HIGHER ORDER JACOBIAN MATRIX THEORY AND INVARIANTS OF SINGULARITIES

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Dedicated to Professor Shing Tung Yau on the occasion of his 75th birthday

Abstract

The higher order Nash blow-up of an algebraic variety replaces singular points with limits of certain spaces carrying higher order data associated to the variety at non-singular points. Inspired by the higher order Nash blow-up, we develop higher order Jacobian matrix theory, a generalization of the Jacobian matrix theory in the usual sense. The most significant contribution of this work is the higher order generalization of similarity transformation. It bridges the two long-standing, distinct research directions in the study of higher-order Jacobian matrices: the one from the higher order Nash blowup of a hypersurface, and the one representing a higher-order tangent map of a morphism. As an application, we use it to construct different contact and right invariants of isolated singularities, generalizing the Tjurina algebra and the Milnor algebra in singularity theory respectively. It is worth mentioning that the Tjurina algebra and the Milnor algebra were constructed using first-order derivatives, but now we are constructing invariants using arbitrary higher-order derivatives. The higher order Jacobian matrix theory also has potential applications and inspirations in multiple branches of mathematics: inspiring the establishment of higher order Hessian matrix theory and projective invariants, calculating explicit expression for the inverse of an automorphism of a power series ring, and performing explicit calculations in finite determinacy.

Keywords. higher order Nash blow-up, higher order Jacobian matrix, singularities, Milnor algebra, Tjurina algebra.

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Contents

1. Introduction

1.1. Higher Order Nash Blow-up.

One of the central problems in singularity theory is the classification of singularities. A goal is to find enough invariants associated to isolated singularities that one can distinguish among them up to contact equivalence (see Definition 1.11). However, not many effective invariants are known. Moreover, most of the known invariants, for example, the geometric genus, are in general hard to compute. In this article, we shall introduce many new invariants of isolated singularities which are more easily calculated. The algebra of germs of holomorphic functions at the origin of $\mathbb{C}^l$ is denoted as $\mathcal{O}_l$. $\mathcal{O}_l$ has a unique maximal ideal $\mathfrak{m}$ (the set of germs of holomorphic functions which vanish at the origin) and can be naturally identified with the algebra of convergent power series in l indeterminates with complex coefficients.

Let $(V, 0)$ be an isolated hypersurface singularity defined by a holomorphic function $f : (\mathbb{C}^l, 0) \to (\mathbb{C}, 0)$. The multiplicity $\mathrm{mult}(f)$ of the singularity $(V, 0)$ is defined to be the order of the lowest nonvanishing term in the power series expansion of f at 0. For any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^l, 0)$ defined by f, the second author considered the Lie algebra of derivations of the moduli algebra

$$A(V) := \mathcal{O}_l/\langle f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_l}\rangle, \text{i.e.}, L(V) := \mathrm{Der}(A(V), A(V)).$$

and showed that $L(V)$ is a finite dimensional solvable Lie algebra ([**Ya86**], [**Ya91**]). $L(V)$ is called the Yau algebra of V in [**Kh06**] and [**Yu96**] in order to distinguish it from other Lie algebras arising in the context of an isolated hypersurface singularity ([**Al-Ma92**], [**A-V-Z12**]). The Yau algebra plays an important role in singularities [**Se-Ya90**]. In this paper, we will introduce a series of new derivation Lie algebras which are generalizations of the Yau algebra.

The classical Nash blow-up of an algebraic variety can be viewed as the parameter space of the tangent spaces of smooth points and their limits. It is natural to ask whether we can get a smooth variety by Nash blow-ups. There has been much work on this problem, such as González-Sprinberg [**Go82**], Hironaka [**Hi83**], Nobile [**No75**], Rebassoo [**Re77**], and Spivakovsky [**Sp90**], etc. Recall the famous theorem of A. Nobile:

Theorem 1.1 ([**No75**]). *Let X be a variety over an algebraically closed field of characteristic zero, then the Nash blow-up of X is an isomorphism if and only if X is non-singular.*

Theorem 1.1 (Nobile's theorem) fails over fields of prime characteristic. We state here a recently proved theorem for prime characteristic under a normality condition.

Theorem 1.2 (Theorem 3.10 in [**Du-Nu22**]). *Let X be a normal variety over an algebraically closed field $\mathbb{K}$ of dimension d. Suppose that $\mathbb{K}$ has prime characteristic p. If $\mathbf{Nash}_1(X) \cong X$, then X is non-singular.*

This article mainly focuses on higher order Nash blow-up with the definition given below.

Definition 1.3 (Higher Order Nash blow-up [**Ya07**]). Let X be a variety of dimension d, $x \in X$, $x^{(n)} := Spec(\mathcal{O}_{X,x}/m_x^{n+1})$ its n-th infinitesimal neighborhood and $\mathbf{Hilb}_{\binom{d+n}{n}}(X)$ is the Hilbert scheme of length $\binom{d+n}{n}$ points of X. If X is smooth at x, then $x^{(n)}$ is an Artinian subscheme of X of length $\binom{d+n}{n}$. Therefore, it corresponds to a point

$$[x^{(n)}] \in \mathbf{Hilb}_{\binom{d+n}{n}}(X),$$

which induced the following morphism of schemes,

$$\sigma_n : X_{sm} \longrightarrow \mathbf{Hilb}_{\binom{d+n}{n}}(X)$$

where X_{sm} denotes the smooth locus of X. The graph of σ_n is canonically isomorphic to X_{sm}. We define the n-th order *Nash blow-up* of X (called *the higher order Nash blow-up* with order n of X), denoted by the higher order Nash blow-up $\mathbf{Nash}_n(X)$, to be the closure of the graph Γ_{σ_n} with reduced scheme structure in $X \times_k \mathbf{Hilb}_{\binom{d+n}{n}}(X)$, together with the projection

$$\pi_n : \mathbf{Nash}_n(X) \longrightarrow X,$$

which is projective and birational. Moreover, it is an isomorphism over X_{sm}.

In order to compute the higher order Nash blow-up, especially in the hypersurface case, the higher-order Jacobian matrix of a polynomial was defined in [**Du17**] and was rediscovered and developed further in [**B-P-D20**] and [**B-J-N19**].

In this paper, we introduce different kinds of higher order Jacobian matrices on top of the $\mathrm{Jac}_n(F)$ in [**Du17**] which is redefined in Definition 1.14 for simplicity.

Let $F \in \mathbb{C}[x_1, \cdots, x_l]$. The ideal which gives the higher order Nash blow-up with order n of a hypersurface $V(F)$ is the ideal generated by the maximal minors of $\mathrm{Jac}_n(F)$ ([**Du17**]). With this in mind it is natural to introduce the following higher order Nash blow-up local algebra for an isolated hypersurface singularity.

Definition 1.4 ([**M-H-Y-Z23**])**.** With the notations above.

Let $F \in \mathbb{C}[x_1, \cdots, x_l]$ and $\mathrm{Jac}_n(F)$ be the Jacobian matrix with order n. Let $\mathcal{J}_n(F) \subset \mathbb{C}[x_1, \cdots, x_l]$ be the ideal generated by the maximal minors of $\mathrm{Jac}_n(F)$. Then we define a new higher order Nash blow-up local algebra of V to be: $\mathcal{M}_n(V) := \mathcal{O}_l/\langle F, \mathcal{J}_n(F)\rangle$. We use $d_n(V)$ to denote the dimension of $\mathcal{M}_n(V)$.

Remark 1.5. If $(V, 0)$ is an isolated hypersurface singularity, then it is easy to see that the $\mathcal{M}_1(V)$ is exactly the moduli algebra $A(V)$, moreover, $\mathcal{M}_n(V)$ is Artinian (cf. Corollary 2.2, [**Du17**]).

In [**M-H-Y-Z23**], the authors made the following conjecture.

Conjecture 1.6 ([**M-H-Y-Z23**])**.** *Let $(V, 0)$ be an isolated hypersurface singularity defined by a polynomial $F(x_1, \cdots, x_l) \in \mathbb{C}[\boldsymbol{x}]$. Then $\mathcal{M}_n(V)$ is a contact invariant of $(V, 0)$, i.e. it depends only on the isomorphism class of the germ $(V, 0)$.*

In [**M-H-Y-Z23**], Hussain-Ma-Yau-Zuo verified the Conjecture 1.6 for $l = 2, n = 2$. In [**F-H-Y-Z25**], the Conjecture 1.6 for $l = 2, n = 3$ is verified.

We introduce general notations and conventions of the paper.

Definition 1.7. We define $\boldsymbol{x} := (x_1, x_2, \cdots, x_l)$, $\boldsymbol{x}' := (x_1, x_2, \cdots, x_{l'})$ and $\boldsymbol{x}'' := (x_1, x_2, \cdots, x_{l''})$ where $l, l', l'' \in \mathbb{N}_+$.

Definition 1.8. Suppose A is a finite set. We define $\#A$ as the cardinality of A.

Definition 1.9. For the two elements i, j in a set A, $\delta_{i,j}$ means Kronecker delta.

Definition 1.10. For $\alpha := (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$, we define $\alpha! := \alpha_1! \cdot \alpha_2! \cdot \cdots \cdot \alpha_l!$ and $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_l$.

The concepts of contact equivalence and right equivalence are classical for analytic functions in singularity theory (cf. [**G-L-S07**]). We recall them and generalize them to $\mathbb{C}[[\boldsymbol{x}]]$.

Definition 1.11. Let $F, G \in \mathbb{C}[[\boldsymbol{x}]]$, resp. $F, G \in \mathbb{C}\{\boldsymbol{x}\}$.

(1) F is called *right equivalent* to G, $F \overset{\mathrm{r}}{\sim} G$, if there exists some $h \in Aut(\mathbb{C}[[\boldsymbol{x}]])$, resp. $h \in Aut(\mathbb{C}\{\boldsymbol{x}\})$, and $F = h(G)$.

(2) F is called *contact equivalent* to G, $F \overset{\mathrm{c}}{\sim} G$, if there exists some $u \in (\mathbb{C}[[\boldsymbol{x}]])^*$, resp. $u \in (\mathbb{C}\{\boldsymbol{x}\})^*$, and some $h \in Aut(\mathbb{C}[[\boldsymbol{x}]])$, resp. $h \in Aut(\mathbb{C}\{\boldsymbol{x}\})$, such that $F = u \cdot h(G)$.

Remark 1.12. When $F, G \in \mathcal{O}_l$, $F \overset{\mathrm{r}}{\sim} G$ iff F and G define, up to a change of coordinates in $\left(\mathbb{C}^l, \mathbf{0}\right)$, the same map germs $\left(\mathbb{C}^l, \mathbf{0}\right) \to (\mathbb{C}, 0)$, while $F \overset{\mathrm{c}}{\sim} G$ iff F and G have, up to coordinate change, the same zero-fibre.

Definition 1.13. Assume that $F \in \mathbb{C}[[\boldsymbol{x}]]$ (resp. $F \in \mathbb{C}\{\boldsymbol{x}\}$) and $I^{(F)}$ is some ideal in $\mathbb{C}[[\boldsymbol{x}]]$ (resp. $\mathbb{C}\{\boldsymbol{x}\}$) with generators solely associated with components of F and their various orders of partial derivatives.

(1) If we have $\frac{\mathbb{C}[[\boldsymbol{x}]]}{h\left(I^{(F)}\right)} \cong \frac{\mathbb{C}[[\boldsymbol{x}]]}{I^{(h(F))}}$, resp. $\frac{\mathbb{C}\{\boldsymbol{x}\}}{h\left(I^{(F)}\right)} \cong \frac{\mathbb{C}\{\boldsymbol{x}\}}{I^{(h(F))}}$, for any $h \in Aut(\mathbb{C}[[\boldsymbol{x}]])$, resp. $h \in Aut(\mathbb{C}\{\boldsymbol{x}\})$, the algebra $\frac{\mathbb{C}[[\boldsymbol{x}]]}{I^{(F)}}$, resp. $\frac{\mathbb{C}\{\boldsymbol{x}\}}{I^{(F)}}$, is a *right invariant*.

(2) If we have $\frac{\mathbb{C}[[\boldsymbol{x}]]}{h\left(I^{(F)}\right)} \cong \frac{\mathbb{C}[[\boldsymbol{x}]]}{I^{(u \cdot (h(F)))}}$, resp. $\frac{\mathbb{C}\{\boldsymbol{x}\}}{h\left(I^{(F)}\right)} \cong \frac{\mathbb{C}\{\boldsymbol{x}\}}{I^{(u \cdot h(F))}}$, for any $h \in Aut(\mathbb{C}[[\boldsymbol{x}]])$, resp. $h \in Aut(\mathbb{C}\{\boldsymbol{x}\})$, and any $u \in \left(\mathbb{C}[[\boldsymbol{x}]]^*\right)^{l'}$, resp. $u \in \left(\mathbb{C}\{\boldsymbol{x}\}^*\right)^{l'}$, the algebra $\frac{\mathbb{C}[[\boldsymbol{x}]]}{I^{(F)}}$ (resp. $\frac{\mathbb{C}\{\boldsymbol{x}\}}{I^{(F)}}$) is a *contact invariant*.

1.2. Higher Order Jacobian Matrix Theory.

The theory of Jacobian matrix has long been regarded as a classical approach, yet it falls short in accurately describing higher order partial derivatives, thus limiting its effectiveness in studying non-linear problems. The development of higher order Jacobian matrix theory aims to compensate for the deficiency.

Higher order Jacobian matrices can be seen as a generalization of Jacobian matrices that contains more than linear information. Undoubtedly, akin to classical Jacobian matrix theory, higher order Jacobian matrix theory is fundamental and holds immense potential for significant contributions across various mathematical disciplines in the future. We show several direct applications and indirect inspirations from higher order Jacobian matrix theory in Subsection 1.5.

The initial motivation for higher order Jacobian matrix theory is to prove Conjecture 1.6. In this process, we want to study the relationship between the matrix $\mathrm{Jac}_n(F)$ and the matrix $\mathrm{Jac}_n(G)$ in Definition 1.14 defined below when F and G are contact equivalent under Definition 1.11.

In this paper, we introduce matrices QJac_n, TJac_n and ATJac_n which are also the generalizations of the Jacobian matrix.

Although the higher order Jacobian matrix arises from higher order Nash blow-up, we shall define the matrices for any F in $\mathbb{C}[[\boldsymbol{x}]]$ and any $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$.

We introduce matrices QJac_n's, TJac_n's and ATJac_n's in this subsection. For matrices Jac_n's and QJac_n's, we introduce Definition 1.14. For matrices TJac_n's and ATJac_n's, we introduce Definition 1.17.

Definition 1.14. Consider $F \in \mathbb{C}[[\boldsymbol{x}]]$. Let $n \in \mathbb{N}_+$. The matrix $\mathrm{Jac}_n(F)$ is defined to be a matrix whose rows are labeled by multi-indices $\{\boldsymbol{a} \in \mathbb{N}^l \colon |\boldsymbol{a}| \le n-1\}$ and columns by $\{\boldsymbol{b} \in \mathbb{N}^l \colon 1 \le |\boldsymbol{b}| \le n\}$. The $(\boldsymbol{a}, \boldsymbol{b})$-entry of $\mathrm{Jac}_n(F)$ is

$$\begin{cases} \frac{1}{(\boldsymbol{b}-\boldsymbol{a})!}\frac{\partial^{|\boldsymbol{b}-\boldsymbol{a}|}F}{\partial \boldsymbol{x}^{\boldsymbol{b}-\boldsymbol{a}}}, & \text{if } \boldsymbol{b} \ge \boldsymbol{a} \\ 0, & \text{otherwise} \end{cases}.$$

Arrange the labels by graded lexicographical order.

Let $n \in \mathbb{N}$. For the definition of matrix $\mathrm{QJac}_n(F)$, simply replace $\{\boldsymbol{a} \in \mathbb{N}^l \colon |\boldsymbol{a}| \le n-1\}$ (resp. $\{\boldsymbol{b} \in \mathbb{N}^l \colon 1 \le |\boldsymbol{b}| \le n\}$) by $\{\boldsymbol{a} \in \mathbb{N}^l \colon |\boldsymbol{a}| \le n\}$ (resp. $\{\boldsymbol{b} \in \mathbb{N}^l \colon |\boldsymbol{b}| \le n\}$) in Definition 1.14.

Remark 1.15. For $F \in \mathbb{C}[[\boldsymbol{x}]]$ and $n \in \mathbb{N}_+$, $\mathrm{Jac}_n(F)$ is of size $\binom{l+n-1}{l} \times \left(\binom{l+n}{l} - 1\right)$ and $\mathrm{QJac}_n(F)$ is of size $\binom{l+n}{l} \times \binom{l+n}{l}$.

We give some examples of $\mathrm{Jac}_n(F)$'s and $\mathrm{QJac}_n(F)$'s in Subsection 4.1.

The following two kinds of matrices TJac_n and ATJac_n are used for studying the relationship between $\mathrm{QJac}_n(F)$ and $\mathrm{QJac}_n(G)$ in Definition 1.14 when F and G are contact equivalent or right equivalent. (The TJac_n matrices also hold significant representational value, as discussed in Subsection 1.3.)

Before we go further, it is needed to refer to ${}^{(h)}\gamma_{(S)}^{(R)}$ which is defined in Definition 1.16.

Definition 1.16. Let $R \in \mathbb{Z}^{l'}$, $S \in \mathbb{Z}^{l}$ and the $\mathbb{C}$-algebra homomorphism $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$. We define

$$ {}^{(h)}\gamma_{(S)}^{(R)} $$

$$:= \begin{cases} \sum_{\substack{K \in \mathbb{N}^{l'};\\ K \le R}} \left((-1)^{|K|} \cdot \binom{R}{K} \cdot h\left((\boldsymbol{x}')^{K} \right) \cdot \frac{\partial^{|S|}\left(h\left((\boldsymbol{x}')^{R-K} \right) \right)}{\partial \boldsymbol{x}^{S}} \right), & R \in \mathbb{N}^{l'} \text{ and } S \in \mathbb{N}^{l} \\ 0, & \text{otherwise} \end{cases}. $$

In particular, ${}^{(h)}\gamma_{(S)}^{(R)}$ is equal to 1 when $R = (0, 0, \cdots, 0)$ and $S = (0, 0, \cdots, 0)$, and equal to 0 when $R = (0, 0, \cdots, 0)$ and $|S| > 0$.

Definition 1.17. Consider the $\mathbb{C}$-algebra homomorphism $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$. Let $n \in \mathbb{N}_{+}$. The matrix $\mathrm{TJac}_n(h)$ is defined to be a matrix whose rows are labeled by multi-indices $\{\boldsymbol{a} \in \mathbb{N}^{l'} \colon 1 \le |\boldsymbol{a}| \le n\}$ and columns by $\{\boldsymbol{b} \in \mathbb{N}^{l} \colon 1 \le |\boldsymbol{b}| \le n\}$. The $(\boldsymbol{a}, \boldsymbol{b})$-entry of $\mathrm{TJac}_n(h)$ is $\frac{{}^{(h)}\gamma_{(\boldsymbol{b})}^{(\boldsymbol{a})}}{\boldsymbol{b}!}$. Arrange the labels by graded lexicographical order.

Let $n \in \mathbb{N}_{+}$. For the definition of matrix $\mathrm{ATJac}_n(h)$, simply replace $\{\boldsymbol{a} \in \mathbb{N}^{l'} \colon 1 \le |\boldsymbol{a}| \le n\}$ (resp. $\{\boldsymbol{b} \in \mathbb{N}^{l} \colon 1 \le |\boldsymbol{b}| \le n\}$ by $\{\boldsymbol{a} \in \mathbb{N}^{l'} \colon |\boldsymbol{a}| \le n\}$ (resp. $\{\boldsymbol{b} \in \mathbb{N}^{l} \colon |\boldsymbol{b}| \le n\}$) in Definition 1.17.

Let $i, j \in \mathbb{N}_{+}$. For the definition of matrix ${}^{(h)}A_{i,j}$, simply replace $\{\boldsymbol{a} \in \mathbb{N}^{l'} \colon 1 \le |\boldsymbol{a}| \le n\}$ (resp. $\{\boldsymbol{b} \in \mathbb{N}^{l} \colon 1 \le |\boldsymbol{b}| \le n\}$ by $\{\boldsymbol{a} \in \mathbb{N}^{l'} \colon |\boldsymbol{a}| = i\}$ (resp. $\{\boldsymbol{b} \in \mathbb{N}^{l} \colon |\boldsymbol{b}| = j\}$) in Definition 1.17.

Remark 1.18. For the $\mathbb{C}$-algebra homomorphism $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$, we know $\mathrm{TJac}_n(h)$ is a matrix of the size $\left(\binom{n+l'}{l'} - 1 \right) \times \left(\binom{n+l}{l} - 1 \right)$ and $\mathrm{ATJac}_n(h)$ is a matrix of the size $\binom{n+l'}{l'} \times \binom{n+l}{l}$ for arbitrary $n \in \mathbb{N}_{+}$.

We give some examples of $\mathrm{TJac}_n(h)$ and $\mathrm{ATJac}_n(h)$ in Subsection 4.2.

Theorems A, B, and C are foundational for Theorems D and E, as they establish matrix expressions for the chain and Leibniz rules within and between the QJac_n and ATJac_n matrix categories. In addition, Theorem 2.10 is a key link in higher order Jacobian matrix theory.

For the matrix expression of the chain rule, we have Theorems A and B.

