

A NEW SERIES OF LOCAL ALGEBRAS FOR SINGULARITIES AND THEIR DERIVATION LIE ALGEBRAS

GUORUI MA, SIYONG TAO, STEPHEN S.-T. YAU, AND HUIQING ZUO

ABSTRACT. In this paper, we introduce a new series of invariants for singularities. Let $V_f = (V(f), \mathbf{0})$ be an isolated hypersurface singularity defined by a germ of holomorphic function $f \in \mathbb{C}\{z_1, \dots, z_n\}$ at the origin $\mathbf{0}$ of $\mathbb{C}^n$. Fix an integer $k > 0$, let $\mathcal{T}_k(f)$ be the ideal of $\mathbb{C}\{z_1, \dots, z_n\}$ generated by the i -th order partial derivatives of f for $0 \leq i \leq k$. The k -th order Tjurina algebra $A_k(V_f)$ is the local $\mathbb{C}$ -algebra $\mathbb{C}\{z_1, \dots, z_n\}/\mathcal{T}_k(f)$, which is a biholomorphic invariant of V_f . When f is a weighted homogeneous polynomial, a new conjecture about the non-existence of non-zero negative weight derivations on $A_k(V_f)$ is proposed. We verify this conjecture up to three variables.

Keywords. derivations, k -th order Tjurina algebras, isolated singularities, invariants. MSC(2010). 14B05, 32S05.

1. INTRODUCTION

In singularity theory, one of the main tasks is to find intrinsic characteristics that remain invariant under biholomorphic transformations and can thus be used to distinguish isolated singularities. Most of the well-established invariants of singularities are hard to compute, e.g., the geometric genus, monodromy and so on. Therefore, one of the goals of this paper is to introduce a series of invariants which can be computed and compared easily.

Definition 1.1. Let $\mathcal{O}_n$ be the ring of germs of holomorphic functions at $\mathbf{0} = (0, \dots, 0)$. A germ of a holomorphic function at $\mathbf{0}$ is the equivalence class of a holomorphic function f defined in an open neighborhood U of $\mathbf{0}$. Given two functions f_1 and f_2 defined in open neighborhoods U_1 and U_2 of $\mathbf{0}$, respectively, they are equivalent if there exists a smaller neighborhood $U_3 \subseteq U_1 \cap U_2$ of $\mathbf{0}$ such that $f_1|_{U_3} = f_2|_{U_3}$.

We identify $\mathcal{O}_n$ with $\mathbb{C}\{z_1, \dots, z_n\}$, the subring of convergent power series in $\mathbb{C}[[z_1, \dots, z_n]]$, by the Taylor series expansion of holomorphic functions at $\mathbf{0}$. $\mathcal{O}_n$ is a local $\mathbb{C}$ -algebra with the unique maximal ideal $\mathfrak{m} = \langle z_1, \dots, z_n \rangle \subseteq \mathcal{O}_n$. $\mathfrak{m}$ can be viewed as the ideal of germs of holomorphic functions that vanish at $\mathbf{0}$.

For a germ of holomorphic function $f \in \mathcal{O}_n$, suppose that f is holomorphic on a neighborhood U of $\mathbf{0}$ and $f(\mathbf{0}) = 0$. We denote it by $f : (\mathbb{C}^n, \mathbf{0}) \rightarrow (\mathbb{C}, 0)$. f defines an analytic set

$$V(f) := \{(x_1, \dots, x_n) \in U : f(x_1, \dots, x_n) = 0\} \subseteq U.$$

It is well known that

- (1) If $f \in \mathfrak{m}^2$, then $\mathbf{0}$ is a singular point of V .
- (2) $V(f) \setminus \{\mathbf{0}\}$ is non-singular if and only if the local $\mathbb{C}$ -algebra

$$\frac{\mathcal{O}_n}{\langle f, \partial_{z_1} f, \dots, \partial_{z_n} f \rangle} \tag{1}$$

is Artinian, i.e., it is a finite-dimensional $\mathbb{C}$ -vector space.

Corresponding author: Stephen S.-T. Yau, Huaqing Zuo.

Yau is supported by Tsinghua University Education Foundation.

Zuo is supported by NSFC Grant 12271280 and BJNSF Grant 1252009.

Data availability: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

We say that $V_f = (\mathbf{V}(f), \mathbf{0})$ is an isolated hypersurface singularity defined by $f \in \mathcal{O}_n$ if $\mathbf{0}$ is the only singular point of $\mathbf{V}(f)$. We call the local $\mathbb{C}$ -algebra in (1) the moduli algebra of V_f and denote it by $A(V_f)$. The Mather-Yau theorem states that isolated hypersurface singularities can be completely described by their moduli algebras and also by $\mathbb{C}$ -algebras

$$B(V_f) := \frac{\mathcal{O}_n}{\langle f, z_i \partial_{z_j} f : 1 \leq i, j \leq n \rangle}.$$

Theorem 1.2. ([14, Theorem, Page 243]) Suppose $V_f = (\mathbf{V}(f), \mathbf{0})$ and $V_g = (\mathbf{V}(g), \mathbf{0})$ are isolated hypersurface singularities in $\mathbb{C}^n$ defined by the germs of holomorphic functions f and $g \in \mathcal{O}_n$, respectively. The following conditions are equivalent.

- (1) V_f is biholomorphically equivalent to V_g .
- (2) $A(V_f)$ is isomorphic to $A(V_g)$ as a $\mathbb{C}$ -algebra.
- (3) $B(V_f)$ is isomorphic to $B(V_g)$ as a $\mathbb{C}$ -algebra.

Definition 1.3. Let $\mathbb{K}$ be a field and A a $\mathbb{K}$ -algebra via a ring homomorphism $\eta : \mathbb{K} \rightarrow A$. A $\mathbb{K}$ -derivation on A is a map (of abelian groups) $D : A \rightarrow A$ satisfying

- (1) $D \circ \eta = 0$.
- (2) For any $a, b \in A$, $D(ab) = bD(a) + aD(b)$.

The set of all $\mathbb{K}$ -derivations on A is denoted by $\text{Der}(A)$.

If D and $D' \in \text{Der}(A)$, then $[D, D'] := D \circ D' - D' \circ D \in \text{Der}(A)$. So $\text{Der}(A)$ becomes a Lie algebra with this bracket. For the isolated hypersurface singularity $V_f = (\mathbf{V}(f), \mathbf{0})$ in $\mathbb{C}^n$, the third author [22, 23] introduced the Yau algebra $L(V_f) := \text{Der}(A(V_f))$ and proved that $L(V_f)$ is a finite-dimensional and solvable Lie algebra (cf. [23, Theorem 2]). The Yau algebra holds significance in singularity theory, as highlighted in [19].

Definition 1.4. For a polynomial

$$f = \sum_{(\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n} c_{\alpha_1, \dots, \alpha_n} z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \quad c_{\alpha_1, \dots, \alpha_n} \in \mathbb{C},$$

we define $\text{Supp}(f) := \{z_1^{\alpha_1} \cdots z_n^{\alpha_n} : c_{\alpha_1, \dots, \alpha_n} \neq 0\}$.

Fix an n -tuple $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{N}^n$ such that all $w_i \geq 1$, we say that f is a weighted homogeneous polynomial with respect to the weight type $\mathbf{w}$ of weighted degree $\deg_{\mathbf{w}} f := d$ if for all $z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in \text{Supp}(f)$, there is $\sum_{i=1}^n \alpha_i w_i = d$. We set $\deg_{\mathbf{w}} 0 = +\infty$.

Fix a weight type $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{N}^n$ of positive integers. With loss of generality we assume that $w_1 \geq w_2 \geq \dots \geq w_n \geq 1$. For a finitely generated $\mathbb{C}$ -algebra

$$A = \frac{\mathbb{C}[z_1, \dots, z_n]}{\langle f_1, \dots, f_r \rangle}, \quad (2)$$

where every f_i is a weighted homogeneous polynomial with respect to $\mathbf{w}$. A is a graded algebra. It decomposes as $A = \bigoplus_{d \in \mathbb{N}} A_{\mathbf{w}, d}$. $A_{\mathbf{w}, d}$ is the $\mathbb{C}$ -vector space of all $g + \langle f_1, \dots, f_r \rangle$ where g is a weighted homogeneous polynomial with respect to $\mathbf{w}$ of weighted degree d .

We say that a derivation D on A has weighted degree $d_0 \in \mathbb{Z}$ with respect to $\mathbf{w}$, if

$$D(A_{\mathbf{w}, d}) \subseteq A_{\mathbf{w}, d+d_0}$$

for every $d \in \mathbb{N}$. If $d_0 < 0$, we call D a negative weight derivation. Here we set $A_{\mathbf{w}, d} = 0$ if $d < 0$. Take $A = \mathbb{C}[z_1, \dots, z_n]$, for a derivation $D = \sum_{i=1}^n p_i \partial_{z_i}$ on A with $p_i \in A$, if D has weighted degree d_0 and $p_i \neq 0$, then $D(z_i) = p_i \in A_{\mathbf{w}, w_i+d_0}$, p_i is a weighted homogeneous polynomial.

Let $\text{Der}_{\mathbf{w},d}(A) = \{D \in \text{Der}(A) : D \text{ has weighted degree } d \text{ with respect to } \mathbf{w}\}$. There is a natural structure of graded A -module on $\text{Der}(A)$.

$$\text{Der}(A) = \bigoplus_{d \in \mathbb{Z}} \text{Der}_{\mathbf{w},d}(A).$$

Now, we are concerned with the conditions which will give $\text{Der}_{\mathbf{w},<0}(A) =: \bigoplus_{d<0} \text{Der}_{\mathbf{w},d}(A)$, i.e., there are no non-zero derivations on A of negative weighted degree. There are several cases to consider.

- Wahl [21, Conjecture 1.4] conjectured that if A is an isolated complete intersection singularity of dimension ≥ 2 , then $\text{Der}_{\mathbf{w},<0}(A) = 0$.
- The Wahl Conjecture was first solved by Aleksandrov in [1, Section 6]. Aleksandrov conjectured that $\text{Der}_{\mathbf{w},<0}(A) = 0$ when A is an Artinian complete intersection.
- If we substitute $\mathbb{C}$ with $\mathbb{Q}$ and assume that A is an Artinian complete intersection with even weights $w_i > 0$, this is the Halperin Conjecture.

Halperin Conjecture. ([10, Chapter 39]) Suppose F is a rational elliptic space with a non-zero Euler-Poincaré characteristic, and $F \rightarrow E \rightarrow B$ is a Serre fibration of simply-connected spaces. The Serre spectral sequence for this fibration collapses at the E_2 page.

