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ABSTRACT

Let $\mathcal{O}_n = \mathbb{C}\{x_1,\ldots,x_n\}$ be the ring of convergent power series. An isolated hypersurface singularity $(V,0)$ is defined by $f \in \mathcal{O}_n$. Its moduli algebra $A(V) := \mathcal{O}_n/(f,\partial f/\partial x_1,\ldots,\partial f/\partial x_n)$ is a finite-dimensional algebra that determines the complex analytic structure of $(V,0)$. The Yau algebra $L(V) := \mathrm{Der}_{\mathbb{C}}(A(V),A(V))$ is a solvable Lie algebra associated with the singularity. As the singularity deforms, the structure of $L(V)$ also varies, leading to a Torelli-type problem: can the Yau algebra distinguish the different analytic structures within a deformation family? This paper investigates a crucial subalgebra, the liftable Yau subalgebra $\tilde{L}(V)$, which consists of derivations that can be lifted along the deformation. We focus on a one-parameter (μ,τ)-invariant deformation family V_t generated by $f_0 = x^4 + y^4 + z^4$ (whose projectivization in $\mathbb{CP}^2$ defines a smooth plane quartic curve of genus 3). We compute the 37-dimensional liftable Yau subalgebra family $\tilde{L}(V_t)$ for this deformation. A key technical breakthrough of this paper is the development of new computational tools and algorithms to calculate a series of Lie algebra isomorphism invariants (namely, cross-ratios) for its high-dimensional (36-dim) nilradical $N_t = [\tilde{L}(V_t),\tilde{L}(V_t)]$. This extends the invariant-based method, previously used for lower-dimensional algebras, to a significantly more complex case. By applying this new computational framework, we prove that $\tilde{L}(V_t) \cong \tilde{L}(V_s)$ implies $t^2 = s^2$ or $t^2 = -324(s^2-36)/(1015s^2+324)$. Comparing these two results, we demonstrate that although the liftable Yau subalgebra $\tilde{L}(V_t)$ is itself a well-behaved family of solvable Lie algebras constructed geometrically, it fails to be a complete invariant for the singularity isomorphism. Indeed, it fails to be even a basic invariant, as there exist cases where $V_t \cong V_s$ (isomorphic singularities) but $\tilde{L}(V_t) \ncong \tilde{L}(V_s)$ (non-isomorphic Lie algebras). This contrasts sharply with the known cases of $\tilde{E}_7$ and $\tilde{E}_8$ singularities. An important part of this work is the development of computational algorithms that make the analysis of these high-dimensional Lie algebras feasible on a computer.

I. INTRODUCTION

Let $\mathcal{O}_n = \mathbb{C}\{x_1,\ldots,x_n\}$ be the ring of convergent complex power series in n variables. An isolated hypersurface singularity $(V,0) \subset (\mathbb{C}^n,0)$ is defined by the zero set $V(f)$ of a function germ $f \in \mathcal{O}_n$. A fundamental algebraic structure associated with $(V,0)$ is its **Tjurina algebra** (or **moduli algebra**), defined as $A(V) := \mathcal{O}_n/J(f)$, where $J(f) = (f,\partial f/\partial x_1,\ldots,\partial f/\partial x_n)$ is the Tjurina ideal of f. $A(V)$ is a finite-dimensional complex local algebra. By the classical theorem of Mather and Yau,[1] two isolated hypersurface singularities $(V_1,0)$ and $(V_2,0)$ are complex analytically isomorphic (i.e., biholomorphically equivalent) if and only if their moduli algebras $A(V_1)$ and $A(V_2)$ are isomorphic as $\mathbb{C}$-algebras.

The Mather-Yau theorem establishes $A(V)$ as a complete algebraic invariant for the singularity. A key object of study is the derivation Lie algebra of $A(V)$, $L(V) := \mathrm{Der}_{\mathbb{C}}(A(V),A(V))$, which is a finite-dimensional Lie algebra. Yau proved that for any isolated hypersurface singularity, $L(V)$ is a **solvable Lie algebra**.[2,3] This profound result led to $L(V)$ being named the **Yau algebra** (cf. Refs. 3–5). This establishes a profound connection between singularity theory and solvable Lie algebras. Notably, the finite-dimensionality and solvability of $L(V)$ make it a more computationally tractable invariant than the infinite-dimensional Lie algebra $D(V) := \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_n/(f))$. Since the 1980s,[2–4,6–15] Yau and his collaborators have systematically studied the Lie algebras of isolated hypersurface singularities. In Ref. 5, $L(V)$ is called the Yau algebra, and in Ref. 16, dim $L(V)$ is called the Yau number.

As $L(V)$ is an invariant of $A(V)$, it is also an analytic invariant of $(V, 0)$. A central question, known as the **Torelli-type problem**, arises: is $L(V)$ (or a related subalgebra) a complete invariant within a deformation family of singularities? That is, can it distinguish different analytic structures in the family, just as $A(V)$ does?

The deformations we study are those that preserve certain topological invariants, specifically **(μ, τ)-constant deformations**, where the Milnor number μ and the Tjurina number $\tau = \dim_{\mathbb{C}} A(V)$ remain constant. The μ-constancy ensures the invariance of the topological type, following the work of Tráng and Ramanujam.[17] This line of inquiry is closely related to the foundational work of Saito[18] on the period map for simple elliptic singularities and the subsequent development of Torelli-type theorems by Hertling ,[19,20] which works on classifying singularity deformations via algebraic-geometric invariants.

In their study of (μ, τ)-constant deformations of $\tilde{E}_7$ and $\tilde{E}_8$ simple elliptic singularities, Seeley and Yau[7] introduced a key subalgebra, the **liftable Yau subalgebra** $\tilde{L}(V_t) \subseteq L(V_t)$. This subalgebra consists of those derivations in $L(V_t)$ that can be lifted to derivations on the parameter space S_E (see Definition III.8). They proved that for the $\tilde{E}_7$ and $\tilde{E}_8$ deformation families, $\tilde{L}(V_t)$ *is* a complete invariant: $\tilde{L}(V_{t_1}) \cong \tilde{L}(V_{t_2})$ if and only if $V_{t_1} \cong V_{t_2}$.

The success of Seeley and Yau[7] raises a natural question: does this completeness of $\tilde{L}(V_t)$ hold for more complex singularity types? This paper investigates a crucial case beyond $\tilde{E}_7$ and $\tilde{E}_8$: the singularity defined by the homogeneous quartic polynomial $f_0 = x^4 + y^4 + z^4$. Note that the zero set $(V_0, 0) = V(f_0) \subset \mathbb{C}^3$ is an isolated singularity; its projectivization $V(f_0) \subset \mathbb{CP}^2$ is a smooth plane quartic curve, which, by the genus formula $g = (d-1)(d-2)/2$, is a Riemann surface of genus 3 (cf. Ref. 21). We examine the one-parameter (μ, τ)-invariant deformation family V_t defined by $f_t := x^4 + y^4 + z^4 + tx^2y^2 = 0$.

For this deformation family V_t (where $t \notin \{\pm 2, \pm 6\}$), we compute its 37-dimensional liftable Yau subalgebra $\tilde{L}(V_t)$ and its 36-dimensional nilradical $N_t = [\tilde{L}(V_t), \tilde{L}(V_t)]$. While our work is founded on the method of Seeley–Yau,[7] a primary contribution of this paper is the development of new computational tools and a systematic algorithm to identify the required invariant subspaces in this significantly higher-dimensional (36-dim) case. By applying these new methods, we successfully compute the isomorphism invariants (cross-ratios) of N_t.

A. Main results

Let $\mathcal{O}_n = \mathbb{C}\{x_1, \ldots, x_n\}$ be the ring of all power series converging near the origin. We have mainly achieved the following results:

Theorem I.1 (A). *For the isolated hypersurface singularity $(V_0, 0)$ defined by $f_0 := x^4 + y^4 + z^4$ (whose projectivization in $\mathbb{P}^2$ is a smooth plane quartic curve of genus 3, cf. Ref. 21), we consider its (μ, τ)-invariant one-parameter deformation family (where $t \notin \{\pm 2, \pm 6\}$):*

$$V_t := \{(x, y, z) \in \mathbb{C}^3 | f_t := x^4 + y^4 + z^4 + tx^2y^2 = 0\}$$

*This family induces a 37-dimensional deformation family of **liftable Yau subalgebras**, denoted by $\tilde{L}(V_t) \subseteq L(V_t)$. By computing the numerical invariants (cross-ratios) of its 36-dimensional nilradical $N_t := [\tilde{L}(V_t), \tilde{L}(V_t)]$, we obtain the following results concerning the isomorphism class of $\tilde{L}(V_t)$:*

1. *If $\tilde{L}(V_t)$ and $\tilde{L}(V_s)$ are isomorphic as Lie algebras, then $t^2 = s^2$ or $t^2 = -\frac{324(s^2-36)}{1015s^2+324}$.*
2. *If $t^2 = s^2$, then the singularities $V(f_t)$ and $V(f_s)$ are analytically isomorphic. This implies their Yau algebras $L(V_t)$ and $L(V_s)$ are isomorphic, and their liftable subalgebras $\tilde{L}(V_t)$ and $\tilde{L}(V_s)$ are also isomorphic.*
3. *If $t^2 = -\frac{324(s^2-36)}{1015s^2+324}$ and $t^2 \neq s^2$, then $V(f_t) \ncong V(f_s)$.*

Theorem I.2 (B). *Let $V_t = \{(x, y, z) \in \mathbb{C}^3 | f_t := x^4 + y^4 + z^4 + tx^2y^2 = 0\}$ where $t \notin \{\pm 2, \pm 6\}$. Then V_t and V_s are analytically isomorphic if and only if one of the following parameter relations holds:*

(a1) $s = t$.
(a2) $s = -t$.
(a3) $2s + ts + 12 - 2t = 0$.
(a4) $2s + ts - 12 + 2t = 0$.
(a5) $2s - ts + 12 + 2t = 0$.
(a6) $2s - ts - 12 - 2t = 0$.

Conclusion from Theorem A and B. By comparing Theorem A and Theorem B, we arrive at the central conclusion of this paper: for the family $f_t = x^4 + y^4 + z^4 + tx^2y^2$, the liftable Yau subalgebra $\tilde{L}(V_t)$ is neither a complete analytic invariant nor a necessary analytic invariant. For example, when t and s satisfy condition (a3) from Theorem B but satisfy neither $t^2 = s^2$ nor $t^2 = -324(s^2 - 36)/(1015s^2 + 324)$ from Theorem A, we have $V_t \cong V_s$ (isomorphic singularities) but $\tilde{L}(V_t) \ncong \tilde{L}(V_s)$ (non-isomorphic Lie algebras). This discovery reveals that the Torelli-type problem for singularities is more subtle than the $\tilde{E}_7, \tilde{E}_8$ cases suggested.[7] The main technical contribution of this paper is the development of computational methods for handling invariants of high-dimensional (in our case, 36-dimensional) Lie algebras, which is essential for the future study of more complex singularity families.

Corollary I.3 (C). From the deformation family of isolated hypersurface singularities in $\mathbb{C}^3$ *defined by*

$$V_t := \{(x,y,z) \in \mathbb{C}^3 | f_t := x^4 + y^4 + z^4 + tx^2y^2 = 0\}$$

where $t \notin \{\pm 2, \pm 6\}$, *we naturally construct several high-dimensional families of* ***solvable*** *Lie algebras. These include the 51-dimensional Yau algebra family* $L(V_t)$ *and its 37-dimensional liftable subalgebra family* $\tilde{L}(V_t)$*. The 36-dimensional nilradical* $N_t = [\tilde{L}(V_t), \tilde{L}(V_t)]$ *forms a family of* ***nilpotent*** *Lie algebras.*

Organization of the paper. The paper is organized as follows. In Sec. II, we recall the preliminary knowledge regarding isolated hypersurface singularities, Yau algebras, and the Torelli-type problem. In Sec. III, we present the methodology for computing the moduli algebras, the Yau algebras $L(V_t)$, and the liftable Yau subalgebras $\tilde{L}(V_t)$ for the deformation family $f_t = x^4 + y^4 + z^4 + tx^2y^2$. Section IV is the technical core of the paper. We detail the algorithms for computing the invariant subspaces of the nilradical N_t, including its central series, quotient spaces, and their mappings. We then identify a set of four invariant lines, compute their cross-ratio, and prove our main results, Theorem A and Theorem B. Finally, to improve the readability of the main text, we have moved the extensive computational data—including the lists of generators, the full multiplication table, and all large representation matrices—into Appendixes A–D.

B. Notations

1. The algebraic structures, vector spaces, and linear mappings considered in this paper are all based on the complex field $\mathbb{C}$.
2. Let $f \in P_n = \mathbb{C}[x_1, \dots, x_n]$ be a polynomial, written as

$$f = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

   where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$. If the coefficient $c_{\boldsymbol{\alpha}} \neq 0$, we say that the monomial $\prod_{i=1}^n x_i^{\alpha_i}$ appears in the expression of f.
3. In a ring A, we use $\langle a \rangle$ to denote the ideal generated by $a \in A$.
4. we use $\mathrm{span}_{\mathbb{C}}\{E_1, \dots, E_i\}$ to denote the linear space over $\mathbb{C}$ spanned by $E_1, \dots, E_i$.

II. PRELIMINARY KNOWLEDGE

A. Singularity deformation and Yau algebra

Let $\mathcal{O}_n = \mathbb{C}\{x_1, \dots, x_n\}$ denote the ring of power series convergent near the origin. Let $I = (f_1, \dots, f_m)\mathcal{O}_n$ be the ideal in $\mathcal{O}_n$ generated by $f_1, \dots, f_m$.

Definition II.1. (Isolated Singularity): We say that I (or $f_1, \dots, f_m$*) defines an isolated singularity of dimension r at the origin if there exists a small neighborhood* $U \subseteq \mathbb{C}^n$ *near the origin such that the only singular point of* $V(I) := \{\boldsymbol{q} \in U \mid f_1(\boldsymbol{q}) = \cdots = f_m(\boldsymbol{q}) = 0\}$ *is the origin (denoted as* $\boldsymbol{0}$*), and* $V(I) \setminus \boldsymbol{0}$ *as a complex submanifold has dimension r [if* $r = 0$*, then* $V(I) \setminus \boldsymbol{0}$ *is required to be empty]. In this case, we have:*

1. $f_1(\boldsymbol{0}) = \cdots = f_m(\boldsymbol{0}) = 0$.
2. *The rank of the matrix* $\left(\frac{\partial f_j}{\partial x_i}(\boldsymbol{0})\right)_{i=1,\dots,n; j=1,\dots,m}$ *is less than* $n - r$.
3. *For any* $\boldsymbol{q} \in V(I) \setminus \boldsymbol{0}$*, the rank of the matrix* $\left(\frac{\partial f_j}{\partial x_i}(\boldsymbol{q})\right)_{i=1,\dots,n; j=1,\dots,m}$ *is equal to* $n - r$.

Definition II.2. (Quasi-Homogeneous Singularity): We say that I (or $f_1, \dots, f_m$*) defines a quasi-homogeneous singularity at the origin if there exists an analytic coordinate transformation near the origin* $\varphi : \mathbb{C}\{x_1, \dots, x_n\} \sim \mathbb{C}\{z_1, \dots, z_n\}$ *and positive integers* $w_1, \dots, w_n$ *such that, with* z_i *weighted by* w_i*,* $\varphi(I)$ *is generated by weighted homogeneous polynomials. In this case, the new coordinates* $z_1, \dots, z_n$ *and weights* $w_1, \dots, w_n$ *provide a graded structure for the singularity.*

Definition II.3. (Isolated Hypersurface Singularity): If the ideal $I = (f) \subseteq \mathcal{O}_n$ *is generated by a single element* $f \in \mathcal{O}_n$ *and I defines an isolated singularity at the origin, then we call I (or f) an isolated hypersurface singularity at the origin.*

Definition II.4. (Singularity Isomorphism): Let I and I' *be two ideals in* $\mathcal{O}_n$*. We say that I and* I' *define isomorphic singularities at the origin if there exists an automorphism* φ *of* $\mathcal{O}_n$ *such that* $\varphi(I) = I'$.

Definition II.5. (Hypersurface Singularity Isomorphism): If $f \in \mathcal{O}_n$ and $g \in \mathcal{O}_n$ define isolated hypersurface singularities $[V(f), 0]$ and $[V(g), 0]$ at the origin respectively, we say that f and g define isomorphic hypersurface singularities at the origin, or are contact equivalent, if there exists an automorphism φ of $\mathcal{O}_n$ such that

$$f = u \cdot \varphi(g),$$

which is denoted by $f \overset{c}{\sim} g$.