Theorem A. *For $G \in \mathbb{C}[[\boldsymbol{x}']]$ and the $\mathbb{C}$-algebra homomorphism $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$, under Definitions 1.14 and 1.17 then*

$$ h\left(\mathrm{QJac}_n(G) \right) \cdot \mathrm{ATJac}_n(h) = \mathrm{ATJac}_n(h) \cdot \mathrm{QJac}_n(h(G)) $$

for any $n \in \mathbb{N}_{+}$.

Theorem B. *For the $\mathbb{C}$-algebra homomorphism $g : \mathbb{C}[[\boldsymbol{x}'']] \to \mathbb{C}[[\boldsymbol{x}']]$ and the $\mathbb{C}$-algebra homomorphism $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$, under Definition 1.17 then*

$$\mathrm{ATJac}_n(h \circ g) = h(\mathrm{ATJac}_n(g)) \cdot \mathrm{ATJac}_n(h)$$

and

$$\mathrm{TJac}_n(h \circ g) = h(\mathrm{TJac}_n(g)) \cdot \mathrm{TJac}_n(h)$$

for any $n \in \mathbb{N}_+$ and $n \in \mathbb{N}$ respectively.

For the matrix expression of the Leibniz rule, we have Theorem C.

Theorem C. *For $u, G \in \mathbb{C}[[\boldsymbol{x}]]$, under Definition 1.14, then*

$$\mathrm{QJac}_n(uG) = \mathrm{QJac}_n(u) \cdot \mathrm{QJac}_n(G)$$

for any $n \in \mathbb{N}_+$.

It is noteworthy that our definitions and properties of the aforementioned four types of matrices can be seamlessly extended to the case of convergent power series rings. What's more, Theorems A, B and C in higher order Jacobian matrix theory also works for the formal power series rings over $\mathcal{R}$ and the $\mathcal{R}$-algebra homomorphisms between them where $\mathcal{R}$ is an integral domain.

1.3. Representational Remarks on the TJac_n matrix.

For the concept of higher order Jacobian matrices, there has been different attempts. As mentioned before, the Jac_n matrix was constructed in [**B-P-D20**], [**B-J-N19**] and [**Du17**] from the higher order Nash blowup of a hypersurface. On the other hand, the TJac_n matrix was constructed from another background in [**C-D-G21**] (the matrix is named D_x^n in [**C-D-G21**]), representing a higher-order tangent map of a morphism. Crucially, we bridge these research directions by introducing the variations of matrices (i.e., $\mathrm{QJac}_n(h)$ and $\mathrm{ATJac}_n(h)$) and establishing a higher-order similarity transformation formula for these variations for the first time (i.e., Theorem A).

Furthermore, we present distinct algorithms for computing higher-order Jacobian matrices and explore the relationships among the elements of the $\mathrm{TJac}_n(h)$ matrices in Section 2. The theorems in Section 2 are more intrinsic, while the matrix equations are representative. From these intrinsic results, we directly derive the representative results (Theorems A, B and C). Then from the three theorems we directly obtain Theorems D and E.

Here we give an explanation to show that our construction of the TJac_n matrix is also the same matrix representation, using Theorem 2.10 in Section 2.

For linear endomorphism $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$, we have

$$\begin{bmatrix} h(x_1) \\ h(x_2) \\ \vdots \\ h(x_{l'}) \end{bmatrix} = \mathrm{Jac}(h) \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_l \end{bmatrix} = \mathrm{Jac}(h)\,|_0 \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_l \end{bmatrix}$$

by the property of Jacobian matrix. For general endomorphism $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$, what is the representation of the transformation from the elements

$$x_1, x_2, \cdots, x_l, x_1^2, x_1x_2, \cdots, x_1x_l, \cdots, x_l^2, \cdots$$

to the elements

$$h(x_1), h(x_2), \cdots, h(x_{l'}), (h(x_1))^2,$$
$$h(x_1)\,h(x_2), \cdots, h(x_1)\,h(x_{l'}), \cdots, (h(x_{l'}))^2, \cdots ?$$

Theorem 1.20 gives us the answer to the question.

Definition 1.19. For the $\mathbb{C}$-algebra homomorphism $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$ and $n \in \mathbb{N}$, we define the $\binom{n+l'}{l'} \times 1$ column vector

$$\mathrm{BAS}_n(h) := \begin{bmatrix} \mathrm{Bas}^{(1)}(h) \\ \mathrm{Bas}^{(2)}(h) \\ \vdots \\ \mathrm{Bas}^{(n)}(h) \end{bmatrix}$$

where $\mathrm{Bas}^{(k)}(h)$ is a column vector with entries labeled by $\{I \in \mathbb{N}^{l'} : |I| = k\}$ arranged by graded lexicographical order. The I-entry of $\mathrm{Bas}^{(k)}(h)$ is $h\left((\boldsymbol{x}')^I\right)$.

Theorem 1.20. *For* $F = \sum_{I\in\mathbb{N}^l} a_I \boldsymbol{x}^I \in \mathbb{C}[[\boldsymbol{x}]]$ *and* $n \in \mathbb{N}$, *define the* n-jet truncation *by*

$$\mathrm{Jet}^{(n)}(F) := \sum_{\substack{I\in\mathbb{N}^l \\ |I|\le n}} a_I \boldsymbol{x}^I \in \mathbb{C}[\boldsymbol{x}].$$

For matrices C *over* $\mathbb{C}[[\boldsymbol{x}]]$, *we extend this* entrywise *as*

$$\mathrm{Jet}^{(n)}(C) := \left(\mathrm{Jet}^{(n)}(C_{ij})\right).$$

(1) For the $\mathbb{C}$*-algebra homomorphisms* $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$ *and* $id \in Aut(\mathbb{C}[[\boldsymbol{x}]])$, *under Definitions 1.17 and 1.19, we have*

$$\mathrm{Jet}^{(n)}(\mathrm{BAS}_n(h)) = \mathrm{TJac}_n(h)\,|_0 \cdot \mathrm{BAS}_n(id)$$

for arbitrary $n \in \mathbb{N}_+$.

It follows that:

(2) For the $\mathbb{C}$-algebra homomorphisms $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$ and $h' \in Aut(\mathbb{C}[[\boldsymbol{x}]])$, under Definitions 1.17 and 1.19, we have

$$\mathrm{Jet}^{(n)}(\mathrm{BAS}_n(h)) = \mathrm{TJac}_n\left(h \circ (h')^{-1}\right)|_0 \cdot \mathrm{Jet}^{(n)}\left(\mathrm{BAS}_n(h')\right)$$

for arbitrary $n \in \mathbb{N}_+$.

Proof. (1) From Definition 1.9 and Theorem 2.10, for arbitrary $n \in \mathbb{N}_+$ and $P = (p_1, p_2, \cdots, p_{l'}) \in \mathbb{N}^{l'}$ with $1 \leq |P| \leq n$, we have

$$\mathrm{Jet}^{(n)}\left(h\left((\boldsymbol{x}')^P\right)\right) = \mathrm{Jet}^{(n)}\left(\prod_{i=1}^{l'}\prod_{k_i=1}^{p_i} h(x_i)\right)$$

$$= \mathrm{Jet}^{(n)}\left(\prod_{i=1}^{l'}\prod_{k_i=1}^{p_i}\sum_{I^{(i,k_i)}\in\mathbb{N}^l}\frac{\frac{\partial^{|I^{(i,k_i)}|}h(x_i)}{\partial \boldsymbol{x}^{I^{(i,k_i)}}}|_0}{I^{(i,k_i)}!}\cdot \boldsymbol{x}^{I^{(i,k_i)}}\right)$$

$$= \mathrm{Jet}^{(n)}\left(\prod_{i=1}^{l'}\prod_{k_i=1}^{p_i}\sum_{I^{(i,k_i)}\in\mathbb{N}^l}\frac{{}^{(h)}\gamma^{(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i})}_{(I^{(i,k_i)})}|_0}{I^{(i,k_i)}!}\cdot \boldsymbol{x}^{I^{(i,k_i)}}\right)$$

$$= \sum_{I\in\mathbb{N}^l,|I|\leq n}$$

$$\left(\left(\sum_{\substack{I^{(i,k_i)}\in\mathbb{N}^l,\forall 1\leq i\leq l',\forall 1\leq k_i\leq p_i;\\ \sum_{i=1}^{l'}\sum_{k_i=1}^{p_i} I^{(i,k_i)}=I}}\prod_{i=1}^{l'}\prod_{k_i=1}^{p_i}\frac{{}^{(h)}\gamma^{(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i})}_{(I^{(i,k_i)})}|_0}{I^{(i,k_i)}!}\right)\boldsymbol{x}^I\right)$$

$$= \sum_{I\in\mathbb{N}^l,|I|\leq n}\left(\frac{{}^{(h)}\gamma^{(P)}_{(I)}|_0}{I!}\boldsymbol{x}^I\right) = \sum_{I\in\mathbb{N}^l,1\leq|I|\leq n}\left(\frac{{}^{(h)}\gamma^{(P)}_{(I)}|_0}{I!}\boldsymbol{x}^I\right),$$

which implies the matrix equation

$$\mathrm{Jet}^{(n)}(\mathrm{BAS}_n(h)) = \mathrm{TJac}_n(h)|_0 \cdot \mathrm{BAS}_n(id)$$

holds for the element with row index P. Therefore, we complete the proof.

(2) From Theorem B, we know $\mathrm{TJac}_n(h')|_0$ is invertible and

$$\mathrm{TJac}_n\left((h')^{-1}\right)|_0 = \left(\mathrm{TJac}_n(h')|_0\right)^{-1}.$$

By Theorem B and the result of (1), we complete the proof. q.e.d.

Remark 1.21. In Theorem 1.20, we take the value of the TJac_n matrix at the origin. In polynomial cases, we can naturally obtain such property at other points, which is exactly the equivalent definition (of D_x^n) in [**C-D-G21**].

1.4. On Different Generalizations of the Tjurina Algebra and the Milnor Algebra.

The Tjurina algebra of $f \in \mathbb{C}\{\boldsymbol{x}\}$ (i.e. $\mathbb{C}\{\boldsymbol{x}\}/\langle f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_l}\rangle$) and the Milnor algebra of $f \in \mathbb{C}\{\boldsymbol{x}\}$ (i.e. $\mathbb{C}\{\boldsymbol{x}\}/\langle \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_l}\rangle$) play pivotal roles in singularity theory (c.f. [**A-V-Z12**]).

One application of the Tjurina number $\tau(f)$ and the Milnor number $\mu(f)$ of f (i.e. the dimensions of Tjurina algebra and the Milnor algebra respectively of f) is finite determinacy (cf. Corollary 1.51).

Another application for the Tjurina algebra is the renowned Mather-Yau Theorem.

Theorem 1.22 (Mather-Yau [**G-L-S07**])**.** *We define* $\mathfrak{m} := \langle \boldsymbol{x} \rangle \subset \mathbb{C}\{\boldsymbol{x}\}$ *for simplicity. Let* $f, g \in \mathfrak{m}$. *We define* $j(f) := \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_l}\rangle$ *(similarly define* $j(g)$*).*

The following are equivalent:

(1) $f \overset{c}{\sim} g$.

(2) For all $b \geq 1$,

$$\mathbb{C}\{\boldsymbol{x}\}/\left\langle f, \mathfrak{m}^b j(f)\right\rangle \cong \mathbb{C}\{\boldsymbol{x}\}/\left\langle g, \mathfrak{m}^b j(g)\right\rangle$$

as $\mathbb{C}$*-algebras.*

(3) There is some $b \geq 1$ *such that*

$$\mathbb{C}\{\boldsymbol{x}\}/\left\langle f, \mathfrak{m}^b j(f)\right\rangle \cong \mathbb{C}\{\boldsymbol{x}\}/\left\langle g, \mathfrak{m}^b j(g)\right\rangle$$

as $\mathbb{C}$*-algebras.*

Moreover, if f *has an isolated singularity, then* $f \overset{c}{\sim} g$ *iff* $T_f \cong T_g$, *where* $T_f := \mathbb{C}\{\boldsymbol{x}\}/\langle f, j(f)\rangle$ *is the Tjurina algebra of* f *(similarly define* T_g*).*

Besides, J. Huh also discovers the relationship between Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs ([**Hu12**]).

In this subsection, we will give a glimpse of different generalizations of the Tjurina algebra and the Milnor algebra for higher orders. It is worth mentioning that the Tjurina algebra and the Milnor algebra are constructed using first-order derivatives, but now we are constructing invariants using arbitrary higher-order derivatives.

Based on higher order Jacobian matrices defined in Definitions 1.14 and 1.17, we introduce following ideals and algebras.

Definition 1.23. Let $F \in \mathbb{C}[[\boldsymbol{x}]]$. For $n, k \in \mathbb{N}_+$ satisfying $k \leq \binom{l+n-1}{l}$, we define $\mathcal{J}_n^{(k)}(F)$ to be the ideal in $\mathbb{C}[[\boldsymbol{x}]]$ generated by all the $(k \times k)$-minors of $\mathrm{Jac}_n(F)$ under Definition 1.14. For $n, k \in \mathbb{N}_+$ satisfying $k \leq \binom{l+n}{l}$, we define $\left(\mathcal{J}_n^{(k)}\right)'(F)$ to be the ideal in $\mathbb{C}[[\boldsymbol{x}]]$ generated by all the $(k \times k)$-minors of $\mathrm{QJac}_n(F)$ under Definition 1.14.

Definition 1.24. Let $F \in \mathbb{C}[[\boldsymbol{x}]]$ and we follow the notations of Definition 1.23. For $n, k \in \mathbb{N}_+$ satisfying $k \leq \binom{l+n-1}{l}$, we define

$$\mathcal{T}_n^{(k)}(F) := \mathbb{C}[[\boldsymbol{x}]]/\langle F, \mathcal{J}_n^{(k)}(F)\rangle.$$

For $n, k \in \mathbb{N}_+$ satisfying $k \leq \binom{l+n}{l}$, we define

$$\left(\mathcal{T}_n^{(k)}\right)'(F) := \mathbb{C}[[\boldsymbol{x}]]/\left(\mathcal{J}_n^{(k)}\right)'(F).$$

Theorem D. *Assume that $F \in \mathbb{C}[[\boldsymbol{x}]]$ and $n \in \mathbb{N}_+$. Under Definitions 1.13 and 1.24 we have the following statements:*

(1) For any $k \in \mathbb{N}_+$ satisfying $k \leq \binom{l+n-1}{l}$, $\mathcal{T}_n^{(k)}(F)$ is a contact invariant.

(2) For any $k \in \mathbb{N}_+$ satisfying $k \leq \binom{l+n}{l}$, $\left(\mathcal{T}_n^{(k)}\right)'(F)$ is a contact invariant.

For $F \in \mathbb{C}\{\boldsymbol{x}\}$ and $n \in \mathbb{N}_+$, the above statements still hold where the corresponding algebras are $\mathbb{C}\{\boldsymbol{x}\}$ modulo ideals in $\mathbb{C}\{\boldsymbol{x}\}$ generated by the same elements.

Remark 1.25. Theorem D completely proves Conjecture 1.6. Furthermore, the newly introduced algebras are still contact invariants. Note that: $\mathcal{T}_n^{(k)}(F)$ and $\left(\mathcal{T}_n^{(k)}\right)'(F)$ are two different ways to generalize the Tjurina algebra; under Definitions 1.23 and 1.24, the size of minors generating the ideals is arbitrary; F does not need the condition of isolated singularity at origin. In particular, we have introduced a lot of invariants for projective varieties.

In fact, for $F \in \mathbb{C}[[\boldsymbol{x}]]$ and any $k \in \mathbb{N}_+$ satisfying $k \leq \binom{l+n-1}{l}$, the ideals have the relationship $\left(\mathcal{J}_n^{(k)}\right)'(F) \subset \langle F, \mathcal{J}_n^{(k)}(F)\rangle$. Therefore, the algebra $\left(\mathcal{T}_n^{(k)}\right)'(F)$ is finer than the algebra $\mathcal{T}_n^{(k)}(F)$.

We can also study generalizations of the Milnor algebra.

Parallel to $\left(\mathcal{T}_n^{(k)}\right)'(F)$ and $\mathcal{T}_n^{(k)}(F)$, we define two kinds of totally new algebras $\left(\mathcal{M}_n^{(k)}\right)'(F)$ and $\left(\mathcal{M}_n^{(k)}\right)''(F)$ related to the matrices $\mathrm{TJac}_n(F)$ and $\mathrm{ATJac}_n(F)$ introduced in this paper.

Definition 1.26. Let $F \in \mathbb{C}[[\boldsymbol{x}]]$. F can be treated as the $\mathbb{C}$-algebra homomorphism

$$F' : \mathbb{C}[[x_1]] \to \mathbb{C}[[\boldsymbol{x}]]$$

by letting $F'(x_1) := F$. For $n, k \in \mathbb{N}_+$ satisfying $k \le n$, we define $\mathcal{T}\mathcal{J}_n^{(k)}(F)$ to be the ideal in $\mathbb{C}[[\boldsymbol{x}]]$ generated by all the $(k \times k)$-minors of $\mathrm{TJac}_n(F')$ under Definition 1.17.

Definition 1.27. Let $F \in \mathbb{C}[[\boldsymbol{x}]]$. F can be treated as the $\mathbb{C}$-algebra homomorphism

$$F' : \mathbb{C}[[x_1]] \to \mathbb{C}[[\boldsymbol{x}]]$$

by letting $F'(x_1) := F$. For $n, k \in \mathbb{N}_+$ satisfying $k \le n+1$, we define $\mathcal{A}\mathcal{T}\mathcal{J}_n^{(k)}(F)$ to be the ideal in $\mathbb{C}[[\boldsymbol{x}]]$ generated by all the $(k \times k)$-minors of $\mathrm{ATJac}_n(F')$ under Definition 1.17.

Definition 1.28. Let $F \in \mathbb{C}[[\boldsymbol{x}]]$. For $n, k \in \mathbb{N}_+$ satisfying $k \le n$, we define

$$\left(\mathcal{M}_n^{(k)}\right)'(F) := \mathbb{C}[[\boldsymbol{x}]]/\mathcal{T}\mathcal{J}_n^{(k)}(F)$$

under Definition 1.26.

Definition 1.29. Let $F \in \mathbb{C}[[\boldsymbol{x}]]$. For $n, k \in \mathbb{N}_+$ satisfying $k \le n+1$, we define

$$\left(\mathcal{M}_n^{(k)}\right)''(F) := \mathbb{C}[[\boldsymbol{x}]]/\mathcal{A}\mathcal{T}\mathcal{J}_n^{(k)}(F)$$

under Definition 1.27.

Theorem E. *Let $F \in \mathbb{C}[[\boldsymbol{x}]]$ and $n \in \mathbb{N}_+$. Under Definitions 1.13, 1.28 and 1.29, we have the following statements:*

(1) For any $k \in \mathbb{N}_+$ satisfying $k \le n$, $\left(\mathcal{M}_n^{(k)}\right)'(F)$ is a right invariant.

(2) For any $k \in \mathbb{N}_+$ satisfying $k \le n+1$, $\left(\mathcal{M}_n^{(k)}\right)''(F)$ is a right invariant.

For $F \in \mathbb{C}\{\boldsymbol{x}\}$ and $n \in \mathbb{N}_+$, the above statements still hold where the corresponding algebras are $\mathbb{C}\{\boldsymbol{x}\}$ modulo ideals in $\mathbb{C}\{\boldsymbol{x}\}$ generated by the same elements.

Remark 1.30. Parallel to Theorem D, Theorem E states $\left(\mathcal{M}_n^{(k)}\right)'(F)$ and $\left(\mathcal{M}_n^{(k)}\right)''(F)$ are right invariants under Definition 1.13. Note that $\left(\mathcal{M}_n^{(k)}\right)'(F)$ and $\left(\mathcal{M}_n^{(k)}\right)''(F)$ are two different ways to generalize the Milnor algebra; Under Definitions 1.28 and 1.29, the size of minors generating the ideals is arbitrary; F does not need the condition of isolated singularity at the origin. In particular, we have introduced a lot of invariants for projective varieties.

Remark 1.31. For $1 \leq k \leq n$, one can easily verify $\left(\mathcal{M}_n^{(k)}\right)'(F) = \left(\mathcal{M}_n^{(k+1)}\right)''(F)$ under Definitions 1.28 and 1.29.

It is clear that $\mathcal{T}_n^{(k)}(F)$ and $\left(\mathcal{M}_n^{(k)}\right)'(F)$ can be viewed as generalized Tjurina algebra and Milnor algebra respectively. However, both $\left(\mathcal{T}_n^{(k)}\right)'(F)$ and $\left(\mathcal{M}_n^{(k)}\right)''(F)$ are also different generalizations of Tjurina algebra and Milnor algebra respectively. As the corollary of Theorems D and E, we have the following classical results.

Corollary 1.32 ($n = 1, l' = 1$ case: different versions of classical result)**.** *In cases $F, G \in \mathbb{C}[[\boldsymbol{x}]]$ or $F, G \in \mathbb{C}\{\boldsymbol{x}\}$, we have*

(1) $F \overset{\mathrm{r}}{\sim} G$ implies that M_F is isomorphic to M_G as analytic algebras. In particular, $\mu(F) = \mu(G)$.

(2) $F \overset{\mathrm{c}}{\sim} G$ implies that T_F is isomorphic to T_G as analytic algebras. In particular, $\tau(F) = \tau(G)$.

