldsymbol{x}) \in \mathbb{C}\{\boldsymbol{x}\}$ with

$$Y(\boldsymbol{x}) - Y^{(j)}(\boldsymbol{x}) \in \langle \boldsymbol{x} \rangle^{2^j} \cdot \langle \frac{\partial F}{\partial y}(\boldsymbol{x}, Y^{(j)}(\boldsymbol{x}))\rangle = \langle \boldsymbol{x} \rangle^{2^j}$$

such that $F(\boldsymbol{x}, Y(\boldsymbol{x})) = 0$, which implies that the sequence of power series $Y^{(j)}(\boldsymbol{x})$ ($j \in \mathbb{N}$), is formally convergent to $Y(\boldsymbol{x})$.

For instance, we may compute $\sqrt{1+x} - 1$ along the above lines: consider $F(x, y) := (1 + y)^2 - (1 + x) = -x + 2y + y^2$. Then we get $Y^{(0)}(x) = 0$,

$$Y^{(1)}(x) = \frac{x}{2}, \quad Y^{(2)}(x) = \frac{x}{2} - \frac{\frac{x^2}{4}}{x+2} = \frac{x}{2} - \frac{x^2}{8} \cdot \sum_{k=0}^{\infty} \left(\frac{-x}{2}\right)^k, \quad \ldots$$

Plugging in, we get $F\left(x, Y^{(2)}(x)\right) = \frac{1}{64}x^4 +$ higher terms in x. Thus, Newton's lemma shows that

$$\sqrt{1+x} - 1 = \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \ldots$$

is correct up to degree 3.

For the multivariable non-linear cases, S. S. Abhyankar found a method (c.f. [1]).

Theorem 1.11 ([1])**.** *For the field $\mathbb{K}$ with* $\operatorname{char}(\mathbb{K}) = 0$ *and $f \in Aut(\mathbb{K}[[\boldsymbol{x}]])$, one can compute the expression of f^{-1} by*

$$f^{-1}(x_i) = \sum_{I \in \mathbb{N}^l} \left(\frac{1}{I!} \cdot \frac{\partial^{|I|}\left(x_i \cdot \det(\operatorname{Jac}(f)) \cdot \prod_{j=1}^{l}(x_j - f(x_j))\right)}{\partial \boldsymbol{x}^I}\right) \tag{3}$$

for $i = 1, 2, \ldots, l$.

To determine the inverse up to a specified order using S. S. Abhyankar's method, one must calculate higher order terms and then select the relevant lower order terms. This approach leads to redundant computations in practice.

In this article, we present two novel approaches based on higher order Jacobian matrix theory, which can be regarded as nonlinear extensions of the inverse matrix method and the Gaussian elimination method respectively. These approaches concentrate solely on terms of order lower than or equal to the specified target.

As a direct application, we use these two methods to obtain the explicit expression for the implicit function theorem.

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2. Main Results

2.1. **Preliminaries.**

We recall some essential results in higher order Jacobian matrix theory developed by S. Fan, S. S.-T. Yau, and H. Zuo.

2.1.1. *Essential Definitions in Higher Order Jacobian Matrix Theory.*

Before we go further, it is necessary to refer to ${}^{(h)}\Gamma_{(S)}^{(R)}$ which is defined in Definition 2.1. When $\mathcal{R}$ is equal to $\mathbb{C}$, ${}^{(h)}\Gamma_{(S)}^{(R)}$ is indeed $\frac{{}^{(h)}\gamma_{(S)}^{(R)}}{S!}$ in [2]. The definition ${}^{(h)}\Gamma_{(S)}^{(R)}$ is introduced to avoid situations where the denominators are zero.

Definition 2.1 ([3])**.** Let $R \in \mathbb{Z}^{l'}$, $S \in \mathbb{Z}^{l}$ and the $\mathcal{R}$-algebra homomorphism $h : \mathcal{R}\left[[\boldsymbol{x}']\right] \to \mathcal{R}\left[[\boldsymbol{x}]\right]$. We define

$$
{}^{(h)}\Gamma_{(S)}^{(R)} := \begin{cases} \sum_{\substack{K\in\mathbb{N}^{l'};\\ K\le R}} \left((-1)^{|K|}\cdot\binom{R}{K}\cdot h\left((\boldsymbol{x}')^{K}\right)\cdot\frac{\frac{\partial^{|S|}\left(h\left((\boldsymbol{x}')^{R-K}\right)\right)}{\partial \boldsymbol{x}^{S}}}{S!}\right), & R\in\mathbb{N}^{l'} \text{ and } S\in\mathbb{N}^{l},\\ 0, & \text{otherwise,}\end{cases} \tag{4}
$$

where for a_I's ($I \in \mathbb{N}^l$) in $\mathcal{R}$ we define

$$
\frac{\frac{\partial^{|S|}\left(\sum_{I\in\mathbb{N}^l} a_I \boldsymbol{x}^I\right)}{\partial \boldsymbol{x}^S}}{S!} := \sum_{\substack{I\in\mathbb{N}^l;\\ I\ge S}} \left(\binom{I}{S}\cdot a_I \boldsymbol{x}^{I-S}\right). \tag{5}
$$

In particular, ${}^{(h)}\Gamma_{(S)}^{(R)}$ is equal to 1 when $R = (0, 0, \ldots, 0)$ and $S = (0, 0, \ldots, 0)$, and equal to 0 when $R = (0, 0, \ldots, 0)$ and $|S| > 0$, or $|R| > 0$ and $S = (0, 0, \ldots, 0)$.

Also, for $S \in \mathbb{N}^l$, $1 \le i \le l'$ and $\Delta_i' := \left(\delta_{1,i}, \delta_{2,i}, \ldots, \delta_{l',i}\right)$, we have

$$
{}^{(h)}\Gamma_{(S)}^{(\Delta_i)} = \frac{\frac{\partial^{|S|}(h(x_i))}{\partial \boldsymbol{x}^S}}{S!}. \tag{6}
$$

Remark 2.2 (Taylor expansion for power series over a commutative ring)**.** Under Definition 1.8 and the notations in Definition 2.1, for $h \in End\left(\mathcal{R}\left[[\boldsymbol{x}]\right]\right)$ and $1 \le i \le l$, one can verify

$$
h\left(x_i\right) = \sum_{\substack{S\in\mathbb{N}^l;\\ |S|>0}} \left({}^{(h)}\Gamma_{(S)}^{(\Delta_i)}|_0 \cdot \boldsymbol{x}^S\right) \tag{7}
$$

by taking derivatives on both sides.

Definition 2.3 ([3])**.** Consider the $\mathcal{R}$-algebra homomorphism $h : \mathcal{R}\left[[\boldsymbol{x}']\right] \to \mathcal{R}\left[[\boldsymbol{x}]\right]$.

(1) For $n \in \mathbb{N}_+$, the matrix $\mathrm{TJac}_n\left(h\right)$ is a matrix with rows labeled by $\{\boldsymbol{a} \in \mathbb{N}^{l'} : 1 \le |\boldsymbol{a}| \le n\}$ and columns labeled by $\{\boldsymbol{b} \in \mathbb{N}^l : 1 \le |\boldsymbol{b}| \le n\}$. The $(\boldsymbol{a}, \boldsymbol{b})$-entry of $\mathrm{TJac}_n\left(h\right)$ is ${}^{(h)}\Gamma_{(\boldsymbol{b})}^{(\boldsymbol{a})}$.