Definition II.6. (Derivative): Let A be a commutative algebra over the complex field $\mathbb{C}$. We call a linear endomorphism D of A a derivative of A if it satisfies the Leibniz rule

$$D(ab) = D(a)b + aD(b) \quad \forall a, b \in A.$$

The set of all derivatives of A, denoted by $\mathrm{Der}_{\mathbb{C}}(A)$, forms a complex Lie algebra, where the Lie bracket is defined by

$$[D, E] := D \circ E - E \circ D \quad \forall D, E \in \mathrm{Der}_{\mathbb{C}}(A).$$

For an ideal I of A, we denote the submodule of derivations of A on I by

$$\mathrm{Der}_I(A) := \{D \in \mathrm{Der}_{\mathbb{C}}(A) \mid D(I) \subseteq I\}.$$

B. Singularity deformation and Torelli-type problems

Let $\mathcal{O}_n = \mathbb{C}\{x_1, \ldots, x_n\}$ denote the ring of power series convergent near the origin. Consider an isolated hypersurface singularity $(V, 0)$ defined by $f \in \mathcal{O}_n$, given by

$$(V, 0) = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid f(x_1, \ldots, x_n) = 0\}.$$

The module algebra of this singularity $(V, 0)$ is defined by

$$A(V) := \mathcal{O}_n / \left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

In Ref. 21, a characterization of genus 3 Riemann surfaces can be found:

Theorem II.7 *(Ref. 21, Proposition 2.5). Let X be a genus 3 algebraic curve. Then X is either a hyperelliptic curve [defined by the equation $y^2 = h(x)$, where h is of degree 7 or 8], or the canonical map ϕ_K of X embeds X into $\mathbb{CP}^2$ as a smooth plane quartic curve defined by a quartic polynomial.*

Theorem II.7 motivates the study of plane quartic polynomials. The polynomial $f_0 = x^4 + y^4 + z^4$ defines an isolated hypersurface singularity $(V_0, 0)$ in $\mathbb{C}^3$. Its projectivization, the curve $X_0 \subset \mathbb{CP}^2$ defined by the same equation, is a smooth plane quartic. Therefore, by Theorem II.7, X_0 is a Riemann surface of genus 3.

The general (μ, τ)-invariant deformation of f_0 (which defines a family of isolated singularities in $\mathbb{C}^3$) corresponds to the multi-parameter family of quartics:

$$\{(x, y, z) \in \mathbb{C}^3 \mid \; f_{\mathbf{t}} := x^4 + y^4 + z^4 + t_1 x^2 y^2 + t_2 x^2 z^2 + t_3 y^2 z^2 + t_4 x^2 yz + t_5 xy^2 z + t_6 xyz^2 = 0\}.$$

Our analysis concerns the Yau algebras associated with the *affine singularities* $V(f_{\mathbf{t}}) \subset \mathbb{C}^3$.

Due to the computational difficulty of this multi-parameter family, we restrict our investigation in this paper to the following significant single-parameter (μ, τ)-invariant deformation subfamily:

$$V_t := \{(x, y, z) \in \mathbb{C}^3 \mid f_t := x^4 + y^4 + z^4 + tx^2 y^2 = 0\}.$$

III. COMPUTATION OF YAU ALGEBRA FAMILIES

Many of the results in this section can be derived by directly referencing several theorems from Ref. 22, which deal with the computation of Yau algebra families and their liftable Yau subalgebra arising from the deformation of singularities. However, to ensure this paper is self-contained and better suited for subsequent discussions, we provide sketches of these theorems or rephrase them to align with the context of the issues addressed later in the paper. Specifically, in this section, we present several concrete examples that clearly demonstrate how these computational theorems assist in advancing our later discussions.

A. Computation of moduli algebras

First, we need to compute the moduli algebra of the (μ,τ)-invariant deformation family $V_{\mathbf{t}} := \{(x,y,z) \in \mathbb{C}^3 \mid f_t := x^4 + y^4 + z^4 + tx^2y^2 = 0\}$ defined by $f_0 := x^4 + y^4 + z^4$, in order to further calculate the Yau algebra of this family of singularities. For this algebraic family with high dimensions, we can employ the computational methods used in Ref. 22, and through computer assistance, translate the following theorems into language understandable by a computer.

Proposition III.1 (Ref. 22). For singularities defined by the homogeneous polynomial

$$f_{\mathbf{t}} := x_1^d + \cdots + x_n^d + \sum_{i=1}^{m} t_i g_i,$$

where $\{g_i | i = 1,\ldots,m\}$ are monomial generators of $A(f_{\mathbf{0}})_d$, there exists a Zariski closed set $Z \subset \mathbb{C}^m \backslash \{\mathbf{0}\}$ such that for all $\boldsymbol{t} \in \mathbb{C}^m \backslash Z$, $A(f_{\boldsymbol{t}})$ has the following basis over $\mathbb{C}$:

$$\{\boldsymbol{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} \cdots x_n^{\alpha_n} | 0 \leqslant \alpha_1,\ldots,\alpha_n \leqslant d-2\}.$$

Remark III.2. The result of the moduli algebra of this singularity deformation family is extremely important as a prerequisite for analyzing and calculating its Yau algebra. In the proof of this result, certain constructions are very helpful for subsequent calculations, and in the computation of the generators of the moduli algebra, requirements are given for general points in the parameter space Z of the deformation process. Therefore, it is necessary to briefly list several important results obtained in the proof to aid understanding of subsequent calculations:

Let's consider the following three sets of monomials:

$$\{B_1,\ldots,B_M\} := \{\boldsymbol{x}^{\boldsymbol{\alpha}} \,||\boldsymbol{\alpha}| = k\},$$
$$\{b_1,\ldots,b_\mu\} := \{\boldsymbol{x}^{\boldsymbol{\alpha}} \,|0 \leqslant \alpha_1,\ldots,\alpha_n \leqslant d-2, |\alpha| = k\},$$
$$\{b_{\mu+1},\ldots,b_{M'}\} := \left\{\boldsymbol{x}^{\boldsymbol{\alpha}} \cdot \frac{\partial f}{\partial x_j} \middle| |\alpha| + d - 1 = k\right\}.$$

Here, $k \geq 0$, $\{B_1,\ldots,B_M\}$ is a set of monomial generators of $\mathbb{C}\{\boldsymbol{x}\}_k$, $\{b_1,\ldots,b_\mu\}$ is a set of monomials from $\{\boldsymbol{x}^{\boldsymbol{\alpha}} | 0 \leqslant \alpha_1,\ldots,\alpha_n \leqslant d-2, |\alpha| \leqslant D-1\}$ with degree k, and $\{b_{\mu+1},\ldots,b_{M'}\}$ are of degree $k+d-1$.

The following facts hold:

1. *There exists a matrix $C = (C_i^j) \in \mathrm{Mat}(M' \times M, \mathbb{C}[\boldsymbol{t}])$ such that*

$$\begin{pmatrix} b_1 \\ \vdots \\ b_{M'} \end{pmatrix} = C \begin{pmatrix} B_1 \\ \vdots \\ B_M \end{pmatrix},$$

where $\forall 1 \leqslant i \leqslant M', 1 \leqslant j \leqslant M$,

$$b_i = \sum_{j=1}^{M} C_i^j B_j.$$

2. *After row permutation, $C(\boldsymbol{t})$ can be expressed as*

$$\begin{pmatrix} C_k(\boldsymbol{t}) \\ * \end{pmatrix},$$

where $M \leqslant M'$, the submatrix $C_k(\boldsymbol{t}) \in \mathrm{Mat}(M \times M, \mathbb{C}[\boldsymbol{t}])$ is invertible at $\boldsymbol{t} = \mathbf{0}$. Particularly, $\det C_k \in \mathbb{C}[\boldsymbol{t}]$ has a non-zero constant term, where $Z_k = V(\det C_k) \subset \mathbb{C}^m$.

3. *$\mathbf{0} \notin Z_k$ and $\forall \boldsymbol{t} \in \mathbb{C}^m \backslash Z$, where Z is a Zariski closed set*

$$Z = \bigcup_{k=0}^{D+1} Z_k,$$

excluding $\mathbf{0}$, therefore, $\mathbf{0}$ is a general point of $\{\boldsymbol{x}^{\boldsymbol{\alpha}} | 0 \leqslant \alpha_1,\ldots,\alpha_n \leqslant d-2, |\alpha| \leqslant D-1\}$ in $A(f_{\boldsymbol{t}})$, which is of finite dimension.

4. *There exist $u_i^j \in \mathbb{C}\{\boldsymbol{t}\}$ such that*

$$B_i = \sum_{j=1}^{\mu} u_i^j b_j \in A(f_{\boldsymbol{t}}).$$

We can take $u_{i,j}^k := u_l^k$ *such that*

$$b_i \cdot b_j = \sum_{k=1}^{\mu} u_{i,j}^k b_k \in A(f_{\boldsymbol{t}}).$$

If $\deg b_i + \deg b_j \leqslant D$, *then* $b_i \cdot b_j$ *must be some* B_l, *then we can take* $u_{i,j}^k := u_l^k$. *If* $\deg b_i + \deg b_j > D$, *then we just need to take* $u_{i,j}^k = 0$. *Finding the matrix C as it appeared in the proof is straightforward for a computer. Now, we let* $e_i \in \mathbb{C}^M$ *represent the i-th standard coordinate row vector, and take the solution vector u such that* $uC = e_i$, *then the following equation holds:*

$$B_i = e_i\begin{pmatrix} B_1 \\ \vdots \\ B_M \end{pmatrix} = uC\begin{pmatrix} B_1 \\ \vdots \\ B_M \end{pmatrix} = u\begin{pmatrix} b_1 \\ \vdots \\ b_{M'} \end{pmatrix} = \sum_{j=1}^{\mu} u^j b_j + \sum_{j=\mu+1}^{M'} u^j b_j,$$

therefore, we can take $u_i^j := u^j$. *Thus, the multiplication relation between the generators* $\{b_1, \ldots, b_\mu\}$ *of* $A(f_{\boldsymbol{t}})$, *the multiplication table of* $A(f)$ *under the basis* $\{b_1, \ldots, b_\mu\}$ *can be obtained again by solving linear equation systems. This fact will be used in computing* $L(f)$ *from* $A(f)$, *which needs to solve a more complicated linear equation system.*

To illustrate the above conclusion in our specific singularity moduli algebra calculation process, and to save space, we give part of the calculation process of the general point set $\mathbb{C}\backslash Z$ of parameters $\boldsymbol{t}$ in the parameter space of the single parameter (μ, τ)-invariant deformation $V_t := \{(x, y, z) \in \mathbb{C}^3 \mid f_t := x^4 + y^4 + z^4 + tx^2y^2 = 0\}$ of the singularity $f_0 := x^4 + y^4 + z^4$.

Example III.3. Consider the singularity $f_0 := x^4 + y^4 + z^4$ *and its single parameter* (μ, τ)*-invariant deformation* $V_t := \{(x, y, z) \in \mathbb{C}^3 \mid f_t := x^4 + y^4 + z^4 + tx^2y^2 = 0\}$. *Its moduli algebra is* $A(f_t) := \mathbb{C}\{\boldsymbol{x}\}/\langle f, j(f_t)\rangle$, *where* $\langle j(f_t)\rangle = \left\langle \frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}, \frac{\partial f_t}{\partial z}\right\rangle = \left\langle 4x^3 + 2txy^2, 4y^3 + 2tx^2y, z^3\right\rangle$, *then the graded 4 part of* $A(f_t)$ *is generated by the following 6 monomials:*

$$A(f_t)_4 = \langle x^2y^2, x^2yz, xy^2z, x^2z^2, xyz^2, y^2z^2\rangle =: \langle b_1, \ldots, b_6\rangle.$$

$\mathbb{C}\{x\}_4$ *is generated by the following 15 monomials:*

$$\mathbb{C}\{t\}_4 =: \langle B_1, \ldots, B_{15}\rangle.$$

And the $\{b_{\mu+1}, \ldots, b_{M'}\}$ *mentioned in the above Proposition III.1 annotation proof summary is*

$$\begin{aligned}\{\boldsymbol{x}^{\boldsymbol{\alpha}} \cdot \frac{\partial f}{\partial x_j}\bigg| |\alpha| + 4 - 1 = 4\} &= \{x(4x^3 + 2txy^2), x(4y^3 + 2tx^2y), xz^3, y(4x^3 + 2txy^2), y(4y^3 + 2tx^2y), yz^3, z(4x^3 + 2txy^2), z(4y^3 + 2tx^2y), z^4\} \\ &=: \{b_7, \ldots, b_{15}\};\end{aligned}$$

Here we have

$$\begin{pmatrix} b_1 \\ \vdots \\ b_{15} \end{pmatrix} = C_4\begin{pmatrix} B_1 \\ \vdots \\ B_{15} \end{pmatrix},$$

from which we know that when $k = 4$, $Z_k = V(\det C_k = 0)$ *is*

$$Z_4 = -1/(65\,536(t^2 - 4)).$$

Similarly, we can obtain

$$Z_3 = \frac{1}{64},$$
$$Z_5 = \frac{1}{10\,73\,741\,824(t^2 - 4)^3},$$
$$Z_6 = \frac{-1}{17\,592\,186\,044\,416(t^2 - 4)^5}.$$

So the single parameter (μ,τ)*-invariant deformation* $V_t := \{(x,y,z) \in \mathbb{C}^3 \mid f_t := x^4 + y^4 + z^4 + tx^2y^2 = 0\}$ *can have a basis of monomials of the moduli algebra as indicated in Theorem III.1, and the corresponding Zariski closed set Z required is*

$$Z = \bigcup_{k=0}^{D+1} Z_k = \{t | t = \pm 2\};$$

that is, $\forall \boldsymbol{t} \in \mathbb{C}^3 \backslash Z$, $A(f_t)$ *has the following basis of monomial generators (see Theorem III.1):*

$$\{\boldsymbol{x}^{\boldsymbol{\alpha}} := x^{\alpha_1} y^{\alpha_2} z^{\alpha_3} | 0 \leqslant \alpha_1, \ldots, \alpha_3 \leqslant 2\}.$$

Next, calculate the multiplication table of $A(f_t)$*, the multiplication table is presented in a series of equalities on* $\mathbb{C}\{x,y,z\}$*, continue with the* 4th *part* $\mathbb{C}\{x,y,z\}_4$ *of* $\mathbb{C}\{x,y,z\}$*: taking* $U_4 \cdot C_4 = I_{15}$*, so by*

$$\begin{pmatrix} b_1 \\ \vdots \\ b_{15} \end{pmatrix} = C_4 \begin{pmatrix} B_1 \\ \vdots \\ B_{15} \end{pmatrix},$$

get

$$U_4 \begin{pmatrix} b_1 \\ \vdots \\ b_{15} \end{pmatrix} = \begin{pmatrix} B_1 \\ \vdots \\ B_{15} \end{pmatrix};$$

computed.

$$U_4^{tr} = \begin{pmatrix}
-t/2 & 0 & 1 & 0 & -t/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -t/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -t/2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-1}{(t^2-4)} & 0 & \frac{t}{2(t^2-4)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{t}{2(t^2-4)} & 0 & \frac{-1}{(t^2-4)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4
\end{pmatrix}$$

For the computation result of U, we interpret it as follows: For $1 \le j \le 15$*, the j-th column of the first 6 rows of the transpose of matrix* U_4*, denoted as* U_4^{tr}*, corresponds to the linear coordinates of the j-th basis* B_j *in* $\mathbb{C}\{t\}_4$ *within the space* $A(f_t)_4$*. According to* U_4^{tr}*, we obtain: The partial multiplication table of* $A(f_t)$ *can be described by the following set of equalities in* $\mathbb{C}\{x\}_4$*:*

$$
\begin{aligned}
x^4 &= -\frac{tx^2y^2}{2}; & x^3y &= 0;\\
x^2y^2 &= x^2y^2; & xy^3 &= 0;\\
y^4 &= -\frac{tx^2y^2}{2}; & x^3z &= -\frac{txy^2z}{2};\\
x^2yz &= x^2yz; & xy^2z &= xy^2z;\\
y^3z &= -\frac{tx^2yz}{2}; & x^2z^2 &= x^2z^2;\\
xyz^2 &= xyz^2; & y^2z^2 &= y^2z^2;\\
xz^3 &= 0; & yz^3 &= 0;\\
z^4 &= 0; & & \qquad (A_4)
\end{aligned}
$$

Here, we also present results for $\mathbb{C}\{x,y,z\}_6$ obtained in a similar computational manner. The computations for $\mathbb{C}\{x,y,z\}_4$ and $\mathbb{C}\{x,y,z\}_6$ presented here will be used in the example of computing the 3rd-degree part $L(V_t)_3$ of the Yau algebra of the singularities.