It is one of the most important open problems in rational homotopy theory, also important in differential geometry (cf. [2]). The Halperin conjecture is equivalent to questions above about the non-existence of non-zero negative weight derivations. Assume that the cohomology of such a space F in the Halperin Conjecture has the form

$$H^*(F; \mathbb{Q}) \cong \frac{\mathbb{Q}[x_1, \dots, x_n]}{\langle f_1, \dots, f_n \rangle}.$$

Here $f_1, \dots, f_n$ is a regular sequence. They are weighted homogeneous polynomials with respect to the weight type $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{N}^n$ such that all w_i are positive and even. Moreover, we can choose the weight type $\mathbf{w}$ and f_i to satisfy the following conditions:

$$\deg_{\mathbf{w}} f_1 \geq \dots \geq \deg_{\mathbf{w}} f_n, \quad w_1 \geq \dots \geq w_n, \quad \deg_{\mathbf{w}} f_i \geq 2w_i, i = 1, \dots, n.$$

Meier [13, Theorem A] proved that the Serre spectral sequence for the Serre fibration $F \rightarrow E \rightarrow B$ in the Halperin Conjecture collapses at E_2 if and only if all derivations of negative weighted degree of the graded algebra $H^*(F; \mathbb{Q})$ vanish.

The Halperin Conjecture is verified in several cases. Thomas [20, Lemma 3] proved this conjecture for the case $n = 2$. Lupton [12, Theorem 1] and Chen [7, Theorem 3.1] proved it for the case $n = 3$. Kennard and Wu [11] proved it for the case that $H^*(F; \mathbb{Q})$ has the formal dimension $\sum_{i=1}^n \deg_{\mathbf{w}} f_i - w_i$ at most 20.

In [7], Chen introduced a new weight type for every negative weight derivation on the polynomial ring. It plays an important role in solving the Halperin Conjecture, the Wahl Conjecture, and the Aleksandrov Conjecture. All of these conjectures focus on the negative weight derivations on complete intersection algebras. But Chen's method can be extended to investigate negative derivations on arbitrary finitely generated algebras.

Consider the graded algebra A in (2). We require that the weight type $\mathbf{w} \in \mathbb{N}^n$ satisfying $w_1 \geq \dots \geq w_n \geq 1$.

- If A is Artinian and $\deg_{\mathbf{w}} f_i \geq (r-1)(w_1 w_2)^{n-1}$ for all i , then $\text{Der}_{\mathbf{w},<0}(A) = 0$ (cf. [9, Theorem A]).
- If $\{f_i\}_{i=1}^r$ define a positive-dimensional isolated singularity at the origin of $\mathbb{C}^n$ and $\deg_{\mathbf{w}} f_i \geq (r-1+w_1)(w_1 w_2)^{n-1}$ for all i , $\text{Der}_{\mathbf{w},<0}(A) = 0$ (cf. [3, Theorem A]).

We see that the Halperin Conjecture is true when $H^*(F, \mathbb{Q})$ has relations of large degrees if we replace the field $\mathbb{C}$ by $\mathbb{Q}$.

There are some counterexamples in the noncomplete intersection case when the weighted degrees of f_i are small (cf. [3, Page 196-197]). It is necessary to assume that the degrees of f_i are bounded below. But these bounds tend to infinity when n tends to infinity. When any two of $\{w_i\}_{i=1}^n$ are coprime, [9, Theorem B] and [3, Theorem B] give better bounds which are independent of n . Based on the results, B. Chen [5, Conjecture 1.7] conjectured that if A is Artinian and $\deg_{\mathbf{w}} f_i \geq (r-1)(w_1-1)$ for all i , then $\text{Der}_{\mathbf{w}, <0}(A) = 0$.

On the other hand, in Yau's work on the derivation Lie algebras of the moduli algebras of isolated hypersurface singularities, particularly concerning his micro-local characterization of weighted homogeneous isolated hypersurface singularities, Yau has independently proposed the following conjecture.

Yau Conjecture. Fix a weight type $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{N}^n$ of positive integers. Let $f \in \mathbb{C}[z_1, \dots, z_n]$ be a weighted homogeneous polynomial with respect to $\mathbf{w}$. $V_f = (\mathbf{V}(f), \mathbf{0})$ is the isolated hypersurface singularity defined by f . Then $\text{Der}_{\mathbf{w}, <0}(A(V_f)) = 0$, i.e., $L(V_f)$ is non-negatively graded.

Remark 1. Recall that $\mathcal{O}_n = \mathbb{C}\{z_1, \dots, z_n\}$. Let $R_n := \mathbb{C}[z_1, \dots, z_n]$. $z_1, \dots, z_n$ generate the maximal ideal $\tilde{\mathfrak{m}}$ in R_n and the maximal ideal $\mathfrak{m}$ in $\mathcal{O}_n$, respectively. We have $\tilde{\mathfrak{m}}\mathcal{O}_n = \mathfrak{m}$ and $\mathfrak{m}^N \cap R_n = \tilde{\mathfrak{m}}^N$ for all $N \in \mathbb{N}$.

Suppose that $I = \langle g_1, \dots, g_r \rangle \subseteq R_n$ is a $\tilde{\mathfrak{m}}$ -primary ideal, there exists an integer $N > 0$ such that $\tilde{\mathfrak{m}}^N \subseteq I$. We have a canonical $\mathbb{C}$ -algebra isomorphism

$$\frac{R_n}{I} \cong \frac{\mathcal{O}_n}{I\mathcal{O}_n}, \quad g + I \mapsto g + I\mathcal{O}_n.$$

It is surjective since for any $h \in \mathcal{O}_n$, h can be decomposed as $u_1 + u_2$ with $u_1 \in R_n$ and $u_2 \in \mathfrak{m}^N$. The injectivity follows from $I\mathcal{O}_n \cap R_n = I$. Actually, if $g \in R_n$ and $g = \sum_{i=1}^r g_i h_i$ with each $h_i \in \mathcal{O}_n$. Then write $h_i = u_{i,1} + u_{i,2}$ with $u_{i,1} \in R_n$ and $u_{i,2} \in \mathfrak{m}^N$. We have

$$\sum_{i=1}^r g_i u_{i,2} = g - \sum_{i=1}^r g_i u_{i,1} \in \mathfrak{m}^N \cap R_n = \tilde{\mathfrak{m}}^N \subseteq I.$$

Hence $g \in I$.

We see that the Yau Conjecture is a special case of the Aleksandrov Conjecture. For a weighted homogeneous polynomial $f \in R_n$ defining an isolated singularity $V_f = (\mathbf{V}(f), \mathbf{0})$ at the origin, then $\langle \partial_{z_1} f, \dots, \partial_{z_n} f \rangle = \langle f, \partial_{z_1} f, \dots, \partial_{z_n} f \rangle$ is a $\tilde{\mathfrak{m}}$ -primary ideal in R_n , and

$$A(V_f) \cong \frac{R_n}{\langle \partial_{z_1} f, \dots, \partial_{z_n} f \rangle}$$

is an Artinian complete intersection.

The Yau Conjecture is also of independent interest. It is still open and was proven in the low-dimensional case $n \leq 4$ by explicit calculations (cf. [6, Theorem 2.3], [8, Theorem 2.1]). Assume that the weight type $\mathbf{w}$ satisfies $w_1 \geq \dots \geq w_n$, B. Chen [5, Theorem 1.5] proved that the Yau Conjecture holds if $\deg_{\mathbf{w}} f \geq w_1^2 - w_1 + 1$.

In this paper, we introduce a new series of local $\mathbb{C}$ -algebras and their associated derivation Lie algebras. They are natural generalizations of the moduli algebra and Yau algebra.

Let $V_f = (\mathbf{V}(f), \mathbf{0})$ be an isolated hypersurface singularity in $\mathbb{C}^n$ defined by the germ of a holomorphic function $f \in \mathcal{O}_n$. We define $\mathcal{T}_k(f) \subseteq \mathcal{O}_n$ to be an ideal generated by all i -th order partial derivatives of f with $0 \leq i \leq k$. The new object, the k -th order Tjurina algebras $A_k(V_f)$ is defined to be the Artinian local $\mathbb{C}$ -algebra $\frac{\mathcal{O}_n}{\mathcal{T}_k(f)}$ (see Definition 2.1).

Different kinds of local $\mathbb{C}$ -algebras of isolated hypersurface singularities, e.g.,

$$\frac{\mathcal{O}_n}{\langle f \rangle + \mathfrak{m}^k \langle \partial_{z_1} f, \dots, \partial_{z_n} f \rangle}, \quad \frac{\mathcal{O}_n}{\langle f, \partial_{z_1} f, \dots, \partial_{z_n} f, \det(\partial_{z_i} \partial_{z_j} f)_{1 \leq i, j \leq n} \rangle}$$

have been studied (cf. [4, 16–18]). Moreover, if f is a weighted homogeneous polynomial, then these graded algebras have no non-zero negative weight derivations when $\text{mult}(f) := \max\{N \in \mathbb{N} : f \in \mathfrak{m}^N\} \geq k_0$ for some integer $k_0 > 0$ (cf. [15, Main Theorem]).

Local algebras $\{A_k(V_f)\}_{k \geq 1}$ give more information of singularities. When $k = 1$, $A_1(V_f)$ is just the moduli algebra $A(V_f)$. $A_{k+1}(V_f)$ is a quotient algebra of $A_k(V_f)$. Naturally, we hope to find more connections between the associated derivation Lie algebras of these different algebras, such as their dimensions as $\mathbb{C}$ -vector spaces or the non-existence of non-zero negative weight derivations. Motivated by the Yau conjecture, we proposed the following generalized Yau conjecture.

Conjecture 1. Fix a weight type $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{N}^n$ of positive integers. And let $f \in \mathbb{C}[z_1, \dots, z_n]$ be a weighted homogeneous polynomial with respect to $\mathbf{w}$. There exists a polynomial $F(x) \in \mathbb{Q}[x]$ of degree $n - 1$ such that if $k \geq 2$ and $\text{mult}(f) \geq F(k)$, then there are non-zero negative weight derivation on $A_k(V_f)$.

The main purpose of this paper is to verify the Conjecture for $n = 3$. The case $n = 1$ is trivial, while the case $n = 2$ follows from Corollary 3.2. We hope that the polynomial $F(x) \in \mathbb{Q}[x]$ only depends on n .