(2) For $i, j \in \mathbb{N}$, the matrix ${}^{(h)}A_{i,j}$ is a matrix with rows labeled by $\{\boldsymbol{a} \in \mathbb{N}^{l'} : |\boldsymbol{a}| = i\}$ and columns labeled by $\{\boldsymbol{b} \in \mathbb{N}^l : |\boldsymbol{b}| = j\}$. The $(\boldsymbol{a}, \boldsymbol{b})$-entry of ${}^{(h)}A_{i,j}$ is ${}^{(h)}\Gamma_{(\boldsymbol{b})}^{(\boldsymbol{a})}$.

The labels are arranged by graded lexicographical order.

Remark 2.4. ${}^{(h)}A_{i,j}$ is the zero matrix in the $i > j \geq 1$ case. Therefore, $\mathrm{TJac}_n(h)$ is block upper triangular matrix with the canonical partition by ${}^{(h)}A_{i,j}$'s for arbitrary $n \in \mathbb{N}_+$ and integers $1 \leq i, j \leq n$.

2.1.2. *Essential Results in Higher Order Jacobian Matrix Theory.*

For the matrix expression of the chain rule, we have Theorem 2.5.

Theorem 2.5 ([3]). *For the $\mathcal{R}$-algebra homomorphisms $g : \mathcal{R}[[\boldsymbol{x}'']] \to \mathcal{R}[[\boldsymbol{x}']]$ and $h : \mathcal{R}[[\boldsymbol{x}']] \to \mathcal{R}[[\boldsymbol{x}]]$, under Definition 2.3 then*

$$\mathrm{TJac}_n(h \circ g) = h(\mathrm{TJac}_n(g)) \cdot \mathrm{TJac}_n(h) \tag{8}$$

for any $n \in \mathbb{N}_+$.

Theorem 2.5 is equivalent to the following Theorem 2.6.

Theorem 2.6 ([3]). *For the $\mathcal{R}$-algebra homomorphisms $g : \mathcal{R}[[\boldsymbol{x}'']] \to \mathcal{R}[[\boldsymbol{x}']]$ and $h : \mathcal{R}[[\boldsymbol{x}']] \to \mathcal{R}[[\boldsymbol{x}]]$, $I \in \mathbb{N}^{l''}$ and $J \in \mathbb{N}^l$, we have*

$${}^{(h\circ g)}\Gamma_{(J)}^{(I)} = \sum_{\substack{K\in\mathbb{N}^{l'};\\ |I|\leq|K|\leq|J|}} \left(h\left({}^{(g)}\Gamma_{(K)}^{(I)}\right) \cdot {}^{(h)}\Gamma_{(J)}^{(K)}\right)$$

under Definition 2.1.

In addition, the following Theorem 2.7 is also essential in this article.

Theorem 2.7 ([3]). *Assume that $n \in \mathbb{N}_+$. For the $\mathcal{R}$-algebra homomorphism $h : \mathcal{R}[[\boldsymbol{x}']] \to \mathcal{R}[[\boldsymbol{x}]]$, $J \in \mathbb{N}^l$ and $I, I_1, I_2, \ldots, I_n \in \mathbb{N}^{l'}$ satisfying $I = \sum_{t=1}^n I_t$, we have*

$${}^{(h)}\Gamma_{(J)}^{(I)} = \sum_{\substack{J_t\in\mathbb{N}^l,\forall 1\leq t\leq n;\\ \sum_{t=1}^n J_t=J}} \prod_{i=1}^n {}^{(h)}\Gamma_{(J_i)}^{(I_i)} \tag{9}$$

under Definition 2.1.

2.2. **Main Results in This Paper.**

By Theorem A, we can calculate the inverse of an automorphism of $\mathcal{R}[[\boldsymbol{x}]]$ by taking the inverse of the higher order Jacobian matrices. It can be regarded as a non-linear extension of the inverse matrix method.

Theorem A (Non-linear Extension of the Inverse Matrix Method). *For $f \in Aut(\mathcal{R}[[\boldsymbol{x}]])$, we give the following algorithm for finding the expression of $e := f^{-1}$.*

Under Definition 1.8 and equation (7), we only need to determine coefficients ${}^{(e)}\Gamma_{(S)}^{(\Delta_i)}|_0$'s ($i = 1, 2, \ldots, l$, $\Delta_i \in \mathbb{N}^l$, $S \in \mathbb{N}^l$ and $|S| > 0$). The coefficients are determined by the equation

$$(\mathrm{TJac}_n(e)|_0)_{(1,2,\ldots,l)}^{\left(1,2,\ldots,\binom{l+n}{l}-1\right)} = \left(((\mathrm{TJac}_n(f))|_0)^{-1}\right)_{(1,2,\ldots,l)}^{\left(1,2,\ldots,\binom{l+n}{l}-1\right)} \tag{10}$$

which holds for all integers $n \geq 2$ under Definition 1.6.

Remark 2.8. For $n \in \mathbb{N}_+$, $\mathrm{TJac}_n(f)|_0$ is invertible by the formula

$$\mathrm{TJac}_n(e)|_0 \cdot \mathrm{TJac}_n(f)|_0 = \mathrm{TJac}_n(f)|_0 \cdot \mathrm{TJac}_n(e)|_0 = I \tag{11}$$

from Theorem 2.5.

Based on Theorem 2.5, we also have a non-linear extension of Gaussian elimination method to compute of the inverse of automorphisms.

For $e, f \in Aut\left(\mathcal{R}\left[[\boldsymbol{x}]\right]\right)$ and $1 \le i, j \le l$, we have the Taylor expansion $e\left(x_i\right) = \sum_{S\in\mathbb{N}^l} {}^{(e)}\Gamma_{(S)}^{(\Delta_i)}|_0 \cdot \boldsymbol{x}^S$ and $f\left(x_j\right) = \sum_{R\in\mathbb{N}^l} {}^{(f)}\Gamma_{(R)}^{(\Delta_j)}|_0 \cdot \boldsymbol{x}^R$. Naturally, we may have the following question: if $e = f^{-1}$, how can we express ${}^{(e)}\Gamma_{(S)}^{(\Delta_i)}|_0$'s in terms of ${}^{(f)}\Gamma_{(R)}^{(\Delta_j)}|_0$'s?

In general cases, we present the following Theorem B. It can be regarded as a non-linear extension of the Gauss elimination method. The method presented in Theorem B is more efficient than that in Theorem A, since it avoids calculating the inverse of huge matrix which requires a large amount of computation.

Theorem B (Non-linear Extension of the Gauss Elimination Method)**.** *For $f \in Aut\left(\mathcal{R}\left[[\boldsymbol{x}]\right]\right)$, we give the following algorithm for finding the expression of $e := f^{-1}$.*

Under Definition 1.8 and equation (7), we only need to determine coefficients ${}^{(e)}\Gamma_{(S)}^{(\Delta_i)}|_0$'s ($i = 1, 2, \ldots, l$, $\Delta_i \in \mathbb{N}^l$, $S \in \mathbb{N}^l$ and $|S| > 0$).