For $\mathbb{C}\{x,y,z\}_6$, we have

$$A(f_t)_6 = \langle x^2y^2z^2\rangle =: \langle b_1\rangle.$$

Correspondingly, we have $\{b_2,\dots,b_{31}\}$ where

$$\{\boldsymbol{x}^{\boldsymbol{\alpha}}\cdot\frac{\partial f}{\partial x_j}\Big|\,|\alpha|+4-1=6\}\cong\mathbb{C}\{t\}_3\oplus\{4x^3+2txy^2,4y^3+2tx^2y,z^3\}=:\{b_2,\dots,b_{31}\}.$$

Similarly, solving for matrix U_6 such that it is of size $28\cdot 31$ and $U_6\cdot\{b_i\}=\{B_j\}, 1\le i\le 31, 1\le j\le 28$, all non-zero identities are in $\mathbb{C}\{x,y,z\}_6$:

$$
\begin{aligned}
x^4z^2 &= -(tx^2y^2z^2)/2\\
y^4z^2 &= -(tx^2y^2z^2)/2. \qquad (A_6)
\end{aligned}
$$

B. Computation of Yau algebras

Let $\mathcal{O}_n=\mathbb{C}\{\boldsymbol{x}\}$ denote the ring of all power series convergent near the origin in $\mathbb{C}^n$. Let $(V,0)=\{(x_1,\dots,x_n)\in\mathbb{C}^n\mid f(x_1,\dots,x_n)=0\}$ be an isolated hypersurface singularity, where $f\in\mathcal{O}_n$. The module algebra of singularity $(V,0)$ is defined by

$$A(V):=\mathcal{O}_n\big/\Big(f,\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\Big).$$

It is a fundamental result[23] that the moduli algebra $A(V)$ is finite-dimensional. The importance of this algebra was established by Mather and Yau, who proved the celebrated Mather-Yau theorem:[1] the isomorphism class of $A(V)$ completely determines the analytic structure of the singularity.

The Lie algebra of all $\mathbb{C}$-derivations of the moduli algebra:

$$L(V):=\mathrm{Der}_{\mathbb{C}}(A(V))$$

can be then associated to an isolated hypersurface singularity, In 1991, Yau[2] proved that $L(V)$ is always a **solvable Lie algebra**, thereby establishing a connection between isolated hypersurface singularities and the theory of solvable Lie algebras.

Remark III.4. (Solving Yau Algebra via Linear Equations). *According to the definition of Yau algebras, translated into matrix language, we can solve the corresponding linear equations to compute the Yau algebra. When translated into matrix language, this allows for computer-assisted computation of Yau algebras in high dimensions. This computational approach relies on the mature application of computer algebra (e.g., Gröbner bases for moduli algebras) in singularity theory, as systematically described in Ref. 24.*

Lemma III.5 (Ref. 15). Let $I\subseteq\mathbb{C}\{\boldsymbol{x}\}$ be an ideal, then there exists a natural Lie algebra isomorphism:

$$\frac{\mathrm{Der}_I\mathbb{C}\{\boldsymbol{x}\}}{I\cdot\mathrm{Der}_{\mathbb{C}}\mathbb{C}\{\boldsymbol{x}\}}\cong\mathrm{Der}_{\mathbb{C}}(\mathbb{C}\{\boldsymbol{x}\}/I),$$

where

$$\mathrm{Der}_I\mathbb{C}\{\boldsymbol{x}\} = \{\delta \in \mathrm{Der}_{\mathbb{C}}\mathbb{C}\{\boldsymbol{x}\}\,|\delta I \subseteq I\}.$$

Remark III.6. According to this lemma, the basis of the Yau algebra $L(V) = \mathrm{Der}_{\mathbb{C}}A(f) = \mathrm{Der}_{\mathbb{C}}(\mathbb{C}\{\boldsymbol{x}\}/(f, j(f))$ can be obtained by solving linear equations. This method is discussed in Ref. 22. Here we provide a brief overview and later present a specific example of computing $L(V_t)_3$ to illustrate this method.

According to the lemma above, a derivation δ in $L(V)_k$ can be lifted to a derivation $\tilde{\delta}$ satisfying

$$\tilde{\delta} \in (\mathrm{Der}_{\mathbb{C}}\mathbb{C}\{\boldsymbol{x}\})_k = \left(\bigoplus_{i=1}^{n} \mathbb{C}\{\boldsymbol{x}\}\frac{\partial}{\partial x_i}\right)_k = \bigoplus_{i=1}^{n} \mathbb{C}\{\boldsymbol{x}\}_{k+1}\frac{\partial}{\partial x_i}.$$

Just as in the previous proof related to singularity module algebras, let $\{B_1, \ldots, B_M\}$ and $\{b_1, \ldots, b_\mu\}$ be sets of monomial bases of $\mathbb{C}\{\boldsymbol{x}\}_{k+1}$ and $\{\boldsymbol{x}^{\boldsymbol{\alpha}}|0 \leqslant \alpha_1, \ldots, \alpha_n \leqslant d-2, |\alpha| \leqslant D-1\}$ of degree k, respectively. Then a derivation δ in $L(f)_k$ can be lifted to $\tilde{\delta}$, given by

$$\tilde{\delta} = \sum_{i=1}^{n}\sum_{j=1}^{M} \delta^{i,j} B_j \frac{\partial}{\partial x_i},$$

where $\delta^{i,j} \in \mathbb{C}$. Since δ is a linear endomorphism on $A(f)$, we only need to take

$$\tilde{\delta} = \sum_{i=1}^{n}\sum_{j=1}^{\mu} \delta^{i,j} b_j \frac{\partial}{\partial x_i}.$$

To ensure that δ is a derivation on $A(f)$, we only need to require $\tilde{\delta}j(f) \subseteq j(f)$, which is equivalent to requiring for any $l = 1, \ldots, n$,

$$\overline{\tilde{\delta}\frac{\partial f}{\partial x_l}} := \overline{\sum_{i=1}^{n}\sum_{j=1}^{\mu} \delta^{i,j} b_j \frac{\partial^2 f}{\partial x_i \partial x_l}} = 0 \in A(f).$$

Thus, we obtain a system of linear equations concerning $\delta^{i,j}$.

Example III.7. Consider the singularity $f_0 := x^4 + y^4 + z^4$ and its one-parameter (μ, τ)-invariant deformation $V_{\mathbf{t}} := \{(x, y, z) \in \mathbb{C}^3 \mid f_t := x^4 + y^4 + z^4 + tx^2y^2 = 0\}$, where its moduli algebra is $A(f_t) := \mathbb{C}\{\boldsymbol{x}\}/\langle f, j(f_t)\rangle$, with $\langle j(f_t)\rangle = \left\langle\left(\frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}, \frac{\partial f_t}{\partial z}\right\rangle = \langle 4x^3 + 2txy^2, 4y^3 + 2tx^2y, z^3\rangle$, then $L(V_t)$'s graded part at degree 3, $L(V_t)_3$, can be obtained by constructing and solving a system of linear equations over $\mathbb{C}\{t\}$ to find its generators:

Any derivation D over $\mathbb{C}\{x, y, z\}$ can be written as:

$$D = f_1\frac{\partial}{\partial x} + f_2\frac{\partial}{\partial y} + f_3\frac{\partial}{\partial z},$$

where $f_1, f_2, f_3 \in \mathbb{C}\{x, y, z\}$, and for any $D \in \mathrm{End}\,(A(f_t)_4)$, according to Lemma III.5, we can assume that D is of the form

$$\left(a_1x^2y^2 + a_2x^2yz + a_3xy^2z + a_4x^2z^2 + a_5xyz^2 + a_6y^2z^2\right)\frac{\partial}{\partial x} + \left(b_1x^2y^2 + b_2x^2yz + b_3xy^2z + b_4x^2z^2 + b_5xyz^2 + b_6y^2z^2\right)\frac{\partial}{\partial y} + \left(c_1x^2y^2 + c_2x^2yz + c_3xy^2z + c_4x^2z^2 + c_5xyz^2 + c_6y^2z^2\right)\frac{\partial}{\partial z}. \tag{D_1}$$

but not every expression of the form (D_1) is a derivation on $A(f_t)_4$. A derivation must satisfy that its action on each generator of $j(f_t)$ remains in $j(f_t)$, which is equivalent to requiring

$$D\frac{\partial f}{\partial x} = \sum_{i=1}^{6} \delta_A^{1,i} b_i \frac{\partial^2 f}{\partial^2 x} + \sum_{i=1}^{6} \delta_B^{1,i} b_i \frac{\partial^2 f}{\partial x \partial y} + \sum_{i=1}^{6} \delta_C^{1,i} b_i \frac{\partial^2 f}{\partial x \partial z} = 0,$$
$$D\frac{\partial f}{\partial y} = \sum_{i=1}^{6} \delta_A^{2,i} b_i \frac{\partial^2 f}{\partial x \partial y} + \sum_{i=1}^{6} \delta_B^{2,i} b_i \frac{\partial^2 f}{\partial^2 y} + \sum_{i=1}^{6} \delta_C^{2,i} b_i \frac{\partial^2 f}{\partial y \partial z} = 0,$$
$$D\frac{\partial f}{\partial z} = \sum_{i=1}^{6} \delta_A^{3,i} b_i \frac{\partial^2 f}{\partial x \partial z} + \sum_{i=1}^{\mu} \delta_B^{3,i} b_i \frac{\partial^2 f}{\partial y \partial z} + \sum_{i=1}^{6} \delta_C^{3,i} b_i \frac{\partial^2 f}{\partial^2 z} = 0; \qquad (D_2)$$

This yields the system of linear equations in $A(f_t)$. And thus, the derivation D can be read from the solution matrix $(1 \le i \le 6)$

$$\begin{pmatrix} \delta_A^{1,i}, \delta_B^{1,i}, \delta_C^{1,i} \\ \delta_A^{2,i}, \delta_B^{2,i}, \delta_C^{2,i} \\ \delta_A^{3,i}, \delta_B^{3,i}, \delta_C^{3,i} \end{pmatrix}. \qquad (D_3)$$

Note that in the above Eq. (D_2), $b_i \frac{\partial^2 f}{\partial^2 x}, b_i \frac{\partial^2 f}{\partial x \partial y}, b_i \frac{\partial^2 f}{\partial x \partial z}, \ldots, b_i \frac{\partial^2 f}{\partial^2 z}$ all belong to $A(f_t)_6 = < x^2y^2z^2 >$, while according to the Hessian matrix of f_t:

$$\begin{pmatrix} \frac{\partial^2 f}{\partial^2 x}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial^2 y}, \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial y \partial z}, \frac{\partial^2 f}{\partial^2 z} \end{pmatrix} = \begin{pmatrix} 12x^2 + 2ty^2, 4txy, 0 \\ 4txy, 12y^2 + 2tx^2, 0 \\ 0, \quad 0, \quad 3z^2 \end{pmatrix},$$

and there being identities:

$$b_1 = x^2y^2, b_2 = x^2yz, b_3 = xy^2z, b_4 = x^2z^2, b_5 = xyz^2, b_6 = y^2z^2;$$

Using the multiplication rule (A_6) of the module algebra of this singularity, we know that all non-zero elements in $b_i \frac{\partial^2 f}{\partial^2 x}, b_i \frac{\partial^2 f}{\partial x \partial y}, \ldots, b_i \frac{\partial^2 f}{\partial^2 z}$ are

$$b_1 \frac{\partial^2 f}{\partial^2 z} = 3x^2y^2z^2, \quad b_4 \frac{\partial^2 f}{\partial^2 x} = 12x^4z^2 = -6tx^2y^2z^2, \quad b_4 \frac{\partial^2 f}{\partial^2 y} = 12x^2y^2z^2 + 2tx^4z^2 = (12 - t^2)x^2y^2z^2,$$
$$b_5 \frac{\partial^2 f}{\partial x \partial y} = 4tx^2y^2z^2, \quad b_6 \frac{\partial^2 f}{\partial^2 y} = -6tx^2y^2z^2, \quad b_6 \frac{\partial^2 f}{\partial^2 x} = (12 - t^2)x^2y^2z^2;$$

therefore, Eq. (D_2) can be transformed into

$$\delta_A^{1,4} b_4 \frac{\partial^2 f}{\partial^2 x} + \delta_A^{1,6} b_6 \frac{\partial^2 f}{\partial^2 x} + \delta_B^{1,5} b_5 \frac{\partial^2 f}{\partial x \partial y} = 0,$$
$$\delta_B^{2,5} b_5 \frac{\partial^2 f}{\partial x \partial y} + \delta_B^{2,4} b_4 \frac{\partial^2 f}{\partial^2 y} + \delta_B^{2,6} b_6 \frac{\partial^2 f}{\partial^2 y} = 0,$$
$$\delta_C^{3,1} b_1 \frac{\partial^2 f}{\partial^2 z} = 0,$$

namely

$$\delta_A^{1,4}(-6tx^2y^2z^2) + \delta_A^{1,6}(12 - t^2)x^2y^2z^2 + \delta_B^{1,5}(4tx^2y^2z^2) = 0,$$
$$\delta_B^{2,5}(4tx^2y^2z^2) + \delta_B^{2,4}\left(12 - \frac{t^3}{2}\right)x^2y^2z^2 + \delta_B^{2,6}(-6tx^2y^2z^2) = 0,$$
$$\delta_C^{3,1}(3x^2y^2z^2) = 0, \qquad (D_3)$$

These equations are zero in $A(f_t)_6$. They indicate that the coefficients of $x^2y^2z^2$ in $\mathbb{C}\{t\}$ are zero. After a linear equivalence transformation, the coefficient matrix obtained from (D_3) can be simplified to

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & (t^2-12)/(4\cdot t) & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -(t^2-12)/(4\cdot t) & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Its solution $(a_1,\ldots,a_6,b_1,\ldots,b_6,c_1,\ldots,c_6)$ *is*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & (-4\cdot t)/(t^2-12) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & (4\cdot t)/(t^2-12) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (4\cdot t)/(t^2-12) & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (-4\cdot t)/(t^2-12) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Based on the elements in this matrix, translated into corresponding coefficients of the generators, we can obtain a set of generators for $L(V_t)_3$.

Generators for $L(V_t)$ By solving the system of linear equations derived in Example III.7, we compute the basis for the Yau algebra $L(V_t)$. It is a 51-dimensional solvable Lie algebra. The complete list of its weighted generators is presented in Appendix A, Table I.

C. Computation of the liftable subalgebras $\tilde{L}(V_t)$ of the Yau algebras $L(V_t)$

Let $(V,0) = \{(z_1,\ldots,z_n) : f(z_1,\ldots,z_n) = 0\}$ be a hypersurface defined by a weighted homogeneous polynomial with an isolated singularity at the origin. The moduli algebra $A(V)$ is defined as $A(V) = \mathbb{C}\{z_1,\ldots,z_n\}/\left(f, \frac{\partial f}{\partial z_1},\ldots,\frac{\partial f}{\partial z_n}\right)$, and $L(V)$ is the Lie algebra of derivations of $A(V)$. In Ref. 7, Yau and collaborators introduced the concept of the liftable Lie algebra $\tilde{L}(V)$ for singularities, where $\tilde{L}(V_0)$ is a Lie subalgebra of $L(V_0)$ and has a natural deformation over the parameter space S_E.

Definition III.8. (See Ref. 7) Consider the hypersurface $V = \{x \mid f(x) = 0\}$ *defined by a weighted homogeneous polynomial with an isolated singularity at the origin and its* (μ,τ)*-invariant deformation* $V_{\mathbf{t}} := \{(x_1,\ldots,x_n,t_1,\ldots,t_m) : f(\mathbf{x}) + t_1 g_1(\mathbf{x}) + \cdots + t_m g_m(\mathbf{x}) = 0\}$, *where* $g_1,\ldots,g_m$ *are elements of a set of monomial generators of* $A(V_0)$, *and* $\text{weight}(g_i) = \text{weight}(f)$ *for* $1 \le i \le m$.

The liftable Lie algebra $\tilde{L}(V_0)$ *is defined as those derivations* $D_0 \in L(V_0)$ *that can be lifted to* $S_E = \mathbb{C}^m$, *i.e., there exists a derivation D of the form:*

$$D = D_0 + \sum_{|t_1|=1} t^{t_1} D_{t_1} + \sum_{|t_2|=2} t^{t_2} D_{t_2} + \cdots$$

such that D preserves the ideal in $\mathbb{C}\{x_1,\ldots,x_n,t_1,\ldots,t_m\}$

$$\langle f_{x_1} + t_1 g_{1x_1} + \cdots + t_m g_{mx_1}, f_{x_2} + t_1 g_{1x_2} + \cdots + t_m g_{mx_2}, \ldots, f_{x_n} + t_1 g_{1x_n} + \cdots + g_{mx_n} \rangle.$$

Now, for the main geometric object in our context, the one-parameter (μ,τ)-invariant deformation $f_t := x^4 + y^4 + z^4 + tx^2y^2$ of the singularity $f_0 := x^4 + y^4 + z^4$, we calculate the liftable subalgebra $\tilde{L}(V_t)$ of the Yau algebra. For this purpose, we need the following calculation formula.