At the end of the section, we consider a classical problem of Calabi-Yau manifolds to show the usage of the higher order generalization of the Tjurina and Milnor algebras.

Since the advent of mirror symmetry in string theory, there has been a notable increase in interest and research activities concerning Calabi-Yau manifolds, involving contributions from both physicists and mathematicians. The significant attention that mirror symmetry has attracted from the mathematical community stems from its effective prediction of the number n_k of rational curves of degree k within these manifolds. This conjecture, referred to as the Mirror Conjecture, was elegantly addressed by Lian, Liu, and Yau in their landmark paper ([**L-L-Y97**]). We consider the geometric properties of a specific class of Calabi-Yau manifolds defined by

$$X_s = \left\{(x_1 : \cdots : x_l) \in \mathbb{CP}^{l-1} : x_1^l + \cdots + x_l^l + s x_1 \ldots x_l = 0\right\}.$$

We recall the following results.

Theorem 1.33 ([**Eg-Ho86**])**.** *Let $j(t) = \frac{t^3(t^3+8)^3}{(t^3-1)^3}$ be the j-invariant of the cubic curve $\tilde{\mathrm{C}}_t$ with equation*

$$x_0^3 + x_1^3 + x_2^3 - 3t x_0 x_1 x_2 = 0, t^3 \neq 1.$$

Then $\tilde{\mathrm{C}}_t$ and $\tilde{\mathrm{C}}_{t'}$, are projective equivalent if and only if $j(t) = j(t')$.

Theorem 1.34 (**[Ch-Ya99]**). *For $l \geq 5$, consider the one parameter family of Calabi-Yau manifolds*

$$X_t = \left\{ (x_1 : \cdots : x_l) \in \mathbb{CP}^{l-1} : x_1^l + \cdots + x_l^l + t x_1 \cdots x_l = 0 \right\}.$$

For arbitrary $t_1, t_2 \in \mathbb{C}$, X_{t_1} is projective equivalent to X_{t_2} if and only if $t_1^l = t_2^l$.

When $l = 4$, we have the following conjecture, which has been a several-decades-old problem.

Conjecture 1.35. *Consider the one parameter family of Calabi-Yau manifolds*

$$X_t = \left\{ (x_1 : \cdots : x_4) \in \mathbb{CP}^3 : x_1^4 + x_2^4 + x_3^4 + x_4^4 + t x_1 x_2 x_3 x_4 = 0 \right\}.$$

For arbitrary $t_1, t_2 \in \mathbb{C}$, X_{t_1} is projective equivalent to X_{t_2} if and only if $t_1^4 = t_2^4$.

We demonstrate that Theorems D and E play a role that transcends classical Tjurina and Milnor algebras in the computation of distinguishing equivalence classes below.

We define $F^{(t)} = x_1^4 + x_2^4 + x_3^4 + x_4^4 + t x_1 x_2 x_3 x_4$ where $t \in \mathbb{C}$.

For the $\mathbb{C}$-vector space dimensions of algebras $\mathcal{T}_2^{(k)}\left(F^{(t)}\right)$ ($1 \leq k \leq 5$, $t = 0, 1$) and $\left(\mathcal{T}_2^{(k)}\right)'\left(F^{(t)}\right)$ ($1 \leq k \leq 5$, $t = 0, 1$), we have the following results.

Table 1. Dimensions of the contact invariant $\mathcal{T}_2^{(k)}\left(F^{(t)}\right)$

k	$t = 0$	$t = 1$
1	16	5
2	130	100
3	431	371
4	1012	910
5	2113	1969

Table 2. Dimensions of the contact invariant $\left(\mathcal{T}_2^{(k)}\right)'\left(F^{(t)}\right)$

k	$t = 0$	$t = 1$
1	16	5
2	145	101
3	565	455
4	1535	1315
5	3567	3234

It follows that X_0 and X_1 in Conjecture 1.35 are not projective equivalent using Theorem D.

Also, for the dimensions of $\left(\mathcal{M}_2^{(k)}\right)'\left(F^{(t)}\right)$ $(k = 1, 2,\ t = 0, 1)$, we have the following results.

Table 3. Dimensions of the right invariant $\left(\mathcal{M}_2^{(k)}\right)'\left(F^{(t)}\right)$

k	$t = 0$	$t = 1$
1	16	5
2	630	541

It also follows that X_0 and X_1 in Conjecture 1.35 are not projective equivalent using Theorem E.

However, from the perspective of the Tjurina number and the Milnor number, we cannot obtain the same conclusion, since the Tjurina numbers $\tau\left(F^{(0)}\right)$ and $\tau\left(F^{(1)}\right)$, and the Milnor numbers $\mu\left(F^{(0)}\right)$ and $\mu\left(F^{(1)}\right)$ are equal to 81. It shows that the dimensions of generalized higher order Tjurina algebras and Milnor algebras contain more information than the classical ones.

1.5. Other Applications and Inspirations Related to Higher Order Jacobian Matrix Theory.

Applications and inspirations associated with higher-order Jacobian matrices have become a prominent topic in recent research (cf. [**Ba**], [**C-D-G21**], [**deFe-Do20**], [**Di-St15**], [**Dr-Se24**], etc.).

In addition to presenting novel invariants of hypersurface singularities, the theory of higher-order Jacobian matrices possesses considerable potential. For the convenience of the readers to realize the importance of this theory, we include the applications aimed at addressing specific problems, which appear in [**F-Y-Z2**] - [**F-Y-Z4**].

The three applications referenced in Subsection 1.5 leverage the results from Section 2 to generalize classical findings to their higher-order versions. In practice, the representative results provide us with intuition and direction, while the intrinsic results offer computational approaches. Both are essential.

1.5.1. Higher Order Hessian Matrix Theory and Projective Invariants.

For algebraic varieties determined by homogeneous polynomials of degree two (quadrics), this classification problem can be addressed through quadratic form analysis employing classical Hessian matrix methods. In the broader context of higher-degree hypersurfaces (degree ≥ 3), we develop an innovative theoretical framework extending these classical constructs, which we term “higher order Hessian matrix theory”, as systematically developed in our previous investigation [**F-Y-Z2**]. The main content of this work is as follows.

Definition 1.36 ([**F-Y-Z2**])**.** Let $F \in \mathbb{C}[[\boldsymbol{x}]]$. For $P^{(1)}, P^{(2)} \in \mathbb{N}^l$, we define

(1)
$$\beta_{\left(P^{(1)},P^{(2)}\right)} := \begin{cases} \dfrac{\left(\left(\frac{|P^{(1)}+P^{(2)}|}{2}\right)!\right)^2}{|P^{(1)}+P^{(2)}|!\cdot P^{(1)}!\cdot P^{(2)}!}, & |P^{(1)}|-|P^{(2)}|=0 \\ \dfrac{\left(\frac{|P^{(1)}+P^{(2)}|-1}{2}\right)!\cdot\left(\frac{|P^{(1)}+P^{(2)}|+1}{2}\right)!}{2\cdot|P^{(1)}+P^{(2)}|!\cdot P^{(1)}!\cdot P^{(2)}!}, & |P^{(1)}|-|P^{(2)}|=\pm 1 \\ 0, & \text{otherwise} \end{cases}.$$

For $p, q \in \mathbb{N}_+$, the matrix $H(F)_{p,q}$ is defined to be a matrix whose rows are labeled by multi-indices $\{\boldsymbol{a} \in \mathbb{N}^l : |\boldsymbol{a}| = p\}$ and columns by $\{\boldsymbol{b} \in \mathbb{N}^l : |\boldsymbol{b}| = q\}$. The $(\boldsymbol{a}, \boldsymbol{b})$-entry of $H(F)_{p,q}$ is $\beta_{(\boldsymbol{a},\boldsymbol{b})} \cdot \frac{\partial^{|\boldsymbol{a}+\boldsymbol{b}|}F}{\partial \boldsymbol{x}^{(\boldsymbol{a}+\boldsymbol{b})}}$. Arrange the labels by lexicographical order. Especially, we have $H(F)_{1,1} = \frac{1}{2}\operatorname{Hess}(F)$.

For $n \in \mathbb{N}$, the matrix $\operatorname{AHess}_n(F)$ is defined as

$$\operatorname{AHess}_n(F) := \begin{bmatrix} H(F)_{0,0} & H(F)_{0,1} & \cdots & H(F)_{0,n} \\ H(F)_{1,0} & H(F)_{1,1} & \cdots & H(F)_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ H(F)_{n,0} & H(F)_{n,1} & \cdots & H(F)_{n,n} \end{bmatrix}. \tag{2}$$

We call $\operatorname{AHess}_n(F)$ *the n-th order of associated-Hessian matrix of F.*

Theorem 1.37 ([**F-Y-Z2**])**.** *For $F \in \mathbb{C}[[\boldsymbol{x}']]$ and the linear $\mathbb{C}$-algebra homomorphism $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$, with the notations in Definitions 1.17 and 1.36, we have*

$$\operatorname{AHess}_n(h(F)) = (\operatorname{ATJac}_n(h))^T \cdot h(\operatorname{AHess}_n(F)) \cdot \operatorname{ATJac}_n(h) \tag{3}$$

for arbitrary $n \in \mathbb{N}$, which is equivalent to

$$H(h(F))_{i,j} = \left({}^{(h)}A_{i,i}\right)^T \cdot h\left(H(F)_{i,j}\right) \cdot {}^{(h)}A_{j,j} \tag{4}$$

for arbitrary $i, j \in \mathbb{N}$.

Remark 1.38. Theorem 1.37 can be regarded as higher order conjugate transformation. This result is inspired by the higher order similarity transformation presented in Theorem B.

Building upon Theorem 1.37, significant progress has been made in constructing novel invariants for projective manifolds in [**F-Y-Z2**], with particular emphasis on Calabi-Yau varieties. These invariants—derived through systematic applications of the rank method and determinant method—transcend classical invariants, revealing finer structural properties.

Theorem 1.39 (**[F-Y-Z2]**). *(1) For $F \in \mathbb{C}[[\boldsymbol{x}']]$, $G \in \mathbb{C}[[\boldsymbol{x}]]$ and arbitrary $p, q, k \in \mathbb{N}_+$ with $1 \leq k \leq \binom{\min\{p,q\}+\min\{l,l'\}-1}{\min\{l,l'\}-1}$, we denote the ideal in $\mathbb{C}[[\boldsymbol{x}']]$ generated by all the $k \times k$ minors in $H(F)_{p,q}$ by $I_1^{(k)}$ and the ideal in $\mathbb{C}[[\boldsymbol{x}]]$ generated by all the $k \times k$ minors in $H(G)_{p,q}$ by $I_2^{(k)}$. If there exists linear $\mathbb{C}$-algebra homomorphism $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$, such that $G = h(F)$, we have $I_2^{(k)} \subseteq h\left(I_1^{(k)}\right)$. If $l = l'$ and $h \in Aut(\mathbb{C}[[\boldsymbol{x}]])$, we have $I_2^{(k)} = h\left(I_1^{(k)}\right)$.*

(2) For $F \in \mathbb{C}[[\boldsymbol{x}']]$ and $G \in \mathbb{C}[[\boldsymbol{x}]]$ and for arbitrary $n \in \mathbb{N}$ and $k \in \mathbb{N}_+$ satisfying $1 \leq k \leq \binom{n+\min\{l,l'\}}{\min\{l,l'\}}$, we denote the ideal in $\mathbb{C}[[\boldsymbol{x}']]$ generated by all the $k \times k$ minors in $\mathrm{AHess}_n(F)$ by $J_1^{(k)}$ and the ideal in $\mathbb{C}[[\boldsymbol{x}]]$ generated by all the $k \times k$ minors in $\mathrm{AHess}_n(G)$ by $J_2^{(k)}$. If there exists linear $\mathbb{C}$-algebra homomorphism $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$, such that $G = h(F)$, we have $J_2^{(k)} \subseteq h\left(J_1^{(k)}\right)$. If $l = l'$ and $h \in Aut(\mathbb{C}[[\boldsymbol{x}]])$, we have $J_2^{(k)} = h\left(J_1^{(k)}\right)$.

(3) Assume that $F = \sum_{i=0}^{\infty} F_{(i)}$ (resp. $G = \sum_{i=0}^{\infty} G_{(i)}$) is in $\mathbb{C}[[\boldsymbol{x}]]$ where $F_{(i)}$ (resp. $G_{(i)}$) is zero or a homogeneous polynomial of degree i for arbitrary $i \in \mathbb{N}$. If there exists linear $h \in Aut(\mathbb{C}[[\boldsymbol{x}]])$ such that $G = h(F)$, then for any $n, n' \in \mathbb{N}$ and any $r_0, r_1, r_2, \cdots \in \mathbb{C}$, we have

$$\text{(5)} \quad \operatorname{rank}\left(\mathrm{AHess}_n\left(\sum_{i=0}^{\infty} r_i F_{(i)}\right)|_0\right) = \operatorname{rank}\left(\mathrm{AHess}_n\left(\sum_{i=0}^{\infty} r_i G_{(i)}\right)|_0\right)$$

and

$$\text{(6)} \quad \operatorname{rank}\left(H\left(\sum_{i=0}^{\infty} r_i F_{(i)}\right)_{n,n'}|_0\right) = \operatorname{rank}\left(H\left(\sum_{i=0}^{\infty} r_i G_{(i)}\right)_{n,n'}|_0\right).$$

(4) Assume that $F, G \in \mathbb{R}[[\boldsymbol{x}]]$. If there exists a linear $h \in Aut(\mathbb{R}[[\boldsymbol{x}]])$ such that $G = h(F)$, then for any $n \in \mathbb{N}$, the index of inertia of $H(F)_{n,n}|_0$ (resp. $\mathrm{AHess}_n(F)|_0$) is the same as that of $H(G)_{n,n}|_0$ (resp. $\mathrm{AHess}_n(G)|_0$).

(5) Assume that $F, G \in \mathbb{C}[[\boldsymbol{x}]]$. If there exists a linear $h \in Aut(\mathbb{C}[[\boldsymbol{x}]])$ such that $G = h(F)$, then for any $n_1, n_2 \in \mathbb{N}_+$ satisfying $n_2 \geq n_1$, we have

$$\det\left(\begin{bmatrix} H(G)_{n_1,n_1}|_0 & H(G)_{n_1,n_1+1}|_0 & \cdots & H(G)_{n_1,n_2}|_0 \\ H(G)_{n_1+1,n_1}|_0 & H(G)_{n_1+1,n_1+1}|_0 & \cdots & H(G)_{n_1+1,n_2}|_0 \\ \vdots & \vdots & \ddots & \vdots \\ H(G)_{n_2,n_1}|_0 & H(G)_{n_2,n_1+1}|_0 & \cdots & H(G)_{n_2,n_2}|_0 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} H(F)_{n_1,n_1}|_0 & H(F)_{n_1,n_1+1}|_0 & \cdots & H(F)_{n_1,n_2}|_0 \\ H(F)_{n_1+1,n_1}|_0 & H(F)_{n_1+1,n_1+1}|_0 & \cdots & H(F)_{n_1+1,n_2}|_0 \\ \vdots & \vdots & \ddots & \vdots \\ H(F)_{n_2,n_1}|_0 & H(F)_{n_2,n_1+1}|_0 & \cdots & H(F)_{n_2,n_2}|_0 \end{bmatrix}\right)$$

$$\cdot\left(\det\left(\mathrm{Jac}\left(h\right)\right)\right)^{2\cdot\binom{l+n_2}{l+1}-2\cdot\binom{l+n_1-1}{l+1}}. \tag{7}$$

(6) For $F \in \mathbb{C}[[\boldsymbol{x}]]$ and any $n_1, n_2, n_1', n_2', p, p' \in \mathbb{N}_+$ satisfying $n_2 \geq n_1$ and $n_2' \geq n_1'$, when

$$\det\left(\begin{bmatrix} H\left(F^{p'}\right)_{n_1',n_1'}|_0 & H\left(F^{p'}\right)_{n_1',n_1'+1}|_0 & \cdots & H\left(F^{p'}\right)_{n_1',n_2'}|_0 \\ H\left(F^{p'}\right)_{n_1'+1,n_1'}|_0 & H\left(F^{p'}\right)_{n_1'+1,n_1'+1}|_0 & \cdots & H\left(F^{p'}\right)_{n_1'+1,n_2'}|_0 \\ \vdots & \vdots & \ddots & \vdots \\ H\left(F^{p'}\right)_{n_2',n_1'}|_0 & H\left(F^{p'}\right)_{n_2',n_1'+1}|_0 & \cdots & H\left(F^{p'}\right)_{n_2',n_2'}|_0 \end{bmatrix}\right) \neq 0,$$

$$\frac{\left(\det\left(\begin{bmatrix} H\left(F^{p}\right)_{n_1,n_1}|_0 & \cdots & H\left(F^{p}\right)_{n_1,n_2}|_0 \\ \vdots & \ddots & \vdots \\ H\left(F^{p}\right)_{n_2,n_1}|_0 & \cdots & H\left(F^{p}\right)_{n_2,n_2}|_0 \end{bmatrix}\right)\right)^{\frac{L}{\binom{l+n_2}{l+1}-\binom{l+n_1-1}{l+1}}}}{\left(\det\left(\begin{bmatrix} H\left(F^{p'}\right)_{n_1',n_1'}|_0 & \cdots & H\left(F^{p'}\right)_{n_1',n_2'}|_0 \\ \vdots & \ddots & \vdots \\ H\left(F^{p'}\right)_{n_2',n_1'}|_0 & \cdots & H\left(F^{p'}\right)_{n_2',n_2'}|_0 \end{bmatrix}\right)\right)^{\frac{L}{\binom{l+n_2'}{l+1}-\binom{l+n_1'-1}{l+1}}}}$$

is a invariant under invertible linear transformation, where

$$L = \mathrm{lcm}\left(\binom{l+n_2}{l+1} - \binom{l+n_1-1}{l+1}, \binom{l+n_2'}{l+1} - \binom{l+n_1'-1}{l+1}\right).$$

*In particular, for $F \in \mathbb{C}[\boldsymbol{x}]$ homogeneous of even order m, on condition that $\det\left(H(F)_{\frac{m}{2},\frac{m}{2}}\right) \neq 0$, we have a sequence of invariants $\{j_n'(F)\}_{n\in\mathbb{N}_+}$ (we call it j'-*invariant sequence*) under invertible linear transformation defined by*

$$j_n'(F) := \frac{\left(\det\left(H\left(F^n\right)_{\frac{nm}{2},\frac{nm}{2}}\right)\right)^{\frac{\mathrm{lcm}\left(\binom{l+\frac{nm}{2}-1}{l},\binom{l+\frac{m}{2}-1}{l}\right)}{\binom{l+\frac{nm}{2}-1}{l}}}}{\left(\det\left(H(F)_{\frac{m}{2},\frac{m}{2}}\right)\right)^{\frac{\mathrm{lcm}\left(\binom{l+\frac{nm}{2}-1}{l},\binom{l+\frac{m}{2}-1}{l}\right)}{\binom{l+\frac{m}{2}-1}{l}}}}. \tag{8}$$

(7) Consider germs $(X,0), (Y,0) \subset \left(\mathbb{C}^l, 0\right)$. Assume that X is defined by $F^{(1)}, F^{(2)}, \cdots, F^{(p)} \in \mathbb{C}\{\boldsymbol{x}\}$ and Y is defined by $G^{(1)}, G^{(2)}, \cdots, G^{(q)} \in \mathbb{C}\{\boldsymbol{x}\}$. We also assume that the lowest order among

$$F^{(1)}, F^{(2)}, \cdots, F^{(p)}, G^{(1)}, G^{(2)}, \cdots, G^{(q)}$$

is n. If X and Y are biholomorphically equivalent, we have

$$\left\{\operatorname{rank}\left(H\left(\sum_{i=1}^{p} r_i F^{(i)}\right)_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}|_0\right) \mid \text{for all } r_1, r_2, \cdots, r_p \in \mathbb{C}\right\}$$

$$= \left\{\operatorname{rank}\left(H\left(\sum_{i=1}^{q} s_i G^{(i)}\right)_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}|_0\right) \mid \text{for all } s_1, s_2, \cdots, s_q \in \mathbb{C}\right\}.$$

Especially, when $p = q = 1$, we have

$$\operatorname{rank}\left(H\left(F^{(1)}\right)_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}|_0\right) = \operatorname{rank}\left(H\left(G^{(1)}\right)_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}|_0\right).$$

(8) Assume that the projective manifold X (resp. Y) in $\mathbb{CP}^{l-1}$ is defined by $F^{(1)}, F^{(2)}, \cdots, F^{(p)}$ (resp. $G^{(1)}, G^{(2)}, \cdots, G^{(q)}$), where

$$F^{(1)}, F^{(2)}, \cdots, F^{(p)}, G^{(1)}, G^{(2)}, \cdots, G^{(q)} \in \mathbb{C}[\boldsymbol{x}]$$

are homogeneous. If X and Y are projective equivalent, then for arbitrary $n \in \mathbb{N}_+$, we have

$$\left\{\operatorname{rank}\left(H\left(\sum_{i=1}^{p} r_i F^{(i)}\right)_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}|_0\right) \mid \text{for all } r_1, r_2, \cdots, r_p \in \mathbb{C}\right\}$$

$$= \left\{\operatorname{rank}\left(H\left(\sum_{i=1}^{q} s_i G^{(i)}\right)_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}|_0\right) \mid \text{for all } s_1, s_2, \cdots, s_q \in \mathbb{C}\right\}.$$

Especially, when $p = q = 1$, we have

$$\operatorname{rank}\left(H\left(F^{(1)}\right)_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}|_0\right) = \operatorname{rank}\left(H\left(G^{(1)}\right)_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}|_0\right).$$

As an application of Theorem 1.39, we rigorously confirm Conjecture 1.35. In the course of our proof, we also utilize the following remarkable determinant property associated with ATJac_n and TJac_n matrices.

