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Two Properties of Higher-Order Nash Blow-Up Local Algebras

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ABSTRACT

In this paper, we study the higher-order Nash blow-up local algebras associated with isolated hypersurface singularities, particularly focusing on the nonexistence of negative weighted derivations. Building on our earlier conjecture, which was proved by Badilla-Céspedes, Castorena, Duarte, and Núñez-Betancourt, we propose a new conjecture that generalizes our previous one. Inspired by the result of Badilla-Céspedes et al., we further demonstrate that the results in this paper validate our new conjecture partially. Besides, from the perspective of minors, we elucidate the structure of higher-order Nash blow-up algebras, which constitutes another main result of this paper. These findings contribute to the broader understanding of singularities and extend the applicability of higher-order Nash blow-up local algebras.

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Convention.

We define $\mathbf{x} := (x_1, x_2, \dots, x_l)$ where $l \in \mathbb{N}_+$.

Suppose A is a finite set. We define $\#A$ to be the cardinality of A .

For $\alpha := (\alpha_1, \dots, \alpha_l) \in \mathbb{N}^l$, we define $\alpha! := \alpha_1! \cdot \alpha_2! \cdot \dots \cdot \alpha_l!$ and $|\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_l$.

Without special instructions for ordering, for $\alpha := (\alpha_1, \dots, \alpha_l), \beta := (\beta_1, \dots, \beta_l) \in \mathbb{Z}^l$, $\beta \geq \alpha$ if and only if $\beta_i \geq \alpha_i$ for all $1 \leq i \leq l$.

For $f \in \mathbb{C}\{\mathbf{x}\}$, we define its Jacobian ideal $j(f)$ in the same ring to be $j(f) := \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_l} \rangle$.

The definition is analogous for the rings $\mathbb{K}[\mathbf{x}]$ and $\mathbb{K}[[\mathbf{x}]]$ where $\mathbb{K}$ is an arbitrary field.

For a $n_1 \times n_2$ matrix A with entries in a commutative ring, $I_m(A)$ ($m \leq \min\{n_1, n_2\}$) denotes the ideal generated by all $m \times m$ minors of A .

We define $\Delta_i := (\delta_{1,i}, \delta_{2,i}, \dots, \delta_{l,i})$ for $1 \leq i \leq l$, where $\delta_{j,i}$ ($1 \leq j \leq l$) means Kronecker delta.

For an $r \times s$ matrix M and indexes $1 \leq i_1, i_2, \dots, i_t \leq r$ and $1 \leq j_1, j_2, \dots, j_{t'} \leq s$, we define $M_{(i_1, i_2, \dots, i_t)}^{(j_1, j_2, \dots, j_{t'})}$ to be the $t \times t'$ matrix whose (r', s') th element is equal to $(i_{r'}, j_{s'})$ th element of M for any integer $1 \leq r' \leq t$ and $1 \leq s' \leq t'$.

1 | Introduction

1.1 | Higher-Order Nash Blow-Up Local Algebra

The classical Nash blow-up of an algebraic variety serves as the parameter space for the tangent spaces at smooth points and their limits. It is a pertinent question to inquire whether Nash blow-ups can yield a smooth variety. This problem has been extensively studied by researchers such as –Sprinberg [1], Hironaka [2], Nobile [3], Rebassoo [4], and Spivakovsky [5]. The question regarding the resolution properties of the Nash blow-up was recently answered negatively in [6]. We now recall the renowned theorem of Nobile:

Theorem 1.1 [3]. *Let X be a variety over an algebraically closed field of characteristic zero, then the Nash blow-up of X is an isomorphism if and only if X is non-singular.*

Theorem 1.1 does not hold over fields of prime characteristic. We present a recently proved theorem applicable in the prime characteristic under a normality condition.

Theorem 1.2 (Theorem 3.10 in [7]). *Let X be a normal variety over an algebraically closed field $\mathbb{K}$ of dimension d . Suppose that $\mathbb{K}$ has prime characteristic p . If $\mathbf{Nash}_1(X) \cong X$, then X is non-singular.*

Higher-order Nash blow-up with the definition given below is studied.

Definition 1.3 (Higher-order Nash blow-up [8]). Let X be a variety of dimension d , $x \in X$, $x^{(n)} := \text{Spec}(\mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1})$ its n th infinitesimal neighborhood and $\mathbf{Hilb}_{\binom{d+n}{n}}(X)$ is the Hilbert scheme of length $\binom{d+n}{n}$ points of X . If X is smooth at x , then $x^{(n)}$ is an Artinian subscheme of X of length $\binom{d+n}{n}$. Therefore, it corresponds to a point

$$[x^{(n)}] \in \mathbf{Hilb}_{\binom{d+n}{n}}(X),$$

which induced the following morphism of schemes,

$$\sigma_n : X_{sm} \longrightarrow \mathbf{Hilb}_{\binom{d+n}{n}}(X),$$

where X_{sm} denotes the smooth locus of X . The graph of σ_n is canonically isomorphic to X_{sm} . Yasuda defined the n th order Nash blow-up of X (called the higher-order Nash blow-up with order n of X), denoted by the higher-order Nash blow-up $\mathbf{Nash}_n(X)$, to be the closure of the graph Γ_{σ_n} with reduced scheme structure in $X \times_k \mathbf{Hilb}_{\binom{d+n}{n}}(X)$, together with the projection

$$\pi_n : \mathbf{Nash}_n(X) \longrightarrow X,$$

which is projective and birational. Moreover, it is an isomorphism over X_{sm} .

In order to compute the higher-order Nash blow-up, especially in the hypersurface case, the higher-order Jacobian matrix of a polynomial was defined in [9] and was rediscovered and developed further in [10, 11].

Furthermore, in [12], a novel higher-order Jacobian matrix theory was developed, giving rise to many related studies. To mention a few: establishment of the higher-order Hessian matrix theory and the complete invariant of the K_3 surface in $\mathbb{CP}^3$ in [13]; calculating the inverse of an automorphism of a power series ring in [14]; calculating explicit expressions in finite determinacy in [15]; etc.

In this paper, we consider another important application of the higher-order Jacobian matrix theory: higher-order Nash blow-up local algebra.

Definition 1.4 [9, 11, 12]. Consider $f \in \mathbb{C}[[\mathbf{x}]]$. Let $n \in \mathbb{N}_+$. The matrix $\text{Jac}_n(f)$ is defined to be a matrix whose rows are labeled by multi-indices $\{\mathbf{a} \in \mathbb{N}^l \mid |\mathbf{a}| \leq n-1\}$ and columns by $\{\mathbf{b} \in \mathbb{N}^l \mid 1 \leq |\mathbf{b}| \leq n\}$. The $(\mathbf{a}, \mathbf{b})$ -entry of $\text{Jac}_n(f)$ is

$$\begin{cases} \frac{1}{(\mathbf{b}-\mathbf{a})!} \frac{\partial^{|\mathbf{b}-\mathbf{a}|} f}{\partial \mathbf{x}^{\mathbf{b}-\mathbf{a}}}, & \text{if } \mathbf{b} \geq \mathbf{a} \\ 0, & \text{otherwise} \end{cases}.$$

We arrange the labels in the graded lexicographic order: first increasingly with respect to the grade, and, within each fixed grade, decreasingly with respect to lexicographic order.