By Remark 2.8, we know that $\mathrm{Jet}^{(1)}(f)$ is invertible. We introduce $g, h \in Aut\left(\mathcal{R}\left[[\boldsymbol{x}]\right]\right)$ defined by $h := \left(\mathrm{Jet}^{(1)}(f)\right)^{-1} \circ f$ and $g := h^{-1} = f^{-1} \circ \mathrm{Jet}^{(1)}(f)$ for simplicity. Note that for $S, S' \in \mathbb{N}^l$ such that $|S| = |S'|$, we have

$$
{}^{(g)}\Gamma_{(S)}^{(S')}|_0 = {}^{(h)}\Gamma_{(S)}^{(S')}|_0 = \begin{cases} 1, & S = S', \\ 0, & S \neq S' \end{cases} \tag{12}
$$

from Theorem 2.7.

The coefficients ${}^{(e)}\Gamma_{(S)}^{(\Delta_i)}|_0$'s are determined by the following algorithm.

Step 1: For $i, s = 1, 2, \ldots, l$, we calculate ${}^{(e)}\Gamma_{(\Delta_s)}^{(\Delta_i)}|_0$'s by

$$
\mathrm{Jac}\,(e)\,|_0 = \left(\mathrm{Jac}\,(f)\,|_0\right)^{-1}. \tag{13}
$$

For the non-linear parts ${}^{(e)}\Gamma_{(S)}^{(\Delta_i)}|_0$'s ($S \in \mathbb{N}^l$, $|S| \ge 2$, $i = 1, 2, \ldots, l$), we let $j = 2$ and do the following loop algorithm.

Step 2: For all $S, S' \in \mathbb{N}^l$ satisfying $|S| = |S'| = j$, we find one $1 \le k' \le l$ such that $\Delta_{k'} \le S'$ and calculate

$$
{}^{(e)}\Gamma_{(S)}^{(S')}|_0 = \sum_{\substack{1\le k\le l;\\ \Delta_k \le S;}} \left({}^{(e)}\Gamma_{(\Delta_k)}^{(\Delta_{k'})}|_0 \cdot {}^{(e)}\Gamma_{(S-\Delta_k)}^{(S'-\Delta_{k'})}|_0 \right). \tag{14}
$$

Step 3: For all $S \in \mathbb{N}^l$ satisfying $|S| = j$ and $i = 1, 2, \ldots, l$, we calculate

$$
{}^{(h)}\Gamma_{(S)}^{(\Delta_i)}|_0 = \sum_{\substack{S'\in\mathbb{N}^l;\\ |S'|=j}} \left({}^{(f)}\Gamma_{(S')}^{(\Delta_i)}|_0 \cdot {}^{(e)}\Gamma_{(S)}^{(S')}|_0 \right). \tag{15}
$$

Step 4: If $j > 2$, for all $S, S' \in \mathbb{N}^l$ satisfying $|S| = j$ and $2 \le |S'| < j$, we find one $1 \le k' \le l$ such that $\Delta_{k'} \le S'$ and calculate

$$
{}^{(h)}\Gamma_{(S)}^{(S')}|_0 = \sum_{\substack{K\in\mathbb{N}^l;\\ 1\le|K|\le j-|S'|+1;\\ K\le S}} \left({}^{(h)}\Gamma_{(K)}^{(\Delta_{k'})}|_0 \cdot {}^{(h)}\Gamma_{(S-K)}^{(S'-\Delta_{k'})}|_0 \right). \tag{16}
$$

Step 5: For all $S \in \mathbb{N}^l$ satisfying $|S| = j$ and $i = 1, 2, \ldots, l$, we calculate

$$ {}^{(g)}\Gamma_{(S)}^{(\Delta_i)}|_0 = - \sum_{\substack{S' \in \mathbb{N}^l; \\ 1 \le |S'| < j}} \left({}^{(g)}\Gamma_{(S')}^{(\Delta_i)}|_0 \cdot {}^{(h)}\Gamma_{(S)}^{(S')}|_0 \right). \tag{17} $$

Step 6: For all $S \in \mathbb{N}^l$ satisfying $|S| = j$ and $i = 1, 2, \ldots, l$, we calculate

$$ {}^{(e)}\Gamma_{(S)}^{(\Delta_i)}|_0 = \sum_{k=1}^{l} \left({}^{(e)}\Gamma_{(\Delta_k)}^{(\Delta_i)}|_0 \cdot {}^{(g)}\Gamma_{(S)}^{(\Delta_k)}|_0 \right). \tag{18} $$

Step 7: We increase the value of j by 1, then return to Step 2.

Remark 2.9. From Theorems A and B, we can also know that for $f \in End\left(\mathcal{R}\left[[\boldsymbol{x}]\right]\right)$, $f \in Aut\left(\mathcal{R}\left[[\boldsymbol{x}]\right]\right)$ if and only if $\det\left(\mathrm{Jac}\left(f\right)|_0\right)$ is invertible in $\mathcal{R}$.

In fact, if $f \in Aut\left(\mathcal{R}\left[[\boldsymbol{x}]\right]\right)$, we know

$$ \det\left(\mathrm{Jac}\left(f\right)|_0\right) \cdot \det\left(\mathrm{Jac}\left(f^{-1}\right)|_0\right) = \det\left(\mathrm{Jac}\left(f^{-1}\right)|_0\right) \cdot \det\left(\mathrm{Jac}\left(f\right)|_0\right) = 1, \tag{19} $$

which implies $\det\left(\mathrm{Jac}\left(f\right)|_0\right)$ is invertible in $\mathcal{R}$.

If $\det\left(\mathrm{Jac}\left(f\right)|_0\right)$ is invertible in $\mathcal{R}$, we can obtain the expression of e by the same algorithm as Equations (12) - (18) in Theorem B. In fact, by Theorems 2.6 and 2.7 we have

$$ \begin{cases} \mathrm{Jet}^{(1)}\left(e\right) = \left(\mathrm{Jet}^{(1)}\left(f\right)\right)^{-1}, \\ h = \mathrm{Jet}^{(1)}\left(e\right) \circ f, \\ h \circ g = id, \\ e = g \circ \mathrm{Jet}^{(1)}\left(e\right). \end{cases} \tag{20} $$

It follows that $f \circ e = id$, which implies that equation

$$ \mathrm{TJac}_n\left(e\right)|_0 \cdot \mathrm{TJac}_n\left(f\right)|_0 = I \tag{21} $$

holds for all $n \in \mathbb{N}_+$. We also note that equation (21) is equivalent to the equation

$$ \mathrm{TJac}_n\left(f\right)|_0 \cdot \mathrm{TJac}_n\left(e\right)|_0 = I. \tag{22} $$

Therefore $e \circ f = id$ and f is in $Aut\left(\mathcal{R}\left[[\boldsymbol{x}]\right]\right)$.

We give an example for Theorems A and B in Appendix A. Also, for more complicated cases, we give the MATLAB script of Theorems A and B. One can get the expression of the inverse just by the computer. The complete program list can be found at `https://cloud.tsinghua.edu.cn/d/30647ab11d4848d78dfc/`.

2.3. **Application: Finding the Explicit Expression for the Implicit Function Theorem.**

We give an application of Theorems A and B: finding the explicit expression for the implicit function theorem. We also extend the application to the formal power series rings over an arbitrary field $\mathbb{K}$ case.

In Theorem D, we will give a new algorithm using the novel higher order Jacobian matrix theory. Theorem D can be reduced to Theorem C in special cases.

We show Theorem C first.