Theorem 1.40 ([**F-Y-Z2**])**.** *Under Definition 1.17, for*

$$h \in End\left(\mathbb{C}[[\boldsymbol{x}]]\right)$$

with l variables, we have

$$\det\left({}^{(h)}A_{k,k}\right) = \left(\det\left(\mathrm{Jac}\left(h\right)\right)\right)^{\binom{k+l-1}{l}} \tag{9}$$

and

$$\det\left(\mathrm{ATJac}_k\left(h\right)\right) = \det\left(\mathrm{TJac}_k\left(h\right)\right) = \left(\det\left(\mathrm{Jac}\left(h\right)\right)\right)^{\binom{l+k}{l+1}} \tag{10}$$

for any $k \in \mathbb{N}_+$.

Greuel and his team utilized the SINGULAR program to investigate the Conjecture 1.35, however, no theoretical proof for complete invariant of this family is provided.

Another important application of Theorem 1.39 is the classification of homogeneous fewnomials with isolated singularities.

We recall a related result.

Proposition 1.41 (Proposition 3.1, [**Ya-Zu16**]). *Let $F \in \mathbb{C}[\boldsymbol{x}]$ be a weighted homogeneous fewnomial isolated singularity with* $\mathrm{mult}(F) \geq 3$*. Then F is analytically equivalent to a linear combination of the following three series:*

The Brieskorn type: $x_1^{a_1} + x_2^{a_2} + \cdots + x_{l-1}^{a_{l-1}} + x_l^{a_l}, l \geq 1,$

The chain type: $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{l-1}^{a_{l-1}}x_l + x_l^{a_l}, l \geq 2,$

The loop type: $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{l-1}^{a_{l-1}}x_l + x_l^{a_l}x_1, l \geq 2.$

The fewnomial singularity plays a significant role in mirror symmetry, as pointed out by Ebeling and Takahashi ([**Eb-Ta11**]).

Below are the results pertaining to the cases $l = 4$ and $l = 5$ from [**F-Y-Z2**].

When $l = 4$, let $F \in \mathbb{C}[\boldsymbol{x}]$ be a homogeneous fewnomial of degree 4 with an isolated singularity. We provide all possible forms of F along with their corresponding ranks: $\mathrm{rank}(H(F)_{2,2}|_0)$, $\mathrm{rank}(H(F^2)_{4,4}|_0)$, and $\mathrm{rank}(H(F^3)_{6,6}|_0)$. It is evident that this approach effectively allows for the study of projective inequivalence between different Calabi-Yau manifolds in $\mathbb{CP}^3$ defined by various forms of F.

Table 4. The $l = 4$ case

F	$\mathrm{rank}\left(H(F)_{2,2}\|_0\right)$	$\mathrm{rank}\left(H(F^2)_{4,4}\|_0\right)$	$\mathrm{rank}\left(H(F^3)_{6,6}\|_0\right)$
$x_1^4 + x_2^4 + x_3^4 + x_4^4$	4	22	56
$x_1^3x_2 + x_2^3x_3 + x_3^3x_4 + x_4^4$	7	28	75
$x_1^3x_2 + x_2^3x_3 + x_3^3x_4 + x_4^3x_1$	8	34	76
$x_1^3x_2 + x_2^3x_3 + x_3^4 + x_4^4$	6	23	65
$x_1^3x_2 + x_2^3x_3 + x_3^3x_1 + x_4^4$	7	28	70
$x_1^3x_2 + x_2^4 + x_3^4 + x_4^4$	5	23	60
$x_1^3x_2 + x_2^3x_1 + x_3^4 + x_4^4$	4	22	58
$x_1^3x_2 + x_2^4 + x_3^3x_4 + x_4^4$	6	25	62
$x_1^3x_2 + x_2^3x_1 + x_3^3x_4 + x_4^4$	5	23	66
$x_1^3x_2 + x_2^3x_1 + x_3^3x_4 + x_4^3x_3$	4	22	68

When $l = 5$, let $F \in \mathbb{C}[\boldsymbol{x}]$ be a homogeneous fewnomial of degree 5 with an isolated singularity. We provide all possible forms of F along with their corresponding ranks: $\mathrm{rank}(H(F)_{3,2}|_0)$ and $\mathrm{rank}(H(F^2)_{5,5}|_0)$. It is evident that this approach effectively allows for the study of projective inequivalence between different Calabi-Yau manifolds in $\mathbb{CP}^4$ defined by various forms of F.

Table 5. The $l = 5$ case

F	rank $\left(H(F)_{3,2}\|_0\right)$	rank $\left(H(F^2)_{5,5}\|_0\right)$
$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$	5	45
$x_1^4x_2 + x_2^4x_3 + x_3^4x_4 + x_4^5 + x_5^5$	8	66
$x_1^4x_2 + x_2^4x_3 + x_3^4x_4 + x_4^4x_1 + x_5^5$	9	77
$x_1^4x_2 + x_2^4x_3 + x_3^5 + x_4^5 + x_5^5$	7	56
$x_1^4x_2 + x_2^4x_3 + x_3^4x_1 + x_4^5 + x_5^5$	8	63
$x_1^4x_2 + x_2^5 + x_3^5 + x_4^5 + x_5^5$	6	50
$x_1^4x_2 + x_2^4x_1 + x_3^5 + x_4^5 + x_5^5$	6	51
$x_1^4x_2 + x_2^5 + x_3^4x_4 + x_4^5 + x_5^5$	7	57
$x_1^4x_2 + x_2^4x_1 + x_3^4x_4 + x_4^5 + x_5^5$	7	58
$x_1^4x_2 + x_2^4x_1 + x_3^4x_4 + x_4^4x_3 + x_5^5$	7	67
$x_1^4x_2 + x_2^4x_3 + x_3^4x_1 + x_4^4x_5 + x_5^4x_4$	9	87
$x_1^4x_2 + x_2^4x_3 + x_3^4x_1 + x_4^4x_5 + x_5^5$	9	74
$x_1^4x_2 + x_2^4x_3 + x_3^5 + x_4^4x_5 + x_5^4x_4$	8	70
$x_1^4x_2 + x_2^4x_3 + x_3^5 + x_4^4x_5 + x_5^5$	8	65

Moreover, the classification of projective inequivalence for nonhypersurface cases can be explicitly demonstrated through concrete invariant constructions. To illustrate this principle, we present the following representative cases.

Example 1.42 ([**F-Y-Z2**])**.** Consider projective manifolds in $\mathbb{CP}^3$. Assume that X is defined by $F^{(1)} := x_1^3x_2 + x_2^3x_3 + x_3^3x_4 + x_4^3x_1$ and $F^{(2)} := x_1^3x_2 + x_2^3x_3 + x_3^3x_4 + x_4^4$ and Y is defined by $G^{(1)} := x_1^3x_2 + x_2^4 + x_3^4 + x_4^4$ and $G^{(2)} := x_1^4 + x_2^4 + x_3^4 + x_4^4$. Then X and Y are not projective equivalent.

Example 1.43 ([**F-Y-Z2**])**.** Consider projective manifolds in $\mathbb{CP}^3$. Assume that X is defined by $F^{(1)} := x_1^3x_2 + x_2^4 + x_3^4 + x_4^4$ and $F^{(2)} := x_1^4 + x_2^4 + x_3^3x_4 + x_4^4$ and Y is defined by $G^{(1)} := x_1^3x_2 + x_2^3x_3 + x_3^4 + x_4^4$ and $G^{(2)} := x_1^4 + x_2^4 + x_3^3x_4 + x_4^3x_3$. Then X and Y are not projective equivalent.

Example 1.44 ([**F-Y-Z2**])**.** Consider projective manifolds in $\mathbb{CP}^4$. Assume that X is defined by $F^{(1)} := x_1^4x_2 + x_2^4x_3 + x_3^4x_1 + x_4^5 + x_5^5$ and $F^{(2)} := x_1^4x_2 + x_2^4x_3 + x_3^5 + x_4^5 + x_5^5$ and Y is defined by $G^{(1)} := x_1^4x_2 + x_2^4x_1 + x_3^5 + x_4^5 + x_5^5$ and $G^{(2)} := x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$. Then X and Y are not projective equivalent.

Example 1.45 ([**F-Y-Z2**])**.** Consider projective manifolds in $\mathbb{CP}^5$. Assume that X is defined by $F^{(1)} := x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^6 + x_6^6$, $F^{(2)} := x_1^5x_2 + x_2^6 + x_3^6 + x_4^6 + x_5^6 + x_6^6$ and $F^{(3)} := x_1^5x_2 + x_2^5x_1 + x_3^6 + x_4^6 + x_5^6 + x_6^6$. We define $G^{(1)} := x_1^5x_2 + x_2^5x_3 + x_3^5x_1 + x_4^5x_5 + x_5^5x_6 + x_6^5x_4$ and $G^{(2)} := x_1^5x_2 + x_2^5x_3 + x_3^5x_1 + x_4^5x_5 + x_5^5x_6 + x_6^6$. Assume that $G^{(1)}$ or $G^{(2)}$ is among the defining polynomials of Y. Then X and Y are not projective equivalent.

By considering the terms with the lowest order, one can also determine the contact inequivalence (i.e. biholomorphic inequivalence), as illustrated by the following example.

Example 1.46 ([**F-Y-Z2**]). Consider germs $(X,0),(Y,0)\subset(\mathbb{C}^6,0)$. Assume X is defined by $F^{(1)},F^{(2)},\cdots,F^{(m_1)}\in\mathbb{C}\{x_1,x_2,x_3,x_4,x_5,x_6\}$ where $m_1\geq 3$, $\mathrm{Jet}^{(6)}\left(F^{(1)}\right)=x_1^6+x_2^6+x_3^6+x_4^6+x_5^6+x_6^6$, $\mathrm{Jet}^{(6)}\left(F^{(2)}\right):=x_1^5x_2+x_2^6+x_3^6+x_4^6+x_5^6+x_6^6$, $\mathrm{Jet}^{(6)}\left(F^{(3)}\right):=x_1^5x_2+x_2^5x_1+x_3^6+x_4^6+x_5^6+x_6^6$ and $\mathrm{Jet}^{(6)}\left(F^{(i)}\right)=0$ for $i=4,5,\cdots,m_1$. Assume that $G^{(1)},G^{(2)}\in\mathbb{C}\{x_1,x_2,x_3,x_4,x_5,x_6\}$ with $\mathrm{Jet}^{(6)}\left(G^{(1)}\right):=x_1^5x_2+x_2^5x_3+x_3^5x_1+x_4^5x_5+x_5^5x_6+x_6^5x_4$ and $\mathrm{Jet}^{(6)}\left(G^{(2)}\right):=x_1^5x_2+x_2^5x_3+x_3^5x_1+x_4^5x_5+x_5^5x_6+x_6^6$. If $G^{(1)}$ or $G^{(2)}$ vanish on Y, then X and Y are not biholomorphically equivalent.

1.5.2. Calculating Explicit Expression for the Inverse of an Automorphism of a Power Series Ring.

The implicit function theorem is a fundamental result that finds applications across various branches of mathematics. In many cases, deriving an explicit expression is essential. More generally, for a given automorphism of a power series ring over an arbitrary commutative ring with unity, our objective is to compute the inverse expression. While the linear terms are determined by the inverse of the Jacobian matrix, the computations for the nonlinear terms become increasingly challenging.

In the univariate case, one can utilize Newton's Lemma to explicitly obtain the implicit function, or more broadly, the inverse of an automorphism of formal power series.

Lemma 1.47 (Newton's lemma [**G-L-S07**]). *Let $F\in\mathbb{C}\{\boldsymbol{x},y\}$ and $k\in\mathbb{N}_+$. Let $\overline{Y}(\boldsymbol{x})\in\mathbb{C}\{\boldsymbol{x}\}$ be such that, for $D:=\frac{\partial F}{\partial y}(\boldsymbol{x},\overline{Y}(\boldsymbol{x}))$, we have*

$$F(\boldsymbol{x},\overline{Y}(\boldsymbol{x}))\in\langle\boldsymbol{x}\rangle^k\cdot\langle D\rangle^2\subset\mathbb{C}\{\boldsymbol{x}\}.$$

Then there exists a $Y(\boldsymbol{x})\in\mathbb{C}\{\boldsymbol{x}\}$ with $Y(\boldsymbol{x})-\overline{Y}(\boldsymbol{x})\in\langle\boldsymbol{x}\rangle^k\cdot\langle D\rangle$ such that $F(\boldsymbol{x},Y(\boldsymbol{x}))=0$.

For the multivariable nonlinear cases, S. S. Abhyankar introduced a method to tackle the challenge of deriving explicit expressions (cf. [**Ab74**]).

Theorem 1.48 ([**Ab74**]). *For the field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K})=0$ and $f\in Aut(\mathbb{K}[[\boldsymbol{x}]])$, one can compute the expression of f^{-1} by*

(11)
$$f^{-1}(x_i)=\sum_{I\in\mathbb{N}^l}\left(\frac{1}{I!}\cdot\frac{\partial^{|I|}\left(x_i\cdot\det(\mathrm{Jac}(f))\cdot\prod_{j=1}^{l}(x_j-f(x_j))\right)}{\partial\boldsymbol{x}^I}\right)$$

for $i=1,2,\cdots,l$.

To determine the inverse up to a specified order using S. S. Abhyankar's method, one must calculate higher-order terms and then select the relevant lower-order terms. This approach often leads to redundant computations in practice.

In [**F-Y-Z3**], we introduce two novel approaches for determining the inverse of an automorphism of a formal power series ring over an arbitrary commutative ring with unity, grounded in higher order Jacobian matrix theory. These methods can be conceived as nonlinear extensions of the inverse matrix method and the Gaussian elimination method, respectively, and they avoid the redundant computations that are common in traditional approaches.

By employing these two methods, one can effectively derive the explicit expression in the implicit function theorem.

1.5.3. Performing Explicit Calculations in Finite Determinacy.

In singularity theory, the isolated hypersurface singularity is closely related to finite determinacy.

We have the following definitions for finite determinacy.

Definition 1.49. For any positive integer k, $F \in \mathbb{C}\{\boldsymbol{x}\}$ is called *right k-determined*, resp. *contact k-determined*, in $\mathbb{C}\{\boldsymbol{x}\}$, if for each $G \in \mathbb{C}\{\boldsymbol{x}\}$ with $\mathrm{Jet}^{(k)}(F) = \mathrm{Jet}^{(k)}(G)$, we have $F \overset{\mathrm{r}}{\sim} G$, resp. $F \overset{\mathrm{c}}{\sim} G$, in $\mathbb{C}\{\boldsymbol{x}\}$. The minimal such k is called *the right determinacy*, resp. *the contact determinacy*, in $\mathbb{C}\{\boldsymbol{x}\}$ of F. F is called *finitely right determined*, resp. *finitely contact determined*, in $\mathbb{C}\{\boldsymbol{x}\}$, if f is right k-determined, resp. contact k determined, in $\mathbb{C}\{\boldsymbol{x}\}$ for some $k \in \mathbb{N}$. We also have parallel definitions in $\mathbb{K}[[\boldsymbol{x}]]$.

Here are some famous results of finite determinacy.

Theorem 1.50 (Finite determinacy theorem [**G-L-S07**]). *Let $F \in \mathfrak{m} := \langle \boldsymbol{x} \rangle \subset \mathbb{C}\{\boldsymbol{x}\}$.*

(1) F is right k-determined if

$$\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \cdot j(F).$$

(2) F is contact k-determined if

$$\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \cdot j(F) + \mathfrak{m} \cdot \langle F \rangle.$$

With the method of highest corner, Theorem 1.50 (1) also works for the $\mathbb{K}[[\boldsymbol{x}]]$ case with $char(\mathbb{K}) = 0$ (cf. [**Gr-Pf02**]).

Also, we have the following relationship between the Milnor number, Tjurina number and the finite determinacy.

Corollary 1.51 ([**G-L-S07**]). *If $f \in \mathbb{C}\{\boldsymbol{x}\}$, $f(\mathbf{0}) = 0$, has an isolated singularity with Milnor number μ and Tjurina number τ, then:*

(1) f is right $(\mu+1)$-determined.

(2) f is contact $(\tau+1)$-determined.

In finite determinacy, to compute the explicit forms of h and u in Definition 1.11 is also an interesting question. For this question, [**F-Y-Z4**] gives a matrix method by higher order Jacobian matrix theory.

1.6. Statement.

This paper is part of Fan's doctoral thesis and was finished in early 2023. After this paper was finished, Q. Thuong Lê, T. Yasuda informed the second author that they independently proved Conjecture 1.6 ([**Le-Ya25**]), which is only a small part of our Theorem D. (They also discover some results similar to Theorem C and apply it to the proof of the conjecture.) Recently, D. Duarte etc. also informed the second author that they prove a special case of Conjecture 1.6 ([**B-C-D25**]). It is worth mentioning that Theorem D presented in this paper not only provides a complete proof for Conjecture 1.6, but also introduces many generalizations of the Tjurina algebra which are contact invariants.

In [**C-D-G21**], the authors present an equivalent definition of the TJac_n matrix and establish the same property as the second equation in Theorem B. Through a distinct algorithm derived from our original perspective, we independently derive this property. This approach provides a straightforward implementation of the abstract definitions outlined in [**C-D-G21**]. The equivalence between the two definitions are discussed in Remark 1.21.

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2. Proof of Theorems A, B and C

2.1. The Structure of Section 2.

We will prove Theorems A, B and C in this section.

In Subsection 2.2, in addition to the calculation method of ${}^{(h)}\gamma_{(S)}^{(R)}$ mentioned in Definition 1.16, we also provide two other calculation methods.

In Subsections 2.3, 2.4 and 2.5, we use these calculation methods to obtain the following results:

1. In Subsection 2.3, we prove Corollary 2.8 using the calculation of ${}^{(h)}\gamma_{(S)}^{(R)}$ by the inclusion-exclusion principle (Definition 1.16);
2. In Subsection 2.4, we show Theorem 2.10 using the calculation of ${}^{(h)}\gamma_{(S)}^{(R)}$ by ρ (Theorem 2.3);
3. In Subsection 2.5, we establish Theorem 2.11 using the calculation of ${}^{(h)}\gamma_{(S)}^{(R)}$ by the exact expression (Theorem 2.4).

On the basis above, we prove Theorems A, B and C in Subsections 2.6 and 2.7.

In Subsection 2.6, we derive the matrix expression of the chain rule (Theorems A and B) for higher order Jacobian matrices.

In Subsection 2.7, we give a matrix expression of the Leibniz rule (Theorem C) for higher order Jacobian matrices.

2.2. Three Methods for Calculating ${}^{(h)}\gamma_{(S)}^{(R)}$.

We introduce the following definition based on the inclusion-exclusion principle in Definition 1.16. In this subsection, we introduce other methods of calculating ${}^{(h)}\gamma_{(S)}^{(R)}$ to better apply this new concept. First we introduce two lemmas.

Lemma 2.1. *For $l' \in \mathbb{N}_+$ and $B \in \mathbb{N}^{l'}$ satisfying $|B| > 0$, we have*

$$\sum_{\substack{K\in\mathbb{N}^{l'};\\ K\le B}} \left((-1)^{|K|}\cdot\binom{B}{K}\right) = 0.$$

Proof. Let $B = (b_1, b_2, \cdots, b_{l'})$, we have the expansion

$$\prod_{t=1}^{l'} (1-x_t)^{b_t} = \sum_{\substack{K\in\mathbb{N}^{l'};\\ K\le B}} \left((-1)^{|K|}\cdot\binom{B}{K}\cdot(\boldsymbol{x}')^K\right).$$

Letting $\boldsymbol{x}' = (1, 1, \cdots, 1)$, we complete the proof. q.e.d.

Lemma 2.2. *For the $\mathbb{C}$-algebra homomorphism $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$, $R := (r_1, r_2, \cdots, r_{l'}) \in \mathbb{N}^{l'}$ and $S := (s_1, s_2, \cdots, s_l) \in \mathbb{N}^l$, $|R| > 0$, $|S| > 0$, we have*

$$\frac{\partial^{|S|}\left(h\left((\boldsymbol{x}')^R\right)\right)}{\partial \boldsymbol{x}^S}$$

$$= \sum_{\rho:\left\{(d,i_d)\in\mathbb{N}_+^2:1\le d\le l,1\le i_d\le s_d\right\}\to\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le r_t\right\}} \prod_{t=1}^{l'}\prod_{q_t=1}^{r_t}$$

$$\frac{\partial^{\left|\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)\right|} h(x_t)}{\partial \boldsymbol{x}^{\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)}}.$$

Proof. When $|S| = 1$, the equation is trivial.