Main Theorem. Let $V_f = (\mathbf{V}(f), \mathbf{0})$ be an isolated hypersurface singularity defined by the germ of a weighted homogeneous polynomial $f \in \mathbb{C}[z_1, z_2, z_3]$ with respect to $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{N}^3$. Assume that $w_1 \geq w_2 \geq w_3 \geq 1$. If

$$k \geq 2, \quad \text{mult}(f) \geq \frac{k^2 + 3k - 2}{2}, \quad (3)$$

then there are no non-zero negative weight derivations on $A_k(V_f)$.

2. PRELIMINARY

Definition 2.1. Let $V_f = (\mathbf{V}(f), \mathbf{0})$ be a hypersurface singularity defined by the germ of a holomorphic function $f \in \mathcal{O}_n$. For an n -tuple $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we set

$$|\alpha| := \sum_{i=1}^n \alpha_i, \quad \partial^\alpha f := \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n} f.$$

Fix an integer $k \geq 1$.

- (1) The k -th order Jacobian ideal of f is the following ideal of $\mathcal{O}_n$:

$$\text{Jac}_k(f) := \langle \partial^\alpha f : \alpha \in \mathbb{N}^n, |\alpha| = k \rangle.$$

- (2) The k -th order Milnor algebra is the analytic algebra:

$$M_k(V_f) := \frac{\mathcal{O}_n}{\mathcal{J}_k(f)}, \quad \mathcal{J}_k(f) := \sum_{i=1}^k \text{Jac}_i(f).$$

- (3) The k -th order Tjurina ideal is the analytic algebra:

$$A_k(V_f) := \frac{\mathcal{O}_n}{\mathcal{T}_k(f)}, \quad \mathcal{T}_k(f) := \langle f \rangle + \sum_{i=1}^k \text{Jac}_i(f).$$

If $V_f = (\mathbf{V}(f), \mathbf{0})$ is an isolated hypersurface singularity, then $\mathcal{T}_k(f)$ is an $\mathfrak{m} = \langle z_1, \dots, z_n \rangle$ -primary ideal, this is because $\mathcal{T}_k(f) \supseteq \langle f, \partial_{z_1} f, \dots, \partial_{z_n} f \rangle$. So $A_k(V_f)$ is an Artinian local $\mathbb{C}$ -algebra. We denote its dimension as a $\mathbb{C}$ -vector space by $\tau_k(V_f)$.

For any monomial $z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in \text{Supp}(f)$, $0 \neq \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n} f \notin \mathfrak{m}$, it is a unit in $\mathcal{O}_n$. Hence if $k > \text{mult}(f)$, $M_k(V_f) = A_k(V_f) = 0$.

Definition 2.2. Let $f, g \in \mathcal{O}_n$ be holomorphic functions at $\mathbf{0}$.

- (1) f is called right equivalent to g , if there exists a $\mathbb{C}$ -algebra automorphism φ of $\mathcal{O}_n$ such that $f = \varphi(g)$
- (2) f is called contact equivalent to g , if there exists a $\mathbb{C}$ -algebra automorphism φ of $\mathcal{O}_n$ and a unit $u \in \mathcal{O}_n^\times$ such that $f = u\varphi(g)$.

Theorem 2.3. Suppose that $V_f = (V(f), \mathbf{0})$ and $V_g = (V(g), \mathbf{0})$ are isolated hypersurface singularities defined by the germs of holomorphic functions f and $g \in \mathcal{O}_n$, respectively.

- (1) If f is right equivalent to g , then $M_k(V_f) \cong M_k(V_g)$ for all $k \geq 1$.
- (2) If f is contact equivalent to g , then $A_k(V_f) \cong A_k(V_g)$ for all $k \geq 1$.

Proof. If $f = \varphi(g)$ where $\varphi \in \text{Aut}_{\mathbb{C}}(\mathcal{O}_n)$, by the Leibniz rule,

$$\begin{aligned} \partial_{z_i}(f) &= \partial_{z_i}(\varphi(g)) = \partial_{z_i}(g(\varphi(z_1), \dots, \varphi(z_n))) \\ &= \sum_{j=1}^n (\partial_{z_i}g)(\varphi(z_1), \dots, \varphi(z_n)) \partial_{z_i}\varphi(z_j) = \sum_{j=1}^n \varphi(\partial_{z_i}g) \partial_{z_i}\varphi(z_j). \end{aligned}$$

Hence $\text{Jac}(f) = \text{Jac}(\varphi(g)) \subseteq \varphi(\text{Jac}(g))$. We shall prove that

$$\mathcal{J}_i(f) = \mathcal{J}_i(\varphi(g)) \subseteq \varphi(\mathcal{J}_i(g)). \quad (4)$$

(4) holds for $i = 1$. Suppose that it holds for $i = 1, \dots, l-1$, consider $\partial^\beta f \in \text{Jac}_k(f)$ with $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ and $|\beta| = l$. Assume that $\beta_{i_0} > 0$, let $\gamma = (\beta_1, \dots, \beta_{i_0} - 1, \dots, \beta_n)$. $|\gamma| = l-1$. By induction, $\partial^\gamma f \in \text{Jac}_{l-1}(f) \subseteq \varphi(\mathcal{J}_{l-1}(g))$. Then

$$\partial^\gamma f = \sum_{\alpha \in \mathbb{N}^n, 1 \leq |\alpha| \leq l-1} h_\alpha \varphi(\partial^\alpha g) = \sum_{\alpha \in \mathbb{N}^n, 1 \leq |\alpha| \leq l-1} \varphi(\tilde{h}_\alpha \partial^\alpha g).$$

Here $h_\alpha \in \mathcal{O}_n$ and $\tilde{h}_\alpha = \varphi^{-1}(h_\alpha)$. We obtain

$$\begin{aligned} \partial^\beta f &= \partial_{z_j} \partial^\gamma f = \sum_{\alpha \in \mathbb{N}^n, 1 \leq |\alpha| \leq l-1} \partial_{z_{i_0}} \varphi(\tilde{h}_\alpha \partial^\alpha g) \\ &= \sum_{\alpha \in \mathbb{N}^n, 1 \leq |\alpha| \leq l-1} \sum_{j=1}^n \varphi(\partial_{z_j}(\tilde{h}_\alpha \partial^\alpha g)) \partial_{z_{i_0}} \varphi(z_j). \end{aligned}$$

Since $\partial_{z_j}(\tilde{h}_\alpha \partial^\alpha g) \in \text{Jac}_{|\alpha|}(g) + \text{Jac}_{|\alpha|+1}(g)$, we see that $\partial^\beta f \in \varphi(\mathcal{J}_l(g))$ for all $\beta \in \mathbb{N}^n$ such that $|\beta| = l$. Hence $\mathcal{J}_l(f) = \mathcal{J}_{l-1}(f) + \text{Jac}_k(f) \subseteq \varphi(\mathcal{J}_l(g))$. Thus (4) holds for all i . By symmetry, $g = \varphi^{-1}(f)$, $\mathcal{J}_i(g) \subseteq \varphi^{-1}(\mathcal{J}_i(f))$. We see that $\varphi(\mathcal{J}_i(g)) = \mathcal{J}_i(f)$. φ induces the isomorphism

$$\bar{\varphi} : M_k(V_g) \rightarrow M_k(V_f), \quad h + \mathcal{J}_k(g) \mapsto \varphi(h) + \mathcal{J}_k(f).$$

To prove (2), if $f = u\varphi(g)$ for a $\mathbb{C}$ -algebra automorphism $\varphi \in \text{Aut}_{\mathbb{C}}(\mathcal{O}_n)$ and a unit u of $\mathcal{O}_n^\times$, then by the product rule, for any $\beta \in \mathbb{N}^n$ such that $|\beta| = k$,

$$\begin{aligned} \partial^\beta f &= \sum_{\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \alpha_i \leq \beta_i} \binom{\beta}{\alpha} (\partial^\alpha u) (\partial^{\beta-\alpha} \varphi(g)), \\ \partial^\beta \varphi(g) &= \sum_{\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \alpha_i \leq \beta_i} \binom{\beta}{\alpha} (\partial^\alpha u^{-1}) (\partial^{\beta-\alpha} f). \end{aligned}$$

Here $\binom{\beta}{\alpha} = \prod_{i=1}^n \binom{\beta_i}{\alpha_i}$. It is obvious to see that $\mathcal{T}_k(f) = \mathcal{T}_k(\varphi(g))$. Combining the result of (1) we have $\mathcal{T}_k(\varphi(g)) = \varphi(g) + \mathcal{J}_k(\varphi(g)) = \varphi(\langle g \rangle + \mathcal{J}_k(g)) = \varphi(\mathcal{T}_k(g))$. Hence

$$A_k(V_g) = \frac{\mathcal{O}_n}{\mathcal{T}_k(g)} \cong \frac{\mathcal{O}_n}{\mathcal{T}_k(g)} = \frac{\mathcal{O}_n}{\mathcal{T}_k(\varphi(g))} = A_k(V_f).$$

The second isomorphism is given by $h + \mathcal{T}_k(g) \rightarrow \varphi(h) + \mathcal{T}_k(\varphi(g))$. Thus we proved this theorem. $\square$

Remark 2. For a polynomial f in $\mathbb{C}[z_1, \dots, z_n]$, we define the k -th order Jacobian ideal of f to be the ideal

$$J_k(f) := \langle \partial^\alpha f : |\alpha| = k \rangle \subseteq \mathbb{C}[z_1, \dots, z_n].$$

For $f, g \in \mathbb{C}[z_1, \dots, z_n]$, if there exists a $\mathbb{C}$ -algebra automorphism $\varphi \in \text{Aut}_{\mathbb{C}}(\mathbb{C}[z_1, \dots, z_n])$ satisfying $f = \varphi(g)$, one can prove that

$$\sum_{i=1}^k J_i(f) = \varphi \left(\sum_{i=1}^k J_i(g) \right)$$

for all $k \geq 1$ in a similar way.

Based on Theorem 2.3, for a hypersurface isolated singularity $V_f = (\mathbf{V}(f), \mathbf{0})$ defined by $f \in \mathcal{O}_n$, it is natural to introduce the associated k -th order derivation Lie algebra $\mathcal{L}_k(V_f) := \text{Der}(A_k(V_f))$. We denote its dimension as a finite-dimensional $\mathbb{C}$ -vector space by $\delta_k(V_f)$. $\delta_k(V_f)$ is also a new numerical analytic invariant. We have reasons to believe that these new Lie algebras and numerical invariants play important roles in the study of singularities, because $A_k(V_f)$ provides more information than the moduli algebra $A(V_f)$. Also, there are some sharp upper estimate conjectures for the $(k$ -th) Yau number of isolated hypersurface singularities defined by weighted homogeneous polynomials, (cf. [16, Theorem A to Theorem E]), which have been proved for a large class of singularities. We wonder that is there any some sharp upper estimate conjecture for $\delta_k(V_f)$.