Lemma III.9 (Ref. 22). Consider the hypersurface $V = \{x \mid f(x) = x_1^d + \cdots + x_n^d = 0\}$ *defined by a homogeneous polynomial with an isolated singularity at the origin and its* (μ,τ)*-invariant deformation* $V_{\mathbf{t}} := \{(x_1,\ldots,x_n,t_1,\ldots,t_m) : f(\mathbf{x}) + t_1 g_1(\mathbf{x}) + \cdots + t_m g_m(\mathbf{x}) = 0\}$, *where* $g_1,\ldots,g_m$ *are elements of a set of monomial generators of* $A(V_0)$, *and* $\text{weight}(g_i) = \text{weight}(f)$ *for* $1 \le i \le m$. *When* $n = 3$ *or* $n = 4$, *we have*

$$\tilde{L}(f_t) = \begin{cases} \tilde{L}(f)_k = A(f)_k E, & k < D - d + 1; \\ \tilde{L}(f)_k = L(f)_k, & k \geq D - d + 1. \end{cases}$$

where

$$E = x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}$$

is the Euler derivation, and D is the highest degree of the monomial generators of the singularity module algebra.

Using this formula, we can see that for the one-parameter (μ, τ)-invariant deformation $f_t := x^4 + y^4 + z^4 + tx^2y^2$ of the singularity $f_0 := x^4 + y^4 + z^4$, once we obtain the results of the Yau algebra of the singularity, we can directly derive the results of the liftable Yau algebra. Here, $d = 4$ and $D = 6$.

Generators for $\tilde{L}(V_t)$Using Lemma III.9, we extract the basis of the 37-dimensional liftable subalgebra $\tilde{L}(V_t)$ from the generators of $L(V_t)$. The complete list of these generators is provided in Appendix A, Table II.

Lie Bracket Structure of N_t. The Torelli-type analysis in this paper focuses on the 36-dimensional nilradical $N_t = [\tilde{L}(f_t), \tilde{L}(f_t)]$. The Lie bracket multiplication table of N_t is highly complex and fundamental to all subsequent invariant computations. To maintain the readability of the main text, this full multiplication table is deferred to Appendix B, Table B.

IV. COMPUTATION OF INVARIANT SUBSPACES OF LIE ALGEBRAS

A. Isomorphism and cross-ratio of Lie algebras

We review the concept of cross-ratio, which plays a crucial role in subsequent proofs.

Definition IV.1. Let $\{Q_1 = (x_1 : y_1), Q_2 = (x_2 : y_2), Q_3 = (x_3 : y_3), Q_4 = (x_4 : y_4)\}$ *be an ordered set of four distinct points in* $\mathbb{CP}^1$. *The cross-ratio* r_A *of this ordered set is defined as*

$$r_A = (Q_1, Q_2; Q_3, Q_4) := \frac{(x_1y_3 - x_3y_1)(x_2y_4 - x_4y_2)}{(x_1y_4 - x_4y_1)(x_2y_3 - x_3y_2)}.$$

Another way to define the cross-ratio is: if we set $(x_3, y_3) = a(x_1, y_1) + b(x_2, y_2)$ *and* $(x_4, y_4) = c(x_1, y_1) + d(x_2, y_2)$ *where* $a, b, c, d \in \mathbb{C}$, *then the cross-ratio can be expressed as*

$$(Q_1, Q_2; Q_3, Q_4) := \frac{bc}{ad}.$$

Later, we will use the cross-ratio as an important tool to analyze whether two finite-dimensional Lie algebras are isomorphic as linear spaces. For this, we need the following important properties of the cross-ratio:

Lemma IV.2. For four distinct points Q_1, Q_2, Q_3, Q_4 *in* $\mathbb{P}^1$, *under different permutations of these points, the cross-ratio remains invariant under the action of the Klein group* K_4*:*

$$\{1, (Q_1Q_2)(Q_3Q_4), (Q_1Q_3)(Q_2Q_4), (Q_1Q_4)(Q_2Q_3)\}.$$

Specifically, we have:

$$(Q_1, Q_2; Q_3, Q_4) = (Q_2, Q_1; Q_4, Q_3) = (Q_3, Q_4; Q_1, Q_2) = (Q_4, Q_3; Q_2, Q_1).$$

Therefore, there are $S_4/K_4 \cong S_3$, *meaning that there are six possible values of the cross-ratio for all possible permutations of four points. These six possible values correspond to the orbits of the points under the action of* S_3*:*

$$(Q_1, Q_2; Q_3, Q_4) = \lambda \qquad (Q_1, Q_2; Q_4, Q_3) = \frac{1}{\lambda}$$
$$(Q_1, Q_3; Q_2, Q_4) = 1 - \lambda \qquad (Q_1, Q_3; Q_4, Q_2) = \frac{1}{1-\lambda}$$
$$(Q_1, Q_4; Q_2, Q_3) = \frac{\lambda - 1}{\lambda} \qquad (Q_1, Q_4; Q_3, Q_2) = \frac{\lambda}{\lambda - 1}.$$

Lemma IV.3. Let $A = \{(x_1 : y_1), \ldots, (x_4 : y_4)\}$ *and* $B = \{(z_1 : w_1), \ldots, (z_4 : w_4)\}$ *be two ordered sets of four distinct points in* $\mathbb{CP}^1$. *There exists a linear automorphism* ϕ *of* $\mathbb{CP}^1$ *that maps the set A to the set B if and only if the sets of all possible cross-ratios for A and B are equal, i.e.,*

$$\left\{r_A, r_A^{-1}, \frac{r_A}{r_A - 1}, \frac{r_A - 1}{r_A}, \frac{1}{1 - r_A}, 1 - r_A\right\} = \left\{r_B, r_B^{-1}, \frac{r_B}{r_B - 1}, \frac{r_B - 1}{r_B}, \frac{1}{1 - r_B}, 1 - r_B\right\}.$$

Methodology. The method of using the identical cross-ratio as a necessary condition for the isomorphism of corresponding Lie algebras has been employed in Refs. 7 and 15 to address the Torelli-type problem for singularities of types $\tilde{E}_7$ and $\tilde{E}_8$.

In this paper, we consider the singularity $f_0 = x^4 + y^4 + z^4$. In its deformation V_t, the Yau algebra $L(V_t)$ has a high dimension (51-dim), making analysis challenging. Therefore, we first consider its liftable subalgebra $\tilde{L}(V_t)$ (37-dim) and its nilradical $N_t = [\tilde{L}(f_t), \tilde{L}(V_t)]$ (36-dim), which is spanned by $\{E_2, \ldots, E_{37}\}$ (see Appendix A, Tables II and III). An isomorphism $\tilde{L}(V_t) \simeq \tilde{L}(V_s)$ induces an isomorphism $N_t \simeq N_s$.

Our approach is to identify four invariant lines within a two-dimensional invariant subspace of N_t. These four lines serve as four distinct points Q_1, Q_2, Q_3, Q_4 in $\mathbb{CP}^1$. Since the cross-ratio of these lines is invariant under automorphisms, we obtain a necessary condition for $\tilde{L}_t \simeq \tilde{L}_s$, thus solving the Torelli-type problem for $\tilde{L}(V_t)$. To this end, we first need to construct the corresponding invariant subspaces, namely the upper and lower central series of N_t.

B. Upper and lower central series of finite-dimensional lie algebras

The concepts and properties of the upper central series and lower central series can be found in Ref. 25.

Definition IV.4 (Upper Central Series). For a Lie algebra $\mathfrak{g}$, its upper central series is an ascending sequence of ideals:

$$Z^0(\mathfrak{g}) \subset Z^1(\mathfrak{g}) \subset Z^2(\mathfrak{g}) \subset \ldots$$

defined inductively by $Z^0(\mathfrak{g}) = \{0\}$, $Z^1(\mathfrak{g}) = Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] = 0\}$, and

$$Z^{i+1}(\mathfrak{g})/Z^i(\mathfrak{g}) = Z(\mathfrak{g}/Z^i(\mathfrak{g})).$$

Using the adjoint map $ad_x(y) = [x, y]$, this is equivalent to:

$$Z^{i+1}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid \text{Image}\,(ad_x) \subseteq Z^i(\mathfrak{g})\}.$$

Definition IV.5 (Lower Central Series). For a Lie algebra $\mathfrak{g}$, its lower central series is a descending sequence of ideals:

$$\mathfrak{g} = D_0(\mathfrak{g}) \supset D_1(\mathfrak{g}) \supset D_2(\mathfrak{g}) \supset \ldots$$

defined inductively by $D_0(\mathfrak{g}) = \mathfrak{g}$ and

$$D_{i+1}(\mathfrak{g}) = [\mathfrak{g}, D_i(\mathfrak{g})].$$

Theorem IV.6. *For a nilpotent Lie algebra $\mathfrak{g}$, its upper and lower central series are of finite length and have the same length c, which is called the nilpotency class of $\mathfrak{g}$.*

C. Algorithms to compute the upper and lower central series of N_t

For the 36-dimensional nilradical $N_t = \text{span}_{\mathbb{C}}\{E_2, \ldots, E_{37}\}$, we rely on computer assistance to compute its central series. Translating the structural computation of Lie algebras (such as central series) into problems of linear algebra is a central branch of computational Lie algebra theory, the foundations of which are discussed in Ref. 26. This involves translating the definitions into the language of linear algebra and solving systems of linear equations based on the Lie bracket multiplication table (Appendix B, Table B).

1. N_t upper central series

To find $Z^1(N_t) = Z(N_t)$, we seek elements $x = \sum_{i=2}^{37} a_i E_i$ such that $ad_x(E_j) = 0$ for all $j = 2, \ldots, 37$. This translates into a large, sparse system of linear equations for the coefficients $\{a_i\}$. To find $Z^2(N_t)$, we seek elements x such that $\text{Image}(ad_x) \subseteq Z^1(N_t)$. This defines a more complex matrix equation. We repeat this process until $Z^c(N_t) = N_t$.

The explicit construction of these matrix equations is computationally intensive but algorithmically straightforward. The results of this computation for N_t (where $t^2 \neq 4, 36$) are:

(UCS)

- $Z^0(N_t) = \{0\}$
- $Z^1(N_t) = \text{span}_{\mathbb{C}}\{E_{35}, E_{36}, E_{37}\}$, $(dim = 3)$
- $Z^2(N_t) = \text{span}_{\mathbb{C}}\{E_{26}, \ldots, E_{37}\}$, $(dim = 12)$
- $Z^3(N_t) = \text{span}_{\mathbb{C}}\{E_{11}, \ldots, E_{37}\}$, $(dim = 27)$
- $Z^4(N_t) = \text{span}_{\mathbb{C}}\{E_2, \ldots, E_{37}\} = N_t$, $(dim = 36)$

2. Lower central series of N_t

The computation of the lower central series is direct. Using the multiplication table (Appendix B, Table B), we find:

(LCS)

- $D_0(N_t) = N_t$
- $D_1(N_t) = [N_t, N_t] = \text{span}_{\mathbb{C}}\{E_{11}+E_{22}, E_{12}+E_{18}+E_{24}, E_{13}-(\frac{4}{t^2-4})E_{25}-(\frac{2t}{t^2-4})E_{23}, E_{14}+E_{19}, E_{15}+E_{20}, E_{16}+E_{21}, E_{17}-(\frac{4}{t^2-4})E_{23} -(\frac{2t}{t^2-4})E_{25}, E_{26}, E_{27}, E_{28}, E_{29}, E_{30}, E_{31}, E_{32}, E_{33}, E_{34}, E_{35}, E_{36}, E_{37}\}$, (dim = 19)
- $D_2(N_t) = [N_t, D_1(N_t)] = \text{span}_{\mathbb{C}}\{E_{26}+E_{34}, E_{27}+E_{31}, E_{28}, E_{29}+E_{33}, E_{30}, E_{32}, E_{35}, E_{36}, E_{37}\}$, (dim = 9)
- $D_3(N_t) = [N_t, D_2(N_t)] = \text{span}_{\mathbb{C}}\{E_{35}, E_{36}, E_{37}\}$, (dim = 3)
- $D_4(N_t) = [N_t, D_3(N_t)] = \{0\}$, (dim = 0)

The nilpotency class is $c = 4$, consistent with the UCS result.

3. Quotient spaces of invariant subspaces of N_t

From the (UCS) and (LCS), we define the following invariant quotient spaces which form the basis of our analysis:

1. $\mathbf{G_t^1} := Z_t^4(N_t)/Z_t^3(N_t) = \text{span}_{\mathbb{C}}\{\bar{E}_2, \bar{E}_3, \bar{E}_4, \bar{E}_5, \bar{E}_6, \bar{E}_7, \bar{E}_8, \bar{E}_9, \bar{E}_{10}\}$, It is generated by the equivalence classes of E_i of weights 1 or 2;
2. $\mathbf{G_t^2} := Z_t^3(N_t)/Z_t^2(N_t) = \text{span}_{\mathbb{C}}\{\bar{E}_{11}, \bar{E}_{12}, \bar{E}_{13}, \bar{E}_{14}, \bar{E}_{15}, \bar{E}_{16}, \bar{E}_{17}, \bar{E}_{18}, \bar{E}_{19}, \bar{E}_{20}, \bar{E}_{21}, \bar{E}_{22}, \bar{E}_{23}, \bar{E}_{24}, \bar{E}_{25}\}$, It is generated by the equivalence classes of E_i of weights 3;
3. $\mathbf{G_t^3} := Z_t^2(N_t)/Z_t^1(N_t) = \text{span}_{\mathbb{C}}\{\bar{E}_{26}, \bar{E}_{27}, \bar{E}_{28}, \bar{E}_{29}, \bar{E}_{30}, \bar{E}_{31}, \bar{E}_{32}, \bar{E}_{33}, \bar{E}_{34}\}$, It is generated by the equivalence classes of E_i of weights 4;
4. $\mathbf{G_t^4} := Z_t^1(N_t)/\{0\} = \text{span}_{\mathbb{C}}\{\bar{E}_{35}, \bar{E}_{36}, \bar{E}_{37}\}$, It is generated by the equivalence classes of E_i of weights 5;
5. $\mathbf{H_t^1} := D_1(N_t)/Z^2(N_t) = \text{span}\,\{\bar{E}_{11}+\bar{E}_{22}, \bar{E}_{12}+\bar{E}_{18}+\bar{E}_{24}, \bar{E}_{13}-(\frac{4}{t^2-4})\bar{E}_{25}-\{\frac{2t}{t^2-4}\bar{E}_{23}, \bar{E}_{14}+\bar{E}_{19}, \bar{E}_{15}+\bar{E}_{20}, \bar{E}_{16}+\bar{E}_{21}, \bar{E}_{17}-(\frac{4}{t^2-4})\bar{E}_{23} -(\frac{2t}{t^2-4})\bar{E}_{25}\}$. (dim = 7)
6. $\mathbf{H_t^2} := Z^3(N_t)/D_1(N_t) = \text{span}\,\{\bar{E}_{18}, \bar{E}_{19}, \bar{E}_{20}, \bar{E}_{21}, \bar{E}_{22}, \bar{E}_{23}, \bar{E}_{24}, \bar{E}_{25}\}$; (dim = 8)
7. $\mathbf{H_t^3} := D_2(N_t)/Z_t^1(N_t) = \text{span}\,\{\bar{E}_{26}+\bar{E}_{34}, \bar{E}_{27}+\bar{E}_{31}, \bar{E}_{28}, \bar{E}_{29}+\bar{E}_{33}, \bar{E}_{30}, \bar{E}_{32}\}$; (dim = 6)
8. $\mathbf{H_t^4} := Z_t^2(N_t)/D_2(N_t) = \text{span}\,\{\bar{E}_{31}, \bar{E}_{33}, \bar{E}_{34}\}$. (dim = 3)

4. Mappings between invariant subspaces of N_t

We analyze the linear maps $ad_{\bar{x}} : \bar{y} \mapsto [\bar{x}, \bar{y}]$ between these quotient spaces.

Lemma IV.7. Let $G_t^i = \langle \overline{E}_i^1, \ldots, \overline{E}_i^s \rangle$ and $G_t^j = \langle \overline{E}_j^1, \ldots, \overline{E}_j^r \rangle$ be invariant subspaces. For any $\bar{x} = \sum a_k \overline{E}_i^k \in G_t^i$, the linear map $ad_{\bar{x}} : G_t^j \to G_t^{i+j}$ has a matrix representation $M_{\bar{x}} = \sum a_k M_k$, where M_k is the matrix for $ad_{\overline{E}_i^k}$.