If the equation holds when $1 \le |S| \le s_0 - 1$ for some $s_0 \ge 2$, we consider the $|S| = s_0$ case. There exists $1 \le k \le l$ such that $s_k > 0$. By induction, we have

$$\frac{\partial^{|S|-1}\left(h\left((\boldsymbol{x}')^R\right)\right)}{\partial \boldsymbol{x}^{S-(\delta_{1,k},\delta_{2,k},\cdots,\delta_{l,k})}}$$

$$= \sum_{\rho:\left\{(d,i_d)\in\mathbb{N}_+^2:1\le d\le l,1\le i_d\le s_d-\delta_{d,k}\right\}\to\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le r_t\right\}} \prod_{t=1}^{l'}\prod_{q_t=1}^{r_t}$$

$$\frac{\partial^{\left|\left(\sum_{i_1=1}^{s_1-\delta_{1,k}}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l-\delta_{l,k}}\delta_{(t,q_t),\rho(l,i_l)}\right)\right|}h\left(x_t\right)}{\partial\boldsymbol{x}^{\left(\sum_{i_1=1}^{s_1-\delta_{1,k}}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l-\delta_{l,k}}\delta_{(t,q_t),\rho(l,i_l)}\right)}}.$$

Below, for simplicity, we denote

$$T_1=\left(\sum_{i_1=1}^{s_1-\delta_{1,k}}\delta_{(t,q_t),\rho(1,i_1)}+\delta_{1,k}\cdot\delta_{(t,q_t),(j,w_j)},\cdots,\right.$$
$$\left.\sum_{i_l=1}^{s_l-\delta_{l,k}}\delta_{(t,q_t),\rho(l,i_l)}+\delta_{l,k}\cdot\delta_{(t,q_t),(j,w_j)}\right)$$

and

$$T_2=\left(\sum_{i_1=1}^{s_1-\delta_{1,k}}\delta_{(t,q_t),\rho(1,i_1)}+\delta_{1,k}\cdot\delta_{(t,q_t),\rho(k,s_k)},\cdots,\right.$$
$$\left.\sum_{i_l=1}^{s_l-\delta_{l,k}}\delta_{(t,q_t),\rho(l,i_l)}+\delta_{l,k}\cdot\delta_{(t,q_t),\rho(k,s_k)}\right).$$

Acting $\frac{\partial}{\partial x_k}$ on both sides of the equation, we obtain

$$\frac{\partial^{|S|}\left(h\left(\left(\boldsymbol{x'}\right)^R\right)\right)}{\partial\boldsymbol{x}^S}$$
$$=\sum_{\rho:\left\{(d,i_d)\in\mathbb{N}_+^2:1\le d\le l,1\le i_d\le s_d-\delta_{d,k}\right\}\to\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le r_t\right\}}$$
$$\sum_{j=1}^{l'}\sum_{w_j=1}^{r_j}\prod_{t=1}^{l'}\prod_{q_t=1}^{r_t}\frac{\partial^{|T_1|}h\left(x_t\right)}{\partial\boldsymbol{x}^{T_1}}$$
$$=\sum_{\rho:\left\{(d,i_d)\in\mathbb{N}_+^2:1\le d\le l,1\le i_d\le s_d\right\}\to\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le r_t\right\}}$$
$$\prod_{t=1}^{l'}\prod_{q_t=1}^{r_t}\frac{\partial^{|T_2|}h\left(x_t\right)}{\partial\boldsymbol{x}^{T_2}}$$
$$=\sum_{\rho:\left\{(d,i_d)\in\mathbb{N}_+^2:1\le d\le l,1\le i_d\le s_d\right\}\to\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le r_t\right\}}$$
$$\prod_{t=1}^{l'}\prod_{q_t=1}^{r_t}\frac{\partial^{\left|\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)\right|}h\left(x_t\right)}{\partial\boldsymbol{x}^{\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)}}$$

with the second equation obtained by assigning $\rho\left(k,s_k\right)=\left(j,w_j\right)$ where $1\le j\le l$ and $1\le w_j\le r_j$. Therefore the proof is complete. q.e.d.

In Theorem 2.3 below, we obtain the calculation method of ${}^{(h)}\gamma_{(S)}^{(R)}$ by ρ.

Theorem 2.3. *For the $\mathbb{C}$-algebra homomorphism $h:\mathbb{C}[[\boldsymbol{x}']]\to\mathbb{C}[[\boldsymbol{x}]]$, $R:=(r_1,r_2,\cdots,r_{l'})\in\mathbb{N}^{l'}$, $S:=(s_1,s_2,\cdots,s_l)\in\mathbb{N}^l$, $|R|>0$ and $|S|>0$, we have*

$$\begin{aligned}&{}^{(h)}\gamma_{(S)}^{(R)}\\=&\sum_{\substack{\rho:\left\{(d,i_d)\in\mathbb{N}_+^2:1\le d\le l,1\le i_d\le s_d\right\}\to\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le r_t\right\};\\ \operatorname{Im}\rho=\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le r_t\right\}}}\prod_{t=1}^{l'}\prod_{q_t=1}^{r_t}\\&\frac{\partial^{\left|\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)\right|}h(x_t)}{\partial\boldsymbol{x}^{\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)}}.\end{aligned}$$

In particular, when

$$|R|>|S|>0,$$

we have ${}^{(h)}\gamma_{(S)}^{(R)}=0$.

Proof. By Lemmas 2.1 and 2.2 and the inclusion-exclusion principle, the equation in Definition 1.16 can be written in the form of

$$\begin{aligned}{}^{(h)}\gamma_{(S)}^{(R)}=&\sum_{\substack{K\in\mathbb{N}^{l'};\\K\le R}}\left((-1)^{|K|}\cdot\binom{R}{K}\cdot h\left((\boldsymbol{x}')^K\right)\cdot\frac{\partial^{|S|}\left(h\left((\boldsymbol{x}')^{R-K}\right)\right)}{\partial\boldsymbol{x}^S}\right)\\=&\sum_{\substack{K=(k_1,k_2,\cdots,k_{l'})\in\mathbb{N}^{l'};\\K\le R}}\left[(-1)^{|K|}\cdot\binom{R}{K}\cdot h\left((\boldsymbol{x}')^K\right)\right.\\&\cdot\left(\sum_{\rho:\left\{(d,i_d)\in\mathbb{N}_+^2:1\le d\le l,1\le i_d\le s_d\right\}\to\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le r_t-k_t\right\}}\prod_{t=1}^{l'}\prod_{q_t=1}^{r_t-k_t}\right.\\&\left.\left.\frac{\partial^{\left|\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)\right|}h(x_t)}{\partial\boldsymbol{x}^{\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)}}\right)\right]\\=&\sum_{\substack{K=(k_1,k_2,\cdots,k_{l'})\in\mathbb{N}^{l'};\\K\le R;}}\ \sum_{1\le p_t^{(1)}<p_t^{(2)}<\cdots<p_t^{(k_t)}\le r_t,\forall 1\le t\le l'\text{ satisfying }k_t\ne0}\\&\sum_{\substack{\rho:\left\{(d,i_d)\in\mathbb{N}_+^2:1\le d\le l,1\le i_d\le s_d\right\}\to\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le r_t\right\};\\\left(\cup_{1\le t\le l'\text{ satisfying }k_t\ne0}\left\{\left(t,p_t^{(1)}\right),\left(t,p_t^{(2)}\right),\cdots,\left(t,p_t^{(k_t)}\right)\right\}\right)\cap\operatorname{Im}\rho=\emptyset}}\end{aligned}$$

$$\prod_{t=1}^{l'}\left((-1)^{k_t}\cdot\prod_{q_t=1}^{r_t}\frac{\partial^{\left|\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)\right|}h\left(x_t\right)}{\partial \boldsymbol{x}^{\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)}}\right)$$

$$=\sum_{\substack{\rho:\left\{(d,i_d)\in\mathbb{N}_+^2:1\le d\le l,1\le i_d\le s_d\right\}\to\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le r_t\right\};\\ b_t=\#\left(\left\{(t,j_t)\in\mathbb{N}_+^2:1\le j_t\le r_t\right\}\setminus\text{ Im }\rho\right),\forall 1\le t\le l'}}$$

$$\prod_{t=1}^{l'}\left[\left(\sum_{k_t=0}^{b_t}(-1)^{k_t}\cdot\binom{b_t}{k_t}\right)\right.$$

$$\left.\cdot\left(\prod_{q_t=1}^{r_t}\frac{\partial^{\left|\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)\right|}h\left(x_t\right)}{\partial \boldsymbol{x}^{\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)}}\right)\right]$$

$$=\sum_{\substack{\rho:\left\{(d,i_d)\in\mathbb{N}_+^2:1\le d\le l,1\le i_d\le s_d\right\}\to\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le r_t\right\};\\ b_t=\#\left(\left\{(t,j_t)\in\mathbb{N}_+^2:1\le j_t\le r_t\right\}\setminus\text{ Im }\rho\right),\forall 1\le t\le l'}}$$

$$\left[\left(\sum_{0\le k_t\le b_t,\forall 1\le t\le l'}\left((-1)^{\sum_{t=1}^{l'}k_t}\cdot\prod_{t=1}^{l'}\binom{b_t}{k_t}\right)\right)\cdot\right.$$

$$\left.\left(\prod_{t=1}^{l'}\prod_{q_t=1}^{r_t}\frac{\partial^{\left|\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)\right|}h\left(x_t\right)}{\partial \boldsymbol{x}^{\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)}}\right)\right]$$

$$=\sum_{\substack{\rho:\left\{(d,i_d)\in\mathbb{N}_+^2:1\le d\le l,1\le i_d\le s_d\right\}\to\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le r_t\right\};\\ \text{Im }\rho=\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le r_t\right\}}}\prod_{t=1}^{l'}\prod_{q_t=1}^{r_t}$$

$$\frac{\partial^{\left|\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)\right|}h\left(x_t\right)}{\partial \boldsymbol{x}^{\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)}}.$$

In particular, when

$$|R|>|S|>0,$$

we have ${}^{(h)}\gamma_{(S)}^{(R)}=0$.

Therefore the proof is complete. q.e.d.

From the latter part of this section, we obtain the exact expression of ${}^{(h)}\gamma_{(S)}^{(R)}$ in Theorem 2.4, which is the third calculation method.

Theorem 2.4. *For the $\mathbb{C}$-algebra homomorphism $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$, $R := (r_1, r_2, \cdots, r_{l'}) \in \mathbb{N}^{l'}$, $S \in \mathbb{N}^l$ and $|R| > 0$, we have*

$$^{(h)}\gamma_{(S)}^{(R)} = \sum_{\substack{\sum_{i=1}^{l'}\sum_{q_i=1}^{r_i} \boldsymbol{a}^{(i)}(q_i)=S;\\ \boldsymbol{a}^{(i)}(q_i)\in\mathbb{N}^l, \forall 1\le i\le l', \forall 1\le q_i\le r_i,\ if\ r_i>0;\\ |\boldsymbol{a}^{(i)}(q_i)|>0, \forall 1\le i\le l', \forall 1\le q_i\le r_i,\ if\ r_i>0}} \left(\frac{S!}{\prod_{t=1}^{l'}\prod_{q_t=1}^{r_t}\boldsymbol{a}^{(t)}(q_t)!} \cdot \prod_{t=1}^{l'}\prod_{q_t=1}^{r_t} \frac{\partial^{|\boldsymbol{a}^{(t)}(q_t)|}\left(h\left(x_t\right)\right)}{\partial \boldsymbol{x}^{\boldsymbol{a}^{(t)}(q_t)}}\right).$$

In particular, when $S = (0, 0, \cdots, 0)$, we have $^{(h)}\gamma_{(S)}^{(R)} = 0$.

Proof. It follows immediately from Theorem 2.10. q.e.d.

Remark 2.5. From Definition 1.16, Theorems 2.3 and 2.4, we have obtained three methods to calculate the new definition $^{(h)}\gamma_{(S)}^{(R)}$. Namely, the calculation by the inclusion-exclusion principle, the calculation by ρ and the calculation by the exact expression.

In the next three subsections, we will develop three different branches by the three methods respectively.

2.3. Applications for the Calculation of $^{(h)}\gamma_{(S)}^{(R)}$ by the Inclusion-exclusion Principle.

The calculation of $^{(h)}\gamma_{(S)}^{(R)}$ by the inclusion-exclusion principle is useful mainly in studying the partial derivative properties of $^{(h)}\gamma_{(S)}^{(R)}$ by the Leibniz rule, since the formula does not include new concepts and is easy to calculate derivatives. To describe the relationships in swapping the order between homomorphisms and derivatives in Corollary 2.8 is the main goal in this subsection, which ultimately contributes to the construction of the equation of matrices in Theorem A.

We first introduce a result, which is fundamental in this subsection.

Theorem 2.6. *For the $\mathbb{C}$-algebra homomorphism $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$, $R = (r_1, r_2, \cdots, r_{l'}) \in \mathbb{N}^{l'}$ and $S \in \mathbb{N}^l$, we have*

$$\frac{\partial\left(^{(h)}\gamma_{(S)}^{(R)}\right)}{\partial x_j} = {}^{(h)}\gamma_{\left(S+\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)\right)}^{(R)}$$
$$-\sum_{i=1}^{l'}\left(r_i \cdot {}^{(h)}\gamma_{\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)}^{\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)} \cdot {}^{(h)}\gamma_{(S)}^{\left(R-\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)\right)}\right)$$

for $1 \le j \le l$ under Definition 1.16.

Proof. Assume that $S := (s_1, s_2, \cdots, s_l) \in \mathbb{N}^l$.

The equation is obvious when $|R| = 0$ by Definition 1.16. We discuss the $|R| \ge 1$ case.

Acting $\frac{\partial}{\partial x_j}$ on both sides of the equation

$$ {}^{(h)}\gamma^{(R)}_{(S)} = \sum_{\substack{K\in\mathbb{N}^{l'};\\ K\le R}} \left((-1)^{|K|}\cdot\binom{R}{K}\cdot h\left((\boldsymbol{x'})^{K}\right)\cdot\frac{\partial^{|S|}\left(h\left((\boldsymbol{x'})^{R-K}\right)\right)}{\partial\boldsymbol{x}^{S}} \right) $$

in Definition 1.16, we obtain

$$ \begin{aligned} \frac{\partial\left({}^{(h)}\gamma^{(R)}_{(S)}\right)}{\partial x_j} = &\sum_{\substack{K=(k_1,k_2,\cdots,k_{l'})\in\mathbb{N}^{l'};\\ K\le R}} \Bigg((-1)^{|K|}\cdot\binom{R}{K}\cdot \\ &\Bigg(\sum_{i=1}^{l'}\Bigg(\Bigg(k_i\cdot {}^{(h)}\gamma^{(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i})}_{(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j})}\cdot\prod_{1\le t\le l' \text{ satisfying } k_t\ge\delta_{i,t}}\left(h\left(x_t\right)\right)^{k_t-\delta_{i,t}} \\ &\cdot\frac{\partial^{|S|}\left(h\left((\boldsymbol{x'})^{R-K}\right)\right)}{\partial\boldsymbol{x}^{S}}\Bigg) \\ &+h\left((\boldsymbol{x'})^{K}\right)\cdot\frac{\partial^{|S|+1}\left(h\left((\boldsymbol{x'})^{R-K}\right)\right)}{\partial\boldsymbol{x}^{S+\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)}}\Bigg)\Bigg) \\ = &\sum_{\substack{K=(k_1,k_2,\cdots,k_{l'})\in\mathbb{N}^{l'};\\ K\le R}}\Bigg[(-1)^{|K|}\cdot\binom{R}{K}\cdot\sum_{1\le i\le l' \text{ satisfying } k_i\ge 1} \\ &\Bigg(\Bigg(k_i\cdot {}^{(h)}\gamma^{(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i})}_{(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j})}\cdot\prod_{1\le t\le l' \text{ satisfying } k_t\ge\delta_{i,t}}\left(h\left(x_t\right)\right)^{k_t-\delta_{i,t}} \\ &\cdot\frac{\partial^{|S|}\left(h\left((\boldsymbol{x'})^{R-K}\right)\right)}{\partial\boldsymbol{x}^{S}}\Bigg)\Bigg] + {}^{(h)}\gamma^{(R)}_{\left(S+\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)\right)} \\ = &\sum_{i=1}^{l'}\sum_{\substack{K=(k_1,k_2,\cdots,k_{l'})\in\mathbb{N}^{l'};\\ \left(\delta_{i,1},\delta_{i,2},\cdots,\delta_{i,l'}\right)\le K\le R}}\Bigg[(-1)^{|K|}\cdot\binom{R}{K}\cdot k_i\cdot {}^{(h)}\gamma^{(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i})}_{(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j})}\cdot \\ &h\left((\boldsymbol{x'})^{K-\left(\delta_{i,1},\delta_{i,2},\cdots,\delta_{i,l'}\right)}\right)\cdot\frac{\partial^{|S|}\left(h\left((\boldsymbol{x'})^{R-K}\right)\right)}{\partial\boldsymbol{x}^{S}}\Bigg]+{}^{(h)}\gamma^{(R)}_{\left(S+\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)\right)} \end{aligned} $$

$$=\sum_{i=1}^{l'}\sum_{\substack{K=(k_1,k_2,\cdots,k_{l'})\in\mathbb{N}^{l'};\\ K\le R-\left(\delta_{i,1},\delta_{i,2},\cdots,\delta_{i,l'}\right)}}$$

$$\left[(-1)^{|K|+1}\cdot\binom{R}{K+\left(\delta_{i,1},\delta_{i,2},\cdots,\delta_{i,l'}\right)}\cdot(k_i+1)\cdot{}^{(h)}\gamma^{\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)}_{\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)}\cdot\right.$$

$$\left. h\left(\left(\boldsymbol{x}'\right)^{K}\right)\cdot\frac{\partial^{|S|}\left(h\left(\left(\boldsymbol{x}'\right)^{R-K-\left(\delta_{i,1},\delta_{i,2},\cdots,\delta_{i,l'}\right)}\right)\right)}{\partial\boldsymbol{x}^{S}}\right]$$

$$+{}^{(h)}\gamma^{(R)}_{\left(S+\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)\right)}$$

$$={}^{(h)}\gamma^{(R)}_{\left(S+\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)\right)}$$

$$-\sum_{i=1}^{l'}\left(r_i\cdot{}^{(h)}\gamma^{\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)}_{\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)}\cdot{}^{(h)}\gamma^{\left(R-\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)\right)}_{(S)}\right).$$

Therefore the proof is complete. q.e.d.

With this result, we can study the derivatives of the composite functions.

Theorem 2.7. *For $G\in\mathbb{C}\left[\left[\boldsymbol{x}'\right]\right]$ and the $\mathbb{C}$-algebra homomorphism $h:\mathbb{C}\left[\left[\boldsymbol{x}'\right]\right]\to\mathbb{C}\left[\left[\boldsymbol{x}\right]\right]$ and $S\in\mathbb{N}^{l}$ satisfying $|S|\ge 1$, we have*

$$\frac{\partial^{|S|}h\left(G\right)}{\partial\boldsymbol{x}^{S}}=\sum_{\substack{W\in\mathbb{N}^{l'};\\ 1\le|W|\le|S|}}\left(h\left(\frac{\frac{\partial^{|W|}G}{\partial\left(\boldsymbol{x}'\right)^{W}}}{W!}\right)\cdot{}^{(h)}\gamma^{(W)}_{(S)}\right)$$

under Definition 1.16.

Proof. When $|S|=1$, it is clear that for any $1\le j\le l$ and $S=\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)$, the formula holds.

Assume that the formula still holds for any $1\le|S|\le s_0-1$ where $s_0\in\mathbb{N}_{+}$ and $s_0\ge 2$. We prove that the formula also holds for the $|S|=s_0$ case.