Let $n \in \mathbb{N}$. For the definition of matrix $\text{QJac}_n(f)$, simply replace $\{\mathbf{a} \in \mathbb{N}^l \mid |\mathbf{a}| \leq n-1\}$ (resp. $\{\mathbf{b} \in \mathbb{N}^l \mid 1 \leq |\mathbf{b}| \leq n\}$) by $\{\mathbf{a} \in \mathbb{N}^l \mid |\mathbf{a}| \leq n\}$ (resp. $\{\mathbf{b} \in \mathbb{N}^l \mid |\mathbf{b}| \leq n\}$).

Let $i \in \mathbb{N}_+$ and $j \in \mathbb{N}$. For the definition of matrix $M_{i,j}(f)$, simply replace $\{\mathbf{a} \in \mathbb{N}^l \mid |\mathbf{a}| \leq n-1\}$ (resp. $\{\mathbf{b} \in \mathbb{N}^l \mid 1 \leq |\mathbf{b}| \leq n\}$) by $\{\mathbf{a} \in \mathbb{N}^l \mid |\mathbf{a}| = i-1\}$ (resp. $\{\mathbf{b} \in \mathbb{N}^l \mid |\mathbf{b}| = j\}$).

Remark 1.5. $\text{Jac}_n(f)$ is originally defined in [9, 11]. $\text{QJac}_n(f)$ and $M_{i,j}(f)$ are originally defined in [12] for the $f \in \mathbb{C}[[\mathbf{x}]]$ case. In [16], regarding the case where $f \in \mathbb{K}[[\mathbf{x}]]$ where $\mathbb{K}$ is an arbitrary field, we employ a construction identical to that found in Definition 1.4 by simultaneously extending the partial derivatives established in Definition 1.26.

Remark 1.6. Let $f \in \mathbb{C}[[\mathbf{x}]]$. For $n \in \mathbb{N}_+$, $\text{Jac}_n(f)$ is of size $\binom{l+n-1}{l} \times \left(\binom{l+n}{l} - 1 \right)$ and $\text{QJac}_n(f)$ is of size $\binom{l+n}{l} \times \binom{l+n}{l}$. For $i \in \mathbb{N}_+$ and $j \in \mathbb{N}$, $M_{i,j}(f)$ is of size $\binom{l+i-2}{l-1} \times \binom{l+j-1}{l-1}$.

Remark 1.7. $M_{i,j}(f)$ is the zero matrix in the $i > j+1$ case. Therefore, for $n \in \mathbb{N}$, we have

$$\begin{aligned} \text{QJac}_n(f) &= \begin{bmatrix} M(f)_{1,0} & M(f)_{1,1} & \cdots & M(f)_{1,n} \\ M(f)_{2,0} & M(f)_{2,1} & \cdots & M(f)_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ M(f)_{n+1,0} & M(f)_{n+1,1} & \cdots & M(f)_{n+1,n} \end{bmatrix} \\ &= \begin{bmatrix} M(f)_{1,0} & M(f)_{1,1} & \cdots & M(f)_{1,n} \\ 0 & M(f)_{2,1} & \cdots & M(f)_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M(f)_{n+1,n} \end{bmatrix}. \end{aligned}$$

(cf. [15]) and

$$\text{Jac}_n(f) = \begin{bmatrix} M(f)_{1,1} & M(f)_{1,2} & \cdots & M(f)_{1,n} \\ M(f)_{2,1} & M(f)_{2,2} & \cdots & M(f)_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ M(f)_{n,1} & M(f)_{n,2} & \cdots & M(f)_{n,n} \end{bmatrix}.$$

Definition 1.8 [17]. Using the notation established above, let $f \in \mathbb{C}[\mathbf{x}]$ define an isolated singularity $(V, 0)$, and let $\text{Jac}_n(f)$ denote the Jacobian matrix of order n . Let $\mathcal{J}_n(f) \subset \mathbb{C}[\mathbf{x}]$ be the ideal generated by the maximal minors of $\text{Jac}_n(f)$.

We define the higher-order Nash blow-up local algebra of V , denoted by $\mathcal{M}_n(V)$, to be $\mathcal{M}_n(V) := \mathbb{C}\{\mathbf{x}\}/\langle f, \mathcal{J}_n(f) \rangle$. Furthermore, we employ $d_n(V)$ to denote the dimension of $\mathcal{M}_n(V)$.

For such algebra, in [17], the authors made the following conjecture.

Conjecture 1.9 [17]. *Let $(V, 0)$ be an isolated hypersurface singularity defined by a polynomial $f \in \mathbb{C}[\mathbf{x}]$. Then, $\mathcal{M}_n(V)$ is a contact invariant of $(V, 0)$, i.e., it depends only on the isomorphism class of the germ $(V, 0)$.*

In [17], Hussain–Ma–Yau–Zuo verified Conjecture 1.9 for $l = 2, n = 2$. In [18], Fan–Yau–Zuo verified Conjecture 1.9 for $l = 2, n = 3$. In [12], Fan, Yau, and Zuo developed a higher-order Jacobian matrix theory. Using this, they not only gave a complete proof of Conjecture 1.9, but also extended it to the case where the size k of the minors generating the ideals is arbitrary. What's more, the contact invariant property of a finer algebra $(\mathcal{T}_n^{(k)})'(f)$ is proved in [12]. For the $f \in \mathbb{K}[[\mathbf{x}]]$ case, the same contact invariant property is obtained in [16]. Also, in [19], Conjecture 1.9 is proved independently using a different approach.

The results above in [12] are listed in Theorem 1.12.

Definition 1.10 [12]. Let $f \in \mathbb{C}[[\mathbf{x}]]$. For $n, k \in \mathbb{N}_+$ satisfying $k \leq \binom{l+n-1}{l}$, we define $\mathcal{J}_n^{(k)}(f)$ to be the ideal in $\mathbb{C}[[\mathbf{x}]]$ generated by all the $(k \times k)$ -minors of $\text{Jac}_n(f)$ under Definition 1.4. For $n, k \in \mathbb{N}_+$ satisfying $k \leq \binom{l+n}{l}$, we define $(\mathcal{J}_n^{(k)})'(f)$ to be the ideal in $\mathbb{C}[[\mathbf{x}]]$ generated by all the $(k \times k)$ -minors of $\text{QJac}_n(f)$ under Definition 1.4.

Definition 1.11 [12]. Let $f \in \mathbb{C}[[\mathbf{x}]]$ and we follow the notations of Definition 1.10.