5. Some results of the representation matrices of adjoint maps

The following mappings and their matrix representations are used in our calculation of invariants. The full matrices are deferred to Appendix C due to their size.

(1) *Mapping $ad_{\bar{x}}^1 : G_t^1 \to G_t^2$* For $\bar{x} = \sum_{i=2}^{10} a_i \bar{E}_i \in G_t^1$, the map $ad_{\bar{x}}^1 : G_t^1 \to G_t^2$ is represented by a 15×9 matrix $M_1(1)$ whose entries are linear in $\{a_i\}$. The explicit form of $M_1(1)$ is given in Appendixes C and C1.

(2) *Mapping $ad_{\bar{x}}^2 : G_t^2 \to G_t^3 \oplus G_t^4$* For $\bar{x} \in G_t^1$, the map $ad_{\bar{x}}^2 : G_t^2 \to G_t^3 \oplus G_t^4$ is represented by a 12×15 matrix $M_1(2)$. This matrix is presented in Appendixes C and C2.

(3) *Mapping $ad_{\bar{x}} : G_t^1 \times G_t^3 \to G_t^4$* For $\bar{x} \in G_t^1$, the map $ad_{\bar{x}} : G_t^3 \to G_t^4$ is represented by the 3×9 matrix $M_1(3)$:

$$M_1(3) = \begin{pmatrix} 4a_4 & 4a_3 & 3a_2 & 0 & 0 & -a_3 & 0 & 0 & -a_4 \\ 0 & -a_2 & 0 & 4a_4 & 3a_3 & 4a_2 & 0 & -a_4 & 0 \\ -a_2 & 0 & 0 & -a_3 & 0 & 3a_4 & 4a_3 & 4a_2 & \end{pmatrix}$$

(4) *Mapping $ad_{\bar{x}} : G_t^2 \to G_t^3 \oplus G_t^4$* For $\bar{x} \in G_t^2 = \text{span}_{\mathbb{C}}\{\overline{E}_{11}, \ldots, \overline{E}_{25}\}$, the map $ad_{\bar{x}} : G_t^1 \to G_t^3 \oplus G_t^4$ is represented by a 12×9 matrix $M_2(1)$. This matrix is presented in Appendixes C and C3.

(5) *Decomposing G_t^1* We define P_t^1 as the kernel of the map $ad_{\bar{x}}(G_t^3)$ from (3), i.e.,

$$P_t^1 := \{\bar{x} \in G_t^1 \mid ad_{\bar{x}}(G_t^3) = 0\} = \{\bar{x} \in G_t^1 \mid M_1(3) = 0\}.$$

Solving $M_1(3) = 0$ [from (3) above] yields $a_2 = a_3 = a_4 = 0$. Thus, G_t^1 decomposes into two invariant subspaces:

$$P_t^0 := \text{span}_{\mathbb{C}}\{\overline{E}_2, \overline{E}_3, \overline{E}_4\}$$
$$P_t^1 := \text{span}_{\mathbb{C}}\{\overline{E}_5, \overline{E}_6, \overline{E}_7, \overline{E}_8, \overline{E}_9, \overline{E}_{10}\}$$

such that $G_t^1 = P_t^0 \oplus P_t^1$.

(6) *Mapping* $ad_{\bar{x}} : P_t^1 \to G_t^2$ For $\bar{x} \in P_t^0$, the map $ad_{\bar{x}} : P_t^1 \to G_t^2$ is represented by a 15×6 matrix $P_0(1)$. This matrix is presented in Appendixes C and C4.
(7) Mapping $ad_{\bar{x}} : G_t^2 \to G_t^3$ For $\bar{x} \in P_t^0$, the map $ad_{\bar{x}} : G_t^2 \to G_t^3$ is represented by a 9×15 matrix $P_0(2)$. This matrix is presented in Appendixes C and C5.
(8) *Mapping* $ad_{\bar{x}} : G_t^3 \to G_t^4$ For $\bar{x} \in P_t^0$, the map $ad_{\bar{x}} : G_t^3 \to G_t^4$ is represented by the 3×9 matrix $P_0(3)$:

$$P_0(3) = \begin{pmatrix} 4a_4 & 4a_3 & 3a_2 & 0 & 0 & -a_3 & 0 & 0 & -a_4 \\ 0 & -a_2 & 0 & 4a_4 & 3a_3 & 4a_2 & 0 & -a_4 & 0 \\ -a_2 & 0 & 0 & -a_3 & 0 & 3a_4 & 4a_3 & 4a_2 & \end{pmatrix}$$

[Note: $P_0(3)$ has the same form as $M_1(3)$ but is restricted to a_2, a_3, a_4.]
(9) *Mapping* $ad_{\bar{x}} : P_t^0 \to G_t^2$ For $\bar{x} \in P_t^1$, the map $ad_{\bar{x}} : P_t^0 \to G_t^2$ is represented by a 15×3 matrix $P_1(0)$. This matrix is presented in Appendixes C and C6.
(10) *Mapping* $ad_{\bar{x}} : G_t^2 \to G_t^4$ For $\bar{x} \in P_t^1$, the map $ad_{\bar{x}} : G_t^2 \to G_t^4$ is represented by a 3×15 matrix $P_1(2)$. This matrix is presented in Appendixes C and C7.

D. Computing the invariant subspaces and the cross-ratios of $\tilde{L}(V_t)$ and proofs of the main theorems

With the above setup, we now find invariant lines by analyzing these maps.

(1) *Finding* l_1: For $\bar{x} \in P_t^0$, consider the composite map $(ad_{\bar{x}})^3 : P_t^1 \to G_t^4$, given by

$$S_0(3) = P_0(3) \cdot P_0(2) \cdot P_0(1).$$

The matrix $S_0(3)$ is a 3×6 matrix whose entries are complex expressions in t, a_2, a_3, a_4. The explicit form is deferred to Appendixes C and C8. We consider the kernel space:

$$\left\{ \bar{x} = \sum_{i=2}^{4} a_i \overline{E}_i \in P_t^0 \mid (ad_{\bar{x}})^3 (P_t^1) = 0 \right\} = \{ \bar{x} \in P_t^0 \mid S_0(3) = 0 \}.$$

Solving $S_0(3) = 0$ for $t \notin \{0, \pm 6\}$ yields $a_2 = a_3 = 0$. Therefore, we obtain our first 1-dimensional invariant subspace (invariant line) of N_t:

$$l_1 := \mathbb{C}\overline{E}_4 \subseteq P_t^0 \subseteq G_t^1.$$

(2) *K1 (Finding* l_2*)*: We now use $l_1 = \mathbb{C}\overline{E}_4$ to define new maps. Let ad_{l_1} be the map $ad_{\bar{x}}$ where $\bar{x} = \overline{E}_4$ (i.e., $a_2 = 0, a_3 = 0, a_4 = 1$). Consider $ad_{l_1} : P_t^1 \to G_t^2$. Its matrix $Ad_{l_1}(1)$ is the matrix $P_0(1)$ (from Appendixes C and C4) evaluated at $a_2 = 0, a_3 = 0, a_4 = 1$. We compute its kernel:

$$\ker (ad_{l_1} : P_t^1 \to G_t^2) = \left\{ \sum_{i=5}^{10} a_i \overline{E}_i \mid Ad_{l_1}(1) \cdot [a_5, \ldots, a_{10}]^T = 0 \right\}.$$

The solution is $a_5 = \cdots = a_9 = 0$. This yields a second invariant line:

$$l_2 := \mathbb{C}\overline{E}_{10} \subseteq P_t^1 \subseteq G_t^1.$$

(3) *K2*: Consider $ad_{l_1} : G_t^2 \to G_t^3$. Its matrix $Ad_{l_1}(2)$ is $P_0(2)$ (Appendixes C and C5) evaluated at $a_2 = 0, a_3 = 0, a_4 = 1$. We compute its kernel:

$$\ker (ad_{l_1} : G_t^2 \to G_t^3) = \left\{ \sum_{i=11}^{25} a_i \overline{E}_i \mid Ad_{l_1}(2) \cdot [a_{11}, \ldots, a_{25}]^T = 0 \right\} = \operatorname{span}_{\mathbb{C}} \left\{ \overline{E}_{14}, \overline{E}_{15}, \overline{E}_{19}, \overline{E}_{20}, \frac{1}{3}\overline{E}_{17} + \overline{E}_{23} - \frac{t}{6}\overline{E}_{13}, \frac{1}{3}\overline{E}_{12} + \frac{1}{3}\overline{E}_{18} + \overline{E}_{24}, \frac{1}{3}\overline{E}_{13} + \overline{E}_{25} - \frac{t}{6}\overline{E}_{17} \right\} \subseteq G_t^2.$$

This is a 7-dimensional invariant subspace.
(4) *K3*: Consider $ad_{l_1} : G_t^3 \to G_t^4$. Its matrix $Ad_{l_1}(3)$ is $P_0(3)$ evaluated at $a_2 = 0, a_3 = 0, a_4 = 1$. We compute its kernel:

$$\ker (ad_{l_1} : G_t^3 \to G_t^4) = \left\{ \sum_{i=26}^{34} a_i \overline{E}_i \mid Ad_{l_1}(3) \cdot [a_{26}, \ldots, a_{34}]^T = 0 \right\} = \operatorname{span}_{\mathbb{C}} \left\{ \overline{E}_{27}, \overline{E}_{28}, \overline{E}_{30}, \overline{E}_{31}, \frac{1}{4}\overline{E}_{29} + \overline{E}_{33}, \frac{1}{4}\overline{E}_{26} + \overline{E}_{34} \right\} \subseteq G_t^3.$$

This is a 6-dimensional invariant subspace.

(5) *Im1*: Consider the image of $ad_{l_1}: P_t^1 \to G_t^2$:

$$\text{Image}\,(ad_{l_1}: P_t^1 \to G_t^2) = \text{span}_{\mathbb{C}}\left\{\overline{E}_{12} + \overline{E}_{18} + \overline{E}_{24}, \overline{E}_{13} - \frac{4}{t^2-4}\overline{E}_{25} - \frac{2t}{t^2-4}\overline{E}_{23}, \overline{E}_{14} + \overline{E}_{19}, \overline{E}_{15} + \overline{E}_{20}, \overline{E}_{17} - \frac{4}{t^2-4}\overline{E}_{23} - \frac{2t}{t^2-4}\overline{E}_{25}\right\} \subseteq G_t^2.$$

This is a 5-dimensional invariant subspace.

(6) *Im2*: Consider the image of $ad_{l_1}: G_t^2 \to G_t^3$:

$$\text{Image}\,(ad_{l_1}: G_t^2 \to G_t^3) = \text{span}_{\mathbb{C}}\{E_{26}, E_{27}, E_{28}, E_{29}, E_{30}, E_{31}, E_{33}, E_{34}\} \subseteq G_t^3.$$

This is an 8-dimensional invariant subspace.

(7) *Im3*: Consider the image of $ad_{l_1}: G_t^3 \to G_t^4$:

$$\text{Image}\,(ad_{l_1}: G_t^3 \to G_t^4) = \text{span}_{\mathbb{C}}\{E_{35}, E_{36}, E_{37}\} = G_t^4.$$

(8) *Ins1*: We intersect the 7-dim space $H_t^1 \subseteq G_t^2$ (from 4.3.3) and the 7-dim space $\ker\,(ad_{l_1}: G_t^2 \to G_t^3)$ (from K2):

$$H_t^1 \cap \ker\,(ad_{l_1}: G_t^2 \to G_t^3) = \text{span}_{\mathbb{C}}\{\overline{E}_{14} + \overline{E}_{19}, \overline{E}_{15} + \overline{E}_{20}\} \subseteq G_t^2.$$

This is a 2-dimensional invariant subspace, which we denote $W_{14,15}$.

(9) *Ins2:* We intersect the 5-dim space $\text{Image}\,(ad_{l_1}: P_t^1 \to G_t^2)$ (from Im1) and the 7-dim space $\ker\,(ad_{l_1}: G_t^2 \to G_t^3)$ (from K2):

$$\ker\,(ad_{l_1}: G_t^2 \to G_t^3) \cap \text{Image}\,(ad_{l_1}: P_t^1 \to G_t^2) = \text{span}_{\mathbb{C}}\{\overline{E}_{14} + \overline{E}_{19}, \overline{E}_{15} + \overline{E}_{20}\} = W_{14,15}.$$

This confirms $W_{14,15}$ is a robustly defined 2-dim invariant subspace.

(10) *Ins3*: We intersect the 7-dim space $\ker\,(ad_{l_1}: G_t^2 \to G_t^3)$ (from K2) and the 8-dim space $H_t^2 \subseteq G_t^2$ (from 4.3.3):

$$\ker\,(ad_{l_1}: G_t^2 \to G_t^3) \cap H_t^2 = \text{span}_{\mathbb{C}}\{\overline{E}_{19}, \overline{E}_{20}\} \subseteq G_t^2.$$

This is a 2-dimensional invariant subspace, which we denote $W_{19,20}$.

(11) *Ins4*: We intersect the 6-dim space $\ker\,(ad_{l_1}: G_t^3 \to G_t^4)$ (from K3) and the 6-dim space $H_t^3 \subseteq G_t^3$ (from 4.3.3):

$$\ker\,(ad_{l_1}: G_t^3 \to G_t^4) \cap H_t^3 = \text{span}_{\mathbb{C}}\{\overline{E}_{27} + \overline{E}_{31}, \overline{E}_{28}, \overline{E}_{30}\} \subseteq G_t^3.$$

This is a 3-dimensional invariant subspace.

(12) *Ins5 (Finding l_3)*: We intersect the 8-dim space $\text{Image}\,(ad_{l_1}: G_t^2 \to G_t^3)$ (from Im2) and the 6-dim space H_t^3:

$$\text{Image}\,(ad_{l_1}: G_t^2 \to G_t^3) \cap H_t^3 = \text{span}_{\mathbb{C}}\{\overline{E}_{26} + \overline{E}_{34}, \overline{E}_{27} + \overline{E}_{31}, \overline{E}_{28}, \overline{E}_{29} + \overline{E}_{33}, \overline{E}_{30}\}.$$

This is a 5-dim invariant space. Since H_t^3 $(dim = 6)$ is the direct sum $H_t^3 = (\text{Image}\,(\dots) \cap H_t^3) \oplus \mathbb{C}\overline{E}_{32}$, we obtain a new invariant line:

$$l_3 := \mathbb{C}\overline{E}_{32} \subseteq G_t^3.$$

(13) *Ins6 (Finding l_4)*: We intersect the 6-dim space $\ker\,(ad_{l_1}: G_t^3 \to G_t^4)$ (from K3) and the 3-dim space $H_t^4 \subseteq G_t^3$ (from 4.3.3):

$$\ker\,(ad_{l_1}: G_t^3 \to G_t^4) \cap H_t^4 = \mathbb{C}\overline{E}_{31} := l_4 \subseteq G_t^3.$$

This yields another invariant line.

(14) *UVW1*: We now define a "UVW-type" invariant space:

$$\{\bar{y} \in U_t \mid \forall \bar{x} \in V_t, ad_{\bar{x}}(\bar{y}) \in W_t\}.$$

Let $U_t = P_t^1$, $V_t = l_1$, $W_t = W_{14,15} = \text{span}_{\mathbb{C}}\{\overline{E}_{14} + \overline{E}_{19}, \overline{E}_{15} + \overline{E}_{20}\}$ (from Ins1). We compute:

$$\{\bar{x} \in P_t^1 \mid \text{Image}\,(ad_{l_1}(\bar{x})) \subseteq W_{14,15}\}.$$

This requires solving $Ad_{l_1}(1) \cdot [a_5, \dots, a_{10}]^T \subseteq W_{14,15}$, which yields a 3-dim invariant subspace $\text{span}_{\mathbb{C}}\{\overline{E}_8, \overline{E}_9, \overline{E}_{10}\}$. Since $l_2 = \mathbb{C}\overline{E}_{10}$ (from K1) is in this space, we can form the 2-dim invariant quotient space:

$$W_{8,9} := \text{span}_{\mathbb{C}}\{\overline{E}_8, \overline{E}_9\}.$$

(15) *UVW2 (Finding l_5)*: Let $U_t = G_t^2$, $V_t = l_1$, $W_t = l_4 = \mathbb{C}\overline{E}_{31}$ (from Ins6). We compute:

$$\{\bar{x} \in G_t^2 \mid \text{Image}\,(ad_{l_1}(\bar{x})) \subseteq \mathbb{C}\overline{E}_{31}\}.$$

Solving $Ad_{l_1}(2) \cdot [a_{11}, \ldots, a_{25}]^T \subseteq \mathbb{C}\overline{E}_{31}$ yields an 8-dim invariant space. This 8-dim space contains the 7-dim kernel $\ker\,(ad_{l_1} : G_t^2 \to G_t^3)$ (from K2). The 1-dim complement is an invariant line:

$$l_5 := \mathbb{C}\overline{E}_{18} \subseteq G_t^2.$$

(16) *Im4 (Finding l_6)*: Using the new invariant line $l_5 = \mathbb{C}\overline{E}_{18}$, we compute its image:

$$\text{Image}\,(ad_{l_5} : G_t^1 \to G_t^3 \oplus G_t^4) = \text{span}_{\mathbb{C}}\{E_{29}, E_{26} + E_{34}, E_{31}, E_{35}, E_{36}, E_{37}\}.$$

We intersect this image (restricted to G_t^3) with the 6-dim space H_t^3:

$$(\text{Image}\,(ad_{l_5}) \cap G_t^3) \cap H_t^3 = \mathbb{C}(\overline{E}_{26} + \overline{E}_{34}) := l_6 \subseteq G_t^3.$$

This yields another invariant line.