When $|S|=s_0$, there exists some $1\le j\le l$ such that the j-th component of S is no less than 1. By Theorems 2.3 and 2.6, it follows that

$$\frac{\partial^{|S|}h\left(G\right)}{\partial\boldsymbol{x}^{S}}=\frac{\partial\left(\frac{\partial^{|S|-1}h(G)}{\partial\boldsymbol{x}^{S-\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)}}\right)}{\partial x_j}$$

$$=\sum_{\substack{W\in\mathbb{N}^{l'};\\ 1\le|W|\le s_0-1}}\left(\frac{\frac{\partial\left(h\left(\frac{\partial^{|W|}G}{\partial\left(\boldsymbol{x}'\right)^{W}}\right)\right)}{\partial x_j}}{W!}\cdot{}^{(h)}\gamma^{(W)}_{\left(S-\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)\right)}\right.$$

$$+h\left(\frac{\frac{\partial^{|W|}G}{\partial(\boldsymbol{x}')^{W}}}{W!}\right)\cdot\frac{\partial\left({}^{(h)}\gamma^{(W)}_{\left(S-\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)\right)}\right)}{\partial x_j}\Bigg)$$

$$=\sum_{i=1}^{l'}\sum_{\substack{W\in\mathbb{N}^{l'};\\1\le|W|\le s_0-1}}\left(h\left(\frac{\frac{\partial^{|W|+1}G}{\partial(\boldsymbol{x}')^{W+\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)}}}{W!}\right)\right.$$

$$\left.\cdot{}^{(h)}\gamma^{\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)}_{\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)}\cdot{}^{(h)}\gamma^{(W)}_{\left(S-\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)\right)}\right)$$

$$+\sum_{\substack{W=(w_1,w_2,\cdots,w_{l'})\in\mathbb{N}^{l'};\\1\le|W|\le s_0-1}}\left[h\left(\frac{\frac{\partial^{|W|}G}{\partial(\boldsymbol{x}')^{W}}}{W!}\right)\cdot\right.$$

$$\left.\left({}^{(h)}\gamma^{(W)}_{(S)}-\sum_{i=1}^{l'}w_i\cdot{}^{(h)}\gamma^{\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)}_{\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)}\cdot{}^{(h)}\gamma^{\left(W-\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)\right)}_{\left(S-\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)\right)}\right)\right]$$

$$=\sum_{i=1}^{l'}\sum_{\substack{W\in\mathbb{N}^{l'};\\1\le|W|\le s_0-1}}\left(h\left(\frac{\frac{\partial^{|W|+1}G}{\partial(\boldsymbol{x}')^{W+\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)}}}{W!}\right)\right.$$

$$\left.\cdot{}^{(h)}\gamma^{\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)}_{\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)}\cdot{}^{(h)}\gamma^{(W)}_{\left(S-\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)\right)}\right)$$

$$-\sum_{i=1}^{l'}\sum_{\substack{W\in\mathbb{N}^{l'};\\1\le|W|\le s_0-1;\\w_i\ge1}}\left(h\left(\frac{\frac{\partial^{|W|}G}{\partial(\boldsymbol{x}')^{W}}}{\left(W-\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)\right)!}\right)\right.$$

$$\left.\cdot{}^{(h)}\gamma^{\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)}_{\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)}\cdot{}^{(h)}\gamma^{\left(W-\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)\right)}_{\left(S-\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)\right)}\right)$$

$$+\sum_{\substack{W\in\mathbb{N}^{l'};\\1\le|W|\le s_0-1}}\left(\frac{h\left(\frac{\partial^{|W|}G}{\partial(\boldsymbol{x}')^{W}}\right)}{W!}\cdot{}^{(h)}\gamma^{(W)}_{(S)}\right)$$

$$=\sum_{i=1}^{l'}\sum_{\substack{W\in\mathbb{N}^{l'};\\|W|\le s_0-1}}\left(h\left(\frac{\frac{\partial^{|W|+1}G}{\partial(\boldsymbol{x}')^{W+\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)}}}{W!}\right)\right.$$

$$\left.\cdot{}^{(h)}\gamma^{\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)}_{\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)}\cdot{}^{(h)}\gamma^{(W)}_{\left(S-\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)\right)}\right)$$

$$
-\sum_{i=1}^{l'} \sum_{\substack{W\in\mathbb{N}^{l'};\\ |W|\le s_0-2}} \left(h\left(\frac{\frac{\partial^{|W|+1}G}{\partial(\boldsymbol{x'})^{W+\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)}}}{W!} \right) \right.
$$

$$
\left. \cdot {}^{(h)}\gamma^{\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)}_{\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)} \cdot {}^{(h)}\gamma^{(W)}_{\left(S-\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)\right)} \right)
$$

$$
+ \sum_{\substack{W\in\mathbb{N}^{l'};\\ 1\le|W|\le s_0-1}} \left(\frac{h\left(\frac{\partial^{|W|}G}{\partial(\boldsymbol{x'})^{W}}\right)}{W!} \cdot {}^{(h)}\gamma^{(W)}_{(S)} \right)
$$

$$
=\sum_{i=1}^{l'} \sum_{\substack{W\in\mathbb{N}^{l'};\\ |W|=s_0-1}} \left(h\left(\frac{\frac{\partial^{|W|+1}G}{\partial(\boldsymbol{x'})^{W+\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)}}}{W!} \right) \right.
$$

$$
\left. \cdot {}^{(h)}\gamma^{\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)}_{\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)} \cdot {}^{(h)}\gamma^{(W)}_{\left(S-\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)\right)} \right)
$$

$$
+ \sum_{\substack{W\in\mathbb{N}^{l'};\\ 1\le|W|\le s_0-1}} \left(\frac{h\left(\frac{\partial^{|W|}G}{\partial(\boldsymbol{x'})^{W}}\right)}{W!} \cdot {}^{(h)}\gamma^{(W)}_{(S)} \right)
$$

$$
=\sum_{i=1}^{l'} \sum_{\substack{W=(w_1,w_2,\cdots,w_{l'})\in\mathbb{N}^{l'};\\ |W|=s_0;\\ w_i\ge 1}}
$$

$$
\left(\frac{h\left(\frac{\partial^{|W|}G}{\partial(\boldsymbol{x'})^{W}}\right)}{W!} \cdot \left(w_i \cdot {}^{(h)}\gamma^{\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)}_{\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)} \cdot {}^{(h)}\gamma^{\left(W-\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)\right)}_{\left(S-\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)\right)} \right) \right)
$$

$$
+ \sum_{\substack{W\in\mathbb{N}^{l'};\\ 1\le|W|\le s_0-1}} \left(\frac{h\left(\frac{\partial^{|W|}G}{\partial(\boldsymbol{x'})^{W}}\right)}{W!} \cdot {}^{(h)}\gamma^{(W)}_{(S)} \right)
$$

$$
= \sum_{\substack{W=(w_1,w_2,\cdots,w_{l'})\in\mathbb{N}^{l'};\\ |W|=s_0;\\ w_i\ge 1}}
$$

$$
\left(\frac{h\left(\frac{\partial^{|W|}G}{\partial(\boldsymbol{x'})^{W}}\right)}{W!} \cdot \sum_{i=1}^{l'} \left(w_i \cdot {}^{(h)}\gamma^{\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)}_{\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)} \cdot {}^{(h)}\gamma^{\left(W-\left(\delta_{1,i},\delta_{2,i},\cdots,\delta_{l',i}\right)\right)}_{\left(S-\left(\delta_{1,j},\delta_{2,j},\cdots,\delta_{l,j}\right)\right)} \right) \right)
$$

$$+ \sum_{\substack{W \in \mathbb{N}^{l'}; \\ 1 \le |W| \le s_0 - 1}} \left(\frac{h\left(\frac{\partial^{|W|} G}{\partial (\boldsymbol{x}')^W}\right)}{W!} \cdot {}^{(h)}\gamma_{(S)}^{(W)} \right)$$

$$= \sum_{\substack{W \in \mathbb{N}^{l'}; \\ 1 \le |W| \le s_0}} \left(\frac{h\left(\frac{\partial^{|W|} G}{\partial (\boldsymbol{x}')^W}\right)}{W!} \cdot {}^{(h)}\gamma_{(S)}^{(W)} \right) - \sum_{\substack{W = (w_1, w_2, \cdots, w_{l'}) \in \mathbb{N}^{l'}; \\ |W| = s_0; \\ w_i \ge 1}} \left[\frac{h\left(\frac{\partial^{|W|} G}{\partial (\boldsymbol{x}')^W}\right)}{W!} \cdot \right.$$

$$\left. \left({}^{(h)}\gamma_{(S)}^{(W)} - \sum_{i=1}^{l'} \left(w_i \cdot {}^{(h)}\gamma_{(\delta_{1,j}, \delta_{2,j}, \cdots, \delta_{l,j})}^{(\delta_{1,i}, \delta_{2,i}, \cdots, \delta_{l',i})} \cdot {}^{(h)}\gamma_{(S - (\delta_{1,j}, \delta_{2,j}, \cdots, \delta_{l,j}))}^{(W - (\delta_{1,i}, \delta_{2,i}, \cdots, \delta_{l',i}))} \right) \right) \right]$$

$$= \sum_{\substack{W \in \mathbb{N}^{l'}; \\ 1 \le |W| \le s_0}} \left(\frac{h\left(\frac{\partial^{|W|} G}{\partial (\boldsymbol{x}')^W}\right)}{W!} \cdot {}^{(h)}\gamma_{(S)}^{(W)} \right)$$

$$- \sum_{\substack{W = (w_1, w_2, \cdots, w_{l'}) \in \mathbb{N}^{l'}; \\ |W| = s_0; \\ w_i \ge 1}} \left(\frac{h\left(\frac{\partial^{|W|} G}{\partial (\boldsymbol{x}')^W}\right)}{W!} \cdot \frac{\partial \left({}^{(h)}\gamma_{(S - (\delta_{1,j}, \delta_{2,j}, \cdots, \delta_{l,j}))}^{(W)} \right)}{\partial x_j} \right)$$

$$= \sum_{\substack{W \in \mathbb{N}^{l'}; \\ 1 \le |W| \le s_0}} \left(h \left(\frac{\frac{\partial^{|W|} G}{\partial (\boldsymbol{x}')^W}}{W!} \right) \cdot {}^{(h)}\gamma_{(S)}^{(W)} \right).$$

We've completed the proof of the formula for the $|S| = s_0$ case.
Therefore, we complete the proof. q.e.d.

We may transform the formula in Theorem 2.7 slightly to cater for the proof of Theorem A.

Corollary 2.8. *For $G \in \mathbb{C}[[\boldsymbol{x}']]$ and the $\mathbb{C}$-algebra homomorphism $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$, $S \in \mathbb{N}^l$, we have*

$$\frac{\partial^{|S|} h(G)}{\partial \boldsymbol{x}^S} = \sum_{\substack{W \in \mathbb{N}^{l'}; \\ |W| \le |S|}} \left(h \left(\frac{\frac{\partial^{|W|} G}{\partial (\boldsymbol{x}')^W}}{W!} \right) \cdot {}^{(h)}\gamma_{(S)}^{(W)} \right)$$

under Definition 1.16.

Proof. The $|S| = 0$ case is obvious. For the $|S| \ge 1$ case, we know ${}^{(h)}\gamma_{(S)}^{(0,0,\cdots,0)} = 0$ by Definition 1.16. By Theorem 2.7, the proof is complete. q.e.d.

2.4. Applications for the Calculation of ${}^{(h)}\gamma_{(S)}^{(R)}$ by ρ.

Generally speaking, the calculation of ${}^{(h)}\gamma_{(S)}^{(R)}$ by ρ is useful in decomposing ${}^{(h)}\gamma_{(S)}^{(R)}$'s with larger $|R|$ and $|S|$ into those with smaller $|R|$ and $|S|$. The main result in this subsection is Theorem 2.10.

Theorem 2.9. *For the $\mathbb{C}$-algebra homomorphism $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$, $R \in \mathbb{Z}^{l'}$, $S \in \mathbb{Z}^l$ and $I \in \mathbb{N}^{l'}$ satisfying $I \leq R$, we have*

$$\frac{{}^{(h)}\gamma_{(S)}^{(R)}}{S!} = \sum_{\substack{J\in\mathbb{N}^l;\\ J\leq S}} \left(\frac{{}^{(h)}\gamma_{(J)}^{(I)}}{J!} \cdot \frac{{}^{(h)}\gamma_{(S-J)}^{(R-I)}}{(S-J)!}\right).$$

Proof. We only need to verify the case when $R \in \mathbb{N}^{l'}$, $S \in \mathbb{N}^l$, $|R| > 0$ and $|S| > 0$ are satisfied at the same time.

Assume that $R := (r_1, r_2, \cdots, r_{l'}) \in \mathbb{Z}^{l'}$, $S := (s_1, s_2, \cdots, s_l) \in \mathbb{Z}^l$ and $I = \left(i^{(1)}, i^{(2)}, \cdots, i^{(l')}\right) \in \mathbb{N}^{l'}$.

For the $|I| = 0$ case and the $|I| = |R|$ case, the equation holds by Definition 1.16.

We consider the $0 < |I| < |R|$ case. From Theorem 2.3, we obtain

$$\begin{aligned}
&{}^{(h)}\gamma_{(S)}^{(R)}\\
&= \sum_{\substack{\rho:\left\{(d,i_d)\in\mathbb{N}_+^2:1\leq d\leq l,1\leq i_d\leq s_d\right\}\to\left\{(t,j_t)\in\mathbb{N}_+^2:1\leq t\leq l',1\leq j_t\leq r_t\right\};\\ \operatorname{Im}\rho=\left\{(t,j_t)\in\mathbb{N}_+^2:1\leq t\leq l',1\leq j_t\leq r_t\right\}}} \prod_{t=1}^{l'}\prod_{q_t=1}^{r_t} \\
&\qquad \frac{\partial^{\left|\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)\right|} h(x_t)}{\partial \boldsymbol{x}^{\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)}}\\
&= \sum_{\substack{\rho:\left\{(d,i_d)\in\mathbb{N}_+^2:1\leq d\leq l,1\leq i_d\leq s_d\right\}\to\left\{(t,j_t)\in\mathbb{N}_+^2:1\leq t\leq l',1\leq j_t\leq r_t\right\};\\ \operatorname{Im}\rho=\left\{(t,j_t)\in\mathbb{N}_+^2:1\leq t\leq l',1\leq j_t\leq r_t\right\}}}\\
&\left[\left(\prod_{t=1}^{l'}\prod_{q_t=1}^{i^{(t)}} \frac{\partial^{\left|\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)\right|} h(x_t)}{\partial \boldsymbol{x}^{\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)}}\right)\right.\\
&\left.\cdot\left(\prod_{t=1}^{l'}\prod_{q_t=i^{(t)}+1}^{r_t} \frac{\partial^{\left|\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)\right|} h(x_t)}{\partial \boldsymbol{x}^{\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)}}\right)\right]\\
&= \sum_{\substack{J=\left(j^{(1)},j^{(2)},\cdots,j^{(l)}\right)\in\mathbb{N}^l;\\ |I|\leq|J|\leq|S|-|R|+|I|;\\ J\leq S}}
\end{aligned}$$

$$\sum_{\substack{\rho:\left\{(d,i_d)\in\mathbb{N}_+^2:1\le d\le l,1\le i_d\le s_d\right\}\to\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le r_t\right\};\\ \mathrm{Im}\ \rho=\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le r_t\right\};\\ \#\left(\{(d,1),(d,2),\cdots,(d,s_d)\}\cap\rho^{-1}\left(\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le i^{(t)}\right\}\right)\right)=j^{(d)},\forall 1\le d\le l}}$$

$$\left[\left(\prod_{t=1}^{l'}\prod_{q_t=1}^{i^{(t)}}\frac{\partial^{\left|\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)\right|}h\left(x_t\right)}{\partial\boldsymbol{x}^{\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)}}\right)\right.$$

$$\left.\cdot\left(\prod_{t=1}^{l'}\prod_{q_t=i^{(t)}+1}^{r_t}\frac{\partial^{\left|\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)\right|}h\left(x_t\right)}{\partial\boldsymbol{x}^{\left(\sum_{i_1=1}^{s_1}\delta_{(t,q_t),\rho(1,i_1)},\cdots,\sum_{i_l=1}^{s_l}\delta_{(t,q_t),\rho(l,i_l)}\right)}}\right)\right]$$

$$=\sum_{\substack{J=\left(j^{(1)},j^{(2)},\cdots,j^{(l)}\right)\in\mathbb{N}^l;\\ |I|\le|J|\le|S|-|R|+|I|;\\ J\le S}}\left[\left(\frac{S!}{J!\cdot(S-J)!}\right)\right.$$

$$\cdot\left(\sum_{\substack{\rho^{(1)}:\left\{(d,i_d)\in\mathbb{N}_+^2:1\le d\le l,1\le i_d\le j^{(d)}\right\}\to\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le i^{(t)}\right\};\\ \mathrm{Im}\ \rho^{(1)}=\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le i^{(t)}\right\}}}\right.$$

$$\left.\prod_{t=1}^{l'}\prod_{q_t=1}^{i^{(t)}}\frac{\partial^{\left|\left(\sum_{i_1=1}^{j^{(1)}}\delta_{(t,q_t),\rho^{(1)}(1,i_1)},\cdots,\sum_{i_l=1}^{j^{(l)}}\delta_{(t,q_t),\rho^{(1)}(l,i_l)}\right)\right|}h\left(x_t\right)}{\partial\boldsymbol{x}^{\left(\sum_{i_1=1}^{j^{(1)}}\delta_{(t,q_t),\rho^{(1)}(1,i_1)},\cdots,\sum_{i_l=1}^{j^{(l)}}\delta_{(t,q_t),\rho^{(1)}(l,i_l)}\right)}}\right)$$

$$\cdot\left(\sum_{\substack{\rho^{(2)}:\left\{(d,i_d)\in\mathbb{N}_+^2:1\le d\le l,1\le i_d\le s_d-j^{(d)}\right\}\to\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le r_t-i^{(t)}\right\};\\ \mathrm{Im}\ \rho^{(2)}=\left\{(t,j_t)\in\mathbb{N}_+^2:1\le t\le l',1\le j_t\le r_t-i^{(t)}\right\}}}\right.$$

$$\left.\left.\prod_{t=1}^{l'}\prod_{q_t=1}^{r_t-i^{(t)}}\frac{\partial^{\left|\left(\sum_{i_1=1}^{s_1-j^{(1)}}\delta_{(t,q_t),\rho^{(2)}(1,i_1)},\cdots,\sum_{i_l=1}^{s_l-j^{(l)}}\delta_{(t,q_t),\rho^{(2)}(l,i_l)}\right)\right|}h\left(x_t\right)}{\partial\boldsymbol{x}^{\left(\sum_{i_1=1}^{s_1-j^{(1)}}\delta_{(t,q_t),\rho^{(2)}(1,i_1)},\cdots,\sum_{i_l=1}^{s_l-j^{(l)}}\delta_{(t,q_t),\rho^{(2)}(l,i_l)}\right)}}\right)\right]$$

$$=S!\cdot\sum_{\substack{J\in\mathbb{N}^l;\\ |I|\le|J|\le|S|-|R|+|I|;\\ J\le S}}\left(\frac{{}^{(h)}\gamma_{(J)}^{(I)}}{J!}\cdot\frac{{}^{(h)}\gamma_{(S-J)}^{(R-I)}}{(S-J)!}\right)$$

$$=S!\cdot\sum_{\substack{J\in\mathbb{N}^l;\\ J\le S}}\left(\frac{{}^{(h)}\gamma_{(J)}^{(I)}}{J!}\cdot\frac{{}^{(h)}\gamma_{(S-J)}^{(R-I)}}{(S-J)!}\right).$$

Therefore the proof is complete. q.e.d.

Theorem 2.10. *Assume that $n \in \mathbb{N}_+$. For the $\mathbb{C}$-algebra homomorphism $h : \mathbb{C}[[\boldsymbol{x}']] \to \mathbb{C}[[\boldsymbol{x}]]$, $J \in \mathbb{N}^l$ and $I, I_1, I_2, \cdots, I_n \in \mathbb{N}^{l'}$ satisfying $I = \sum_{t=1}^{n} I_t$, we have*

$$\frac{{}^{(h)}\gamma_{(J)}^{(I)}}{J!} = \sum_{\substack{J_t \in \mathbb{N}^l, \forall 1 \le t \le n; \\ \sum_{t=1}^{n} J_t = J}} \prod_{i=1}^{n} \frac{{}^{(h)}\gamma_{(J_i)}^{(I_i)}}{J_i!}$$

under Definition 1.16.

Proof. When $n = 1$, the theorem holds.

Assume that the theorem holds for all $1 \le n \le n_0 - 1$ where $n_0 \ge 2$. We have

$$\frac{{}^{(h)}\gamma_{(J)}^{(I)}}{J!} = \sum_{\substack{J'_t \in \mathbb{N}^l, \forall 1 \le t \le n_0 - 1; \\ \sum_{t=1}^{n_0-1} J'_t = J}} \prod_{i=1}^{n_0-1} \frac{{}^{(h)}\gamma_{\left(J'_i\right)}^{\left(I'_i\right)}}{J'_i!}$$

for any $I'_1, I'_2, \cdots, I'_{n_0-1} \in \mathbb{N}^{l'}$ satisfying that $I = \sum_{t=1}^{n_0-1} I'_t$.

We prove the $n = n_0$ case. For any $I_1, I_2, \cdots, I_{n_0} \in \mathbb{N}^{l'}$ satisfying that $I = \sum_{t=1}^{n_0} I_t$, for $1 \le t' \le n_0 - 1$, we define

$$I'_{t'} := \begin{cases} I_{t'}, & t' \ne n_0 - 1 \\ I_{n_0-1} + I_{n_0}, & t' = n_0 - 1 \end{cases}.$$

By the previous assumption and Theorem 2.9, we have

$$\begin{aligned}
\frac{{}^{(h)}\gamma_{(J)}^{(I)}}{J!} &= \sum_{\substack{J'_t \in \mathbb{N}^l, \forall 1 \le t \le n_0 - 1; \\ \sum_{t=1}^{n_0-1} J'_t = J}} \prod_{i=1}^{n_0-1} \frac{{}^{(h)}\gamma_{\left(J'_i\right)}^{\left(I'_i\right)}}{J'_i!} \\
&= \sum_{\substack{J'_t \in \mathbb{N}^l, \forall 1 \le t \le n_0 - 1; \\ \sum_{t=1}^{n_0-1} J'_t = J}} \\
&\left[\left(\prod_{i=1}^{n_0-2} \frac{{}^{(h)}\gamma_{\left(J'_i\right)}^{\left(I'_i\right)}}{J'_i!} \right) \cdot \sum_{\substack{J_{n_0-1}, J_{n_0} \in \mathbb{N}^l; \\ J_{n_0-1} + J_{n_0} = J'_{n_0-1}}} \left(\frac{{}^{(h)}\gamma_{\left(J_{n_0-1}\right)}^{\left(I_{n_0-1}\right)}}{J_{n_0-1}!} \cdot \frac{{}^{(h)}\gamma_{\left(J_{n_0}\right)}^{\left(I_{n_0}\right)}}{J_{n_0}!} \right) \right] \\
&= \sum_{\substack{J_t \in \mathbb{N}^l, \forall 1 \le t \le n_0 - 2; \\ \sum_{t=1}^{n_0-2} J_t \le J}}
\end{aligned}$$

$$\left[\left(\prod_{i=1}^{n_0-2}\frac{{}^{(h)}\gamma_{(J_i)}^{(I_i)}}{J_i!}\right)\cdot\sum_{\substack{J_{n_0-1},J_{n_0}\in\mathbb{N}^l;\\ J_{n_0-1}+J_{n_0}=J-\sum_{t=1}^{n_0-2}J_t}}\left(\frac{{}^{(h)}\gamma_{(J_{n_0-1})}^{(I_{n_0-1})}}{J_{n_0-1}!}\cdot\frac{{}^{(h)}\gamma_{(J_{n_0})}^{(I_{n_0})}}{J_{n_0}!}\right)\right]$$

$$=\sum_{\substack{J_t\in\mathbb{N}^l,\forall 1\le t\le n_0;\\ \sum_{t=1}^{n_0}J_t=J}}\prod_{i=1}^{n_0}\frac{{}^{(h)}\gamma_{(J_i)}^{(I_i)}}{J_i!}.$$

Therefore, we complete the proof. q.e.d.