Theorem C. *Assume that $l \geq l'$ and the given homomorphism $f : \mathbb{K}[[\boldsymbol{x}']] \to \mathbb{K}[[\boldsymbol{x}]]$ satisfies*

$$\det \begin{pmatrix} \frac{\partial f(x_1)}{\partial x_1} & \frac{\partial f(x_1)}{\partial x_2} & \cdots & \frac{\partial f(x_1)}{\partial x_{l'}} \\ \frac{\partial f(x_2)}{\partial x_1} & \frac{\partial f(x_2)}{\partial x_2} & \cdots & \frac{\partial f(x_2)}{\partial x_{l'}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(x_{l'})}{\partial x_1} & \frac{\partial f(x_{l'})}{\partial x_2} & \cdots & \frac{\partial f(x_{l'})}{\partial x_{l'}} \end{pmatrix} |_0 \neq 0$$

where $\mathbb{K}$ is a field. We define $\boldsymbol{y}' := (y_1, y_2, \ldots, y_{l'})$. For the system of equations

$$f(x_i) = y_i$$

where $i = 1, 2, \ldots, l'$, there exists a $\mathbb{K}$-algebra homomorphism

$$g : \mathbb{K}\left[\left[\boldsymbol{y}'\right]\right] \to \mathbb{K}\left[\left[\boldsymbol{y}', x_{l'+1}, x_{l'+2}, \ldots, x_l\right]\right]$$

such that

$$g(y_i) = x_i$$

where $i = 1, 2, \ldots, l'$.

Consider $f_0 \in End(\mathbb{K}[[\boldsymbol{x}]])$ defined by

$$f_0(x_i) := \begin{cases} f(x_i), & i = 1, 2, \ldots, l', \\ x_i, & i = l'+1, l'+2, \ldots, l. \end{cases} \tag{23}$$

It is easy to see that $f_0 \in Aut(\mathbb{K}[[\boldsymbol{x}]])$. One can determine the coefficients of $g(y_i) = f_0^{-1}(y_i)$'s ($i = 1, 2, \ldots, l'$) in terms of variables $y_1, y_2, \ldots, y_{l'}, x_{l'+1}, x_{l'+2}, \ldots, x_l$ by either Theorem A or B.

Especially, it is also correct in the case of convergent power series rings over $\mathbb{C}$.

We give an example for Theorem C in Appendix A.

For general cases, we have Theorem D.

Theorem D. *Assume that $l \geq l'$ and the homomorphism $f : \mathbb{K}[[\boldsymbol{x}']] \to \mathbb{K}[[\boldsymbol{x}]]$ satisfies*

$$\det \begin{pmatrix} \frac{\partial f(x_1)}{\partial x_1} & \frac{\partial f(x_1)}{\partial x_2} & \cdots & \frac{\partial f(x_1)}{\partial x_{l'}} \\ \frac{\partial f(x_2)}{\partial x_1} & \frac{\partial f(x_2)}{\partial x_2} & \cdots & \frac{\partial f(x_2)}{\partial x_{l'}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(x_{l'})}{\partial x_1} & \frac{\partial f(x_{l'})}{\partial x_2} & \cdots & \frac{\partial f(x_{l'})}{\partial x_{l'}} \end{pmatrix} |_0 \neq 0$$

where $\mathbb{K}$ is a field. We define $\boldsymbol{y}' := (y_1, y_2, \ldots, y_{l'})$. For the system of equations (System A)

$$f(x_i) = 0$$

where $i = 1, 2, \ldots, l'$, there always exists a $\mathbb{K}$-algebra homomorphism

$$h : \mathbb{K}\left[\left[\boldsymbol{x}'\right]\right] \to \mathbb{K}[[x_{l'+1}, x_{l'+2}, \ldots, x_l]]$$

such that

$$h(x_i) = x_i$$

for all $i = 1, 2, \ldots, l'$.

To obtain the coefficients of h, we may introduce another system of equations (System B)

$$f(x_i) = y_i$$

where $i = 1, 2, \ldots, l'$. There exists a $\mathbb{K}$-algebra homomorphism

$$g : \mathbb{K}\left[\left[\boldsymbol{y}'\right]\right] \to \mathbb{K}\left[\left[\boldsymbol{y}', x_{l'+1}, x_{l'+2}, \ldots, x_l\right]\right]$$

such that

$$g\left(y_i\right)=x_i$$

where $i=1,2,\ldots,l'$ *in System B. The coefficients of* $g\left(y_i\right)$*'s* ($i=1,2,\ldots,l'$) *in terms of variables* $y_1,y_2,\ \ldots,y_{l'},x_{l'+1},x_{l'+2},\ldots,x_l$ *are determined by Theorem C.*

Then in System A, for any $i=1,2,\ldots,l'$*, the coefficients of* $h\left(x_i\right)$*'s* ($i=1,2,\ldots,l'$) *in terms of variables* $x_{l'+1},x_{l'+2},\ldots,x_l$ *are determined by the equation*

$$h\left(x_i\right)=\left(g\left(y_i\right)\right)|_{\left(\boldsymbol{y}',x_{l'+1},x_{l'+2},\ldots,x_l\right)=\left(0,0,\ldots,0,x_{l'+1},x_{l'+2},\ldots,x_l\right)}.$$

Especially, it is also correct in the case of convergent power series rings over $\mathbb{C}$.

We give an example for Theorem D in Appendix A.

3. Proof of Theorems A - D

In this section, we will prove Theorems A - D.

Proof of Theorem A. It follows immediately from equation (11). □

For the proof of Theorem B, we need the following lemma.

Lemma 3.1. *Assume that* $g,h\in Aut\left(\mathcal{R}\left[\left[\boldsymbol{x}\right]\right]\right)$ *satisfy* $g=h^{-1}$ *and* $\mathrm{Jac}\left(h\right)|_0=I$. *For* $S\in\mathbb{N}^l$ *satisfying* $|S|\geq 2$, *and integer* $1\leq i\leq l$, *we have*

$$^{(g)}\Gamma^{(\Delta_i)}_{(S)}|_0=-\sum_{\substack{S'\in\mathbb{N}^l;\\ 1\leq|S'|<|S|}}\left({}^{(g)}\Gamma^{(\Delta_i)}_{(S')}|_0\cdot{}^{(h)}\Gamma^{(S')}_{(S)}|_0\right). \tag{24}$$

Proof. From Theorem 2.6, we have

$$\sum_{\substack{S'\in\mathbb{N}^l;\\ 1\leq|S'|\leq|S|}}\left({}^{(g)}\Gamma^{(\Delta_i)}_{(S')}|_0\cdot{}^{(h)}\Gamma^{(S')}_{(S)}|_0\right)=0. \tag{25}$$

From Theorem 2.7, for $S,S'\in\mathbb{N}^l$ such that $|S|=|S'|$, we have

$$^{(h)}\Gamma^{(S')}_{(S)}|_0=\begin{cases}1, & S=S',\\ 0, & S\neq S'.\end{cases}$$

Therefore, we prove equation (24). □

Proof of Theorem B. It follows immediately from Theorems 2.6 and 2.7, Lemma 3.1, and the fact that

$$\mathrm{TJac}_j\left(\phi^{-1}\right)|_0=\left(\mathrm{TJac}_j\left(\phi\right)|_0\right)^{-1} \tag{26}$$

for all $\phi\in Aut\left(\mathcal{R}\left[\left[\boldsymbol{x}\right]\right]\right)$ and $j\in\mathbb{N}_+$, and the relations

$$\begin{cases}h=\left(\mathrm{Jet}^{(1)}\left(f\right)\right)^{-1}\circ f=\left(\mathrm{Jet}^{(1)}\left(e\right)\right)\circ f,\\ g=h^{-1},\\ e=h^{-1}\circ\left(\mathrm{Jet}^{(1)}\left(f\right)\right)^{-1}=g\circ\left(\mathrm{Jet}^{(1)}\left(e\right)\right).\end{cases} \tag{27}$$