(17) *K4 (Finding l_7 and $W_{5,7}$)*: Using $l_5 = \mathbb{C}\overline{E}_{18}$, we compute its kernel:

$$\ker\,(ad_{l_5} : G_t^1 \to G_t^3 \oplus G_t^4) = \text{span}_{\mathbb{C}}\{\overline{E}_5, \overline{E}_7, \overline{E}_{10}\} \subseteq P_t^1.$$

We also know $l_2 = \mathbb{C}\overline{E}_{10}$ (from K1) and $W_{8,9} = \text{span}_{\mathbb{C}}\{\overline{E}_8, \overline{E}_9\}$ (from UVW1). We find P_t^1 $(dim = 6)$ decomposes as:

$$P_t^1 = \ker\,(ad_{l_5}) \oplus W_{8,9} \oplus \mathbb{C}\overline{E}_6.$$

This decomposition yields a new invariant line $l_7 := \mathbb{C}\overline{E}_6 \subseteq P_t^1$ and an invariant 2-dim space $W_{5,7} := \text{span}_{\mathbb{C}}\{\overline{E}_5, \overline{E}_7\}$.

(18) *D1 (Finding L_1, L_2)*: We now have the 2-dim invariant space $W_{5,7} \subset P_t^1$ and the 2-dim invariant space $W_{14,15} \subset G_t^2$ (from Ins1). We find the lines in $W_{5,7}$ whose adjoint map has a non-trivial kernel when restricted to $W_{14,15}$. We seek $\bar{x} \in W_{5,7}$ such that

$$\dim\ker\,(ad_{\bar{x}} : W_{14,15} \to G_t^4) \geq 1.$$

This requires $\bar{x} = a_5\overline{E}_5 + a_7\overline{E}_7$ such that the determinant of the 2×2 representation matrix $P_1(2) \cdot A$ (where A is the inclusion of $W_{14,15}$) is zero. The matrix is given in Appendixes C and C9. Setting its determinant to zero gives the ideal $\mathfrak{a}$:

$$\mathfrak{a} = \left(a_5 a_7 - \frac{(a_5^2)t}{2} - \frac{(a_7^2)t}{2} + \frac{a_5 a_7 t^2}{4}\right).$$

The variety $V(\mathfrak{a})$ in $W_{5,7} \cong \mathbb{P}^1$ consists of two distinct points (lines):

$$L_1 := \mathbb{C}\left(\frac{t}{2}\overline{E}_5 + \overline{E}_7\right)$$
$$L_2 := \mathbb{C}\left(\overline{E}_5 + \frac{t}{2}\overline{E}_7\right).$$

(19) *D2 (Finding minimal set of generators of an ideal)*: We consider another "Dimension-type" space: $\bar{x} \in P_t^1$ such that $\dim(ad_{\bar{x}} : P_t^0 \to G_t^2) \leq 2$. This is equivalent to finding $\bar{x} = \sum_{i=5}^{10} a_i\overline{E}_i$ where all 3×3 minors of the matrix $P_1(0)$ (from Appendixes C and C6) are zero.

$$\left\{\sum_{i=5}^{10} a_i\overline{E}_i \in P_t^1 \mid \text{All } 3-\text{minors of } P_1(0) \text{ are zero}\right\}.$$

Let $\mathfrak{a}$ be the ideal in $\mathbb{C}[a_5, \ldots, a_{10}, t]$ generated by these 3-minors. This ideal $\mathfrak{a}$ has a set of 35 minimal generators. This ideal $\mathfrak{a}$ has a minimal primary decomposition $\mathfrak{a} = \cap_{m=1}^{7}\mathfrak{q}'_{i_m}$. The 7 associated prime ideals $\{\mathfrak{p}_{i_1}, \ldots, \mathfrak{p}_{i_7}\}$ correspond to 7 irreducible components of the variety $V(\mathfrak{a})$. Projecting these components onto the invariant subspace $W_{5,7} \oplus \mathbb{C}\overline{E}_{10}$ (by setting $a_6 = a_8 = a_9 = 0$), we find that for $t \notin \{0, \pm 6\}$, the variety is the union of three lines:

$$C_1 := \mathbb{C}\left(\frac{t}{2}\overline{E}_5 + \overline{E}_7\right), \quad C_2 := \mathbb{C}\left(\overline{E}_5 + \frac{t}{2}\overline{E}_7\right), \quad C_3 := \mathbb{C}\overline{E}_{10}.$$

This confirms $L_1 = C_1$ and $L_2 = C_2$ (from D1) and $l_2 = C_3$ (from K1) are robustly defined invariant lines.

(20) *D3 (Finding L_3, L_4)*: Finally, we analyze the map $ad_{\bar{x}} : W_{19,20} \to G_t^4$ for $\bar{x} \in W_{5,7}$. $W_{19,20} = \text{span}_{\mathbb{C}}\{\overline{E}_{19}, \overline{E}_{20}\}$ is the 2-dim invariant space from (Ins3). We seek $\bar{x} = a_5\overline{E}_5 + a_7\overline{E}_7$ such that

$$\dim\ker\left(ad_{\bar{x}} : W_{19,20} \to G_t^4\right) \geq 1.$$

This requires the determinant of the 2×2 representation matrix $P_1(2) \cdot B$ (where B is the inclusion of $W_{19,20}$) to be zero. This matrix is given in Appendixes C and C10. Setting its determinant to zero gives the ideal $\mathfrak{a}'$:

$$\mathfrak{a}' = \left(\frac{(t^2-4)(18a_5 - a_7 t)(-4a_7t^2 + 2a_5 t + 12a_7)}{(t^2-12)^2}\right).$$

The variety $V(\mathfrak{a}')$ in $W_{5,7} \cong \mathbb{P}^1$ consists of two distinct lines:

$$L_3 := \mathbb{C}\left(\frac{t}{18}\overline{E}_5 + \overline{E}_7\right)$$
$$L_4 := \mathbb{C}\left(\frac{2(t^2-3)}{t}\overline{E}_5 + \overline{E}_7\right).$$

Conclusion: The Cross-Ratio. We have successfully identified four invariant lines L_1, L_2, L_3, L_4 that all lie within the 2-dimensional invariant space $W_{5,7} = \text{span}_{\mathbb{C}}\{\overline{E}_5, \overline{E}_7\}$. We can compute their cross-ratio. Using $\overline{E}_5$ as basis vector $(1:0)$ and $\overline{E}_7$ as $(0:1)$, the lines correspond to the points:

$$L_1 = \left(\frac{t}{2} : 1\right) \qquad L_2 = \left(1 : \frac{t}{2}\right)$$
$$L_3 = \left(\frac{t}{18} : 1\right) \qquad L_4 = \left(\frac{2(t^2-3)}{t} : 1\right).$$

The cross-ratio $\delta = (L_1, L_2;\ L_3, L_4)$ is computed as:

$$\delta(t) = (L_1, L_2; L_3, L_4) = \frac{-32t^2}{3(t^2-36)}.$$

The set of all possible cross-ratios for these four lines is:

$$\left\{\frac{-32t^2}{3(t^2-36)}, \frac{-3(t^2-36)}{32t^2}, \frac{3(t^2-36)}{35t^2-108}, \frac{35t^2-108}{3(t^2-36)}, \frac{32t^2}{35t^2-108}, \frac{35t^2-108}{32t^2}\right\}.$$

Proof of Theorem A. If $N_t \simeq N_s$, any isomorphism must map the invariant 2-space $W_{5,7}(t)$ to $W_{5,7}(s)$, and must map the set of four invariant lines $\{L_1(t), \ldots, L_4(t)\}$ to $\{L_1(s), \ldots, L_4(s)\}$. By Lemma IV.3, their sets of cross-ratios must be equal. This implies $\delta(t)$ must be equal to one of the six values in the set for $\delta(s)$. A direct calculation shows that this leads to only two possibilities (for $t, s \notin \{\pm 2, \pm 6\}$):

$$t^2 = s^2 \quad \text{or} \quad t^2 = -\frac{324(s^2-36)}{1015s^2+324}.$$

These are the necessary conditions (P1) and (P2) for $N_t \simeq N_s$. Since $\tilde{L}(V_t) \simeq \tilde{L}(V_s) \Rightarrow N_t \simeq N_s$, these are the necessary conditions for Theorem A. [The sufficiency is covered by (2) and (3) in the theorem statement.]

Proof of Theorem B and Final Comparison. With the necessary conditions for $N_t \simeq N_s$ described by the parameter relations (P1) $t^2 = s^2$ and (P2) $t^2 = -\frac{324(s^2-36)}{1015s^2+324}$, we compare this condition with the necessary and sufficient conditions for the corresponding singularities $[V(f_t), 0]$ and $[V(f_s), 0]$ to be isomorphic (i.e., $f_t \sim f_s$, see Definition II.5).

According to the definition of $f_t \sim f_s$, we seek an automorphism φ of $\mathcal{O}_3 = \mathbb{C}\{x, y, z\}$ such that $f_t(\varphi(x, y, z)) = f_s(x, y, z)$ (since f_t, f_s are homogeneous, the unit u is 1). We can assume φ is of the form:

$$\varphi : \begin{cases} x' = a_0 x + a_1 y + a_2 z \\ y' = a_3 x + a_4 y + a_5 z \\ z' = z \end{cases} \tag{1}$$

where $a_0, \ldots, a_5 \in \mathbb{C}$. We must determine if solutions for a_i exist under the conditions (P1) and (P2). This is equivalent to analyzing the ideals $\mathfrak{I}_1$ and $\mathfrak{I}_2$ in the coefficient ring $\mathbb{C}\{t, s, a_0, \ldots, a_5\}$:

$$\begin{cases} \mathfrak{I}_1 = \langle t^2 - s^2, \mathfrak{co} \rangle \\ \mathfrak{I}_2 = \langle t^2(1015s^2 + 324) + 324(s^2 - 36), \mathfrak{co} \rangle \end{cases} \tag{S*}$$

where $\mathfrak{I} = \langle \mathfrak{co} \rangle$ is the ideal generated by the coefficients of $F = f_t(x', y', z') - f_s(x, y, z)$. The condition $f_t \sim f_s$ is equivalent to the variety $V(\mathfrak{I})$ being non-empty.

The ideal $\mathfrak{I}$ has a minimal primary decomposition $\mathfrak{I} = \cap_{i=1}^{42} \mathfrak{i}_i$. Excluding the components where $t, s \in \{\pm 2, \pm 6\}$ (which are excluded in our analysis), we are left with $\mathfrak{I}' = \cap_{i=1}^{30} \mathfrak{i}_i$. To provide evidence for this decomposition without overburdening the main text, representative examples of these ideals $\mathfrak{i}_i$ are presented in Appendix D. The analysis of the zero sets of these 30 ideals $\mathfrak{i}_i$ [which contain generators such as $(s \pm t)$, $(2s + ts + 12 - 2t)$, etc.] shows that $V(\mathfrak{I}')$ is non-empty if and only if s and t satisfy one of the six relations listed in Theorem B. This proves Theorem B.

Now we return to analyze the ideals in (S*) based on this result:

(1) **Case (P1): $t^2 = s^2$.** This ideal $\langle t^2 - s^2 \rangle$ corresponds exactly to conditions (a1) $s = t$ and (a2) $s = -t$ of Theorem B. Therefore, if $t^2 = s^2$, an isomorphism φ exists, and $V_t \simeq V_s$. For example, if $s = \zeta t$ ($\zeta^2 = 1$), taking ξ such that $\xi^2 = \zeta$, the automorphism $\varphi : (x, y, z) \mapsto (x, \xi y, z)$ maps f_t to f_s. This proves part (2) of Theorem A.

(2) **Case (P2):** $t^2 = -\frac{324(s^2-36)}{1015s^2+324}$. We consider the ideal $\mathfrak{I}_2$. By calculation, it can be verified that the ideal $\mathfrak{g} := \langle t^2(1015s^2 + 324) + 324(s^2 - 36) \rangle$ is incompatible with any of the decomposition ideals $\mathfrak{i}_i$ of $\mathfrak{I}'$ that correspond to the isomorphism conditions (a3)–(a6) of Theorem B. This implies that $V(\mathfrak{I}_2)$ is empty. Therefore, if (P2) holds (and $t^2 \neq s^2$), no such automorphism φ exists. Thus, $V_t \not\simeq V_s$. This proves part (3) of Theorem A.

Remark IV.8. Furthermore, we consider the (full) Yau algebra $L(V_t)$. By the Mather-Yau theorem,[1] *$V_t \simeq V_s \Leftrightarrow A(V_t) \simeq A(V_s)$, which implies $L(V_t) \simeq L(V_s)$. According to the analysis above, when $V_t \simeq V_s$ under one of the conditions (a3)–(a6) of Theorem B, we know $L(V_t) \simeq L(V_s)$. However, these conditions (a3)–(a6) are incompatible with the Lie algebra isomorphism conditions (P1) and (P2) for N_t. Since in our case $N_t \simeq N_s \Leftrightarrow \tilde{L}(V_t) \simeq \tilde{L}(V_s)$, this provides examples where the full Yau algebras $L(V_t)$ and $L(V_s)$ are isomorphic, but their liftable subalgebras $\tilde{L}(V_t)$ and $\tilde{L}(V_s)$ are not.*

(1) If (P1) $t^2 = s^2$ holds, then (a1) or (a2) from Theorem B also holds. Thus $V_t \simeq V_s$. This proves part (2) of Theorem A.

(2) If (P2) $t^2 = -\frac{324(s^2-36)}{1015s^2+324}$ holds (and $t^2 \neq s^2$), we must check if this relation is compatible with any of the isomorphism conditions (a3)–(a6) from Theorem B. By calculation, it can be verified that the ideal $\mathfrak{g} = \langle t^2(1015s^2 + 324) + 324(s^2 - 36) \rangle$ is incompatible with any of the ideals i_i from the decomposition (J^*) of $\mathfrak{I}'$. This implies that when (P2) holds, $V_t \not\simeq V_s$. This proves part (3) of Theorem A.

Remark IV.9. The comparison of Theorem A and Theorem B yields the central conclusion (as stated in the Introduction): when $V_t \simeq V_s$ via one of the conditions (a3)–(a6) of Theorem B, we know $L(V_t) \simeq L(V_s)$ (by Mather-Yau). However, since (a3)–(a6) are incompatible with (P1) and (P2), we have $\tilde{L}(V_t) \not\simeq \tilde{L}(V_s)$. This provides a concrete example where the Yau algebras are isomorphic but their liftable subalgebras are not.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Zhiwen Liu: Data curation (lead); Formal analysis (equal); Investigation (equal); Methodology (equal); Software (equal); Validation (equal); Visualization (equal); Writing – original draft (equal). **Stephen S.-T. Yau**: Conceptualization (equal); Funding acquisition (lead); Methodology (equal); Resources (lead); Supervision (lead); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

TABLE I. Generators of the Yau algebra $L(V_t)$.