Also, we may ask does the Torelli type theorem holds for these new algebras, i.e., suppose that $V_{\mathbf{t}}$ is a (μ, τ) -constant family of isolated hypersurface singularities with parameters $\mathbf{t} = (t_1, \dots, t_m)$, if $V_{\mathbf{t}}$ and $V_{\mathbf{s}}$ are biholomorphically equivalent, do we have $A_k(V_{\mathbf{t}}) \cong A_k(V_{\mathbf{s}})$ for a fixed k ? Whether the family of isolated complex hypersurface singularities can be distinguished by means of their new Lie algebras?

In this paper, we focus on weighted homogeneous isolated hypersurface singularities.

Let $V_f = (\mathbf{V}(f), \mathbf{0})$ be an isolated hypersurface singularity defined by a weighted homogeneous polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ with respect to the weight type $\mathbf{w} = (w_1, \dots, w_n)$. Suppose that $\mathbf{w} \in \mathbb{N}^n$ and all $w_i \geq 1$. For any $\alpha \in \mathbb{N}^n$, if $\partial^\alpha f \neq 0$, then $\partial^\alpha f$ is a weighted homogeneous polynomial of weighted degree $\deg_{\mathbf{w}} f - \sum_{i=1}^n \alpha_i w_i$. Thus

$$\partial^\alpha f = \frac{1}{\deg_{\mathbf{w}} f - \sum_{i=1}^n \alpha_i w_i} \sum_{j=1}^n w_j z_j \partial_{z_j} \partial^\alpha f \in \text{Jac}_{|\alpha|+1}(f).$$

This implies that $\langle f \rangle \subseteq \text{Jac}_1(f) \subseteq \dots \subseteq \text{Jac}_k(f)$ and

$$M_k(V_f) = A_k(V_f) = \frac{\mathcal{O}_n}{\text{Jac}_k(f)}. \quad (5)$$

3. PROOF OF MAIN THEOREM

Let $R_n := \mathbb{C}[z_1, \dots, z_n]$. Fix a weight type $\mathbf{w} = (w_1, \dots, w_n)$ of positive integers. We suppose that $w_1 \geq \dots \geq w_n$ without loss of generality. Let $f \in R_n$ be a non-zero weighted homogeneous polynomial of multiplicity ≥ 3 . f defines an isolated hypersurface singularity $V_f = (\mathbf{V}(f), \mathbf{0})$ at the origin $\mathbf{0} = (0, \dots, 0)$ of $\mathbb{C}^n$. By [8, Lemma 2.1], f is contact equivalent to a weighted homogeneous polynomial of one of the following two forms which has the same weighted degree and multiplicity as f .

$$z_1^m + \sum_{i=1}^m a_i(z_2, \dots, z_n) z_1^{m-i}, \quad a_i \in \mathbb{C}[z_2, \dots, z_n]. \quad (\text{Case 1})$$

$$z_1^m z_{i_0} + \sum_{i=1}^m a_i(z_2, \dots, z_n) z_1^{m-i}, \quad a_i \in \mathbb{C}[z_2, \dots, z_n], 2 \leq i_0 \leq n. \quad (\text{Case 2})$$

Fix an integer $2 \leq k < \text{mult}(f)$, combining with Remark 1, Theorem 2.3, and (5), we only need to prove that there are no non-zero negative weight derivations on the finitely generated local graded $\mathbb{C}$ -algebra which is isomorphic to $A_k(V_f)$:

$$\tilde{A}_k(f) := \frac{R_n}{J_k(f)}, \quad J_k(f) = \langle \partial^\alpha f : \alpha \in \mathbb{N}^n, |\alpha| = k \rangle \subseteq R_n.$$

We can directly assume that f is already of the form (Case 1) or (Case 2).

Let D be a negative weight derivation on $\tilde{A}_k(V_f)$ with respect to the weight type $\mathbf{w}$. For $1 \leq i \leq n$, choose $p_i \in R_n$ such that $D(z_i + J_k(f)) = p_i + J_k(f)$, then D is induced by

$$\tilde{D} = \sum_{i=1}^n p_i \partial_{z_i} \in \text{Der}(R_n),$$

i.e., for any $h \in R_n$, $D(h + J_k(f)) = \tilde{D}(h) + J_k(f)$. $\tilde{D}$ is a negative weight derivation on R_n , and $\tilde{D}$ preserves $J_k(f)$, i.e., $\tilde{D}(J_k(f)) \subseteq J_k(f)$. To prove that there are no non-zero negative weight derivations on $\tilde{A}_k(f) \cong A_k(V_f)$, it suffices to prove the following statement under certain conditions on k and $\text{mult}(f)$.

Statement 1. Fix a weight type $\mathbf{w} = (w_1, \dots, w_n)$ of positive integers such that $w_1 \geq \dots \geq w_n \geq 1$. Consider the weighted homogeneous polynomial

$$f = \sum_{i=0}^m a_i(z_2, \dots, z_n)z_1^{m-i},$$

with $a_0 = 1$ or z_{i_0} ($i_0 \geq 2$). Suppose that f defines an isolated singularity $V_f = (V(f), \mathbf{0})$ at $\mathbf{0} \in \mathbb{C}^n$. For some integer $2 \leq k < \text{mult}(f)$, if a negative weight derivation $\tilde{D} = \sum_{i=1}^n p_i \partial_{z_i}$ on R_n preserves $J_k(f)$, then all $p_i \in J_k(f)$.

Note that $w_1 \geq \dots \geq w_n$, if $\tilde{D} = \sum_{i=1}^n p_i \partial_{z_i}$ is a negative weight derivation and $p_i \neq 0$, then p_i is a weighted homogeneous polynomial with weighted degree $< w_i$. This implies that $p_i \in \mathbb{C}[z_{i+1}, \dots, z_n]$. If $\tilde{D}$ preserves $J_k(f)$, then $p_i \notin \mathbb{C} \setminus \{0\}$ for all i . Otherwise, since $\langle z_1, \dots, z_n \rangle^{\tau_k(V_f)+1} \subseteq J_k(f)$, we obtain

$$\underbrace{\tilde{D} \circ \dots \circ \tilde{D}}_{(\tau_k(V_f)+1)\text{-times}} (z_i^{\tau_k(V_f)+1}) = (\tau_k(V_f) + 1)! p_i^{\tau_k(V_f)+1} \in J_k(f),$$

which contradicts the fact that $k < \text{mult}(f)$ and $J_k(f) \neq R_n$. So $p_n = 0$.

Statement 1 is true for all $2 \leq k < \text{mult}(f)$ when $w_1 = \dots = w_n$. In this case, if $p_i \neq 0$, then $\deg_{\mathbf{w}} p_i < w_i$, but $p_i \notin \mathbb{C} \setminus \{0\}$, it is impossible. If a homogeneous polynomial $f \in R_n$ defines an isolated singularity $V_f = (V(f), \mathbf{0})$ at $\mathbf{0}$, there are no non-zero negative weight derivations on $A_k(V_f)$ for all $2 \leq k < \text{mult}(f)$.

From now on we always assume that the weight type $\mathbf{w}$ satisfies $w_1 \geq \dots \geq w_n$ and $w_1 > w_n$. In particular, when $n = 3$, to prove Statement 1, we only need to consider the following three cases.

$$\tilde{D} = p(z_2, z_3) \partial_{z_1}, \quad \tilde{D}(J_k(f)) \subseteq J_k(f). \quad (\text{Case A})$$

$$\tilde{D} = q(z_3) \partial_{z_2}, \quad \tilde{D}(J_k(f)) \subseteq J_k(f). \quad (\text{Case B})$$

$$\tilde{D} = p(z_2, z_3) \partial_{z_1} + q(z_3) \partial_{z_2}, \quad \tilde{D}(J_k(f)) \subseteq J_k(f). \quad (\text{Case C})$$

In (Case C), we require that $p \neq 0$ and $q \neq 0$.

Let $\tilde{D} = \sum_{i=1}^{n-1} p_i \partial_{z_i} \in \text{Der}(R_n)$ be a negative weight derivation preserving $J_k(f)$. For every $0 \leq \alpha \leq k$, $\tilde{D}(\partial_{z_1}^{k-\alpha} \partial_{z_2}^\alpha f) \in J_k(f)$. If $\tilde{D}(\partial_{z_1}^{k-\alpha} \partial_{z_2}^\alpha f) \neq 0$, then

$$\deg_{\mathbf{w}} \tilde{D}(\partial_{z_1}^{k-\alpha} \partial_{z_2}^\alpha f) < \deg_{\mathbf{w}} \partial_{z_1}^{k-\alpha} \partial_{z_2}^\alpha f.$$

So $\tilde{D}(\partial_{z_1}^{k-\alpha}\partial_{z_2}^\alpha f)$ must lie in the ideal generated by weighted homogeneous polynomials in $J_k(f)$ with smaller degrees, i.e.,

$$\begin{aligned} \tilde{D}(\partial_{z_1}^{k-\alpha}\partial_{z_2}^\alpha f) &\in \langle \partial^\alpha f : \alpha \in \mathbb{N}^n, |\alpha| = k, \deg_{\mathbf{w}} \partial^\alpha f \leq \deg_{\mathbf{w}} \tilde{D}(\partial_{z_1}^{k-\alpha}\partial_{z_2}^\alpha f) \rangle \\ &\subseteq \langle \partial^\alpha f : \alpha \in \mathbb{N}^n, |\alpha| = k, \deg_{\mathbf{w}} \partial^\alpha f < \deg_{\mathbf{w}} \partial_{z_1}^{k-\alpha}\partial_{z_2}^\alpha f \rangle. \end{aligned}$$

For those $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ such that $|\alpha| = k$ and $\alpha_1 \leq k - \alpha$, if $\partial^\alpha f \neq 0$, then

$$\begin{aligned} \deg_{\mathbf{w}} \partial^\alpha f &= \deg_{\mathbf{w}} f - \sum_{i=1}^n \alpha_i w_i \geq \deg_{\mathbf{w}} f - \alpha_1 w_1 - \left(\sum_{i=2}^n \alpha_i \right) w_2 \\ &\geq \deg_{\mathbf{w}} f - (k - \alpha) w_1 - \alpha w_2 = \deg_{\mathbf{w}} \partial_{z_1}^{k-\alpha} \partial_{z_2}^\alpha f. \end{aligned}$$

Hence we obtain

$$\tilde{D}(\partial_{z_1}^{k-\alpha}\partial_{z_2}^\alpha f) \in \langle \partial^\alpha f : \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, |\alpha| = k, \alpha_1 < k - \alpha \rangle. \quad (6)$$

Especially, when $\alpha = 0$, any non-zero generator $\partial^\alpha f$ of $J_k(f)$ with $|\alpha| = k$ has weighted degree greater than $\deg_{\mathbf{w}} \partial_{z_1}^k f$. Hence $\tilde{D}(\partial_{z_1}^k f) = \partial_{z_1}^k \tilde{D}(f) = 0$. This is because each p_i in $\tilde{D}$ does not depend on z_1 . Note that

$$\tilde{D}(f) = \tilde{D}(a_0)z_1^m + \sum_{i=1}^m (\tilde{D}(a_i) + (m-i+1)p_1 a_{i-1})z_1^{m-i},$$

$\partial_{z_1}^k \tilde{D}(f) = 0$ implies that $\tilde{D}(a_0) = 0$ and for $1 \leq i \leq m-k$

$$\tilde{D}(a_i) + (m-i+1)p_1 a_{i-1} = 0. \quad (7)$$

Proposition 3.1. Keep the notion in Statement 1. For $k < \text{mult}(f)$, if a negative weight derivation $\tilde{D} = p(z_2, \dots, z_n)\partial_{z_1}$ on R_n preserves $J_k(f)$, then $p \in J_k(f)$.