□

Proof of Theorem C. By direct calculation,

$$\det\begin{pmatrix}\frac{\partial f_0(x_1)}{\partial x_1} & \frac{\partial f_0(x_1)}{\partial x_2} & \cdots & \frac{\partial f_0(x_1)}{\partial x_l}\\ \frac{\partial f_0(x_2)}{\partial x_1} & \frac{\partial f_0(x_2)}{\partial x_2} & \cdots & \frac{\partial f_0(x_2)}{\partial x_l}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial f_0(x_l)}{\partial x_1} & \frac{\partial f_0(x_l)}{\partial x_2} & \cdots & \frac{\partial f_0(x_l)}{\partial x_l}\end{pmatrix}|_0 = \det\begin{pmatrix}\frac{\partial f(x_1)}{\partial x_1} & \frac{\partial f(x_1)}{\partial x_2} & \cdots & \frac{\partial f(x_1)}{\partial x_{l'}}\\ \frac{\partial f(x_2)}{\partial x_1} & \frac{\partial f(x_2)}{\partial x_2} & \cdots & \frac{\partial f(x_2)}{\partial x_{l'}}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial f(x_{l'})}{\partial x_1} & \frac{\partial f(x_{l'})}{\partial x_2} & \cdots & \frac{\partial f(x_{l'})}{\partial x_{l'}}\end{pmatrix}|_0 \neq 0.$$

Therefore, $f_0 \in Aut\left(\mathbb{K}\left[[\boldsymbol{x}]\right]\right)$.

Consider the $\mathbb{K}$-algebra homomorphism

$$g_0 : \mathbb{K}\left[\left[\boldsymbol{y}', x_{l'+1}, x_{l'+2}, \dots, x_l\right]\right] \to \mathbb{K}\left[[\boldsymbol{x}]\right]$$

defined by

$$\begin{cases} g_0\left(y_i\right) = x_i, & i = 1, 2, \dots, l',\\ g_0\left(x_i\right) = x_i, & i = l'+1, l'+2, \dots, l.\end{cases} \tag{28}$$

It follows that $g_0 = f_0^{-1}$ and we can compute $g\left(y_i\right) = g_0\left(y_i\right) = f_0^{-1}\left(y_i\right)$'s $(i = 1, 2, \dots, l')$ in terms of variables $y_1, y_2, \dots, y_{l'}, x_{l'+1}, x_{l'+2}, \dots, x_l$ by either Theorem A or B.

Proof in the convergent power series rings over $\mathbb{C}$ case is similar. □

Proof of Theorem D. By Theorem C, it is clear. □

Appendix A. Examples of Theorems A - D

We show examples of Theorems A - D here to better illustrate the algorithms.

Example A.1 (Example of Theorems A and B)**.** Consider $f \in Aut\left(\mathbb{C}\left\{x_1, x_2\right\}\right)$ satisfying

$$\begin{cases} f\left(x_1\right) = x_1 + x_2 + x_2^2,\\ f\left(x_2\right) = x_1 + 2x_2 + x_1^3.\end{cases}$$

We want to find $e := f^{-1}$. Under Definition 1.8 and equation (7), we only need to determine ${}^{(e)}\Gamma_{(S)}^{(\Delta_i)}|_0$'s ($i = 1, 2$, $\Delta_i \in \mathbb{N}^2$, $S \in \mathbb{N}^2$ and $|S| > 0$) using either Theorem A or B.

With Theorem A method, we get

$$\left(\mathrm{Jac}\left(f\right)\right)|_0 = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix},$$

$$\left(\mathrm{TJac}_2\left(f\right)\right)|_0 = \begin{bmatrix}1 & 1 & 0 & 0 & 1\\ 1 & 2 & 0 & 0 & 0\\ 0 & 0 & 1 & 2 & 1\\ 0 & 0 & 1 & 3 & 2\\ 0 & 0 & 1 & 4 & 4\end{bmatrix},$$

$$\left(\mathrm{TJac}_3\left(f\right)\right)|_0 = \begin{bmatrix}1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 2 & 2\\ 0 & 0 & 1 & 3 & 2 & 0 & 0 & 1 & 2\\ 0 & 0 & 1 & 4 & 4 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1\\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 5 & 2\\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & 8 & 4\\ 0 & 0 & 0 & 0 & 0 & 1 & 6 & 12 & 8\end{bmatrix},$$

$$\cdots$$

It follows that

$$\mathrm{Jac}\,(e)\,|_0 = ((\mathrm{Jac}\,(f))\,|_0)^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix},$$

$$\mathrm{TJac}_2\,(e)\,|_0 = ((\mathrm{TJac}_2\,(f))\,|_0)^{-1} = \begin{bmatrix} 2 & -1 & -2 & 4 & -2 \\ -1 & 1 & 1 & -2 & 1 \\ 0 & 0 & 4 & -4 & 1 \\ 0 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix},$$

$$\mathrm{TJac}_3\,(e)\,|_0 = ((\mathrm{TJac}_3\,(f))\,|_0)^{-1} = \begin{bmatrix} 2 & -1 & -2 & 4 & -2 & 12 & -24 & 18 & -5 \\ -1 & 1 & 1 & -2 & 1 & -10 & 18 & -12 & 3 \\ 0 & 0 & 4 & -4 & 1 & -8 & 20 & -16 & 4 \\ 0 & 0 & -2 & 3 & -1 & 4 & -11 & 10 & -3 \\ 0 & 0 & 1 & -2 & 1 & -2 & 6 & -6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 8 & -12 & 6 & -1 \\ 0 & 0 & 0 & 0 & 0 & -4 & 8 & -5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & -5 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 & -3 & 1 \end{bmatrix}.$$

$$\cdots$$

From the first two rows of the matrices above, we know

$$\begin{bmatrix} {}^{(e)}\Gamma^{(1,0)}_{(1,0)}|_0 & {}^{(e)}\Gamma^{(1,0)}_{(0,1)}|_0 \\ {}^{(e)}\Gamma^{(0,1)}_{(1,0)}|_0 & {}^{(e)}\Gamma^{(0,1)}_{(0,1)}|_0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} {}^{(e)}\Gamma^{(1,0)}_{(2,0)}|_0 & {}^{(e)}\Gamma^{(1,0)}_{(1,1)}|_0 & {}^{(e)}\Gamma^{(1,0)}_{(0,2)}|_0 \\ {}^{(e)}\Gamma^{(0,1)}_{(2,0)}|_0 & {}^{(e)}\Gamma^{(0,1)}_{(1,1)}|_0 & {}^{(e)}\Gamma^{(0,1)}_{(0,2)}|_0 \end{bmatrix} = \begin{bmatrix} -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix},$$

$$\begin{bmatrix} {}^{(e)}\Gamma^{(1,0)}_{(3,0)}|_0 & {}^{(e)}\Gamma^{(1,0)}_{(2,1)}|_0 & {}^{(e)}\Gamma^{(1,0)}_{(1,2)}|_0 & {}^{(e)}\Gamma^{(1,0)}_{(0,3)}|_0 \\ {}^{(e)}\Gamma^{(0,1)}_{(3,0)}|_0 & {}^{(e)}\Gamma^{(0,1)}_{(2,1)}|_0 & {}^{(e)}\Gamma^{(0,1)}_{(1,2)}|_0 & {}^{(e)}\Gamma^{(0,1)}_{(0,3)}|_0 \end{bmatrix} = \begin{bmatrix} 12 & -24 & 18 & -5 \\ -10 & 18 & -12 & 3 \end{bmatrix},$$

$$\cdots$$

One can also obtain ${}^{(e)}\Gamma^{(\Delta_i)}_{(S)}|_0$'s with Theorem B method:

Step 1: The following results are obtained by equation (13):

$$\begin{bmatrix} {}^{(e)}\Gamma^{(1,0)}_{(1,0)}|_0 & {}^{(e)}\Gamma^{(1,0)}_{(0,1)}|_0 \\ {}^{(e)}\Gamma^{(0,1)}_{(1,0)}|_0 & {}^{(e)}\Gamma^{(0,1)}_{(0,1)}|_0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

We let $j = 2$.