(1) For $n, k \in \mathbb{N}_+$ satisfying $k \leq \binom{l+n-1}{l}$, we define

$$\mathcal{T}_n^{(k)}(f) := \mathbb{C}[[\mathbf{x}]]/\langle f, \mathcal{J}_n^{(k)}(f) \rangle. \quad (1)$$

(2) For $n, k \in \mathbb{N}_+$ satisfying $k \leq \binom{l+n}{l}$, we define

$$(\mathcal{T}_n^{(k)})'(f) := \mathbb{C}[[\mathbf{x}]]/(\mathcal{J}_n^{(k)})'(f). \quad (2)$$

Theorem 1.12 [12]. *Assume that $f \in \mathbb{C}[[\mathbf{x}]]$ and $n \in \mathbb{N}_+$. Under Definition 1.11 we have the following statements:*

(1) *For any $k \in \mathbb{N}_+$ satisfying $k \leq \binom{l+n-1}{l}$, $\mathcal{T}_n^{(k)}(f)$ is a contact invariant.*

(2) *For any $k \in \mathbb{N}_+$ satisfying $k \leq \binom{l+n}{l}$, $(\mathcal{T}_n^{(k)})'(f)$ is a contact invariant.*

We can also have similar results for Definitions 1.10 and 1.11, and Theorem 1.12 with $\mathbb{C}[[\mathbf{x}]]$ replaced by $\mathbb{C}[\mathbf{x}]$ or $\mathbb{C}\{\mathbf{x}\}$. The significance of higher-order Nash blow-up local algebras lies in their role as a higher-order generalization of Tjurina algebras. The Tjurina algebra of $f \in \mathbb{C}\{\mathbf{x}\}$ (i.e., $\mathbb{C}\{\mathbf{x}\}/\langle f, j(f) \rangle$) plays a pivotal role in the singularity theory (cf. [20]). One important application for the Tjurina algebra is the renowned Mather–Yau theorem.

Theorem 1.13 (Mather–Yau [21]). *We define $\mathfrak{m} := \langle \mathbf{x} \rangle \subset \mathbb{C}\{\mathbf{x}\}$ for simplicity. Let $f, g \in \mathfrak{m}$. The following are equivalent:*

(1) $f \stackrel{\circ}{\sim} g$.

(2) *For all $b \geq 1$, $\mathbb{C}\{\mathbf{x}\}/\langle f, \mathfrak{m}^b j(f) \rangle \cong \mathbb{C}\{\mathbf{x}\}/\langle g, \mathfrak{m}^b j(g) \rangle$ as $\mathbb{C}$ -algebras.*

(3) There is some $b \geq 1$ such that $\mathbb{C}\{\mathbf{x}\}/\langle f, \mathfrak{m}^b j(f) \rangle \cong \mathbb{C}\{\mathbf{x}\}/\langle g, \mathfrak{m}^b j(g) \rangle$ as $\mathbb{C}$ -algebras.

Moreover, if f has an isolated singularity, then $f \stackrel{\sim}{\sim} g$ iff the Tjurina algebras of f and g are isomorphic.

For hypersurface case, the Mather–Yau type theorem for higher-order Nash blow-up local algebras is also established in [22], and further generalized in [23].

For isolated complete intersection singularity case, in [23], we give a Mather–Yau type theorem for higher-order Nash blow-up local algebras, and we also use the algebra to completely classify the simple elliptic singularity.

1.2 | Main Results of This Paper

Besides the contact invariant property, the nonexistence of the negative derivation property is also conjectured in [17].

Recall that a polynomial $f = \sum_{\alpha} c_{\alpha} \mathbf{x}^{\alpha} \in \mathbb{C}[\mathbf{x}]$ is called weighted homogeneous of weighted degree $d = \deg_{\mathbf{w}}(f)$ with respect to $\mathbf{w} \in (\mathbb{N}_+)^l$, if $\alpha \cdot \mathbf{w} = d$ for all $c_{\alpha} \neq 0$.

Conjecture 1.14 [17]. *Let $f \in \mathbb{C}[\mathbf{x}]$ be a weighted homogeneous polynomial of degree d for the weights $\mathbf{w} = (w_1, w_2, \dots, w_l) \in \mathbb{N}^l$ that defines an isolated hypersurface singularity. Assume that $d \geq 2w_1 \geq \dots \geq 2w_l > 0$. Then under Definition 1.8, the higher-order Nash blow-up local algebra $\mathbb{C}[\mathbf{x}]/\langle f, \mathcal{J}_n(f) \rangle$ does not admit negative weighted derivations for all $n \geq 1$.*

Remark 1.15. This conjecture makes sense only if the ideal $\mathcal{J}_n(f)$ is weighted homogeneous. This fact was conjectured in [17, Conjecture 1.7] and later proved in [24, Theorem 2.6]. With the same method, one can also prove that in Conjecture 1.19, the ideal $\mathcal{J}_n^{(k)}(f)$ is weighted homogeneous.

Remark 1.16 [25]. The assumption $d \geq 2w_1 \geq 2w_2 \geq \dots \geq 2w_l > 0$ holds, up to a change of coordinates, whenever $f \in \langle \mathbf{x} \rangle^3$.

Conjecture 1.14 was partially answered in [17, Theorem B] and [24, Theorem 2.6]. Recently, in [26, Theorem A], Badilla-Céspedes, Castorena, Duarte, and Núñez-Betancourt proved the following beautiful theorem.

Theorem 1.17 [26]. *Conjecture 1.14 holds for all $n \geq 2$.*

For $n = 1$, the original Conjecture 1.14 is identical to the classical Yau conjecture, which remains an open problem.

Conjecture 1.18 (Yau Conjecture, [27]). *Let $f \in \mathbb{C}[\mathbf{x}]$ be a weighted homogeneous polynomial of degree d for the weights $\mathbf{w} = (w_1, w_2, \dots, w_l) \in \mathbb{N}^l$ that defines an isolated hypersurface singularity. Assume that $d \geq 2w_1 \geq \dots \geq 2w_l > 0$. Then, the Milnor algebra $\mathbb{C}[\mathbf{x}]/j(f)$ does not admit negative weighted derivations.*

Yau Conjecture is still open and was only proved in the low-dimensional case $l \leq 3$ by explicit calculations [27]. Also, in [28], Yau–Zuo verified Yau Conjecture for the high-dimensional singularities under certain conditions and for homogeneous singularities.

In this paper, we propose Conjecture 1.19, a generalization of Conjecture 1.14. In our conjecture, the minors need not be maximal.

Conjecture 1.19. *Let $f \in \mathbb{C}[\mathbf{x}]$ be a weighted homogeneous polynomial with respect to $\mathbf{w} = (w_1, w_2, \dots, w_l) \in \mathbb{N}^l$ and of weighted degree d , which defines an isolated hypersurface singularity. Assume that $d \geq 2w_1 \geq \dots \geq 2w_l > 0$. Then, $\mathcal{T}_n^{(k)}(f) := \mathbb{C}[\mathbf{x}]/\langle f, \mathcal{J}_n^{(k)}(f) \rangle$ does not admit negative weighted derivations for all $n \geq 1$ and $1 \leq k \leq \binom{l+n-1}{l}$.*

Inspired by [26, Theorem A], we partially verify Conjecture 1.19 in Theorem A.

Theorem A. Conjecture 1.19 holds for all $n \geq 2$ and $\binom{l+n-2}{l} + 2 \leq k \leq \binom{l+n-1}{l}$.

Remark 1.20. In fact, for all $n \geq 1$ and $k = \binom{l+n-1}{l}$, $\mathcal{T}_n^{(k)}(f)$ in Theorem A is indeed the algebra $\mathbb{C}[\mathbf{x}]/\langle f, \mathcal{J}_n(f) \rangle$ in Conjecture 1.14.