Proof. $f = \sum_{i=0}^m a_i(z_2, \dots, z_n)z_1^{m-i}$. So $k < \text{mult}(f) \leq m+1$. $\tilde{D}(a_0) = 0$. If $k < m$, by (7), $\tilde{D}(a_0) + mp = 0$, thus $p = 0$. If $k = m$, then $\text{mult}(f) = m+1$ and $a_0 = z_{i_0}$ for some $i_0 \geq 2$. $\partial_{z_1}^{m-1}\partial_{z_{i_0}} f \in J_m(f)$, hence

$$\tilde{D}(\partial_{z_1}^{m-1}\partial_{z_{i_0}} f) = p\partial_{z_1}^m \partial_{z_{i_0}} f = m!p \in J_m(f).$$

So $p \in J_m(f)$. □

Proposition 3.1 settles Statement 1 for $n = 2$ and (Case A) for $n = 3$.

Corollary 3.2. Let $f \in \mathbb{C}[z_1, z_2]$ be a weighted homogeneous polynomial. Suppose that f defines an isolated plane curve singularity $V_f = (V(f), (0,0))$ at $(0,0)$ of $\mathbb{C}^2$. For $2 \leq k < \text{mult}(f)$, there are no non-zero negative weight derivations on $A_k(V_f)$.

Lemma 3.3. Let $g \in \mathbb{C}[z_2, \dots, z_n]$ be a weighted homogeneous polynomial with respect to the weight type $\mathbf{v} = (v_2, \dots, v_n) \in \mathbb{N}^{n-1}$. Assume that $v_2 \geq v_3 \geq \dots \geq v_n \geq 1$. Define the partial multiplicity of g as

$$\begin{aligned} \text{mult}_2(g) &= \max\{N \in \mathbb{N} : g \in \langle z_3, \dots, z_n \rangle^N\} \\ &= \min \left\{ \sum_{i=3}^n \alpha_i : z_2^{\alpha_2} \cdots z_n^{\alpha_n} \in \text{Supp}(g) \right\}. \end{aligned}$$

Set $\text{mult}_2(0) = +\infty$. For any $\alpha \in \mathbb{N}$, there is

$$\min \left\{ \frac{\deg_{\mathbf{v}} g - (\alpha-1)v_2}{v_3}, \text{mult}_2(\partial_{z_2}^\alpha g) \right\} \leq \text{mult}_2(g) \leq \frac{\deg_{\mathbf{v}} g}{v_n}.$$

Proof. Consider any monomial $z_2^{\alpha_2} \cdots z_n^{\alpha_n} \in \text{Supp}(g)$.

- If $\alpha_2 \geq \alpha$, then $z_2^{\alpha_2 - \alpha} \cdots z_n^{\alpha_n} \in \text{Supp}(\partial_{z_2}^\alpha g)$. We have

$$\text{mult}_2(\partial_{z_2}^\alpha g) \leq \sum_{i=3}^n \alpha_i.$$

- If $\alpha_2 < \alpha$, since $\deg_{\mathbf{v}} z_2^{\alpha_2} \cdots z_n^{\alpha_n} = \deg_{\mathbf{v}} g$, we have

$$\begin{aligned} \sum_{i=3}^n \alpha_i &\geq \frac{\sum_{i=3}^n \alpha_i v_i}{v_3} = \frac{\deg_{\mathbf{v}} g - \alpha_2 v_2}{v_3} \geq \frac{\deg_{\mathbf{v}} g - (\alpha - 1)\alpha_2}{v_3}, \\ \sum_{i=3}^n \alpha_i &\leq \frac{\sum_{i=3}^n \alpha_i v_i}{v_n} \leq \frac{\deg_{\mathbf{v}} g}{v_n}. \end{aligned}$$

As $\text{mult}_2(g) \leq \sum_{i=3}^n \alpha_i$, we obtain the desired inequality. $\square$

Remark 3. The partial multiplicity has properties analogous to the classical multiplicity. Suppose that $0 \neq g, h \in \mathbb{C}[z_2, \dots, z_n]$.

- (1) $\text{mult}_2(gh) = \text{mult}_2(g) + \text{mult}_2(h)$.
- (2) $\text{mult}_2(g + h) \geq \min\{\text{mult}_2(g), \text{mult}_2(h)\}$.
- (3) $\text{mult}_2(\partial_{z_2}^{\alpha_2} \cdots \partial_{z_n}^{\alpha_n} g) \geq \text{mult}_2(g) - \sum_{i=3}^n \alpha_i$.

Proposition 3.4. Fix a weight type $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{N}^n$ such that $w_1 \geq \cdots \geq w_n \geq 1$. Let $f = \sum_{i=0}^m a_i(z_2, \dots, z_n)z_1^{m-i} \in R_n$ with $a_0 \neq 0$ be a weighted homogeneous polynomial with respect to $\mathbf{w}$. Fix an integer $k < m$. If a non-zero negative weight derivation

$$\tilde{D} = p(z_3, \dots, z_n)\partial_{z_2} \in \text{Der}(R_n)$$

preserves $J_k(f)$, then we have:

- (1) For $0 \leq i \leq m - k$,

$$\text{mult}_2(a_i) \geq \frac{iw_1 + \deg_{\mathbf{w}} a_0}{w_3}.$$

Moreover,

$$\text{mult}_2(a_{m-k+1}) \geq C_{f,k,\mathbf{w}} := \frac{(m-k)w_1 + \deg_{\mathbf{w}} a_0}{w_3}. \quad (8)$$

- (2) Fix an integer $j_1 \geq 1$. For a sequence $C_{f,k,\mathbf{w}} \geq C_2 \geq \cdots \geq C_{j_1}$, we write

$$\{\text{mult}_2(a_{m-k+j})\}_{j=1}^{j_1} \succeq (C_{f,k,\mathbf{w}}, C_2, \dots, C_{j_1})$$

if $\text{mult}_2(a_{m-k+j}) \geq C_j$ for $2 \leq j \leq j_1$. If $\{\text{mult}_2(a_{m-k+j})\}_{j=1}^{j_1} \succeq (C_{f,k,\mathbf{w}}, C_2, \dots, C_{j_1})$, then for $j_1 + 1 \leq j \leq k$,

$$\text{mult}_2(a_{m-k+j}) \geq C_{j_1} - (j - j_1)\text{mult}_2(p) - \frac{(j + j_1 - 1)(j - j_1)}{2}. \quad (9)$$

Proof. For $0 \leq i \leq m - k$, (7) implies that $\partial_{z_2} a_i = 0$. If $a_i \neq 0$, then by Lemma 3.3,

$$\text{mult}_2(a_i) \geq \frac{\deg_{\mathbf{w}} a_i}{w_3} = \frac{iw_1 + \deg_{\mathbf{w}} a_0}{w_3}.$$

For $\alpha > 0$, we rewrite (6) as

$$\begin{aligned} \tilde{D}(\partial_{z_1}^{k-\alpha} \partial_{z_2}^\alpha f) &= p \partial_{z_1}^{k-\alpha} \partial_{z_2}^{\alpha+1} f = \sum_{i=0}^{m-k+\alpha} \frac{(m-i)!}{(m-i-k+\alpha)!} (p \partial_{z_2}^{\alpha+1} a_i) z_1^{m-i-k+\alpha} \\ &= \sum_{\substack{\alpha=(\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \\ |\alpha|=k, \alpha_1 < k-\alpha}} g_\alpha \partial^\alpha f. \end{aligned}$$

Here $g_\alpha \in R_n$. As $\partial^\alpha f = \sum_{i=0}^{m-\alpha_1} \frac{(m-i)!}{(m-i-\alpha_1)!} (\partial_{z_2}^{\alpha_2} \cdots \partial_{z_n}^{\alpha_n} a_i) z_1^{m-i-\alpha_1}$, setting $z_1 = 0$ on both sides of the above equation, we get

$$p \partial_{z_2}^{\alpha+1} a_{m-k+\alpha} \in \langle \partial^\beta a_{m-k+|\beta|} : \beta \in \mathbb{N}^{n-1}, |\beta| < \alpha \rangle. \quad (10)$$

Here for $\beta = (\beta_2, \dots, \beta_n)$, $\partial^\beta = \partial_{z_2}^{\beta_2} \cdots \partial_{z_n}^{\beta_n}$.