Step 2: The following results are obtained by equation (14):

$$\begin{bmatrix} {}^{(e)}\Gamma^{(2,0)}_{(2,0)}|_0 & {}^{(e)}\Gamma^{(2,0)}_{(1,1)}|_0 & {}^{(e)}\Gamma^{(2,0)}_{(0,2)}|_0 \\ {}^{(e)}\Gamma^{(1,1)}_{(2,0)}|_0 & {}^{(e)}\Gamma^{(1,1)}_{(1,1)}|_0 & {}^{(e)}\Gamma^{(1,1)}_{(0,2)}|_0 \\ {}^{(e)}\Gamma^{(0,2)}_{(2,0)}|_0 & {}^{(e)}\Gamma^{(0,2)}_{(1,1)}|_0 & {}^{(e)}\Gamma^{(0,2)}_{(0,2)}|_0 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 1 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix}.$$

Step 3: The following results are obtained by equation (15):

$$\begin{bmatrix} {}^{(h)}\Gamma^{(1,0)}_{(2,0)}|_0 & {}^{(h)}\Gamma^{(1,0)}_{(1,1)}|_0 & {}^{(h)}\Gamma^{(1,0)}_{(0,2)}|_0 \\ {}^{(h)}\Gamma^{(0,1)}_{(2,0)}|_0 & {}^{(h)}\Gamma^{(0,1)}_{(1,1)}|_0 & {}^{(h)}\Gamma^{(0,1)}_{(0,2)}|_0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Step 4: The condition $j > 2$ does not hold. We skip this step.

Step 5: The following results are obtained by equation (17):

$$\begin{bmatrix} {}^{(g)}\Gamma_{(2,0)}^{(1,0)}|_0 & {}^{(g)}\Gamma_{(1,1)}^{(1,0)}|_0 & {}^{(g)}\Gamma_{(0,2)}^{(1,0)}|_0 \\ {}^{(g)}\Gamma_{(2,0)}^{(0,1)}|_0 & {}^{(g)}\Gamma_{(1,1)}^{(0,1)}|_0 & {}^{(g)}\Gamma_{(0,2)}^{(0,1)}|_0 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Step 6: The following results are obtained by equation (18):

$$\begin{bmatrix} {}^{(e)}\Gamma_{(2,0)}^{(1,0)}|_0 & {}^{(e)}\Gamma_{(1,1)}^{(1,0)}|_0 & {}^{(e)}\Gamma_{(0,2)}^{(1,0)}|_0 \\ {}^{(e)}\Gamma_{(2,0)}^{(0,1)}|_0 & {}^{(e)}\Gamma_{(1,1)}^{(0,1)}|_0 & {}^{(e)}\Gamma_{(0,2)}^{(0,1)}|_0 \end{bmatrix} = \begin{bmatrix} -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}.$$

Step 7: We let $j = 3$.

Then we return to Step 2.

Step 2: The following results are obtained by equation (14):

$$\begin{bmatrix} {}^{(e)}\Gamma_{(3,0)}^{(3,0)}|_0 & {}^{(e)}\Gamma_{(2,1)}^{(3,0)}|_0 & {}^{(e)}\Gamma_{(1,2)}^{(3,0)}|_0 & {}^{(e)}\Gamma_{(0,3)}^{(3,0)}|_0 \\ {}^{(e)}\Gamma_{(3,0)}^{(2,1)}|_0 & {}^{(e)}\Gamma_{(2,1)}^{(2,1)}|_0 & {}^{(e)}\Gamma_{(1,2)}^{(2,1)}|_0 & {}^{(e)}\Gamma_{(0,3)}^{(2,1)}|_0 \\ {}^{(e)}\Gamma_{(3,0)}^{(1,2)}|_0 & {}^{(e)}\Gamma_{(2,1)}^{(1,2)}|_0 & {}^{(e)}\Gamma_{(1,2)}^{(1,2)}|_0 & {}^{(e)}\Gamma_{(0,3)}^{(1,2)}|_0 \\ {}^{(e)}\Gamma_{(3,0)}^{(0,3)}|_0 & {}^{(e)}\Gamma_{(2,1)}^{(0,3)}|_0 & {}^{(e)}\Gamma_{(1,2)}^{(0,3)}|_0 & {}^{(e)}\Gamma_{(0,3)}^{(0,3)}|_0 \end{bmatrix} = \begin{bmatrix} 8 & -12 & 6 & -1 \\ -4 & 8 & -5 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 3 & -3 & 1 \end{bmatrix}.$$

Step 3: The following results are obtained by equation (15):

$$\begin{bmatrix} {}^{(h)}\Gamma_{(3,0)}^{(1,0)}|_0 & {}^{(h)}\Gamma_{(2,1)}^{(1,0)}|_0 & {}^{(h)}\Gamma_{(1,2)}^{(1,0)}|_0 & {}^{(h)}\Gamma_{(0,3)}^{(1,0)}|_0 \\ {}^{(h)}\Gamma_{(3,0)}^{(0,1)}|_0 & {}^{(h)}\Gamma_{(2,1)}^{(0,1)}|_0 & {}^{(h)}\Gamma_{(1,2)}^{(0,1)}|_0 & {}^{(h)}\Gamma_{(0,3)}^{(0,1)}|_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 8 & -12 & 6 & -1 \end{bmatrix}.$$

Step 4: The following results are obtained by equation (16):

$$\begin{bmatrix} {}^{(h)}\Gamma_{(3,0)}^{(2,0)}|_0 & {}^{(h)}\Gamma_{(2,1)}^{(2,0)}|_0 & {}^{(h)}\Gamma_{(1,2)}^{(2,0)}|_0 & {}^{(h)}\Gamma_{(0,3)}^{(2,0)}|_0 \\ {}^{(h)}\Gamma_{(3,0)}^{(1,1)}|_0 & {}^{(h)}\Gamma_{(2,1)}^{(1,1)}|_0 & {}^{(h)}\Gamma_{(1,2)}^{(1,1)}|_0 & {}^{(h)}\Gamma_{(0,3)}^{(1,1)}|_0 \\ {}^{(h)}\Gamma_{(3,0)}^{(0,2)}|_0 & {}^{(h)}\Gamma_{(2,1)}^{(0,2)}|_0 & {}^{(h)}\Gamma_{(1,2)}^{(0,2)}|_0 & {}^{(h)}\Gamma_{(0,3)}^{(0,2)}|_0 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 2 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Step 5: The following results are obtained by equation (17):

$$\begin{bmatrix} {}^{(g)}\Gamma_{(3,0)}^{(1,0)}|_0 & {}^{(g)}\Gamma_{(2,1)}^{(1,0)}|_0 & {}^{(g)}\Gamma_{(1,2)}^{(1,0)}|_0 & {}^{(g)}\Gamma_{(0,3)}^{(1,0)}|_0 \\ {}^{(g)}\Gamma_{(3,0)}^{(0,1)}|_0 & {}^{(g)}\Gamma_{(2,1)}^{(0,1)}|_0 & {}^{(g)}\Gamma_{(1,2)}^{(0,1)}|_0 & {}^{(g)}\Gamma_{(0,3)}^{(0,1)}|_0 \end{bmatrix} = \begin{bmatrix} 2 & -6 & 6 & -2 \\ -8 & 12 & -6 & 1 \end{bmatrix}.$$