Weight	Basis
wt-0	$D1 = x\partial_x + y\partial_y \quad D2 = z\partial_z;$
wt-1	$D3 = x^2\partial_x - \frac{4ty^2}{t^2-12}\partial_x, \quad D4 = xy\partial_x + \frac{4tx^2}{t^2-12}\partial_y,$ $D5 = xy\partial_y + \frac{4ty^2}{t^2-12}\partial_x, \quad D6 = y^2\partial_y - \frac{4tx^2}{t^2-12}\partial_y,$ $D7 = xz\partial_x + yz\partial_y, \quad D8 = xz\partial_z, \quad D9 = yz\partial_z, \quad D10 = z^2\partial_z;$
wt-2	$D11 = x^2y\partial_x, \quad D12 = xy^2\partial_x, \quad D13 = zx^2\partial_x - \frac{4ty^2z}{t^2-12}\partial_x,$ $D14 = xyz\partial_x + \frac{4tx^2z}{t^2-12}\partial_y, \quad D15 = x^2y\partial_y, \quad D16 = xy^2\partial_y,$ $D17 = xyz\partial_y + \frac{4ty^2z}{t^2-12}\partial_x, \quad D18 = y^2z\partial_y - \frac{4tx^2z}{t^2-12}\partial_y, \quad D19 = xz^2\partial_x + yz^2\partial_y,$ $D20 = x^2z\partial_z, \quad D21 = xyz\partial_z, \quad D22 = y^2z\partial_z, \quad D23 = xz^2\partial_z,$ $D24 = yz^2\partial_z;$
wt-3	$D25 = x^2y^2\partial_x, \quad D26 = x^2yz\partial_x, \quad D27 = xy^2z\partial_x,$ $D28 = x^2z^2\partial_x - \frac{4ty^2z^2}{t^2-12}\partial_x, \quad D29 = xyz^2\partial_x + \frac{4tx^2z^2}{t^2-12}\partial_y,$ $D30 = x^2y^2\partial_y, \quad D31 = x^2yz\partial_y, \quad D32 = xy^2z\partial_y,$ $D33 = xyz^2\partial_y + \frac{4ty^2z^2}{t^2-12}\partial_x, \quad D34 = y^2z^2\partial_y - \frac{4tx^2z^2}{t^2-12}\partial_y, \quad D35 = x^2yz\partial_z,$ $D36 = xy^2z\partial_z, \quad D37 = x^2z^2\partial_z, \quad D38 = xyz^2\partial_z, \quad D39 = y^2z^2\partial_z;$
wt-4	$D40 = x^2y^2z\partial_x, \quad D41 = x^2yz^2\partial_x, \quad D42 = xy^2z^2\partial_x,$ $D43 = x^2y^2z\partial_y, \quad D44 = x^2yz^2\partial_y, \quad D45 = xy^2z^2\partial_y,$ $D46 = x^2y^2z\partial_z, \quad D47 = x^2yz^2\partial_z, \quad D48 = xy^2z^2\partial_z;$
wt-5	$D49 = x^2y^2z^2\partial_x, \quad D50 = x^2y^2z^2\partial_y, \quad D51 = x^2y^2z^2\partial_z;$

APPENDIX A: GENERATORS OF YAU ALGEBRAS AND SUBALGEBRAS

This appendix contains the complete lists of generators for the Lie algebras discussed in Sec. III, moved here to improve the readability of the main text.

APPENDIX B: LIE BRACKET MULTIPLICATION TABLE FOR N_t

This appendix contains the full Lie bracket multiplication table for the nilradical $N_t = [\tilde{L}(V_t), \tilde{L}(V_t)]$, as referenced in Sec. III. This table is fundamental to all computations in Sec. IV.

APPENDIX C: REPRESENTATIVE COMPUTATION MATRICES

This appendix contains the explicit forms of the large representation matrices used in the computations of Sec. IV. These matrices are difficult to read and are provided here for computational verification. We use \tiny to scale them.

TABLE II. Generators of the liftable Yau algebra $\tilde{L}(V_t)$.

Weight	Basis
wt-0	$E1 = x\partial_x + y\partial_y + z\partial_z;$
wt-1	$E2 = x^2\partial_x + xy\partial_y + xz\partial_z, \quad E3 = y^2\partial_y + xy\partial_x + yz\partial_z,$ $E4 = z^2\partial_z + xz\partial_x + yz\partial_y;$
wt-2	$E5 = x^2y\partial_y + x^2z\partial_z + \frac{txy^2}{2}\partial_x, \quad E6 = x^2y\partial_x + xy^2\partial_y + xyz\partial_z,$ $E7 = xy^2\partial_x + y^2z\partial_z + \frac{tx^2y}{2}\partial_y, \quad E8 = x^2z\partial_x + xz^2\partial_z + xyz\partial_y,$ $E9 = y^2z\partial_y + yz^2\partial_z + xyz\partial_x, \quad E10 = xz^2\partial_x + yz^2\partial_y;$
wt-3	$E11 = x^2y^2\partial_x, \quad E12 = x^2yz\partial_x, \quad E13 = xy^2z\partial_x,$ $E14 = x^2z^2\partial_x + \frac{4ty^2z^2}{(t^2-12)}\partial_x, \quad E15 = xyz^2\partial_x + \frac{4tx^2z^2}{(t^2-12)}\partial_y,$ $E16 = x^2y^2\partial_y, \quad E17 = x^2yz\partial_y, \quad E18 = xy^2z\partial_y,$ $E19 = xyz^2\partial_y + \frac{4ty^2z^2}{(t^2-12)}\partial_x, \quad E20 = y^2z^2\partial_y + \frac{4tx^2z^2}{(t^2-12)}\partial_y, \quad E21 = x^2yz\partial_z,$ $E22 = xy^2z\partial_z, \quad E23 = x^2z^2\partial_z, \quad E24 = xyz^2\partial_z, \quad E25 = y^2z^2\partial_z;$
wt-4	$E26 = x^2y^2z\partial_x, \quad E27 = x^2yz^2\partial_x, \quad E28 = xy^2z^2\partial_x, \quad E29 = x^2y^2z\partial_y,$ $E30 = x^2yz^2\partial_y, \quad E31 = xy^2z^2\partial_y, \quad E32 = x^2y^2z\partial_z, \quad E33 = x^2yz^2\partial_z,$ $E34 = xy^2z^2\partial_z;$
wt-5	$E35 = x^2y^2z^2\partial_x, \quad E36 = x^2y^2z^2\partial_y, \quad E37 = x^2y^2z^2\partial_z.$

1. Matrix $M_1(1)$ for $ad_{\bar{x}}^1 : G_t^1 \to G_t^2$

$$
M_1(1) := \begin{pmatrix}
a5(t/2)-a7 & -a6 & 0 & -a2(t/2) & a3 & a2 & 0 & a4 & 0 \\
-a6 & a5(t/2)-a7 & 0 & a3 & -a2(t/2) & 0 & a2 & 0 & a4 \\
a8(t/2) & -a9 & a5(t/2)-a7 & -a4(t/2) & 0 & a4 & -a2(t/2) & a3 & a2 \\
0 & 0 & 0 & -a10 & 0 & -a8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -a10 & -a9 & 0 & 0 & 0 \\
-a6 & a7(t/2)-a5 & 0 & a3 & a2 & -a3(t/2) & 0 & 0 & 0 \\
0 & 0 & 0 & -a8 & a9(t/2) & a7(t/2)-a5 & a4 & 0 & -a4(t/2) \\
a2 & -a3(t/2) & 0 & -a9 & -a8 & -a6 & 0 & a4 & 0 \\
0 & a2 & 0 & 0 & -a10 & 0 & -a8 & 0 & 0 \\
0 & 0 & a2 & 0 & 0 & -a10 & -a9 & 0 & 0 \\
0 & 0 & 0 & a7(t/2)-a5 & 0 & a3 & a2 & -a3(t/2) & 0 \\
a5(t/2)-a7 & -a6 & 0 & 0 & 0 & 0 & a8(t/2) & -a9 & -a7 \\
0 & 0 & 0 & -a5 & 0 & a4 & 0 & 0 & a2 \\
-a9 & -a8 & -a6 & 0 & a4 & 0 & a3 & a2 & 0 \\
0 & -a9 & -a7 & 0 & 0 & a4 & 0 & a3 & 0
\end{pmatrix}
$$

TABLE III. Lie bracket multiplication table of the liftable Yau algebra N_t.

Brackets with [E2, Ei]	Brackets with [E3, Ei]
$[E2,E5] = -\frac{t}{2}E11 - \frac{t}{2}E22;$	$[E3,E5] = E16 + E21;$
$[E2,E6] = E16 + E21;$	$[E3,E6] = E11 + E22;$
$[E2,E7] = E11 + E22;$	$[E3,E7] = -\frac{t}{2}E16 - \frac{t}{2}E21;$
$[E2,E8] = E17 + E23 - \frac{t}{2}E13;$	$[E3,E8] = E12 + E18 + E24;$
$[E2,E9] = E12 + E18 + E24;$	$[E3,E9] = E13 + E25 - \frac{t}{2}E17;$
$[E2,E10] = E14 + E19;$	$[E3,E10] = E15 + E20;$
$[E2,E11] = -E32;$	$[E3,E12] = 3E26;$
$[E2,E12] = -E29 - E33;$	$[E3,E14] = \frac{9(t^2-4)}{t^2-12}E27;$
$[E2,E13] = 2E26 - E34;$	$[E3,E15] = \frac{8t}{t^2-12}E30 + \frac{5t^2-36}{t^2-12}E28;$
$[E2,E14] = -\frac{3(t^2-4)}{t^2-12}E30 - \frac{t(t^2-4)}{t^2-12}E28;$	$[E3,E16] = -E32;$
$[E2,E15] = 2E27 - \frac{E31(7t^2-12)}{t^2-12};$	$[E3,E17] = 2E29 - E33;$
$[E2,E18] = 3E29;$	$[E3,E18] = -E26 - E34;$
$[E2,E19] = \frac{8t}{t^2-12}E28 + \frac{5t^2-36}{t^2-12}E30;$	$[E3,E19] = 2E31 - \frac{E27(7t^2-12)}{t^2-12};$
$[E2,E20] = \frac{9(t^2-4)}{t^2-12}E31;$	$[E3,E20] = -\frac{3(t^2-4)}{t^2-12}E28 - \frac{t(t^2-4)}{t^2-12}E30;$
$[E2,E22] = 3E32;$	$[E3,E21] = 3E32;$
$[E2,E23] = -\frac{3t}{2}E34;$	$[E3,E23] = 3E33;$
$[E2,E24] = 3E33;$	$[E3,E24] = 3E34;$
$[E2,E25] = 3E34;$	$[E3,E25] = -\frac{3t}{2}E33;$
$[E2,E26] = -E37;$	$[E3,E27] = 4E35;$
$[E2,E27] = -E36;$	$[E3,E29] = -E37;$
$[E2,E28] = 3E35;$	$[E3,E30] = 3E36;$
$[E2,E31] = 4E36;$	$[E3,E31] = -E35;$
$[E2,E34] = 4E37;$	$[E3,E33] = 4E37;$
Brackets with [E4, Ei]	**Brackets with [E5, Ei]**
$[E4,E5] = E17 + E23 - \frac{t}{2}E13;$	$[E5,E13] = -2E37;$
$[E4,E6] = E12 + E18 + E24;$	$[E5,E14] = -\frac{t(t^2-4)}{2(t^2-12)}E35;$
$[E4,E7] = E13 + E25 - \frac{t}{2}E17;$	$[E5,E15] = -\frac{8(t^2-3)}{t^2-12}E36;$
$[E4,E8] = E14 + E19;$	$[E5,E19] = \frac{4t}{t^2-12}E35;$
$[E4,E9] = E15 + E20;$	$[E5,E20] = \frac{9(t^2-4)}{t^2-12}E36;$
$[E4,E11] = 3E26;$	$[E5,E23] = -\frac{3t}{2}E37;$
$[E4,E12] = 3E27;$	$[E5,E25] = 3E37;$
$[E4,E13] = 3E28;$	
$[E4,E16] = 3E29;$	
$[E4,E17] = 3E30;$	

TABLE III. (*Continued*.)

Brackets with [E4, Ei]	Brackets with [E5, Ei]
$[E4, E18] = 3E31;$	
$[E4, E21] = 2E33 - E29;$	
$[E4, E22] = 2E34 - E26;$	
$[E4, E23] = \frac{t}{2}E28 - E30;$	
$[E4, E24] = -E27 - E31;$	
$[E4, E25] = \frac{t}{2}E30 - E28;$	
$[E4, E26] = 4E35;$	
$[E4, E29] = 4E36;$	
$[E4, E32] = 3E37;$	
$[E4, E33] = -E36;$	
$[E4, E34] = -E35;$	
Brackets with [E6, Ei]	**Brackets with [E7, Ei]**
$[E6, E12] = -E37;$	$[E7, E14] = \frac{9(t^2-4)}{t^2-12}E35;$
$[E6, E14] = -\frac{3(t^2-4)}{t^2-12}E36;$	$[E7, E15] = \frac{4t}{t^2-12}E36;$
$[E6, E15] = \frac{4(t^2-6)}{t^2-12}E35;$	$[E7, E17] = -2E37;$
$[E6, E18] = -E37;$	$[E7, E19] = -\frac{8(t^2-3)}{t^2-12}E35;$
$[E6, E19] = \frac{4(t^2-6)}{t^2-12}E36;$	$[E7, E20] = -\frac{t(t^2-4)}{2(t^2-12)}E36;$
$[E6, E20] = -\frac{3(t^2-4)}{t^2-12}E35;$	$[E7, E23] = 3E37;$
$[E6, E24] = 3E37;$	$[E7, E25] = -\frac{3t}{2}E37;$
Brackets with [E8, Ei]	**Brackets with [E9, Ei]**
$[E8, E11] = -E37;$	$[E9, E12] = 3E35;$
$[E8, E12] = -E36;$	$[E9, E16] = -E37;$
$[E8, E13] = 2E35;$	$[E9, E17] = 2E36;$
$[E8, E18] = 3E36;$	$[E9, E18] = -E35;$
$[E8, E22] = 2E37;$	$[E9, E21] = 2E37;$
$[E8, E23] = \frac{t}{2}E35;$	$[E9, E23] = -E36;$
$[E8, E24] = -E36;$	$[E9, E24] = -E35;$
$[E8, E25] = -E35;$	$[E9, E25] = \frac{t}{2}E36;$
Brackets with [E10, Ei]	
$[E10, E11] = 3E35;$	
$[E10, E16] = 3E36;$	
$[E10, E21] = -2E36;$	
$[E10, E22] = -2E35;$	

2. Matrix $M_1(2)$ for $ad_{\tilde{x}}^2 : G_t^2 \to G_t^3 \oplus G_t^4$

$$M_1(2) := \left(\begin{array}{cccccccc}
3a4 & 3a3 & 2a2 & 0 & 0 & 0 & 0 & -a3 \\
0 & 3a4 & 0 & a3((6t^2)/(t^2-12)+3) & 2a2 & 0 & 0 & 0 \\
0 & 0 & 3a4 & -a2(t+(8t)/(t^2-12)) & a3((2t^2)/(t^2-12)+3) & 0 & 0 & 0 \\
0 & -a2 & 0 & 0 & 0 & 3a4 & 2a3 & 3a2 \\
0 & 0 & 0 & -a2((2t^2)/(t^2-12)+1) & (8a3t)/(t^2-12) & 0 & 3a4 & 0 \\
0 & 0 & 0 & 0 & -a2((6t^2)/(t^2-12)+1) & 0 & 0 & 3a4 \\
-a2 & 0 & 0 & 0 & 0 & -a3 & 0 & 0 \\
0 & -a2 & 0 & 0 & 0 & 0 & -a3 & 0 \\
0 & 0 & -a2 & 0 & 0 & 0 & 0 & -a3 \\
3a10 & 3a9 & 2a8 & \begin{array}{c} a7((6t^2)/(t^2-12)+3) \\ -a5(t/2+(4t)/(t^2-12)) \end{array} & a6((2t^2)/(t^2-12)+2) & 0 & 0 & -a9 \\
0 & -a8 & 0 & -a6((2t^2)/(t^2-12)+1) & \begin{array}{c} (4a7t)/(t^2-12) \\ -a5((6t^2)/(t^2-12)+2) \end{array} & 3a10 & 2a9 & 3a8 \\
-a8 & -a6 & -2a5 & 0 & 0 & -a9 & -2a7 & -a6
\end{array}\right.$$

$$\left.\begin{array}{ccccccc}
0 & 0 & 0 & -a4 & 0 & 0 & 0 \\
-a3((6t^2)/(t^2-12)+1) & 0 & 0 & 0 & 0 & -a4 & 0 \\
(8a2t)/(t^2-12) & -a3((2t^2)/(t^2-12)+1) & 0 & 0 & (a4t)/2 & 0 & -a4 \\
0 & 0 & -a4 & 0 & 0 & 0 & 0 \\
a2((2t^2)/(t^2-12)+3) & -a3(t+(8t)/(t^2-12)) & 0 & 0 & -a4 & 0 & (a4t)/2 \\
2a3 & a2((6t^2)/(t^2-12)+3) & 0 & 0 & 0 & -a4 & 0 \\
0 & 0 & 3a3 & 3a2 & 0 & 0 & 0 \\
0 & 0 & 2a4 & 0 & 3a3 & 3a2 & -(3a3t)/2 \\
0 & 0 & 0 & 2a4 & -(3a2t)/2 & 3a3 & 3a2 \\
\begin{array}{c} (4a5t)/(t^2-12) \\ -a7((6t^2)/(t^2-12)+2) \end{array} & -a6((2t^2)/(t^2-12)+1) & 0 & -2a10 & (a8t)/2 & -a9 & -a8 \\
a6((2t^2)/(t^2-12)+2) & \begin{array}{c} a5((6t^2)/(t^2-12)+3) \\ -a7(t/2+(4t)/(t^2-12)) \end{array} & -2a10 & 0 & -a9 & -a8 & (a9t)/2 \\
0 & 0 & 2a9 & 2a8 & 3a7-(3a5t)/2 & 3a6 & 3a5-(3a7t)/2
\end{array}\right).$$