Especially, when $\alpha = 1$, as $\partial_{z_2} a_i = 0$ for $i \leq m-k$, we have

$$\tilde{D}(\partial_{z_1}^{k-1} \partial_{z_2} f) = p \partial_{z_2}^2 a_{m-k+1} \in \langle \partial_{z_1}^k f \rangle.$$

However, $\partial_{z_1}^k f$ contains $a_0 z_1^m$ as the highest order z_1 -term and $a_{m-k+1} \in \mathbb{C}[z_2, \dots, z_n]$, so $\partial_{z_2}^2 a_{m-k+1} = 0$. If $a_{m-k+1} \neq 0$, then by Lemma 3.3,

$$\text{mult}_2(a_{m-k+1}) \geq \frac{\deg_{\mathbf{w}} a_{m-k+1} - w_2}{w_3} = C_{f,k,\mathbf{w}} + \frac{w_1 - w_2}{w_3} \geq C_{f,k,\mathbf{w}}.$$

We will prove (9) by induction on j . It is obvious if $j = j_1$. Assume that (9) holds for $j = j_1, j_1 + 1, \dots, j_2$. Since $\frac{(j_2+j_1-1)(j_2-j_1)}{2} = \sum_{l=j_1}^{j_2-1} l$ and $\text{mult}_2(a_{m-k}) \geq C_{f,k,\mathbf{w}}$, (10) implies that

$$\begin{aligned} & \text{mult}_2(\partial_{z_2}^{j_2+2} a_{m-k+j_2+1}) + \text{mult}_2(p) \\ & \geq \min\{\text{mult}_2(\partial^\beta a_{m-k+|\beta|}) : \beta \in \mathbb{N}^{n-1}, |\beta| \leq j_2\} \\ & \geq \min\{\text{mult}_2(a_{m-k+j}) - j : 0 \leq j \leq j_2\} \\ & = C_{j_1} - (j_2 - j_1) \text{mult}_2(p) - \frac{(j_2 + j_1)(j_2 - j_1 + 1)}{2}. \end{aligned}$$

The second inequality follows from the fact that if $\beta = (\beta_2, \dots, \beta_n)$, then

$$\text{mult}_2(\partial^\beta a_{m-k+|\beta|}) \geq \text{mult}_2(a_{m-k+|\beta|}) - \sum_{i=3}^n \beta_i \geq \text{mult}_2(a_{m-k+|\beta|}) - |\beta|.$$

If $a_{m-k+j_2+1} \neq 0$, then $\deg_{\mathbf{w}} a_{m-k+j_2+1} = C_{f,k,\mathbf{w}} w_3 + (j_2 + 1)w_1$. By Lemma 3.3, we have

$$\text{mult}_2(a_{m-k+j_2+1}) \geq C_{j_1} - (j_2 - j_1) \text{mult}_2(p) - \frac{(j_2 + j_1)(j_2 - j_1 + 1)}{2}.$$

(9) holds for $j = j_2 + 1$. This completes the induction steps and concludes the proof of the proposition. $\square$

Corollary 3.5. Under the assumption of Proposition 3.4, we suppose that the weighted homogeneous polynomial f defines an isolated hypersurface singularity $V_f = (V(f), \mathbf{0})$ at the origin. The weight type $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{N}^n$ satisfies $w_1 \geq \dots \geq w_n \geq 1$ and $w_1 > w_3$. Suppose that

$$C_{f,k,\mathbf{w}} \geq (k-1) \frac{w_2}{w_n} + \frac{k^2 - k + 2}{2}. \quad (11)$$

If a negative weight derivation $\tilde{D} = p(z_3, \dots, z_n) \partial_{z_2}$ on R_n preserves $J_k(f)$, then $p = 0$.

Proof. If $p \neq 0$, then p is a weighted homogeneous polynomial with degree $< w_2$. From Lemma 3.3, $w_2 > \text{mult}_2(p)w_n$. By part (1) of Proposition 3.4, for $1 \leq i \leq m-k$, $\text{mult}_2(a_i) \geq \frac{iw_1}{w_3} > 1$.

$$\text{mult}_2(a_{m-k+1}) \geq C_{f,k,\mathbf{w}} > (k-1) \text{mult}_2(p) + \frac{k^2 - k + 2}{2}.$$

Let $j_1 = 1$. Using part (2) of Proposition 3.4, we obtain that for $1 \leq j \leq k$,

$$\text{mult}_2(a_{m-k+j}) \geq C_{f,k,\mathbf{w}} - (j-1) \text{mult}_2(p) - \frac{j(j-1)}{2} > 1. \quad (12)$$

Hence $\text{mult}_2(a_i) \geq 2$ for every $i \geq 1$. From the definition of the partial multiplicity we see that $a_i(x_2, 0, \dots, 0) = 0$, $\partial_{z_j}(a_i)(x_2, 0, \dots, 0)$ for all $i \geq 1, j \geq 2$ and $x_2 \in \mathbb{C}$. So $f, \partial_{z_1} f, \dots$,

$\partial_{z_n} f$ vanish on the whole z_2 -axis. f does not define an isolated singularity at the origin, a contradiction. $\square$

Claim 1. If f is of the form (Case 1), then $m = \text{mult}(f)$. When

$$\text{mult}(f) \geq (k-1)\frac{w_3}{w_n} + \frac{k^2 + k + 2}{2},$$

(11) holds. Therefore, if a negative weight derivation $\tilde{D} = p(z_3, \dots, z_n)\partial_{z_2}$ on R_n preserves $J_k(f)$, then $p = 0, D = 0$.

Claim 2. If f is of the form (Case 2), then $m = \text{mult}(f) - 1$. When

$$\text{mult}(f) \geq (k-1)\frac{w_3}{w_n} + \frac{k^2 + k + 4}{2},$$

(11) holds. Therefore, if a negative weight derivation $\tilde{D} = p(z_3, \dots, z_n)\partial_{z_2}$ on R_n preserves $J_k(f)$, then $p = 0, D = 0$.

Claim 3. Take $n = 3$, and let $f = z_1^m z_{i_0} + \sum_{i=0}^{m-1} a_i(z_2, z_3)z_1^{m-i}$ be of the form (Case 2). For any a_i , $\text{mult}_2(a_i) = \max\{l \in \mathbb{N} : z_3^l \text{ divides } a_i\}$. Suppose that

$$\text{mult}(f) \geq \frac{k^2 + 3k - 2}{2}.$$

If a negative weight derivation $\tilde{D} = p(z_3)\partial_{z_2}$ on R_3 preserves $J_k(f)$, then $p = 0, D = 0$.

Proof. By (7), $D(z_{i_0}) = 0$ implies that $i_0 = 3$. Then from part (1) of Proposition 3.4, $\text{mult}_2(a_i) \geq 1$ for $0 \leq i \leq m - k$. As $w_1 \geq w_2 > \text{mult}_2(p)w_3$,

$$C_{f,k,w} = \frac{(\text{mult}(f) - k - 1)w_1}{w_3} + 1 > (k-1)\text{mult}_2(p) + \frac{k^2 - k}{2}.$$

Using part (2) of Proposition 3.4, for $1 \leq j \leq k$,

$$\text{mult}_2(a_{m-k+j}) \geq C_{f,k,w} - (j-1)\text{mult}_2(p) - \frac{j(j-1)}{2} > 0.$$

Hence $\text{mult}_2(a_i) \geq 1$ for all $i \geq 0$. z_3 divides f . $f, \partial_{z_1} f, \partial_{z_2} f, \partial_{z_3} f$ vanish on $\{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{C}\}$. This contradicts our assumption that f defines an isolated singularity. $\square$

Claim 4. Take $n = 3$, and let $f = z_1^m + \sum_{i=0}^{m-1} a_i(z_2, z_3)z_1^{m-i}$ be of the form (Case 1). Fix an integer $k \geq 2$. Suppose that

$$m = \text{mult}(f) = \frac{k^2 + 3k - 2}{2}.$$

If a negative weight derivation $\tilde{D} = p(z_3)\partial_{z_2}$ on R_3 preserves $J_k(f)$, then $p = 0, D = 0$.

Proof. If $p \neq 0$, then $(k^2 - k)w_1 < (k^2 - k + 2)w_3$. Otherwise,

$$C_{f,k,w} = \frac{(k-1)w_1}{w_3} + \frac{(k^2 - k)w_1}{2w_3} \geq \frac{(k-1)w_2}{w_3} + \frac{k^2 - k + 2}{2}.$$

(11) holds. $p \neq 0$ contradicts to the conclusion of Corollary 3.5. As $w_2 > \text{mult}_2(p)w_3$, we have $\text{mult}_2(p) = 1$. Let $C_1 = \frac{k^2 + k}{2}$.

$$C_{f,k,w} > \frac{(k-1)w_1}{w_3} + \frac{k^2 - k}{2} > C_1 - 1.$$

We already have $\text{mult}_2(a_i) \geq 2$ for $1 \leq i \leq m - k$. By (12), for $1 \leq j \leq k - 1$,

$$\text{mult}_2(a_{m-k+j}) > C_1 - 1 - (j-1) - \frac{j(j-1)}{2} \geq 2 + k. \quad (13)$$

Since V_f is an isolated singularity at the origin, $\text{mult}_2(a_m) = 1$.

By (13), $\text{mult}_2(a_{m-k+1}) > C_1 - 1$. Actually, $\text{mult}_2(a_{m-k+1}) = C_1$. Otherwise, if $a_{m-k+2} \neq 0$, then

$$\begin{aligned} \text{mult}_2(a_{m-k+2}) &\geq \min\{C_{f,k,w}, \partial_{z_2}^3 a_{m-k+2}\} \\ &\geq \min\{C_{f,k,w}, \min\{\text{mult}_2(a_{m-k+j}) - j : 0 \leq j \leq 1\} - \text{mult}_2(p)\} \\ &\geq \min\{C_{f,k,w}, \min\{C_1, C_1 + 1 - 1\} - 1\} = C_1 - 1, \\ \{\text{mult}_2(a_{m-k+j})\}_{j=1}^2 &\succeq (C_{f,k,w}, C_1 - 1). \end{aligned}$$

$$\text{mult}_2(a_m) \geq C_1 - 1 - (k - 2) - \frac{(k + 1)(k - 2)}{2} \geq 2,$$

which contradicts to the fact that $\text{mult}_2(a_m) = 1$.

By (13), $\text{mult}_2(a_{m-k+2}) > C_1 - 3$. Actually, $\text{mult}_2(a_{m-k+2}) = C_1 - 2$. When $k = 2$, $C_1 = 3$, $\text{mult}_2(a_m) = 1$ holds. When $k \geq 3$, if $\text{mult}_2(a_{m-k+2}) \geq C_1 - 1$ and $a_{m-k+3} \neq 0$,

$$\begin{aligned} \text{mult}_2(a_{m-k+3}) &\geq \min\{C_{f,k,w}, \partial_{z_2}^4 a_{m-k+3}\} \\ &\geq \min\{C_{f,k,w}, \min\{\text{mult}_2(a_{m-k+j}) - j : 0 \leq j \leq 2\} - \text{mult}_2(p)\} \\ &\geq \min\{C_{f,k,w}, \min\{C_1, C_1 + 1 - 1, C_1 - 1 - 2\} - 1\} = C_1 - 4. \\ \{\text{mult}_2(a_{m-k+j})\}_{j=1}^3 &\succeq (C_{f,k,w}, C_1 - 1, C_1 - 4). \end{aligned}$$

$$\text{mult}_2(a_m) \geq C_1 - 4 - (k - 3) - \frac{(k + 2)(k - 3)}{2} \geq 2,$$

a contradiction.