Step 6: The following results are obtained by equation (18):

$$\begin{bmatrix} {}^{(e)}\Gamma_{(3,0)}^{(1,0)}|_0 & {}^{(e)}\Gamma_{(2,1)}^{(1,0)}|_0 & {}^{(e)}\Gamma_{(1,2)}^{(1,0)}|_0 & {}^{(e)}\Gamma_{(0,3)}^{(1,0)}|_0 \\ {}^{(e)}\Gamma_{(3,0)}^{(0,1)}|_0 & {}^{(e)}\Gamma_{(2,1)}^{(0,1)}|_0 & {}^{(e)}\Gamma_{(1,2)}^{(0,1)}|_0 & {}^{(e)}\Gamma_{(0,3)}^{(0,1)}|_0 \end{bmatrix} = \begin{bmatrix} 12 & -24 & 18 & -5 \\ -10 & 18 & -12 & 3 \end{bmatrix}.$$

Step 7: We let $j = 4$.

Then we may return to Step 2 to calculate coefficients of higher order terms.

From either of these two methods, we obtain the expression of $e = f^{-1}$ by equation (7):

$$\begin{cases} \mathrm{Jet}^{(1)}\left(e\left(x_1\right)\right) = 2x_1 - x_2, \\ \mathrm{Jet}^{(1)}\left(e\left(x_2\right)\right) = -x_1 + x_2, \end{cases}$$

$$\begin{cases} \mathrm{Jet}^{(2)}\left(e\left(x_1\right)\right) = 2x_1 - x_2 - 2x_1^2 + 4x_1x_2 - 2x_2^2, \\ \mathrm{Jet}^{(2)}\left(e\left(x_2\right)\right) = -x_1 + x_2 + x_1^2 - 2x_1x_2 + x_2^2, \end{cases}$$

$$\begin{cases}\mathrm{Jet}^{(3)}\left(e\left(x_1\right)\right)=2x_1-x_2-2x_1^2+4x_1x_2-2x_2^2+12x_1^3-24x_1^2x_2+18x_1x_2^2-5x_2^3,\\ \mathrm{Jet}^{(3)}\left(e\left(x_2\right)\right)=-x_1+x_2+x_1^2-2x_1x_2+x_2^2-10x_1^3+18x_1^2x_2-12x_1x_2^2+3x_2^3,\end{cases}$$

$$\cdots$$

Example A.2 (Example of Theorem C)**.** We follow the notations in Theorem C. Consider the system of non-linear equations with coefficients in $\mathbb{C}$:

$$\begin{cases}x_1+x_2+x_1x_3+x_2^3=y_1,\\ x_2+x_1^3=y_2.\end{cases}$$

Can we express x_1 and x_2 in terms of variables y_1, y_2 and x_3? If yes, what are the expressions?

In this case, we define the $\mathbb{C}$-algebra homomorphism $f:\mathbb{C}\{x_1,x_2\}\to\mathbb{C}\{x_1,x_2,x_3\}$ by setting

$$\begin{cases}f\left(x_1\right):=x_1+x_2+x_1x_3+x_2^3,\\ f\left(x_2\right):=x_2+x_1^3,\end{cases}$$

and know

$$\det\begin{pmatrix}\frac{\partial f(x_1)}{\partial x_1} & \frac{\partial f(x_1)}{\partial x_2}\\ \frac{\partial f(x_2)}{\partial x_1} & \frac{\partial f(x_2)}{\partial x_2}\end{pmatrix}|_0=\det\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}|_0=1\neq 0.$$

Therefore we can find x_1 and x_2 in terms of y_1, y_2 and x_3.

There exists a homomorphism $g:\mathbb{C}\{y_1,y_2\}\to\mathbb{C}\{y_1,y_2,x_3\}$ such that

$$\begin{cases}g\left(y_1\right)=x_1,\\ g\left(y_2\right)=x_2.\end{cases}$$

In the following part, we calculate the coefficients of $g\left(y_1\right)$ and $g\left(y_2\right)$ in terms of variables y_1, y_2 and x_3.

We define $f_0\in Aut\left(\mathbb{C}\{x_1,x_2,x_3\}\right)$ by setting

$$\begin{cases}f_0\left(x_1\right):=x_1+x_2+x_1x_3+x_2^3,\\ f_0\left(x_2\right):=x_2+x_1^3,\\ f_0\left(x_3\right):=x_3.\end{cases}$$

In this example, $\mathrm{Jet}^{(k)}$ operates on the power series in terms of variables y_1, y_2 and x_3. By either Theorem A or B, it follows that

$$\begin{cases}\mathrm{Jet}^{(1)}\left(f_0^{-1}\left(y_1\right)\right)=y_1-y_2,\\ \mathrm{Jet}^{(1)}\left(f_0^{-1}\left(y_2\right)\right)=y_2,\\ \mathrm{Jet}^{(1)}\left(f_0^{-1}\left(x_3\right)\right)=x_3,\end{cases}$$

$$\begin{cases}\mathrm{Jet}^{(2)}\left(f_0^{-1}\left(y_1\right)\right)=y_1-y_2-y_1x_3+y_2x_3,\\ \mathrm{Jet}^{(2)}\left(f_0^{-1}\left(y_2\right)\right)=y_2,\\ \mathrm{Jet}^{(2)}\left(f_0^{-1}\left(x_3\right)\right)=x_3,\end{cases}$$

$$\begin{cases}\mathrm{Jet}^{(3)}\left(f_0^{-1}\left(y_1\right)\right)=y_1-y_2-y_1x_3+y_2x_3+y_1^3-3y_1^2y_2+3y_1y_2^2+y_1x_3^2-2y_2^3-y_2x_3^2,\\ \mathrm{Jet}^{(3)}\left(f_0^{-1}\left(y_2\right)\right)=y_2-y_1^3+3y_1^2y_2-3y_1y_2^2+y_2^3,\\ \mathrm{Jet}^{(3)}\left(f_0^{-1}\left(x_3\right)\right)=x_3,\end{cases}$$

$$\cdots$$

Note that $g(y_i) = f_0^{-1}(y_i)$ for $i = 1, 2$. Therefore, we have

$$\begin{cases} \mathrm{Jet}^{(1)}(x_1) = \mathrm{Jet}^{(1)}(g(y_1)) = y_1 - y_2, \\ \mathrm{Jet}^{(1)}(x_2) = \mathrm{Jet}^{(1)}(g(y_2)) = y_2, \end{cases}$$