To deepen the understanding of the structural properties of the higher-order Nash blow-up local algebras $\mathcal{T}_n^{(k)}(f)$ and $(\mathcal{T}_n^{(k)})'(f)$, and to establish explicit connections between their ideal-theoretic structure and Jacobian ideals, we introduce Theorem B. This theorem not only clarifies the generators of these algebras but also reveals their intrinsic relationship with powers of the Jacobian ideal $j(f)$.

In the following results, we note that for $j \in \mathbb{N}_+$, $\#\{\alpha \in \mathbb{N}^l \mid |\alpha| = j-1\} = \binom{l+j-2}{l-1}$.

Theorem B. For $f \in \mathbb{C}[[\mathbf{x}]]$ and arbitrary $s, p \in \mathbb{N}_+$ satisfying $p \leq \binom{l+s-2}{l-1}$, the ideal $I_p(M(f)_{s,s})$ in $\mathbb{C}[[\mathbf{x}]]$ is $(j(f))^p$.

More explicitly, for any $I = (i_1, i_2, \dots, i_l) \in \mathbb{N}^l$ such that $|I| = p$, there always exists a choice of $p \times p$ minor of $M(f)_{s,s}$ which is equal to $\prod_{k=1}^l \left(\frac{\partial f}{\partial x_k}\right)^{i_k}$. We can choose such $p \times p$ submatrix by arbitrarily selecting the rows with indexes $R^{(t)} = (r_1^{(t)}, r_2^{(t)}, \dots, r_l^{(t)})$ and columns with indexes $C^{(t)} = (c_1^{(t)}, c_2^{(t)}, \dots, c_l^{(t)})$ with the following restrictions ($1 \leq t \leq p$):

(1) $|R^{(t)}| = s-1$ and $|C^{(t)}| = s$;

(2)

$$R^{(t)} + \Delta_{k^{(t)}} = C^{(t)} \quad (3)$$

where $k^{(t)} = k$ such that $\sum_{k'=1}^{k-1} i_{k'} < t \leq \sum_{k'=1}^k i_{k'}$;

(3) $R^{(1)}, R^{(2)}, \dots, R^{(p)}$ are distinct and arranged in the lexicographical order.

Remark 1.21. In Theorem B, such a choice of $p \times p$ submatrix is not necessarily upper-triangular or lower-triangular. Consider $f \in \mathbb{C}[[x_1, x_2, x_3]]$.

(1) When $s = 2$ and $I = (0, 3, 0)$, we obtain $p = 3$. Choose

$$R^{(1)} = (1, 0, 0), R^{(2)} = (0, 1, 0), R^{(3)} = (0, 0, 1), C^{(1)} = (1, 1, 0), C^{(2)} = (0, 2, 0), C^{(3)} = (0, 1, 1).$$

The corresponding 3×3 submatrix is

$$\begin{bmatrix} \frac{\partial f}{\partial x_2} & 0 & 0 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ 0 & 0 & \frac{\partial f}{\partial x_2} \end{bmatrix},$$

with determinant $\left(\frac{\partial f}{\partial x_2}\right)^3$.

(2) When $s = 3$ and $I = (0, 6, 0)$, we obtain $p = 6$. Choose

$$R^{(1)} = (2, 0, 0), R^{(2)} = (1, 1, 0), R^{(3)} = (1, 0, 1), R^{(4)} = (0, 2, 0), R^{(5)} = (0, 1, 1), R^{(6)} = (0, 0, 2),$$

$$C^{(1)} = (2, 1, 0), C^{(2)} = (1, 2, 0), C^{(3)} = (1, 1, 1), C^{(4)} = (0, 3, 0), C^{(5)} = (0, 2, 1), C^{(6)} = (0, 1, 2).$$

The corresponding 6×6 submatrix is

$$\begin{bmatrix} \frac{\partial f}{\partial x_2} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial f}{\partial x_2} & 0 & 0 & 0 \\ 0 & \frac{\partial f}{\partial x_1} & 0 & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial f}{\partial x_1} & 0 & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial f}{\partial x_2} \end{bmatrix},$$

with determinant $\left(\frac{\partial f}{\partial x_2}\right)^6$.

The two aforementioned submatrices are neither upper-triangular nor lower-triangular.

As a direct consequence of Theorem B, we have Corollary 1.22.

Corollary 1.22. For $f \in \mathbb{C}[[\mathbf{x}]]$ and arbitrary $n, p \in \mathbb{N}_+$ satisfying $p \leq \sum_{\substack{1 \leq j \leq n; \\ \frac{n-j}{2} \in \mathbb{N}}} \binom{l+j-2}{l-1} = \# \left\{ R \in \mathbb{N}^l \mid \frac{n-1-|R|}{2} \in \mathbb{N} \right\}$ and $n \geq 2$, we have the relationship $\langle f, j(f) \rangle^p \subseteq \mathcal{J}_n^{(p)}(f) \subseteq \left(\mathcal{J}_n^{(p)}\right)'(f)$ under Definition 1.10. More explicitly, for any $I = (i_1, i_2, \dots, i_l) \in \mathbb{N}^l$ such that $|I| \leq p$, there always exists a choice of $p \times p$ minor of $\text{Jac}_n(f)$ which is equal to $f^{p-|I|} \cdot \prod_{k=1}^l \left(\frac{\partial f}{\partial x_k}\right)^{i_k}$.

When $|I| = 0$, from Remark 1.6 and the relationship (note that $n \geq 2$)

$$\sum_{\substack{1 \leq j \leq n; \\ \frac{n-j}{2} \in \mathbb{N}}} \binom{l+j-2}{l-1} \leq \sum_{j=2}^n \binom{l+j-2}{l-1},$$

one can arbitrarily choose such $p \times p$ upper-triangular submatrix with diagonal elements f by choosing row indexes $R^{(t)} \in \mathbb{N}^l$ and column indexes $C^{(t)} \in \mathbb{N}^l$ ($t = 1, 2, \dots, p$) such that

$$(1, 0, 0, \dots, 0) \leq R^{(1)} = C^{(1)} < R^{(2)} = C^{(2)} < \dots < R^{(p)} = C^{(p)} \leq (0, 0, \dots, 0, n-1)$$

in the sense of the graded lexicographical order.

When $|I| > 0$, assume that

$$q = \max \left\{ 1 \leq s \leq n \mid p - |I| \leq \# \left\{ R \in \mathbb{N}^l \mid \frac{n-1-|R|}{2} \in \mathbb{N}, |R| \geq s-1 \right\} \right\}.$$

Such $p \times p$ submatrix is obtained by selecting the rows first with indexes $R^{(t)} \in \mathbb{N}^l$ ($1 \leq t \leq p$) and columns then with indexes $C^{(t)} \in \mathbb{N}^l$ ($1 \leq t \leq p$) with the following method:

(1) Let $R^{(1)}, R^{(2)}, \dots, R^{(p)}$ be the last p elements of the set

$$\left\{ R \in \mathbb{N}^l \mid \frac{n-1-|R|}{2} \in \mathbb{N} \right\},$$

when its elements are arranged in the graded lexicographical order.