When $k \geq 3$, by (13), $\text{mult}_2(a_{m-k+3}) > C_1 - 6$. We claim that $\text{mult}_2(a_{m-k+3}) = C_1 - 5$. If $k = 3, m = 7, C_1 = 6$, $\text{mult}_2(a_7) = 1$ holds. For $k \geq 4$, if $\text{mult}_2(a_{m-k+3}) \geq C_1 - 4$ and $a_{m-k+4} \neq 0$, then

$$\begin{aligned} \text{mult}_2(a_{m-k+4}) &\geq \min\{C_{f,k,w}, \partial_{z_2}^5 a_{m-k+4}\} \\ &\geq \min\{C_{f,k,w}, \min\{\text{mult}_2(a_{m-k+j}) - j : 0 \leq j \leq 3\} - \text{mult}_2(p)\} \\ &\geq \min\{C_{f,k,w}, \min\{C_1, C_1 + 1 - 1, C_1 - 2 - 2, C_1 - 4 - 3\} - 1\} \\ &= C_1 - 8. \\ \{\text{mult}_2(a_{m-k+j})\}_{j=1}^4 &\succeq (C_{f,k,w}, C_1 - 2, C_1 - 4, C_1 - 8). \end{aligned}$$

$$\text{mult}_2(a_m) \geq C_1 - 8 - (k - 4) - \frac{(k + 3)(k - 4)}{2} \geq 2,$$

a contradiction.

By our results, when $k \geq 3$, suppose that $p \neq 0$, then

$$z_2^{d_1} z_3^{C_1} \in \text{Supp}(a_{m-k+1}), \quad z_2^{d_2} z_3^{C_1-2} \in \text{Supp}(a_{m-k+2}), \quad z_2^{d_3} z_3^{C_1-5} \in \text{Supp}(a_{m-k+3}),$$

here $d_i \in \mathbb{N}$. Hence $z_1^{k-1} z_2^{d_1} z_3^{C_1}, z_1^{k-2} z_2^{d_2} z_3^{C_1-2}, z_1^{k-3} z_2^{d_3} z_3^{C_1-5} \in \text{Supp}(f)$. Comparing their weighted degree, we obtain

$$w_1 + 2w_3 = (d_2 - d_1)w_2, \quad w_1 + 3w_3 = (d_3 - d_2)w_2.$$

- If $d_2 - d_1 > 3$, then $w_1 \geq (d_2 - d_1 - 2)w_3 \geq 2w_3$.
- If $d_2 - d_1 < 3$, then $2w_3 \leq (d_2 - d_1 - 1)w_1 \leq 2w_1$.
- If $d_3 - d_2 > 4$, then $w_1 \geq (d_3 - d_2 - 3)w_3 \geq 2w_3$.
- If $d_3 - d_2 < 4$, then $3w_3 \leq (d_3 - d_1 - 1)w_1 \leq 2w_1$.

Combining with $3w_1 \leq 4w_3$, we have $d_2 - d_1 = 3, d_3 - d_2 = 4$. It implies that $w_2 = w_3$, which contradicts to the fact that $w_2 > w_3$.

When $k = 2, m = 4, C_1 = 3$. Suppose that $p \neq 0$, then $\text{mult}_2(a_3) = 3, \text{mult}_2(a_4) = 1$.

$$z_2^{d_1} z_3^3 \in \text{Supp}(a_3), \quad z_2^{d_2} z_3 \in \text{Supp}(a_4),$$

here $d_1, d_2 \in \mathbb{N}$. Hence $z_1^4, z_1 z_2^{d_1} z_3^3, z_2^{d_2} z_3 \in \text{Supp}(f)$. We have

$$3w_1 = d_1 w_2 + 3w_3, \quad w_1 + 2w_3 = (d_2 - d_1)w_2.$$

- If $d_2 - d_1 > 3$, then $w_1 = (d_2 - d_1 - 2)w_3 \geq 2w_3$.
- If $d_2 - d_1 < 3$, then $2w_3 \leq (d_2 - d_1 - 1)w_1 \leq w_1$.
- If $d_1 > 2$, then $3w_1 \geq 6w_3$.
- If $d_1 = 0$, then $w_1 = w_3$.

Combining with $w_1 < 2w_3$, we have $d_2 - d_1 = 3$, $d_1 = 1$ or 2 . But $\partial_{z_2}^2 a_3 = 0$, thus $d_1 = 1$. Then we obtain $\frac{w_1}{w_3} = \frac{11}{8}$, $\frac{w_2}{w_3} = \frac{9}{8}$.

- $z_3^{d_3} \notin \text{Supp}(a_3)$ for any $d_3 \in \mathbb{N}$. Otherwise, $z_1^4, z_1 z_3^{d_3} \in \text{Supp}(f)$, $\frac{w_3}{w_1} = \frac{d_3}{3} \neq \frac{11}{8}$.
- $z_3^{d_4} \notin \text{Supp}(a_4)$ for any $d_4 \in \mathbb{N}$. Otherwise, $z_1^4, z_3^{d_4} \in \text{Supp}(f)$, $\frac{w_1}{w_3} = \frac{d_4}{4} \neq \frac{11}{8}$.

Hence we have $\text{Supp}(a_3) = z_2 z_3^3$ and $\text{Supp}(a_4) = z_2^4 z_3$. So

$$f = z_1^4 + a_1(z_2, z_3)z_1^3 + a_2(z_2, z_3)z_1^2 + c_1 z_1 z_2 z_3^3 + c_2 z_2^4 z_3, \quad c_1, c_2 \in \mathbb{C} \setminus \{0\}.$$

It is easy to observe that $f, \partial_{z_1} f, \partial_{z_2} f, \partial_{z_3} f$ vanish on the whole z_3 -axis, which contradicts to the fact that V_f is an isolated singularity at the origin. Hence $p = 0$. $\square$

The four claims above settle (Case B) of Statement 1 in the setting of $n = 3, k \geq 2$ and

$$\text{mult}(f) \geq \frac{k^2 + 3k - 2}{2}.$$

Proposition 3.6. Fix a weight type $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{N}^n$ such that $w_1 \geq \dots \geq w_n \geq 1$. Let $f = \sum_{i=0}^m a_i(z_2, \dots, z_n)z_1^{m-i}$ with $a_0 \neq 0$ be a weighted homogeneous polynomial with respect to $\mathbf{w}$. Suppose that f is of the form (Case 1) or (Case 2).

When $n > 3$, fix an integer $2 \leq k < m - 1$. When $n = 3$, fix an integer $2 \leq k < m$. If a negative weight derivation

$$\tilde{D} = p(z_2, \dots, z_n)\partial_{z_1} + q(z_{j+1}, \dots, z_n)\partial_{z_j} \in \text{Der}(R_n)$$

with $j \geq 2$, $p \neq 0$ and $q \neq 0$ preserves $J_k(f)$, then there exists a $\mathbb{C}$ -algebra isomorphism $\varphi \in \text{Aut}_{\mathbb{C}}(R_n)$ satisfying:

- $\varphi(f)$ is a weighted homogeneous polynomial with respect to $\mathbf{w}$. And $\deg_{\mathbf{w}} \varphi(f) = \deg_{\mathbf{w}} f$, $\text{mult}(\varphi(f)) = \text{mult}(f)$.
- If f is of the form (Case 1) (resp. (Case 2)), then $\varphi(f)$ is still of the form (Case 1) (resp. (Case 2)).
- The negative weight derivation $\tilde{D}_0 = q(z_3, \dots, z_n)\partial_{z_2}$ preserves $J_k(\varphi(f))$.

Proof. For every $\mathbb{C}$ -algebra isomorphism $\varphi \in \text{Aut}_{\mathbb{C}}(R_n)$, $\tilde{D}^\varphi : a \mapsto \varphi(\tilde{D}(\varphi^{-1}(a)))$ is still a derivation on R_n . As

$$\begin{aligned} \tilde{D}^\varphi(ab) &= \varphi(\tilde{D}(\varphi^{-1}(ab))) = \varphi(\tilde{D}(\varphi^{-1}(a)\varphi^{-1}(b))) \\ &= \varphi(\tilde{D}(\varphi^{-1}(a))\varphi^{-1}(b) + \varphi^{-1}(a)\tilde{D}(\varphi^{-1}(b))) \\ &= \varphi(\tilde{D}(\varphi^{-1}(a)))\varphi(\varphi^{-1}(b)) + \varphi(\varphi^{-1}(a))\varphi(\tilde{D}(\varphi^{-1}(b))) \\ &= \tilde{D}^\varphi(a)b + a\tilde{D}^\varphi(b), \end{aligned}$$

and for any $c \in \mathbb{C}$, $\tilde{D}^\varphi(c) = \varphi(\tilde{D}(c)) = 0$.

Moreover, if $\varphi(f)$ is a weighted homogeneous polynomial, then by (5), $J_k(f) = T_k(\varphi(f))$, $J_k(\varphi(f)) = T_k(\varphi(f))$. From the proof of Theorem 2.3,

$$\begin{aligned} \tilde{D}^\varphi(J_k(\varphi(f))) &= \tilde{D}^\varphi(T_k(\varphi(f))) = \tilde{D}^\varphi(\varphi(T_k(f))) = \varphi(\tilde{D}(T_k(f))) \\ &= \varphi(\tilde{D}(J_k(f))) \subseteq \varphi(J_k(f)) = \varphi(T_k(f)) = T_k(\varphi(f)) = J_k(\varphi(f)). \end{aligned}$$

When $n \geq 3, 2 \leq k \leq m-1$, if f is of the form (Case 1), $a_0 = 1$. By (7) we have

$$mp + D(a_1) = mp + q\partial_{z_j}a_1 = 0.$$

Note that $\deg_{\mathbf{w}} p - w_1 = \deg_{\mathbf{w}} q - w_j = \deg_{\mathbf{w}} \tilde{D}$, then $\deg_{\mathbf{w}} a_1 = \deg_{\mathbf{w}} p - \deg_{\mathbf{w}} q + w_j = w_1$. Consider the $\mathbb{C}$ -algebra automorphism $\varphi \in \text{Aut}_{\mathbb{C}}(R_n)$ given by

$$\varphi : z_1 \mapsto z_1 - \frac{a_1(z_2, \dots, z_n)}{m}, \quad z_i \mapsto z_i, i \geq 2.$$

$\varphi(f)$ is a weighted homogeneous polynomial of the form (Case 1). It has the same weighted degree and multiplicity as f . $\tilde{D}^\varphi = \sum_{i=1}^n \varphi^{-1}(\tilde{D}(\varphi(z_i)))\partial_{z_i} = q\partial_{z_j}$ preserves $J_k(\varphi(f))$.