2.5. Applications for the Calculation of ${}^{(h)}\gamma_{(S)}^{(R)}$ by the Exact Expression.

The calculation of ${}^{(h)}\gamma_{(S)}^{(R)}$ by ρ is useful mainly in revealing the chain rule when h in ${}^{(h)}\gamma_{(S)}^{(R)}$ is the composite of two components. The main result in this subsection is Theorem 2.11.

Theorem 2.11. *For the* $\mathbb{C}$*-algebra homomorphism* $h:\mathbb{C}[[\boldsymbol{x}']]\to\mathbb{C}[[\boldsymbol{x}]]$ *and* $g:\mathbb{C}[[\boldsymbol{x}'']]\to\mathbb{C}[[\boldsymbol{x}']]$, $I\in\mathbb{N}^{l''}$ *and* $J\in\mathbb{N}^l$, *we have*

$$\frac{{}^{(h\circ g)}\gamma_{(J)}^{(I)}}{J!}=\sum_{\substack{K\in\mathbb{N}^{l'};\\ |I|\le|K|\le|J|}}\left(h\left(\frac{{}^{(g)}\gamma_{(K)}^{(I)}}{K!}\right)\cdot\frac{{}^{(h)}\gamma_{(J)}^{(K)}}{J!}\right)$$

under Definition 1.16.

Proof. When $|I|=|J|=0$, both sides of the equation are 1 by Definition 1.16. When $|I|=0$ and $|J|>0$, both sides of the equation are 0 by Definition 1.16. When $|I|>|J|>0$, both sides of the equation are 0 by Theorem 2.3.

Assume $I=\left(i^{(1)},i^{(2)},\cdots,i^{(l'')}\right)\in\mathbb{N}^{l''}$ and $J=\left(j^{(1)},j^{(2)},\cdots,j^{(l)}\right)\in\mathbb{N}^l$.

We consider the $|J|\ge|I|>0$ case. By Theorems 2.3, 2.4, 2.7 and 2.10, we have

$$\frac{{}^{(h\circ g)}\gamma_{(J)}^{(I)}}{J!}=\sum_{\substack{\boldsymbol{a}^{(r)}(q_r)\in\mathbb{N}^l,\forall 1\le r\le l'',1\le q_r\le i^{(r)};\\ |\boldsymbol{a}^{(i)}(q_i)|>0,\forall 1\le r\le l'',1\le q_r\le i^{(r)};\\ \sum_{r=1}^{l''}\sum_{q_r=1}^{i^{(r)}}\boldsymbol{a}^{(r)}(q_r)=J}}\prod_{t=1}^{l''}\prod_{q_t=1}^{i^{(t)}}\frac{\frac{\partial^{|\boldsymbol{a}^{(t)}(q_t)|}(h\circ g)(x_t)}{\partial\boldsymbol{x}^{\boldsymbol{a}^{(t)}(q_t)}}}{\boldsymbol{a}^{(t)}(q_t)!}$$

$$=\sum_{\substack{\boldsymbol{a}^{(r)}(q_r)\in\mathbb{N}^l,\forall 1\le r\le l'',1\le q_r\le i^{(r)};\\ |\boldsymbol{a}^{(i)}(q_i)|>0,\forall 1\le r\le l'',1\le q_r\le i^{(r)};\\ \sum_{r=1}^{l''}\sum_{q_r=1}^{i^{(r)}}\boldsymbol{a}^{(r)}(q_r)=J;}}\prod_{t=1}^{l''}\prod_{q_t=1}^{i^{(t)}}$$

$$\sum_{\substack{\boldsymbol{b}^{(t)}(q_t)\in\mathbb{N}^{l'};\\ 0<|\boldsymbol{b}^{(t)}(q_t)|\le|\boldsymbol{a}^{(t)}(q_t)|}} \left(\frac{h\left(\frac{\partial^{|\boldsymbol{b}^{(t)}(q_t)|}g(x_t)}{\partial(\boldsymbol{x'})^{\boldsymbol{b}^{(t)}(q_t)}}\right)}{\boldsymbol{b}^{(t)}(q_t)!} \cdot \frac{{}^{(h)}\gamma^{\left(\boldsymbol{b}^{(t)}(q_t)\right)}_{\left(\boldsymbol{a}^{(t)}(q_t)\right)}}{\boldsymbol{a}^{(t)}(q_t)!}\right)$$

$$= \sum_{\substack{\boldsymbol{b}^{(r)}(q_r)\in\mathbb{N}^{l'},\forall 1\le r\le l'',\forall 1\le q_r\le i^{(r)};\\ 0<|\boldsymbol{b}^{(r)}(q_r)|\le|J|,\forall 1\le r\le l'',\forall 1\le q_r\le i^{(r)}}} \left[\left(\prod_{t=1}^{l''}\prod_{q_t=1}^{i^{(t)}} \frac{h\left(\frac{\partial^{|\boldsymbol{b}^{(t)}(q_t)|}g(x_t)}{\partial(\boldsymbol{x'})^{\boldsymbol{b}^{(t)}(q_t)}}\right)}{\boldsymbol{b}^{(t)}(q_t)!}\right)\right.$$

$$\left.\cdot\left(\sum_{\substack{\boldsymbol{a}^{(r)}(q_r)\in\mathbb{N}^{l},\forall 1\le r\le l'',\forall 1\le q_r\le i^{(r)};\\ \sum_{r=1}^{l''}\sum_{q_r=1}^{i^{(r)}}\boldsymbol{a}^{(r)}(q_r)=J}} \prod_{t=1}^{l''}\prod_{q_t=1}^{i^{(t)}} \frac{{}^{(h)}\gamma^{\left(\boldsymbol{b}^{(t)}(q_t)\right)}_{\left(\boldsymbol{a}^{(t)}(q_t)\right)}}{\boldsymbol{a}^{(t)}(q_t)!}\right)\right]$$

$$= \sum_{\substack{\boldsymbol{b}^{(r)}(q_r)\in\mathbb{N}^{l'},\forall 1\le r\le l'',\forall 1\le q_r\le i^{(r)};\\ |\boldsymbol{b}^{(r)}(q_r)|>0,\forall 1\le r\le l'',\forall 1\le q_r\le i^{(r)};\\ \sum_{r=1}^{l''}\sum_{q_r=1}^{i^{(r)}}|\boldsymbol{b}^{(r)}(q_r)|\le|J|}}$$

$$\left[\left(\prod_{t=1}^{l''}\prod_{q_t=1}^{i^{(t)}} \frac{h\left(\frac{\partial^{|\boldsymbol{b}^{(t)}(q_t)|}g(x_t)}{\partial(\boldsymbol{x'})^{\boldsymbol{b}^{(t)}(q_t)}}\right)}{\boldsymbol{b}^{(t)}(q_t)!}\right)\cdot\frac{{}^{(h)}\gamma^{\left(\sum_{t=1}^{l''}\sum_{q_t=1}^{i^{(t)}}\boldsymbol{b}^{(t)}(q_t)\right)}_{(J)}}{J!}\right]$$

$$= \sum_{\substack{K\in\mathbb{N}^{l'};\\ |K|\le|J|}}\ \sum_{\substack{\boldsymbol{b}^{(r)}(q_r)\in\mathbb{N}^{l'},\forall 1\le r\le l'',\forall 1\le q_r\le i^{(r)};\\ |\boldsymbol{b}^{(r)}(q_r)|>0,\forall 1\le r\le l'',\forall 1\le q_r\le i^{(r)};\\ \sum_{t=1}^{l''}\sum_{q_t=1}^{i^{(t)}}\boldsymbol{b}^{(t)}(q_t)=K}}$$

$$\left[\left(\prod_{t=1}^{l''}\prod_{q_t=1}^{i^{(t)}} \frac{h\left(\frac{\partial^{|\boldsymbol{b}^{(t)}(q_t)|}g(x_t)}{\partial(\boldsymbol{x'})^{\boldsymbol{b}^{(t)}(q_t)}}\right)}{\boldsymbol{b}^{(t)}(q_t)!}\right)\cdot\frac{{}^{(h)}\gamma^{(K)}_{(J)}}{J!}\right]$$

$$= \sum_{\substack{K\in\mathbb{N}^{l'};\\ |I|\le|K|\le|J|}}$$

$$\left[\left(\sum_{\substack{\boldsymbol{b}^{(r)}(q_r)\in\mathbb{N}^{l'},\forall 1\le r\le l'',\forall 1\le q_r\le i^{(r)};\\ |\boldsymbol{b}^{(r)}(q_r)|>0,\forall 1\le r\le l'',\forall 1\le q_r\le i^{(r)};\\ \sum_{t=1}^{l''}\sum_{q_t=1}^{i^{(t)}}\boldsymbol{b}^{(t)}(q_t)=K}} \prod_{t=1}^{l''}\prod_{q_t=1}^{i^{(t)}} \frac{h\left(\frac{\partial^{|\boldsymbol{b}^{(t)}(q_t)|}g(x_t)}{\partial(\boldsymbol{x'})^{\boldsymbol{b}^{(t)}(q_t)}}\right)}{\boldsymbol{b}^{(t)}(q_t)!}\right)\cdot\frac{{}^{(h)}\gamma_{(J)}^{(K)}}{J!}\right]$$

$$= \sum_{\substack{K\in\mathbb{N}^{l'};\\ |I|\le|K|\le|J|}} \left(h\left(\frac{{}^{(g)}\gamma_{(K)}^{(I)}}{K!}\right)\cdot\frac{{}^{(h)}\gamma_{(J)}^{(K)}}{J!}\right).$$

Therefore, we complete the proof. q.e.d.

2.6. Proof of Theorems A and B.

In this subsection, we will prove Theorems A and B, stating the chain rule in matrix forms.

Proof of Theorem A. By Definition 1.16, Corollary 2.8 and Theorems 2.3 and 2.9, for any $I_1\in\mathbb{N}^{l'}$, $0\le|I_1|\le n$, $I_2\in\mathbb{N}^l$ and $0\le|I_2|\le n$, we have

$$\sum_{\substack{I_3\in\mathbb{N}^{l'};\\ |I_3|\le|I_2|;\\ I_3\ge I_1}} \left(\frac{h\left(\frac{\partial^{|I_3-I_1|}G}{\partial(\boldsymbol{x'})^{I_3-I_1}}\right)}{(I_3-I_1)!}\cdot\frac{{}^{(h)}\gamma_{(I_2)}^{(I_3)}}{I_2!}\right)$$

$$= \sum_{\substack{I_3\in\mathbb{N}^{l'};\\ |I_3|\le|I_2|;\\ I_3\ge I_1}} \sum_{\substack{I_4\in\mathbb{N}^{l};\\ I_4\le I_2}} \left(\frac{h\left(\frac{\partial^{|I_3-I_1|}G}{\partial(\boldsymbol{x'})^{I_3-I_1}}\right)}{(I_3-I_1)!}\cdot\frac{{}^{(h)}\gamma_{(I_4)}^{(I_1)}\cdot{}^{(h)}\gamma_{(I_2-I_4)}^{(I_3-I_1)}}{I_4!\cdot(I_2-I_4)!}\right)$$

$$= \sum_{\substack{I_4\in\mathbb{N}^{l};\\ |I_4|\ge|I_1|;\\ I_4\le I_2}} \sum_{\substack{W\in\mathbb{N}^{l'};\\ |W|\le|I_2|-|I_1|}} \left(\frac{{}^{(h)}\gamma_{(I_4)}^{(I_1)}}{I_4!}\cdot\frac{h\left(\frac{\partial^{|W|}G}{\partial(\boldsymbol{x'})^{W}}\right)\cdot{}^{(h)}\gamma_{(I_2-I_4)}^{(W)}}{W!\cdot(I_2-I_4)!}\right)$$

$$= \sum_{\substack{I_4\in\mathbb{N}^{l};\\ |I_4|\ge|I_1|;\\ I_4\le I_2}} \sum_{\substack{W\in\mathbb{N}^{l'};\\ |W|\le|I_2|-|I_4|}} \left(\frac{{}^{(h)}\gamma_{(I_4)}^{(I_1)}}{I_4!}\cdot\frac{h\left(\frac{\partial^{|W|}G}{\partial(\boldsymbol{x'})^{W}}\right)\cdot{}^{(h)}\gamma_{(I_2-I_4)}^{(W)}}{W!\cdot(I_2-I_4)!}\right)$$

$$= \sum_{\substack{I_4\in\mathbb{N}^{l};\\ |I_4|\ge|I_1|;\\ I_4\le I_2}} \left(\frac{{}^{(h)}\gamma_{(I_4)}^{(I_1)}}{I_4!}\cdot\frac{\frac{\partial^{|I_2-I_4|}h(G)}{\partial\boldsymbol{x}^{I_2-I_4}}}{(I_2-I_4)!}\right).$$

Therefore, the matrix equation holds. q.e.d.

Proof of Theorem B. This is a direct consequence from Theorem 2.11. q.e.d.

2.7. Proof of Theorem C.

In this subsection, we prove Theorem C, stating the Leibniz rule in matrix form.

Lemma 2.12. *Let $G \in \mathbb{C}[[\boldsymbol{x}]]$, $u \in \mathbb{C}[[\boldsymbol{x}]]$ and $C \in \mathbb{N}^l$. We have*

$$\frac{\partial^{|C|}(u \cdot G)}{\partial \boldsymbol{x}^C} = \sum_{\substack{R \in \mathbb{N}^l; \\ 0 \le R \le C}} \left(\binom{C}{R} \cdot \frac{\partial^{|R|} u}{\partial \boldsymbol{x}^R} \cdot \frac{\partial^{|C-R|} G}{\partial \boldsymbol{x}^{C-R}} \right).$$

Proof. We use mathematical induction for $C \in \mathbb{N}^l$.

When $|C| = 0$, the equation is trivial.

For $c_0 \in \mathbb{N}_+$, we assume that the equation holds for all cases satisfying $0 \le |C| \le c_0 - 1$. Consider any case satisfying $|C| = c_0$.

Since $c_0 \ge 1$, there exists some $1 \le t_0 \le l$ satisfying $c_{t_0} \ge 1$. By assumption, we obtain that

$$\frac{\partial^{|C-(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0})|}(u \cdot G)}{\partial \boldsymbol{x}^{C-(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0})}} = \sum_{\substack{R \in \mathbb{N}^l; \\ 0 \le R \le C-(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0})}} \left(\binom{C-(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0})}{R} \cdot \frac{\partial^{|R|} u}{\partial \boldsymbol{x}^R} \cdot \frac{\partial^{|C-(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0})-R|} G}{\partial \boldsymbol{x}^{C-(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0})-R}} \right).$$

Acting $\frac{\partial}{\partial x_{t_0}}$ on both sides of the equation, we get

$$\begin{aligned} &\frac{\partial^{|C|}(u \cdot G)}{\partial \boldsymbol{x}^C} \\ = &\sum_{\substack{R \in \mathbb{N}^l; \\ 0 \le R \le C-(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0})}} \left(\binom{C-(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0})}{R} \cdot \frac{\partial^{|R+(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0})|} u}{\partial \boldsymbol{x}^{R+(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0})}} \cdot \frac{\partial^{|C-(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0})-R|} G}{\partial \boldsymbol{x}^{C-(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0})-R}} \right) \\ &+ \sum_{\substack{R \in \mathbb{N}^l; \\ 0 \le R \le C-(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0})}} \left(\binom{C-(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0})}{R} \cdot \frac{\partial^{|R|} u}{\partial \boldsymbol{x}^R} \cdot \frac{\partial^{|C-R|} G}{\partial \boldsymbol{x}^{C-R}} \right) \\ = &\sum_{\substack{R \in \mathbb{N}^l; \\ (\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0}) \le R \le C}} \end{aligned}$$

$$\left(\binom{C-(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0})}{R-(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0})}\cdot\frac{\partial^{|R|}u}{\partial\boldsymbol{x}^R}\cdot\frac{\partial^{|C-R|}G}{\partial\boldsymbol{x}^{C-R}}\right)$$

$$+\sum_{\substack{R\in\mathbb{N}^l;\\0\le R\le C-\left(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0}\right)}}\left(\binom{C-(\delta_{1,t_0},\delta_{2,t_0},\cdots,\delta_{l,t_0})}{R}\cdot\frac{\partial^{|R|}u}{\partial\boldsymbol{x}^R}\cdot\frac{\partial^{|C-R|}G}{\partial\boldsymbol{x}^{C-R}}\right)$$

$$=\sum_{0\le r_i\le c_i,\forall 1\le i\le l,i\ne t_0}\left[\left(\prod_{1\le t\le l,t\ne t_0}\frac{c_t!}{r_t!\cdot(c_t-r_t)!}\right)\right.$$

$$\cdot\left(\sum_{r_{t_0}=1}^{c_{t_0}-1}\left(\left(\frac{(c_{t_0}-1)!}{(r_{t_0}-1)!\cdot(c_{t_0}-r_{t_0})!}+\frac{(c_{t_0}-1)!}{r_{t_0}!\cdot(c_{t_0}-r_{t_0}-1)!}\right)\right.\right.$$

$$\left.\cdot\frac{\partial^{|(r_1,r_2,\cdots,r_l)|}u}{\partial\boldsymbol{x}^{(r_1,r_2,\cdots,r_l)}}\cdot\frac{\partial^{|C-(r_1,r_2,\cdots,r_l)|}G}{\partial\boldsymbol{x}^{C-(r_1,r_2,\cdots,r_l)}}\right)$$

$$+\left(\frac{\partial^{\left|\left(r_1',r_2',\cdots,r_l'\right)\right|}u}{\partial\boldsymbol{x}^{\left(r_1',r_2',\cdots,r_l'\right)}}\cdot\frac{\partial^{\left|C-\left(r_1',r_2',\cdots,r_l'\right)\right|}G}{\partial\boldsymbol{x}^{C-\left(r_1',r_2',\cdots,r_l'\right)}}\right)\Bigg|_{r_t'=\begin{cases}c_{t_0}, & t=t_0\\ r_t, & 1\le t\le l\text{ and }t\ne t_0\end{cases}}$$

$$\left.\left.+\left(\frac{\partial^{\left|\left(r_1',r_2',\cdots,r_l'\right)\right|}u}{\partial\boldsymbol{x}^{\left(r_1',r_2',\cdots,r_l'\right)}}\cdot\frac{\partial^{\left|C-\left(r_1',r_2',\cdots,r_l'\right)\right|}G}{\partial\boldsymbol{x}^{C-\left(r_1',r_2',\cdots,r_l'\right)}}\right)\Bigg|_{r_t'=\begin{cases}0, & t=t_0\\ r_t, & 1\le t\le l\text{ and }t\ne t_0\end{cases}}\right)\right]$$

$$=\sum_{0\le r_i\le c_i,\forall 1\le i\le l,i\ne t_0}\left[\left(\prod_{1\le t\le l,t\ne t_0}\frac{c_t!}{r_t!\cdot(c_t-r_t)!}\right)\right.$$

$$\cdot\left(\sum_{r_{t_0}=1}^{c_{t_0}-1}\left(\frac{c_{t_0}!}{r_{t_0}!\cdot(c_{t_0}-r_{t_0})!}\cdot\frac{\partial^{|(r_1,r_2,\cdots,r_l)|}u}{\partial\boldsymbol{x}^{(r_1,r_2,\cdots,r_l)}}\cdot\frac{\partial^{|C-(r_1,r_2,\cdots,r_l)|}G}{\partial\boldsymbol{x}^{C-(r_1,r_2,\cdots,r_l)}}\right)\right.$$

$$+\left(\frac{\partial^{\left|\left(r_1',r_2',\cdots,r_l'\right)\right|}u}{\partial\boldsymbol{x}^{\left(r_1',r_2',\cdots,r_l'\right)}}\cdot\frac{\partial^{\left|C-\left(r_1',r_2',\cdots,r_l'\right)\right|}G}{\partial\boldsymbol{x}^{C-\left(r_1',r_2',\cdots,r_l'\right)}}\right)\Bigg|_{r_t'=\begin{cases}c_{t_0}, & t=t_0\\ r_t, & 1\le t\le l\text{ and }t\ne t_0\end{cases}}$$

$$\left.\left.+\left(\frac{\partial^{\left|\left(r_1',r_2',\cdots,r_l'\right)\right|}u}{\partial\boldsymbol{x}^{\left(r_1',r_2',\cdots,r_l'\right)}}\cdot\frac{\partial^{\left|C-\left(r_1',r_2',\cdots,r_l'\right)\right|}G}{\partial\boldsymbol{x}^{C-\left(r_1',r_2',\cdots,r_l'\right)}}\right)\Bigg|_{r_t'=\begin{cases}0, & t=t_0\\ r_t, & 1\le t\le l\text{ and }t\ne t_0\end{cases}}\right)\right]$$

$$=\sum_{\substack{R\in\mathbb{N}^l;\\0\le R\le C}}\left(\binom{C}{R}\cdot\frac{\partial^{|R|}u}{\partial\boldsymbol{x}^R}\cdot\frac{\partial^{|C-R|}G}{\partial\boldsymbol{x}^{C-R}}\right).$$

The formula still holds when $|C|=c_0$. Therefore, the formula holds for any $C\in\mathbb{N}^l$ case and we complete the proof. q.e.d.