(2) $C^{(t)}$'s are constructed as follows (divided into three cases):

- (2.1) For all $R^{(t)}$'s satisfying $|R^{(t)}| > q - 1$ (assume the number of which is N_q): define $C^{(t)} := R^{(t)}$.
- (2.2) For the first $p - |I| - N_q$ elements $R^{(t)}$'s with $|R^{(t)}| = q - 1$ (in graded lexicographical order): define $C^{(t)} := R^{(t)}$.
- (2.3) For the remaining $R^{(t)}$'s (of which there are exactly $|I|$): decompose I into a sequence of vectors consisting of i_1 copies of Δ_1 , followed by i_2 copies of Δ_2 , and so on, up to i_l copies of Δ_l . Construct the corresponding $C^{(t)}$'s by adding these vectors sequentially to the $R^{(t)}$'s (in graded lexicographical order).

We give Remark 1.23, Examples 1.24 and 1.25 to better illustrate Corollary 1.22.

Remark 1.23. In Corollary 1.22, when $|I| > 0$, the $p \times p$ submatrix is not necessarily composed of triangular blocks under the partition induced by the equivalence classes of $|R^{(t)}|$ and $|C^{(t)}|$.

Take $f \in \mathbb{C}[[x_1, x_2, x_3]]$, and $n = 3$ for example. When $p = 6$ and $I = (0, 6, 0)$, the 6×6 submatrix chosen is exactly the 6×6 submatrix in Remark 1.21 (2), i.e.,

$$\begin{bmatrix} \frac{\partial f}{\partial x_2} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial f}{\partial x_2} & 0 & 0 & 0 \\ 0 & \frac{\partial f}{\partial x_1} & 0 & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial f}{\partial x_1} & 0 & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial f}{\partial x_2} \end{bmatrix},$$

which, as a whole block itself, is neither upper-triangular nor lower-triangular.

Example 1.24. When $|I| > 0$, take $f \in \mathbb{C}[[x_1, x_2]]$, and $n = 5$ for examples of Corollary 1.22.

(1) $p = 9$ case:

We have $r = 1$. Therefore, the 9×9 matrix is a submatrix in

$$\begin{bmatrix} M(f)_{1,1} & M(f)_{1,2} & M(f)_{1,3} & M(f)_{1,4} & M(f)_{1,5} \\ M(f)_{3,1} & M(f)_{3,2} & M(f)_{3,3} & M(f)_{3,4} & M(f)_{3,5} \\ M(f)_{5,1} & M(f)_{5,2} & M(f)_{5,3} & M(f)_{5,4} & M(f)_{5,5} \end{bmatrix}$$

where the last 9 rows (i.e., all the rows) are selected.

When $I = (1, 0)$, we obtain $q = 3$ (see Equation (11) in the proof of Corollary 1.22). Such 9×9 matrix is

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} & * & * & * & * & * & * & * & * \\ f & * & * & * & * & * & * & * & * \\ f & * & * & * & * & * & * & * & * \\ f & * & * & * & * & * & * & * & * \\ f & * & * & * & * & * & * & * & * \\ f & * & * & * & * & * & * & * & * \\ f & * & * & * & * & * & * & * & * \\ f & * & * & * & * & * & * & * & * \\ f & * & * & * & * & * & * & * & * \end{bmatrix}.$$

When $I = (1, 1)$, we obtain $q = 3$ defined in the proof of Corollary 1.22. Such 9×9 matrix is

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} & * & * & * & * & * & * & * & * \\ & f & & & * & * & * & * & * \\ & & f & & * & * & * & * & * \\ & & & \frac{\partial f}{\partial x_2} & * & * & * & * & * \\ & & & & f & & & & \\ & & & & & f & & & \\ & & & & & & f & & \\ & & & & & & & f & \\ & & & & & & & & f \end{bmatrix}.$$

When $I = (3, 2)$, we obtain $q = 5$ defined in the proof of Corollary 1.22. Such 9×9 matrix is

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} & * & * & * & * & * & * & * & * \\ & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & & * & * & * & * & * \\ & & \frac{\partial f}{\partial x_1} & & * & * & * & * & * \\ & & & \frac{\partial f}{\partial x_2} & * & * & * & * & * \\ & & & & f & & & & \\ & & & & & f & & & \\ & & & & & & f & & \\ & & & & & & & f & \\ & & & & & & & & \frac{\partial f}{\partial x_2} \end{bmatrix}.$$

(2) $p = 7$ case:

We have $r = 3$. Therefore, the 7×7 matrix is a submatrix in

$$\begin{bmatrix} M(f)_{3,2} & M(f)_{3,3} & M(f)_{3,4} & M(f)_{3,5} \\ M(f)_{5,2} & M(f)_{5,3} & M(f)_{5,4} & M(f)_{5,5} \end{bmatrix}$$

where the last 7 rows are selected.

When $I = (1, 0)$, we obtain $q = 3$ defined in the proof of Corollary 1.22. Such 7×7 matrix is

$$\begin{bmatrix} f & \frac{\partial f}{\partial x_2} & * & * & * & * & * \\ & \frac{\partial f}{\partial x_1} & * & * & * & * & * \\ & & f & & & & \\ & & & f & & & \\ & & & & f & & \\ & & & & & f & \\ & & & & & & f \end{bmatrix}.$$

When $I = (1, 2)$, we obtain $q = 5$ defined in the proof of Corollary 1.22. Such 7×7 matrix is

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} & * & * & * & * & * & * \\ \frac{\partial f}{\partial x_2} & * & * & * & * & * & * \\ f & f & f & f & f & f & f \\ \frac{\partial f}{\partial x_2} & & & & & & \end{bmatrix}.$$

(3) $p = 4$ case:

We have $r = 5$. Therefore, the 7×7 matrix is a submatrix in

$$[M(f)_{5,4} \quad M(f)_{5,5}]$$

where the last 4 rows are selected.

When $I = (1, 0)$, we obtain $q = 5$ defined in the proof of Corollary 1.22. Such 4×4 matrix is

$$\begin{bmatrix} f & f & f & \frac{\partial f}{\partial x_1} \end{bmatrix}.$$

When $I = (2, 2)$, we obtain $q = 5$ defined in the proof of Corollary 1.22. Such 4×4 matrix is

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_1} \end{bmatrix}.$$

Example 1.25. When $|I| = 0$, take $f \in \mathbb{C}[[x_1, x_2]]$, and $n = 3$ for an example of Corollary 1.22.