$$\begin{cases} \mathrm{Jet}^{(2)}(x_1) = \mathrm{Jet}^{(2)}(g(y_1)) = y_1 - y_2 - y_1 x_3 + y_2 x_3, \\ \mathrm{Jet}^{(2)}(x_2) = \mathrm{Jet}^{(2)}(g(y_2)) = y_2, \end{cases}$$

$$\begin{cases} \mathrm{Jet}^{(3)}(x_1) = \mathrm{Jet}^{(3)}(g(y_1)) = y_1 - y_2 - y_1 x_3 + y_2 x_3 + y_1^3 - 3y_1^2 y_2 + 3y_1 y_2^2 + y_1 x_3^2 - 2y_2^3 - y_2 x_3^2, \\ \mathrm{Jet}^{(3)}(x_2) = \mathrm{Jet}^{(3)}(g(y_2)) = y_2 - y_1^3 + 3y_1^2 y_2 - 3y_1 y_2^2 + y_2^3, \end{cases}$$

$$\cdots$$

Example A.3 (Example of Theorem D)**.** We follow the notations in Theorem D. Consider the system of non-linear equations (System A) with coefficients in $\mathbb{C}$:

$$\begin{cases} \sin(x_1 + x_3) + e^{x_2 + x_3^2} - 1 = 0, \\ e^{2x_1 + x_2^2} + \tan\left(-x_3 + x_1^3\right) - 1 = 0. \end{cases}$$

Can we express x_1 and x_2 in terms of variable x_3? If yes, what are the expressions?

In this case, we add two variables y_1, y_2 and construct the system of non-linear equations (System B):

$$\begin{cases} \sin(x_1 + x_3) + e^{x_2 + x_3^2} - 1 = y_1, \\ e^{2x_1 + x_2^2} + \tan\left(-x_3 + x_1^3\right) - 1 = y_2. \end{cases}$$

In System B, we define the $\mathbb{C}$-algebra homomorphism $f : \mathbb{C}\{x_1, x_2\} \to \mathbb{C}\{x_1, x_2, x_3\}$ by setting

$$\begin{cases} f(x_1) := \sin(x_1 + x_3) + e^{x_2 + x_3^2} - 1, \\ f(x_2) := e^{2x_1 + x_2^2} + \tan\left(-x_3 + x_1^3\right) - 1, \end{cases}$$

and know

$$\det \begin{pmatrix} \frac{\partial f(x_1)}{\partial x_1} & \frac{\partial f(x_1)}{\partial x_2} \\ \frac{\partial f(x_2)}{\partial x_1} & \frac{\partial f(x_2)}{\partial x_2} \end{pmatrix} |_0 = \det \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} |_0 = -2 \neq 0.$$

Therefore we can express x_1 and x_2 in terms of y_1, y_2 and x_3 in System B, which implies that we can express x_1 and x_2 in terms of x_3 in System A.

There exists a homomorphism $g : \mathbb{C}\{y_1, y_2\} \to \mathbb{C}\{y_1, y_2, x_3\}$ such that

$$\begin{cases} g(y_1) = x_1, \\ g(y_2) = x_2. \end{cases}$$

In the following part, we calculate the coefficients of $g(y_1)$ and $g(y_2)$ in terms of variables y_1, y_2, x_3.

We define $f_0 \in Aut(\mathbb{C}\{x_1, x_2, x_3\})$ by setting

$$\begin{cases} f_0(x_1) := \sin(x_1 + x_3) + e^{x_2 + x_3^2} - 1, \\ f_0(x_2) := e^{2x_1 + x_2^2} + \tan\left(-x_3 + x_1^3\right) - 1, \\ f_0(x_3) := x_3. \end{cases}$$

In System B, $\mathrm{Jet}^{(k)}$ operates on the power series in terms of variables y_1, y_2 and x_3. By either Theorem A or B, it follows that

$$\begin{cases}\mathrm{Jet}^{(1)}\left(f_0^{-1}(y_1)\right) = \frac{1}{2}y_2 + \frac{1}{2}x_3,\\ \mathrm{Jet}^{(1)}\left(f_0^{-1}(y_2)\right) = y_1 - \frac{1}{2}y_2 - \frac{3}{2}x_3,\\ \mathrm{Jet}^{(1)}\left(f_0^{-1}(x_3)\right) = x_3,\end{cases}$$

$$\begin{cases}\mathrm{Jet}^{(2)}\left(f_0^{-1}(y_1)\right) = \frac{1}{2}y_2 + \frac{1}{2}x_3 - \frac{1}{2}y_1^2 + \frac{1}{2}y_1y_2 + \frac{3}{2}y_1x_3 - \frac{3}{8}y_2^2 - \frac{5}{4}y_2x_3 - \frac{11}{8}x_3^2,\\ \mathrm{Jet}^{(2)}\left(f_0^{-1}(y_2)\right) = y_1 - \frac{1}{2}y_2 - \frac{3}{2}x_3 + \frac{1}{4}y_2^2 + \frac{1}{2}y_2x_3 - \frac{3}{4}x_3^2,\\ \mathrm{Jet}^{(2)}\left(f_0^{-1}(x_3)\right) = x_3,\end{cases}$$

$$\cdots$$

Note that $g(y_i) = f_0^{-1}(y_i)$ for $i = 1, 2$. Therefore, in System B, we have

$$\begin{cases}\mathrm{Jet}^{(1)}(x_1) = \mathrm{Jet}^{(1)}(g(y_1)) = \frac{1}{2}y_2 + \frac{1}{2}x_3,\\ \mathrm{Jet}^{(1)}(x_2) = \mathrm{Jet}^{(1)}(g(y_2)) = y_1 - \frac{1}{2}y_2 - \frac{3}{2}x_3,\end{cases}$$

$$\begin{cases}\mathrm{Jet}^{(2)}(x_1) = \mathrm{Jet}^{(2)}(g(y_1)) = \frac{1}{2}y_2 + \frac{1}{2}x_3 - \frac{1}{2}y_1^2 + \frac{1}{2}y_1y_2 + \frac{3}{2}y_1x_3 - \frac{3}{8}y_2^2 - \frac{5}{4}y_2x_3 - \frac{11}{8}x_3^2,\\ \mathrm{Jet}^{(2)}(x_2) = \mathrm{Jet}^{(2)}(g(y_2)) = y_1 - \frac{1}{2}y_2 - \frac{3}{2}x_3 + \frac{1}{4}y_2^2 + \frac{1}{2}y_2x_3 - \frac{3}{4}x_3^2,\end{cases}$$

$$\cdots$$

In System A, $\mathrm{Jet}^{(k)}$ operates on the power series in terms of variable x_3. By setting $(y_1, y_2, x_3) = (0, 0, x_3)$ in the above equations in System B, in System A we have

$$\begin{cases}\mathrm{Jet}^{(1)}(x_1) = \mathrm{Jet}^{(1)}(h(x_1)) = \frac{1}{2}x_3,\\ \mathrm{Jet}^{(1)}(x_2) = \mathrm{Jet}^{(1)}(h(x_2)) = -\frac{3}{2}x_3,\end{cases}$$

$$\begin{cases}\mathrm{Jet}^{(2)}(x_1) = \mathrm{Jet}^{(2)}(h(x_1)) = \frac{1}{2}x_3 - \frac{11}{8}x_3^2,\\ \mathrm{Jet}^{(2)}(x_1) = \mathrm{Jet}^{(2)}(h(x_2)) = -\frac{3}{2}x_3 - \frac{3}{4}x_3^2,\end{cases}$$

$$\cdots$$

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