Proof of Theorem C. By Lemma 2.12, for any $I_1, I_2 \in \mathbb{N}^l$ satisfying $I_2 \geq I_1$, we have

$$\frac{\frac{\partial^{|I_2-I_1|}(u\cdot G)}{\partial \boldsymbol{x}^{I_2-I_1}}}{(I_2-I_1)!} = \sum_{\substack{I_3\in\mathbb{N}^l;\\ I_1\leq I_3\leq I_2}} \left(\frac{\frac{\partial^{|I_3-I_1|}u}{\partial \boldsymbol{x}^{I_3-I_1}}}{(I_3-I_1)!} \cdot \frac{\frac{\partial^{|I_2-I_3|}G}{\partial \boldsymbol{x}^{I_2-I_3}}}{(I_2-I_3)!} \right).$$

We also note that for any $I_1, I_2 \in \mathbb{N}^l$ not satisfying $I_2 \geq I_1$, we have

$$\sum_{\substack{I_3\in\mathbb{N}^l;\\ I_1\leq I_3\leq I_2}} \left(\frac{\frac{\partial^{|I_3-I_1|}u}{\partial \boldsymbol{x}^{I_3-I_1}}}{(I_3-I_1)!} \cdot \frac{\frac{\partial^{|I_2-I_3|}G}{\partial \boldsymbol{x}^{I_2-I_3}}}{(I_2-I_3)!} \right) = 0.$$

Therefore, the matrix equation holds. q.e.d.

3. Proof of Theorems D and E

Proof of Theorem D. Consider the $F \in \mathbb{C}[[\boldsymbol{x}]]$ case first. Consider $h \in Aut(\mathbb{C}[[\boldsymbol{x}]])$ and $u \in (\mathbb{C}[[\boldsymbol{x}]])^*$ in the following part of the proof.

Proof of (2): From Theorem B, we know that

$$I = \mathrm{ATJac}_n(id) = h\left(\mathrm{ATJac}_n(h^{-1})\right) \cdot \mathrm{ATJac}_n(h).$$

It follows from Theorem A that

$$h\left(\mathrm{QJac}_n(F)\right) \cdot \mathrm{ATJac}_n(h) = \mathrm{ATJac}_n(h) \cdot \mathrm{QJac}_n(h(F)).$$

Therefore,

$$\mathrm{QJac}_n(h(F)) = h\left(\mathrm{ATJac}_n(h^{-1})\right) \cdot h\left(\mathrm{QJac}_n(F)\right) \cdot \mathrm{ATJac}_n(h).$$

From Theorem C, we know that

$$\mathrm{QJac}_n(u \cdot h(F)) = \mathrm{QJac}_n(u) \cdot \mathrm{QJac}_n(h(F)).$$

Therefore,

$$\begin{aligned} &\mathrm{QJac}_n(u \cdot h(F)) \\ &= \mathrm{QJac}_n(u) \cdot h\left(\mathrm{ATJac}_n(h^{-1})\right) \cdot h\left(\mathrm{QJac}_n(F)\right) \cdot \mathrm{ATJac}_n(h). \end{aligned}$$

By the Cauchy-Binet Formula, we get

$$\left(\mathcal{J}_n^{(k)}\right)'(u \cdot h(F)) \subset h\left(\left(\mathcal{J}_n^{(k)}\right)'(F)\right).$$

On the other hand, we have

$$\left(\mathcal{J}_n^{(k)}\right)'\left(h^{-1}\left(u^{-1}\right) \cdot h^{-1}\left(u \cdot h(F)\right)\right) \subset h^{-1}\left(\left(\mathcal{J}_n^{(k)}\right)'(u \cdot h(F))\right),$$

i.e.

$$h\left(\left(\mathcal{J}_n^{(k)}\right)'(F)\right) \subset \left(\mathcal{J}_n^{(k)}\right)'(u \cdot h(F)).$$

Therefore, we get

$$\left(\mathcal{J}_n^{(k)}\right)'(u\cdot h(F)) = h\left(\left(\mathcal{J}_n^{(k)}\right)'(F)\right)$$

and complete the proof of (2).

Proof of (1): From

$$\langle u\cdot h(F), \mathcal{J}_n^{(k)}(u\cdot h(F))\rangle = \langle u\cdot h(F), \left(\mathcal{J}_n^{(k)}\right)'(u\cdot h(F))\rangle$$

$$= \langle h(F), h\left(\left(\mathcal{J}_n^{(k)}\right)'(F)\right)\rangle = h\left(\langle F, \mathcal{J}_n^{(k)}(F)\rangle\right),$$

we complete the proof of (1). By the same token, the statements in the $F\in\mathbb{C}\{\boldsymbol{x}\}$ case can be proven. q.e.d.

Proof of Theorem E. Consider the $F\in\mathbb{C}[[\boldsymbol{x}]]$ case first. F can be treated as the $\mathbb{C}$-algebra homomorphism

$$F' : \mathbb{C}[[x_1]] \to \mathbb{C}[[\boldsymbol{x}]]$$

by letting $F'(x_1) := F$.

Proof of (1):

We have

$$\mathrm{TJac}_n\left(h\circ F'\right) = h\left(\mathrm{TJac}_n(F')\right)\cdot \mathrm{TJac}_n(h).$$

By the Cauchy-Binet Formula, we get

$$\mathcal{T}\mathcal{J}_n^{(k)}(h(F)) \subset h\left(\mathcal{T}\mathcal{J}_n^{(k)}(F)\right).$$

On the other hand, we have

$$\mathcal{T}\mathcal{J}_n^{(k)}(h^{-1}(h(F))) \subset h^{-1}\left(\mathcal{T}\mathcal{J}_n^{(k)}(h(F))\right),$$

i.e.

$$h\left(\mathcal{T}\mathcal{J}_n^{(k)}(F)\right) \subset \mathcal{T}\mathcal{J}_n^{(k)}(h(F)).$$

Therefore, we get

$$\mathcal{T}\mathcal{J}_n^{(k)}(h(F)) = h\left(\mathcal{T}\mathcal{J}_n^{(k)}(F)\right)$$

and complete the proof of (1).

Proof of (2): Simply by replacing TJac_n and $\mathcal{T}\mathcal{J}_n^{(k)}$ in "Proof of (1)" by ATJac_n and $\mathcal{A}\mathcal{T}\mathcal{J}_n^{(k)}$ respectively.

By the same token, the statements in the $F\in\mathbb{C}\{\boldsymbol{x}\}$ case can be proven. q.e.d.

4. Examples

In this section, for $F \in \mathbb{C}\{x_1, x_2\}$ and $i_1, i_2 \cdots, i_s \in \{1, 2\}$, we define

$$F_{i_1, i_2 \cdots, i_s} := \frac{\partial^s F}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_s}}$$

for simplicity.

4.1. Examples of $\mathrm{Jac}_n(F)$'s and $\mathrm{QJac}_n(F)$'s.

Example 4.1. For $F \in \mathbb{C}\{\boldsymbol{x}\}$, we have $\mathrm{Jac}_1(F) = \mathrm{Jac}(F)$.

Example 4.2. For $F \in \mathbb{C}\{x_1, x_2\}$, we have

$$\mathrm{Jac}_2(F) = \begin{bmatrix} F_1 & F_2 & \frac{F_{1,1}}{2} & F_{1,2} & \frac{F_{2,2}}{2} \\ F & 0 & F_1 & F_2 & 0 \\ 0 & F & 0 & F_1 & F_2 \end{bmatrix},$$

$$\mathrm{QJac}_2(F) = \begin{bmatrix} F & F_1 & F_2 & \frac{F_{1,1}}{2} & F_{1,2} & \frac{F_{2,2}}{2} \\ 0 & F & 0 & F_1 & F_2 & 0 \\ 0 & 0 & F & 0 & F_1 & F_2 \\ 0 & 0 & 0 & F & 0 & 0 \\ 0 & 0 & 0 & 0 & F & 0 \\ 0 & 0 & 0 & 0 & 0 & F \end{bmatrix},$$

$$\mathrm{Jac}_3(F)$$

$$= \begin{bmatrix} F_1 & F_2 & \frac{1}{2}F_{1,1} & F_{1,2} & \frac{1}{2}F_{2,2} & \frac{1}{6}F_{1,1,1} & \frac{1}{2}F_{1,1,2} & \frac{1}{2}F_{1,2,2} & \frac{1}{6}F_{2,2,2} \\ F & 0 & F_1 & F_2 & 0 & \frac{1}{2}F_{1,1} & F_{1,2} & \frac{1}{2}F_{2,2} & 0 \\ 0 & F & 0 & F_1 & F_2 & 0 & \frac{1}{2}F_{1,1} & F_{1,2} & \frac{1}{2}F_{2,2} \\ 0 & 0 & F & 0 & 0 & F_1 & F_2 & 0 & 0 \\ 0 & 0 & 0 & F & 0 & 0 & F_1 & F_2 & 0 \\ 0 & 0 & 0 & 0 & F & 0 & 0 & F_1 & F_2 \end{bmatrix},$$

and

$$\mathrm{QJac}_3(F) =$$

$$\begin{bmatrix} F & F_1 & F_2 & \frac{1}{2}F_{1,1} & F_{1,2} & \frac{1}{2}F_{2,2} & \frac{1}{6}F_{1,1,1} & \frac{1}{2}F_{1,1,2} & \frac{1}{2}F_{1,2,2} & \frac{1}{6}F_{2,2,2} \\ 0 & F & 0 & F_1 & F_2 & 0 & \frac{1}{2}F_{1,1} & F_{1,2} & \frac{1}{2}F_{2,2} & 0 \\ 0 & 0 & F & 0 & F_1 & F_2 & 0 & \frac{1}{2}F_{1,1} & F_{1,2} & \frac{1}{2}F_{2,2} \\ 0 & 0 & 0 & F & 0 & 0 & F_1 & F_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & F & 0 & 0 & F_1 & F_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & F & 0 & 0 & F_1 & F_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & F & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & F & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F \end{bmatrix}.$$

4.2. Examples of $\mathrm{TJac}_n(h)$'s and $\mathrm{ATJac}_n(h)$'s.

Example 4.3. For $h \in End(\mathbb{C}\{\boldsymbol{x}\})$, we have

$$\mathrm{TJac}_1(h) = \mathrm{Jac}(h)$$

and

$$\mathrm{ATJac}_1(h) = \mathrm{diag}(1, \mathrm{Jac}(h)).$$

Example 4.4. For $h \in End\left(\mathbb{C}\left\{x_1, x_2\right\}\right)$, we have

$$\mathrm{TJac}_2\left(h\right) = \begin{bmatrix} h_1^{(1)} & h_2^{(1)} & \frac{h_{1,1}^{(1)}}{2} & h_{1,2}^{(1)} & \frac{h_{2,2}^{(1)}}{2} \\ h_1^{(2)} & h_2^{(2)} & \frac{h_{1,1}^{(2)}}{2} & h_{1,2}^{(2)} & \frac{h_{2,2}^{(2)}}{2} \\ 0 & 0 & \left(h_1^{(1)}\right)^2 & 2h_1^{(1)}h_2^{(1)} & \left(h_2^{(1)}\right)^2 \\ 0 & 0 & h_1^{(1)}h_1^{(2)} & h_1^{(1)}h_2^{(2)} + h_1^{(2)}h_2^{(1)} & h_2^{(1)}h_2^{(2)} \\ 0 & 0 & \left(h_1^{(2)}\right)^2 & 2h_1^{(2)}h_2^{(2)} & \left(h_2^{(2)}\right)^2 \end{bmatrix},$$

$$\mathrm{ATJac}_2\left(h\right) = \mathrm{diag}\left(1, \mathrm{TJac}_2\left(h\right)\right),$$

and

$$\mathrm{TJac}_3\left(h\right)$$

$$= \begin{bmatrix} h_1^{(1)} & h_2^{(1)} & \frac{h_{1,1}^{(1)}}{2} & h_{1,2}^{(1)} & \frac{h_{2,2}^{(1)}}{2} & \text{to be continued} \\ h_1^{(2)} & h_2^{(2)} & \frac{h_{1,1}^{(2)}}{2} & h_{1,2}^{(2)} & \frac{h_{2,2}^{(2)}}{2} & \text{to be continued} \\ 0 & 0 & \left(h_1^{(1)}\right)^2 & 2h_1^{(1)}h_2^{(1)} & \left(h_2^{(1)}\right)^2 & \text{to be continued} \\ 0 & 0 & h_1^{(1)}h_1^{(2)} & h_1^{(1)}h_2^{(2)} + h_1^{(2)}h_2^{(1)} & h_2^{(1)}h_2^{(2)} & \text{to be continued} \\ 0 & 0 & \left(h_1^{(2)}\right)^2 & 2h_1^{(2)}h_2^{(2)} & \left(h_2^{(2)}\right)^2 & \text{to be continued} \\ 0 & 0 & 0 & 0 & 0 & \text{to be continued} \\ 0 & 0 & 0 & 0 & 0 & \text{to be continued} \\ 0 & 0 & 0 & 0 & 0 & \text{to be continued} \\ 0 & 0 & 0 & 0 & 0 & \text{to be continued} \end{bmatrix}$$

$$\begin{matrix} \text{continue} & \frac{h_{1,1,1}^{(1)}}{6} & \frac{h_{1,1,2}^{(1)}}{2} & \text{to be continued} \\ \text{continue} & \frac{h_{1,1,1}^{(2)}}{6} & \frac{h_{1,1,2}^{(2)}}{2} & \text{to be continued} \\ \text{continue} & h_{1,1}^{(1)}h_1^{(1)} & h_{1,1}^{(1)}h_2^{(1)} + 2h_{1,2}^{(1)}h_1^{(1)} & \text{to be continued} \\ \text{continue} & \frac{h_{1,1}^{(1)}h_1^{(2)} + h_{1,1}^{(2)}h_1^{(1)}}{2} & \frac{h_{1,1}^{(1)}h_2^{(2)} + 2h_{1,2}^{(1)}h_1^{(2)} + h_{1,1}^{(2)}h_2^{(1)} + 2h_{1,2}^{(2)}h_1^{(1)}}{2} & \text{to be continued} \\ \text{continue} & h_{1,1}^{(2)}h_1^{(2)} & h_{1,1}^{(2)}h_2^{(2)} + 2h_{1,2}^{(2)}h_1^{(2)} & \text{to be continued} \\ \text{continue} & \left(h_1^{(1)}\right)^3 & 3\left(h_1^{(1)}\right)^2 h_2^{(1)} & \text{to be continued} \\ \text{continue} & \left(h_1^{(1)}\right)^2 h_1^{(2)} & \left(h_1^{(1)}\right)^2 h_2^{(2)} + 2h_1^{(1)}h_1^{(2)}h_2^{(1)} & \text{to be continued} \\ \text{continue} & h_1^{(1)}\left(h_1^{(2)}\right)^2 & \left(h_1^{(2)}\right)^2 h_2^{(1)} + 2h_1^{(1)}h_1^{(2)}h_2^{(2)} & \text{to be continued} \\ \text{continue} & \left(h_1^{(2)}\right)^3 & 3\left(h_1^{(2)}\right)^2 h_2^{(2)} & \text{to be continued} \end{matrix}$$

$$\left.\begin{matrix} \text{continue} & \frac{h_{1,2,2}^{(1)}}{2} & \frac{h_{2,2,2}^{(1)}}{6} \\ \text{continue} & \frac{h_{1,2,2}^{(2)}}{2} & \frac{h_{2,2,2}^{(2)}}{6} \\ \text{continue} & h_{2,2}^{(1)}h_1^{(1)} + 2h_{1,2}^{(1)}h_2^{(1)} & h_{2,2}^{(1)}h_2^{(1)} \\ \text{continue} & \frac{h_{2,2}^{(1)}h_1^{(2)} + 2h_{1,2}^{(1)}h_2^{(2)} + h_{2,2}^{(2)}h_1^{(1)} + 2h_{1,2}^{(2)}h_2^{(1)}}{2} & \frac{h_{2,2}^{(1)}h_2^{(2)} + h_{2,2}^{(2)}h_2^{(1)}}{2} \\ \text{continue} & h_{2,2}^{(2)}h_1^{(2)} + 2h_{1,2}^{(2)}h_2^{(2)} & h_{2,2}^{(2)}h_2^{(2)} \\ \text{continue} & 3\left(h_2^{(1)}\right)^2 h_1^{(1)} & \left(h_2^{(1)}\right)^3 \\ \text{continue} & \left(h_2^{(1)}\right)^2 h_1^{(2)} + 2h_1^{(1)}h_2^{(1)}h_2^{(2)} & \left(h_2^{(1)}\right)^2 h_2^{(2)} \\ \text{continue} & h_1^{(1)}\left(h_2^{(2)}\right)^2 + 2h_1^{(2)}h_2^{(1)}h_2^{(2)} & h_2^{(1)}\left(h_2^{(2)}\right)^2 \\ \text{continue} & 3\left(h_2^{(2)}\right)^2 h_1^{(2)} & \left(h_2^{(2)}\right)^3 \end{matrix}\right],$$

where for $j, i_1, i_2 \cdots, i_s \in \{1, 2\}$ we define

$$h_{i_1, i_2 \cdots, i_s}^{(j)} := \frac{\partial^s\left(h\left(x_j\right)\right)}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_s}}$$

for simplicity.

4.3. Examples of Theorem D.

Example 4.5. For $F \in \mathbb{C}\{x_1, x_2\}$, we have:

(1) $\mathcal{T}_2^{(2)}(F)$ is $\mathbb{C}\{x_1, x_2\}$ modulo the ideal generated by F and all the 2×2 minors in

$$\begin{bmatrix} F_1 & F_2 & \frac{F_{1,1}}{2} & F_{1,2} & \frac{F_{2,2}}{2} \\ F & 0 & F_1 & F_2 & 0 \\ 0 & F & 0 & F_1 & F_2 \end{bmatrix},$$

i.e.

$$\langle F_2F_{2,2}, F_1F_{2,2}, F_2F_{1,2}, F_1F_{1,2}, F_2F_{1,1}, F_1F_{1,1}, {F_2}^2, F_1F_2, {F_1}^2 \rangle.$$

(2) $\left(\mathcal{T}_2^{(2)}\right)'(F)$ is $\mathbb{C}\{x_1, x_2\}$ modulo the ideal generated by all the 2×2 minors in

$$\begin{bmatrix} F & F_1 & F_2 & \frac{F_{1,1}}{2} & F_{1,2} & \frac{F_{2,2}}{2} \\ 0 & F & 0 & F_1 & F_2 & 0 \\ 0 & 0 & F & 0 & F_1 & F_2 \\ 0 & 0 & 0 & F & 0 & 0 \\ 0 & 0 & 0 & 0 & F & 0 \\ 0 & 0 & 0 & 0 & 0 & F \end{bmatrix},$$

i.e.

$$\langle F_2F_{2,2}, F_1F_{2,2}, FF_{2,2}, F_2F_{1,2}, F_1F_{1,2}, FF_{1,2},$$
$$F_2F_{1,1}, F_1F_{1,1}, FF_{1,1}, {F_2}^2, F_1F_2, FF_2, {F_1}^2, FF_1, F^2 \rangle.$$

4.4. Examples of Theorem E.

Example 4.6. For $F \in \mathbb{C}\{x_1, x_2\}$, we have:

(1) $\left(\mathcal{M}_2^{(2)}\right)'(F)$ is $\mathbb{C}\{x_1, x_2\}$ modulo the ideal generated by all the 2×2 minors in

$$\begin{bmatrix} F_1 & F_2 & \frac{1}{2}F_{1,1} & F_{1,2} & \frac{1}{2}F_{2,2} \\ 0 & 0 & F_1^2 & 2F_1F_2 & F_2^2 \end{bmatrix},$$

i.e.

$$\langle F_1^3, F_1^2F_2, 2F_1^2F_2, 2F_1F_2^2, F_1F_2F_{1,1} - F_1^2F_{1,2},$$
$$F_1F_2^2, F_2^3, \frac{1}{2}F_2^2F_{1,1} - \frac{1}{2}F_1^2F_{2,2}, F_2^2F_{1,2} - F_1F_2F_{2,2} \rangle.$$

(2) $\left(\mathcal{M}_2^{(2)}\right)''(F)$ is $\mathbb{C}\{x_1, x_2\}$ modulo the ideal generated by all the 2×2 minors in

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & F_1 & F_2 & \frac{1}{2}F_{1,1} & F_{1,2} & \frac{1}{2}F_{2,2} \\ 0 & 0 & 0 & F_1^2 & 2F_1F_2 & F_2^2 \end{bmatrix},$$

i.e.

$$\langle F_1, F_2, F_{1,1}, F_{1,2}, F_{2,2} \rangle.$$

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