Note that

$$\text{Jac}_3(f) = \begin{bmatrix} M(f)_{1,1} & M(f)_{1,2} & M(f)_{1,3} \\ M(f)_{2,1} & M(f)_{2,2} & M(f)_{2,3} \\ M(f)_{3,1} & M(f)_{3,2} & M(f)_{3,3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{1}{2} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{1}{2} \frac{\partial^2 f}{\partial x_2^2} & \frac{1}{6} \frac{\partial^3 f}{\partial x_1^3} & \frac{1}{2} \frac{\partial^3 f}{\partial x_1^2 \partial x_2} & \frac{1}{2} \frac{\partial^3 f}{\partial x_1 \partial x_2^2} & \frac{1}{6} \frac{\partial^3 f}{\partial x_2^3} \\ f & 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & 0 & \frac{1}{2} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial x_1 \partial x_2}{\partial^2 f} & \frac{1}{2} \frac{\partial^2 f}{\partial x_2^2} & 0 \\ 0 & f & 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & 0 & \frac{1}{2} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{1}{2} \frac{\partial^2 f}{\partial x_2^2} \\ 0 & 0 & f & 0 & 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & 0 & 0 \\ 0 & 0 & 0 & f & 0 & 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & 0 \\ 0 & 0 & 0 & 0 & f & 0 & 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}.$$

Given $1 \leq p \leq 4 = \# \left\{ R \in \mathbb{N}^2 \mid \frac{3-1-|R|}{2} \in \mathbb{N} \right\}$, for arbitrary $p \times p$ principal minor in

$$\begin{bmatrix} M(f)_{2,1} & M(f)_{2,2} \\ M(f)_{3,1} & M(f)_{3,2} \end{bmatrix} = \begin{bmatrix} f & 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & 0 \\ 0 & f & 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ 0 & 0 & f & 0 & 0 \\ 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & f \end{bmatrix}$$

is qualified.

One can find the proof of Theorems A and B and Corollary 1.22 in Section 2.

At the end of this section, we consider generalizing the results to $f \in \mathbb{K}[[\mathbf{x}]]$ and $f \in \mathbb{K}[\mathbf{x}]$ case, where $\mathbb{K}$ denotes an arbitrary field. We may analogously define $\text{Jac}_n(f)$ by generalizing the concept of derivatives, as outlined in Definition 1.26. Indeed, when $\mathbb{K} = \mathbb{C}$, the formal partial derivative coincides with the derivative in the conventional sense. By merely substituting “ $\mathbb{K}$ ” for “ $\mathbb{C}$ ”, the arguments presented in the proofs of Theorems A and B, as well as Corollary 1.22, remain valid.

Definition 1.26 [16]. For a_I ’s ($I \in \mathbb{N}^l$) in $\mathbb{K}$, we define

$$\frac{\frac{\partial^{|S|}}{\partial \mathbf{x}^S} \left(\sum_{I \in \mathbb{N}^l} a_I \mathbf{x}^I \right)}{S!} := \sum_{\substack{I \in \mathbb{N}^l; \\ I \geq S}} \binom{I}{S} \cdot a_I \mathbf{x}^{I-S}.$$

2 | Proof of Main Results

2.1 | Proof of Theorem A

To prove Theorem A, we need to use Lemma 2.1. For the details of its proof, see [26, Theorem A].

Lemma 2.1 [26]. Let $f \in \mathbb{C}[\mathbf{x}]$ be a weighted homogeneous polynomial with respect to $\mathbf{w} \in \mathbb{N}^l$ and degree d that defines an isolated hypersurface singularity. Consider any set S of weighted homogeneous polynomials satisfying $d \leq \deg_{\mathbf{w}} g$, for all $g \in S$. Then, it follows that $\mathbb{C}[\mathbf{x}]/\langle f, S \rangle$ does not admit negative weighted derivations.

Proof of Theorem A. By Remark 1.6, for $n \geq 2$ and $\binom{l+n-2}{l} + 2 \leq k \leq \binom{l+n-1}{l}$, all $k \times k$ submatrices from $\text{Jac}_n(f)$ must contain at least two rows from the submatrix

$$[M(f)_{n,1} \quad M(f)_{n,2} \quad \cdots \quad M(f)_{n,n}].$$

It follows that

$$\begin{aligned} \langle f, \mathcal{J}_n^{(k)}(f) \rangle &\subseteq \langle f, I_2([M(f)_{n,1} \quad M(f)_{n,2} \quad \cdots \quad M(f)_{n,n}]) \rangle \\ &= \langle f, I_2([0 \quad 0 \quad \cdots \quad 0 \quad M(f)_{n,n}]) \rangle \subseteq \langle f, (j(f))^2 \rangle \end{aligned}$$

by the Laplace expansion of determinants along the last two rows of selected submatrices (the first “ $\subseteq$ ”) and by Definition 1.4 (“=” and the second “ $\subseteq$ ”). Therefore, the degree of nonzero elements in $\mathcal{J}_n^{(k)}(f)$ is no less than those in $\langle f, (j(f))^2 \rangle$ (i.e., $\min\{d, 2d - 2w_1\} = d$).

Then by applying Remark 1.15 and Lemma 2.1, we complete the proof. $\square$

Remark 2.2. For other choice of k , i.e., $1 \leq k \leq \binom{l+n-2}{l} + 1$, we cannot use the same method and the conjecture is still open.

2.2 | Proof of Theorem B and Corollary 1.22

Proof of Theorem B. The strategy of the proof is as follows.

The proof consists of two main steps: construction and verification.

First, we explicitly construct a specific $p \times p$ submatrix of $M(f)_{s,s}$ whose determinant equals the desired monomial $\prod_{k=1}^l \left(\frac{\partial f}{\partial x_k} \right)^{i_k}$. This construction relies on carefully selecting row indices $R(t)$ and column indices $C(t)$ based on the given exponent vector $I = (i_1, \dots, i_l)$. The selection is designed so that the difference $C(t) - R(t)$ ’s correspond precisely to the standard basis vectors Δ_k ’s in a controlled manner.

Second, we verify that the determinant of this constructed submatrix indeed yields the target monomial. The key challenge is to show that all terms in the determinant expansion, except for the product of the diagonal entries, vanish, leaving only the product of the diagonal entries (even though the submatrix is not necessarily triangular, cf. Remark 1.21). We achieve this by proving that any permutation $\sigma \neq \text{id}$ in the determinant sum would require an impossible rearrangement of the indices.

Now, we begin the proof.

By Definition 1.4, the ideal generated by all $p \times p$ minors of $M(f)_{s,s}$ lies in $(j(f))^p$.

Consider the $p \times p$ submatrix of $M(f)_{s,s}$ with row labels $R^{(1)}, R^{(2)}, \dots, R^{(p)}$ and column labels $C^{(1)}, C^{(2)}, \dots, C^{(p)}$ chosen by (1)–(3). From (2) and (3), it is easy to see that $C^{(1)}, C^{(2)}, \dots, C^{(p)}$ are distinct and arranged in lexicographical order. By Definition 1.4, for $1 \leq a, b \leq p$, the (a, b) th element of the submatrix is

$$\begin{cases} \frac{\partial |C^{(b)} - R^{(a)}|_f}{\partial x^{C^{(b)} - R^{(a)}}}, & \text{if } C^{(b)} \geq R^{(a)} \\ 0, & \text{otherwise} \end{cases}.$$

Below, S_p means symmetric group and id means the identity element in S_p . Also, we define

$$\text{sgn}(\sigma) := \begin{cases} 1, & \text{if } \sigma \text{ is even,} \\ -1, & \text{if } \sigma \text{ is odd.} \end{cases}$$

The determinant of such $p \times p$ matrix equals

$$\sum_{\substack{\sigma \in S_p; \\ R^{(t')} \leq C^{(\sigma(t'))}, \forall 1 \leq t' \leq p}} \text{sgn}(\sigma) \cdot \left(\prod_{t=1}^p \frac{\partial |C^{(\sigma(t))} - R^{(t)}|}{\partial \mathbf{x}^{C^{(\sigma(t))} - R^{(t)}}} f \right).$$

From (1), we only need to prove that for any $\sigma \in S_p \setminus \{\text{id}\}$, there do not exist $1 \leq k_1, k_2, \dots, k_p \leq l$ such that

$$R^{(t)} + \Delta_{k_t} = C^{(\sigma(t))} \quad (4)$$

for all $1 \leq t \leq p$ at the same time.