3. Matrix $M_2(1)$ for $ad_{\tilde{x}} : G_t^2 \to G_t^3 \oplus G_t^4$

The submatrix formed by the first 9 rows of $M_2(1)$ (denoted as $M_2(1)[1:9]$) is

$$M_2(1)_{1:9} := \left(\begin{array}{ccccccccc}
-2a13 & a18-3a12 & a22-3a11 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2a15 & a19((6t^2)/(t^2-12)+1)-a14((6t^2)/(t^2-12)+3) & a24-3a12 & 0 & 0 & 0 & 0 & 0 & 0 \\
a14(t+(8t)/(t^2-12))-(8a19t)/(t^2-12) & a20((2t^2)/(t^2-12)+1)-a15((2t^2)/(t^2-12)+3) & a25-3a13-(a23t)/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
a12-3a18 & -2a17 & a21-3a16 & 0 & 0 & 0 & 0 & 0 & 0 \\
a14((2t^2)/(t^2-12)+1)-a19((2t^2)/(t^2-12)+3) & a20(t+(8t)/(t^2-12))-(8a15t)/(t^2-12) & a23-3a17-(a25t)/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
a15((6t^2)/(t^2-12)+1)-a20((6t^2)/(t^2-12)+3) & -2a19 & a24-3a18 & 0 & 0 & 0 & 0 & 0 & 0 \\
a11-3a22 & a16-3a21 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a12-3a24 & a17-3a23+(3a25t)/2 & -2a21 & 0 & 0 & 0 & 0 & 0 & 0 \\
a13-3a25+(3a23t)/2 & a18-3a24 & -2a22 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

$$M_2(1)_{9:12} := \begin{pmatrix} 0 & 0 & 0 & \begin{matrix} a14(t/2+(4t)/(t^2-12)) \\ -(4a19t)/(t^2-12) \end{matrix} & \begin{matrix} a20((2t^2)/(t^2-12)+1) \\ -a15((2t^2)/(t^2-12)+2) \end{matrix} & \begin{matrix} a19((6t^2)/(t^2-12)+2) \\ -a14((6t^2)/(t^2-12)+3) \end{matrix} \\ 0 & 0 & 0 & \begin{matrix} a15((6t^2)/(t^2-12)+2) \\ -a20((6t^2)/(t^2-12)+3) \end{matrix} & \begin{matrix} a14((2t^2)/(t^2-12)+1) \\ -a19((2t^2)/(t^2-12)+2) \end{matrix} & \begin{matrix} a20(t/2+(4t)/(t^2-12)) \\ -(4a15t)/(t^2-12) \end{matrix} \\ 0 & 0 & 0 & 2a13-3a25+(3a23t)/2 & a12+a18-3a24 & 2a17-3a23+(3a25t)/2 \end{pmatrix}$$

$$\begin{matrix} a25-2a13-(a23t)/2 & a18-3a12+a24 & 2a22-3a11 \\ a12-3a18+a24 & a23-2a17-(a25t)/2 & 2a21-3a16 \\ a11-2a22 & a16-2a21 & 0 \end{matrix}\Bigg).$$

4. Matrix $P_0(1)$ for $ad_{\tilde{x}} : P_t^1 \to G_t^2$

$$\begin{pmatrix} -a2(t/2) & a3 & a2 & 0 & a4 & 0 \\ a3 & -a2(t/2) & 0 & a2 & 0 & a4 \\ -a4(t/2) & 0 & a4 & -a2(t/2) & a3 & a2 \\ 0 & 0 & 0 & -a2(t/2) & a3 & 0 \\ 0 & 0 & 0 & a3 & -a2(t/2) & 0 \\ a3 & a2 & -a3(t/2) & 0 & 0 & 0 \\ a4 & 0 & -a4(t/2) & a2 & -a3(t/2) & 0 \\ -a2(t/2) & a3 & a2 & 0 & a4 & 0 \\ 0 & a2 & 0 & 0 & 0 & a4 \\ 0 & 0 & a2 & 0 & a4 & 0 \\ a3 & a2 & -a3(t/2) & 0 & 0 & 0 \\ 0 & 0 & 0 & a3 & a2 & -a3(t/2) \\ 0 & 0 & 0 & 0 & 0 & a2 \\ 0 & a4 & 0 & a3 & a2 & 0 \\ 0 & 0 & a4 & 0 & a3 & 0 \end{pmatrix}.$$

5. Matrix $P_0(2)$ for $ad_{\tilde{x}} : G_t^2 \to G_t^3$

$$P_0(2) := \begin{pmatrix} 3a4 & 3a3 & 2a2 & 0 & 0 & 0 & 0 & -a3 \\ 0 & 3a4 & 0 & a3((6t^2)/(t^2-12)+3) & 2a2 & 0 & 0 & 0 \\ 0 & 0 & 3a4 & -a2(t+(8t)/(t^2-12)) & a3((2t^2)/(t^2-12)+3) & 0 & 0 & 0 \\ 0 & -a2 & 0 & 0 & 0 & 3a4 & 2a3 & 3a2 \\ 0 & 0 & 0 & -a2((2t^2)/(t^2-12)+1) & (8a3t)/(t^2-12) & 0 & 3a4 & 0 \\ 0 & 0 & 0 & 0 & -a2((6t^2)/(t^2-12)+1) & 0 & 0 & 3a4 \\ -a2 & 0 & 0 & 0 & 0 & -a3 & 0 & 0 \\ 0 & -a2 & 0 & 0 & 0 & 0 & -a3 & 0 \\ 0 & 0 & -a2 & 0 & 0 & 0 & 0 & -a3 \end{pmatrix}$$

$$\left.\begin{matrix} 0 & 0 & 0 & -a4 & 0 & 0 & 0 \\ -a3((6t^2)/(t^2-12)+1) & 0 & 0 & 0 & 0 & -a4 & 0 \\ (8a2t)/(t^2-12) & -a3((2t^2)/(t^2-12)+1) & 0 & 0 & a4(t/2) & 0 & -a4 \\ 0 & 0 & -a4 & 0 & 0 & 0 & 0 \\ a2((2t^2)/(t^2-12)+3) & -a3(t+(8t)/(t^2-12)) & 0 & 0 & -a4 & 0 & a4(t/2) \\ 2a3 & a2((6t^2)/(t^2-12)+3) & 0 & 0 & 0 & -a4 & 0 \\ 0 & 0 & 3a3 & 3a2 & 0 & 0 & 0 \\ 0 & 0 & 2a4 & 0 & 3a3 & 3a2 & -(3a3(t/2) \\ 0 & 0 & 0 & 2a4 & -(3a2(t/2) & 3a3 & 3a2 \end{matrix}\right).$$

6. Matrix $P_1(0)$ for $ad_{\tilde{x}} : P_t^0 \to G_t^2$

$$P_1(0) := \begin{pmatrix} (a5t)/2-a7 & -a6 & -a9 \\ -a6 & (a5t)/2-a7 & -a8 \\ (a8t)/2 & -a9 & (a5t)/2-a7 \\ -a10 & 0 & -a8 \\ 0 & -a10 & -a9 \\ -a6 & (a7t)/2-a5 & 0 \\ -a8 & (a9t)/2 & (a7t)/2-a5 \\ -a9 & -a8 & -a6 \\ 0 & -a10 & 0 \\ -a10 & 0 & -a9 \\ 0 & (a7t)/2-a5 & -a6 \\ 0 & (a5t)/2-a7 & -a6 \\ -a5 & 0 & -a8 \\ -a9 & -a8 & -a6 \\ -a9 & -a7 & 0 \end{pmatrix}.$$

7. Matrix $P_1(2)$ for $ad_{\tilde{x}} : G_t^2 \to G_t^4$

$$P_1(2) := \left(\begin{matrix} 3a10 & 3a9 & 2a8 & \begin{array}{c} a7((6t^2)/(t^2-12)+3) \\ -a5(t/2+(4t)/(t^2-12)) \end{array} & a6((2t^2)/(t^2-12)+2) & 0 & 0 & -a9 \\ 0 & -a8 & 0 & -a6((2t^2)/(t^2-12)+1) & \begin{array}{c} (4a7t)/(t^2-12) \\ -a5((6t^2)/(t^2-12)+2) \end{array} & 3a10 & 2a9 & 3a8 \\ -a8 & -a6 & -2a5 & 0 & 0 & -a9 & -2a7 & -a6 \end{matrix}\right.$$

$$\left.\begin{matrix} \begin{array}{c} (4a5t)/(t^2-12) \\ -a7((6t^2)/(t^2-12)+2) \end{array} & -a6((2t^2)/(t^2-12)+1) & 0 & -2a10 & (a8t)/2 & -a9 & -a8 \\ a6((2t^2)/(t^2-12)+2) & \begin{array}{c} a5((6t^2)/(t^2-12)+3) \\ -a7(t/2+(4t)/(t^2-12)) \end{array} & -2a10 & 0 & -a9 & -a8 & (a9t)/2 \\ 0 & 0 & 2a9 & 2a8 & 3a7-(3a5t)/2 & 3a6 & 3a5-(3a7t)/2 \end{matrix}\right).$$

8. Matrix $S_0(3)$

$S_0(3)(\text{cols }14) =$

$$\begin{pmatrix} -9a2a4^2t & 18a3a4^2 & 18a2a4^2 & \begin{matrix} 12a3^2a4 - a4\left(4a3^2\left(\frac{6t^2}{t^2-12}+1\right)+2a3^2-\frac{24a2^2t}{t^2-12}\right) \\ +a4\left(4a3^2\left(\frac{6t^2}{t^2-12}+3\right)-3a2^2\left(t+\frac{8t}{t^2-12}\right)\right)-6a2^2a4t \end{matrix} \\ 18a3a4^2 & 18a2a4^2 & -9a3a4^2t & \begin{matrix} a4\left(8a2a3+3a2a3\left(\frac{2t^2}{t^2-12}+3\right)+a2a3\left(\frac{6t^2}{t^2-12}+1\right)\right) \\ -a4\left(3a2a3\left(\frac{2t^2}{t^2-12}+1\right)+a2a3\left(\frac{6t^2}{t^2-12}+3\right)\right)+24a2a3a4 \end{matrix} \\ \begin{matrix} 6a3^2a4-a4(6ta2^2-12a3^2) \\ -3a2^2a4t \end{matrix} & 36a2a3a4 & \begin{matrix} 6a2^2a4-a4(-12a2^2+6ta3^2) \\ -3a3^2a4t \end{matrix} & 6a2a3^2+3a2^3t-a2(6ta2^2-12a3^2). \end{pmatrix}$$

$S_0(3)(\text{cols }5-6) =$

$$\begin{pmatrix} \begin{matrix} a4\left(8a2a3+3a2a3\left(\frac{2t^2}{t^2-12}+3\right)+a2a3\left(\frac{6t^2}{t^2-12}+1\right)\right) \\ -a4\left(3a2a3\left(\frac{2t^2}{t^2-12}+1\right)+a2a3\left(\frac{2t^2}{t^2-12}+3\right)\right) \\ +24a2a3a4 \end{matrix} & \begin{matrix} a2\left(4a3^2\left(\frac{6t^2}{t^2-12}+3\right)-3a2^2\left(t+\frac{8t}{t^2-12}\right)\right) \\ -a3\left(3a2a3\left(\frac{2t^2}{t^2-12}+1\right)+a2a3\left(\frac{6t^2}{t^2-12}+3\right)\right) \\ -a2\left(4a3^2\left(\frac{6t^2}{t^2-12}+1\right)+2a3^2-\frac{24a2^2t}{t^2-12}\right) \\ +a3\left(8a2a3+3a2a3\left(\frac{2t^2}{t^2-12}+3\right)+a2a3\left(\frac{6t^2}{t^2-12}+1\right)\right) \end{matrix} \\ \begin{matrix} 12a2^2a4-a4\left(4a2^2\left(\frac{6t^2}{t^2-12}+1\right)+2a2^2-\frac{24a3^2t}{t^2-12}\right) \\ +a4\left(4a2^2\left(\frac{6t^2}{t^2-12}+3\right)-3a3^2\left(t+\frac{8t}{t^2-12}\right)\right) \\ -6a3^2a4t \end{matrix} & \begin{matrix} a3\left(4a2^2\left(\frac{6t^2}{t^2-12}+3\right)-3a3^2\left(t+\frac{8t}{t^2-12}\right)\right) \\ -a2\left(3a2a3\left(\frac{2t^2}{t^2-12}+1\right)+a2a3\left(\frac{6t^2}{t^2-12}+3\right)\right) \\ -a3\left(4a2^2\left(\frac{6t^2}{t^2-12}+1\right)+2a2^2-\frac{24a3^2t}{t^2-12}\right) \\ +a2\left(8a2a3+3a2a3\left(\frac{2t^2}{t^2-12}+3\right)+a2a3\left(\frac{6t^2}{t^2-12}+1\right)\right) \end{matrix} \\ 6a2^2a3+3a3^3t-a3(-12a2^2+6ta3^2) & 0 \end{pmatrix}.$$

9. Matrix $P_1(2) \cdot A$

$$P_1(2)\cdot A = \begin{pmatrix} a7\left(\frac{6t^2}{t^2-12}+3\right)-a7\left(\frac{6t^2}{t^2-12}+2\right)-a5\left(\frac{t}{t^2-12}+\frac{4t}{t^2-12}\right)+\frac{4ta5}{t^2-12} & a6\left(\frac{2t^2}{t^2-12}+2\right)-a6\left(\frac{2t^2}{t^2-12}+1\right) \\ a6\left(\frac{2t^2-12}{+}2\right)-a6\left(\frac{2t^2}{t^2-12}+1\right) & a5\left(\frac{6t^2}{t^2-12}+3\right)-a5\left(\frac{6t^2}{t^2-12}+2\right)-a7\left(\frac{t}{t^2-12}+\frac{4t}{t^2-12}\right)+\frac{4ta7}{t^2-12} \\ \cdots & \cdots \end{pmatrix}.$$

10. Matrix $P_1(2) \cdot B$

$$P_1(2)\cdot B = \begin{pmatrix} (4a5t)/(t^2-12) & a7((6t^2)/(t^2-12)+2) \\ 0 & 0 \\ a5((6t^2)/(t^2-12)+3)-a7(t/2+(4t)/(t^2-12)) & 0 \end{pmatrix}$$

APPENDIX D: REPRESENTATIVE IDEALS FOR ISOMORPHISM CONDITIONS

As stated in the Proof of Theorem B (Sec. IV), the analytic isomorphism $V_t \simeq V_s$ holds if and only if the ideal $\mathfrak{I}' \subset \mathbb{C}\{t, s, a_0, \ldots, a_5\}$ has a non-empty variety. $\mathfrak{I}'$ is the intersection of 30 prime ideals, $\mathfrak{I}' = \cap_{i=1}^{30} \mathfrak{i}_i$, whose generators correspond to the six conditions in Theorem B.

To save space and improve readability, we list only a few representative examples of these ideals $\mathfrak{i}_i$. The full list is available via our computation scripts.

(a1) *Condition* $s = t$. A representative ideal component is:

```
i_1 = < s - t, a0 - 1, a1, a2, a3, a4 - 1, a5 >
```

This corresponds to the identity automorphism $\varphi(x, y, z) = (x, y, z)$.

(a2) *Condition* $s = -t$. A representative ideal component is:

```
i_2 = < s + t, a0 - 1, a1, a2, a3, a4 + 1, a5 >
```

This corresponds to the automorphism $\varphi(x, y, z) = (x, -y, z)$ (assuming $s, t \neq 0$).

(a3) *Condition* $2s + ts + 12 - 2t = 0$. A representative ideal component is:

```
i_3 = < 2s+ts+12-2t, a0, a1 - ((t^2-4) / (2t+12) ), a2, a3 - ((s^2-4) / (2s-12) ), a4, a5 >
```

(Note: The coefficients a_i are functions of s and t.)

(a4–a6) *Other Conditions*. The ideals corresponding to conditions (a4), (a5), and (a6) are similar in structure to i_3, containing the specific parameter relation (e.g., $2s + ts - 12 + 2t = 0$) along with complex constraints on the coefficients $a_0, \ldots, a_5$.

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