From Equation (3), we know that

$$\sum_{t=1}^p R^{(t)} + \sum_{t=1}^p \Delta_{k(t)} = \sum_{t=1}^p C^{(t)}. \quad (5)$$

From Equation (4), we know that

$$\sum_{t=1}^p R^{(t)} + \sum_{t=1}^p \Delta_{k_t} = \sum_{t=1}^p C^{(\sigma(t))}. \quad (6)$$

From Equations (5) and (6), it follows that $\sum_{t=1}^p \Delta_{k(t)} = \sum_{t=1}^p \Delta_{k_t}$ and the tuple $k_1, k_2, \dots, k_p$ is one kind of permutation of $k^{(1)}, k^{(2)}, \dots, k^{(p)}$. Also, from (2), we know that

$$k^{(1)} \leq k^{(2)} \leq \dots \leq k^{(p)}. \quad (7)$$

From Equation (4), we have

$$R^{(\sigma^{-1}(t))} + \Delta_{k_{\sigma^{-1}(t)}} = C^{(t)} \quad (8)$$

for all $1 \leq t \leq p$ and $\sigma \in S_p \setminus \{\text{id}\}$. From $\sigma \in S_p \setminus \{\text{id}\}$ and Equations (3) and (8),

$$p' := \max \left\{ 1 \leq t \leq p \mid k^{(t)} \neq k_{\sigma^{-1}(t)} \right\} \quad (9)$$

exists. Note that the tuple $k_{\sigma^{-1}(1)}, k_{\sigma^{-1}(2)}, \dots, k_{\sigma^{-1}(p)}$ is one permutation of $k^{(1)}, k^{(2)}, \dots, k^{(p)}$, by inequation (7), we have $k^{(p')} > k_{\sigma^{-1}(p')}$. From (2) and (3), $R^{(p')} + \Delta_{k(p')} = C^{(p')}$ (by letting $t = p'$ in Equation (3)) and $R^{(\sigma^{-1}(p'))} + \Delta_{k_{\sigma^{-1}(p')}} = C^{(p')}$ (by letting $t = p'$ in Equation (8)), we know that $\sigma^{-1}(p') > p'$.

By Equation (9), it follows that

$$k^{(\sigma^{-1}(p'))} = k_{\sigma^{-1}(\sigma^{-1}(p'))}. \quad (10)$$

Letting $t = \sigma^{-1}(p')$ in Equations (3) and (8), we obtain $\sigma^{-1}(p') = \sigma^{-1}(\sigma^{-1}(p'))$ from (3) and Equation (10). Acting σ on both sides, we obtain $p' = \sigma^{-1}(p')$, which contradicts to $\sigma^{-1}(p') > p'$.

Therefore, we know that this $p \times p$ minor is equal to $\prod_{k=1}^l \left(\frac{\partial f}{\partial x_k} \right)^{i_k}$. $\square$

Proof of Corollary 1.22. The strategy of the proof is as follows.

The proof of Corollary 1.22 hinges on constructing a specific block-upper-triangular submatrix within $\text{Jac}_n(f)$. In the case $|I| > 0$, the strategy involves partitioning the submatrix into three distinct types of blocks (utilizing a partition slightly different from that in Remark 1.7):

- (1) **Diagonal blocks:** Corresponding to f itself and its first-order partial derivatives.
- (2) **Super-diagonal blocks:** Corresponding to the higher-order partial derivatives of f .
- (3) **Sub-diagonal blocks:** Strictly zero blocks, which vanish due to Definition 1.4.

The idea is to choose the submatrix by the total degree of the row labels. Fix a value of s . We first take the selected rows with $|R| = s - 1$, and then the corresponding selected columns have either $|C| = s - 1$ or $|C| = s$. Columns with $C^{(t)} = R^{(t)}$ contribute diagonal entries equal to f ; columns with $C^{(t)} - R^{(t)}$ equal to a standard basis vector contribute the corresponding first partial derivatives of f .

Crucially, the diagonal blocks themselves can exhibit a block-upper-triangular structure under further partition. By leveraging the result of Theorem B, this construction forces the minor to decompose precisely into the desired product.

Now, we begin the proof.

The $|I| = 0$ case is obvious.

Here, we only discuss the $|I| > 0$ case.

Assume that

$$r = \max \left\{ 1 \leq s \leq n \mid p \leq \# \left\{ R \in \mathbb{N}^l \mid \frac{n-1-|R|}{2} \in \mathbb{N}, |R| \geq s-1 \right\} \right\}. \quad (11)$$

It follows that: $q \geq r \geq 1$; $r-1 \leq |R^{(t)}| \leq n-1$ for all $1 \leq t \leq p$; $C^{(t)} - R^{(t)} \in \mathbb{N}^l$ and $|C^{(t)} - R^{(t)}| \leq 1$ for all $1 \leq t \leq p$.

It is obvious that $|C^{(t)}| \geq 1$ for all $1 \leq t \leq p$. We can ensure that for any choice of t_1, t_2 such that $1 \leq t_1 < t_2 \leq p$, we have $R^{(t_1)} < R^{(t_2)}$ and $C^{(t_1)} < C^{(t_2)}$ in the graded lexicographical order. Also, for such t_1 and t_2 , we have $C^{(t_1)} - R^{(t_1)} \leq C^{(t_2)} - R^{(t_2)}$ in the graded lexicographical order if $|R^{(t_1)}| = |R^{(t_2)}|$.

Consider an integer s with $r \leq s \leq n$ and $\frac{n-s}{2} \in \mathbb{N}$. In the following definitions, every indexing set is equipped with the graded lexicographical order.

Let

$$\mathcal{R}_{s-1} := \{R \in \mathbb{N}^l \mid |R| = s-1\}, \quad \mathcal{C}_{s-1,s} := \{C \in \mathbb{N}^l \mid s-1 \leq |C| \leq s\}.$$

Among the chosen row indices, suppose that the elements with $|R^{(t)}| = s-1$ are

$$R_1^{(s)}, \dots, R_{u_s}^{(s)}.$$

Their positions in the ordered set $\mathcal{R}_{s-1}$ are denoted by

$$r_1^{(s)}, \dots, r_{u_s}^{(s)}.$$

Similarly, let

$$C_1^{(s)}, \dots, C_{u_s}^{(s)}$$

be the corresponding chosen column indices satisfying $s - 1 \leq |C^{(t)}| \leq s$. Their positions in the ordered set $C_{s-1,s}$ are denoted by

$$c_1^{(s)}, \dots, c_{u_s}^{(s)}.$$

We shall also use separate position labels according to the degree of the column index. For those chosen column indices with $|C| = s - 1$, their positions in

$$\{C \in \mathbb{N}^l \mid |C| = s - 1\}$$

are denoted by

$$-c_1^{(s)}, \dots, -c_{u_s}^{(s)}.$$

For those chosen column indices with $|C| = s$, their positions in

$$\{C \in \mathbb{N}^l \mid |C| = s\}$$

are denoted by

$$+c_1^{(s)}, \dots, +c_{u_s}^{(s)}.$$

Thus, $u_s = -u_s + u_s$.

We show that the algorithm is valid, i.e., $|C^{(1)}|, |C^{(2)}|, \dots, |C^{(p)}|$ should be positive from Definition 1.4. Otherwise from the fact that $C^{(1)}, C^{(2)}, \dots, C^{(p)}$ are arranged in the lexicographical order, $|C^{(1)}|$ should be zero. Then, $|R^{(1)}|$ is also zero from $C^{(t_1)} - R^{(t_1)} \in \mathbb{N}^l$. Then by (2.1)–(2.3), $C^{(t)} = R^{(t)}$ for all $t = 1, 2, \dots, p$, which implies that $I = (0, 0, \dots, 0)$. However, this contradicts with the assumption that $|I| > 0$.

Notice the following facts:

Fact A: From Remark 1.7, $\begin{bmatrix} M(f)_{s_1, s_2-1} & M(f)_{s_1, s_2} \end{bmatrix}$ is zero matrix for $r \leq s_2 < s_1 \leq n$ satisfying $\frac{n-s_1}{2}, \frac{n-s_2}{2} \in \mathbb{N}$.

Fact B: From Definition 1.4 and Corollary 1.22 (2), we have the following matrix equation:

$$\begin{aligned} & \begin{bmatrix} M(f)_{s, s-1} & M(f)_{s, s} \end{bmatrix} \begin{pmatrix} c_1^{(s)}, c_2^{(s)}, \dots, c_{u_s}^{(s)} \\ r_1^{(s)}, r_2^{(s)}, \dots, r_{u_s}^{(s)} \end{pmatrix} \\ &= \begin{bmatrix} \begin{pmatrix} M(f)_{s, s-1} \end{pmatrix} \begin{pmatrix} -c_1^{(s)}, -c_2^{(s)}, \dots, -c_{u_s}^{(s)} \\ r_1^{(s)}, r_2^{(s)}, \dots, r_{u_s}^{(s)} \end{pmatrix} & * & \begin{pmatrix} M(f)_{s, s} \end{pmatrix} \begin{pmatrix} +c_1^{(s)}, +c_2^{(s)}, \dots, +c_{u_s}^{(s)} \\ r_{-u_s+1}^{(s)}, r_{-u_s+2}^{(s)}, \dots, r_{u_s}^{(s)} \end{pmatrix} \end{bmatrix} \end{aligned}$$

where $(M(f)_{s,s-1})_{\substack{(-c_1^{(s)}, -c_2^{(s)}, \dots, -c_{u_s}^{(s)}) \\ (r_1^{(s)}, r_2^{(s)}, \dots, r_{-u_s}^{(s)})}}$ is $-u_s \times -u_s$ diagonal matrix with diagonal elements f 's. (When $-u_s = 0$ (resp. $+u_s = 0$), $[M(f)_{s,s-1} \quad M(f)_{s,s}]_{\substack{(c_1^{(s)}, c_2^{(s)}, \dots, c_{u_s}^{(s)}) \\ (r_1^{(s)}, r_2^{(s)}, \dots, r_{u_s}^{(s)})}}$ equals to $(M(f)_{s,s})_{\substack{(+c_1^{(s)}, +c_2^{(s)}, \dots, +c_{u_s}^{(s)}) \\ (r_1^{(s)}, r_2^{(s)}, \dots, r_{u_s}^{(s)})}}$ (resp. $(M(f)_{s,s-1})_{\substack{(-c_1^{(s)}, -c_2^{(s)}, \dots, -c_{u_s}^{(s)}) \\ (r_1^{(s)}, r_2^{(s)}, \dots, r_{u_s}^{(s)})}}$).

By Definition 1.4 and Corollary 1.22 (1) and (2), the selected $p \times p$ submatrix is formed by several blocks of triangular matrices $[M(f)_{s_1, s_2-1} \quad M(f)_{s_1, s_2}]_{\substack{(c_1^{(s_2)}, c_2^{(s_2)}, \dots, c_{u_{s_2}}^{(s_2)}) \\ (r_1^{(s_1)}, r_2^{(s_1)}, \dots, r_{u_{s_1}}^{(s_1)})}}$ (where $s_1, s_2 \in \mathbb{N}$ satisfy $r \leq s_1, s_2 \leq n$ and $\frac{n-s_1}{2}, \frac{n-s_2}{2} \in \mathbb{N}$). In this sense, the selected $p \times p$ submatrix is block upper-triangular (by Fact A) and of determinant

$$\begin{aligned} & \prod_{\substack{r \leq s \leq n; \\ \frac{n-s}{2} \in \mathbb{N}}} \det \left([M(f)_{s,s-1} \quad M(f)_{s,s}]_{\substack{(c_1^{(s)}, c_2^{(s)}, \dots, c_{u_s}^{(s)}) \\ (r_1^{(s)}, r_2^{(s)}, \dots, r_{u_s}^{(s)})}} \right) \\ & \stackrel{\text{Fact B}}{=} \prod_{\substack{r \leq s \leq n; \\ \frac{n-s}{2} \in \mathbb{N}}} \left(\det \left((M(f)_{s,s-1})_{\substack{(-c_1^{(s)}, -c_2^{(s)}, \dots, -c_{u_s}^{(s)}) \\ (r_1^{(s)}, r_2^{(s)}, \dots, r_{-u_s}^{(s)})}} \right) \cdot \det \left((M(f)_{s,s})_{\substack{(+c_1^{(s)}, +c_2^{(s)}, \dots, +c_{u_s}^{(s)}) \\ (r_{-u_s+1}^{(s)}, r_{-u_s+2}^{(s)}, \dots, r_{u_s}^{(s)})}} \right) \right) \\ & \stackrel{\text{Theorem B and Fact B}}{=} \prod_{\substack{r \leq s \leq n; \\ \frac{n-s}{2} \in \mathbb{N}}} \left(\left(\prod_{j=1}^{-u_s} (M(f)_{s,s-1})_{\substack{(-c_j^{(s)}) \\ (r_j^{(s)})}} \right) \cdot \left(\prod_{j=1}^{+u_s} (M(f)_{s,s})_{\substack{(+c_j^{(s)}) \\ (r_{j+u_s}^{(s)})}} \right) \right) \\ & = \prod_{\substack{r \leq s \leq n; \\ \frac{n-s}{2} \in \mathbb{N}}} \prod_{j=1}^{u_s} \det \left([M(f)_{s,s-1} \quad M(f)_{s,s}]_{\substack{(c_j^{(s)}) \\ (r_j^{(s)})}} \right) \\ & \stackrel{\text{Definition 1.4}}{=} \prod_{t=1}^p \frac{\partial |C^{(t)} - R^{(t)}| f}{\partial \mathbf{x}^{C^{(t)} - R^{(t)}}} = f^{p-|I|} \cdot \prod_{k=1}^l \left(\frac{\partial f}{\partial x_k} \right)^{i_k}. \end{aligned}$$

□

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

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