In (Case 2), $a_0 = z_{i_0}$, by (7), $D(a_0) = 0$ means that $i_0 \neq j$.

$$mpz_{i_0} + q\partial_{z_j}a_1 = 0.$$

We claim that z_{i_0} divides $\partial_{z_j}a_1$. If $\partial_{z_j}a_1 = z_{i_0}g$ with g , choose h such that $\partial_{z_j}h = g$, then

$$\deg_{\mathbf{w}} h = w_j + \deg_{\mathbf{w}} g = w_j + \deg_{\mathbf{w}} a_1 - w_j - w_{i_0} = w_1.$$

Here $g, h \in \mathbb{C}[z_2, \dots, z_n]$. Consider the $\mathbb{C}$ -algebra automorphism $\varphi \in \text{Aut}_{\mathbb{C}}(R_n)$ given by

$$\varphi : z_1 \mapsto z_1 - \frac{h(z_2, \dots, z_n)}{m}, \quad z_i \mapsto z_i, i \geq 2.$$

$\varphi(f)$ is a weighted homogeneous polynomial of the form (Case 2). $\varphi(f)$ and f have the same weighted degree and multiplicity. $mp = -q\partial_{z_j}a_1$. $\tilde{D}^\varphi = q\partial_{z_j}$ preserves $J_k(\varphi(f))$.

Now we prove that z_{i_0} divides $\partial_{z_j}a_1$. If not, then z_{i_0} divides q . Suppose that $q = z_{i_0}q_1$. Rewrite f and $\partial_{z_j}a_1$ in the form

$$f = \sum_{i=0}^{m_1} b_i z_{i_0}^{m_1-i}, \quad m_1 \geq 1, b_i \in \mathbb{C}[z_1, \dots, \widehat{z_{i_0}}, \dots, z_n], b_{m_1} \neq 0$$

$$\partial_{z_j}a_1 = \sum_{i=0}^{m_1} c_i z_{i_0}^{m_1-i}, \quad c_i \in \mathbb{C}[z_2, \dots, \widehat{z_{i_0}}, \dots, z_n].$$

Here $\widehat{z_{i_0}}$ means omitted. Hence $\tilde{D}(\partial_{z_1}^k f) = 0$ implies that

$$\sum_{i=0}^{m_1} \left(-\frac{1}{m} (\partial_{z_1}^{k+1} b_i) \left(\sum_{l=0}^{m_1} c_l z_{i_0}^{m_1-l} \right) + (\partial_{z_1}^k \partial_{z_j} b_i) z_{i_0} \right) \cdot z_{i_0}^{m_1-i} = 0.$$

Comparing the z_{i_0} -components on both sides, we obtain

$$(\partial_{z_1}^{k+1} b_{m_1}) c_{m_1} = 0,$$

$$(\partial_{z_1}^{k+1} b_{m_1-1}) c_{m_1} + (\partial_{z_1}^{k+1} b_{m_1}) c_{m_1-1} = m \partial_{z_1}^k \partial_{z_j} b_{m_1}.$$

z_{i_0} does not divide $\partial_{z_j}a_1$, so $c_{m_1} \neq 0$. Hence $\partial_{z_1}^{k+1} b_{m_1} = 0$, and $\partial_{z_1}^k b_{m_1} \in \mathbb{C}[z_2, \dots, z_n]$, so does $\partial_{z_1}^k \partial_{z_j} b_{m_1}$. f is of the form (Case 2), b_{m_1-1} contains z_1^m as the highest z_1 -order term. $(\partial_{z_1}^{k+1} b_{m_1-1}) c_{m_1}$ contains $c_{m_1} z_1^{m-k-1} \neq 0$ as the highest z_1 -order term, but

$$(\partial_{z_1}^{k+1} b_{m_1-1}) c_{m_1} = m \partial_{z_1}^k \partial_{z_j} b_{m_1} \in \mathbb{C}[z_2, \dots, z_n].$$

It is impossible when $k < m-1$.

If $n = 3, k = m-1$, then we have $j = 2, i_0 = 3$. $f = z_1^{k+1} z_3 + \sum_{i=1}^{k+1} a_i(z_2, z_3) z_1^{k+1-i}$. The negative weight derivation $\tilde{D} = p(z_2, z_3) \partial_{z_1} + q(z_3) \partial_{z_2}$ preserves $J_k(f)$. $p, q \neq 0$. We still have $(k+1)pz_3 + q\partial_{z_2}a_1 = 0$. If z_3 does not divide $\partial_{z_2}a_1$, suppose that $q = z_3q_1$ and

$$a_1 = cz_2^d + h_1(z_2, z_3)z_3, \quad d \geq 1, c \in \mathbb{C} \setminus \{0\}, h_1 \in \mathbb{C}[z_2, z_3].$$

Actually $d \geq 2$. Otherwise, $\deg_{\mathbf{w}} z_1^k z_2 = \deg_{\mathbf{w}} z_1^{k+1} z_3$, which contradicts $w_1 \geq w_2 \geq w_3$. By (6),

$$\tilde{D}(\partial_{z_1}^{k-1} \partial_{z_2} f) = (k-1)!((kq\partial_{z_2}^2 a_1)z_1 + kp\partial_{z_2} a_1 + q\partial_{z_2}^2 a_2) \in \langle \partial_{z_1}^k f \rangle.$$

As $\partial_{z_1}^k f = (k+1)!z_1z_3 + k!a_1$, there is some $h_2 \in \mathbb{C}[z_2, z_3]$ such that $\tilde{D}(\partial_{z_1}^{k-1}\partial_{z_2}f) = h_2\partial_{z_1}^k f$. Then we obtain

$$(k+1)h_2z_3 = q\partial_{z_2}^2 a_1, \quad kh_2a_1 = kp\partial_{z_2}a_1 + q\partial_{z_2}^2 a_2.$$

$q = z_3q_1$, $q_1 \in \mathbb{C}[z_3]$, then $h_2 = \frac{q_1\partial_{z_2}^2 a_1}{k+1}$, $p = \frac{-q_1\partial_{z_2}a_1}{k+1}$. We get

$$ka_1\partial_{z_2}^2 a_1 = -k(\partial_{z_2}a_1)^2 + (k+1)z_3\partial_{z_2}^2 a_2.$$

Comparing the z_1^{2d-2} -components on both sides, we have $kc^2d(d-1) = -\frac{kc^2d^2}{3}$. But this is impossible, as $d \geq 2$. Hence z_3 divides $\partial_{z_2}a_1$. $\square$

Proposition 3.6 settles (Case C) of Statement 1 in the setting of $n = 3, k \geq 2$ and

$$\text{mult}(f) \geq \frac{k^2 + 3k - 2}{2}.$$

For $f = \sum_{i=0}^m a_i z_1^{m-i}$, we have $m \geq \text{mult}(f) - 1 \geq k + 1$. If a negative weight derivation

$$\tilde{D} = p(z_2, z_3)\partial_{z_1} + q(z_3)\partial_{z_2} \in \text{Der}(R_n)$$

with $p \neq 0$ and $q \neq 0$ preserves $J_k(f)$, then there exists a weighted homogeneous polynomial $f_1 \in R_3$ which has the same form, weighted degree, multiplicity as f such that:

- f_1 also defines an isolated singularity at the origin of $\mathbb{C}^3$.
- The non-zero negative weight derivation $\tilde{D}_0 = q(z_3)\partial_{z_2}$ on R_3 preserves $J_k(f_1)$.

From Claim 1 to Claim 4 we conclude that $q = 0$, which leads to a contradiction with our assumption. Hence (Case C) of Statement 1 cannot occur under the above conditions.

Remark 4. The condition $\text{mult}(f) \geq \frac{k^2+3k-2}{2}$ is necessary. Let $f = z_1^3 + z_1z_2z_3^2 + z_2^3z_3 + z_3^5$ be the weighted homogeneous polynomial in $\mathbb{C}[z_1, z_2, z_3]$ with respect to the weight type $\mathbf{w} = (5, 4, 3)$.

$$\begin{aligned} \partial_{z_1}^2 f &= 6z_1, & \partial_{z_1}\partial_{z_2}f &= z_3^2, & \partial_{z_1}\partial_{z_3}f &= 2z_2z_3. \\ \partial_{z_2}^2 f &= 6z_2z_3, & \partial_{z_2}\partial_{z_3}f &= 2z_1z_3 + 3z_2^2, & \partial_{z_3}^2 f &= 2z_1z_2 + 20z_3^3. \end{aligned}$$

It is easy to check that $\tilde{D} = z_3\partial_{z_2}$ induces a negative weight derivation of $A_2(V_f)$.

REFERENCES

- [1] A. G. Aleksandrov, *Cohomology of a quasihomogeneous complete intersection*, Izv. Akad. Nauk SSSR Ser. Mat. **49** (1985), no. 3, 467–510, 672 (Russian). MR0794953
- [2] Igor Belegarde and Vitali Kapovitch, *Obstructions to nonnegative curvature and rational homotopy theory*, J. Amer. Math. Soc. **16** (2003), no. 2, 259–284, DOI 10.1090/S0894-0347-02-00418-6. MR1949160
- [3] Bingyi Chen, Hao Chen, Stephen S.-T. Yau, and Huaiqing Zuo, *The nonexistence of negative weight derivations on positive dimensional isolated singularities: generalized Wahl conjecture*, J. Differential Geom. **115** (2020), no. 2, 195–224, DOI 10.4310/jdg/1589853625. MR4100703
- [4] Bingyi Chen, Naveed Hussain, Stephen S.-T. Yau, and Huaiqing Zuo, *Variation of complex structures and variation of Lie algebras II: New Lie algebras arising from singularities*, J. Differential Geom. **115** (2020), no. 3, 437–473, DOI 10.4310/jdg/1594260016. MR4120816
- [5] Bingyi Chen and Stephen Shing-Toung Yau, *On nonexistence of negative weight derivations on moduli algebras: Yau’s conjecture*, Mat. Sb. **217** (2026), no. 1, 114–138, DOI 10.4213/sm10201 (Russian, with Russian summary). MR5045737
- [6] Hao Chen, *On negative weight derivations of the moduli algebras of weighted homogeneous hypersurface singularities*, Math. Ann. **303** (1995), no. 1, 95–107, DOI 10.1007/BF01460981. MR1348357
- [7] Hao Chen, *Nonexistence of negative weight derivations on graded Artin algebras: a conjecture of Halperin*, J. Algebra **216** (1999), no. 1, 1–12, DOI 10.1006/jabr.1998.7750. MR1694610
- [8] Hao Chen, Yi-Jing Xu, and Stephen S.-T. Yau, *Nonexistence of negative weight derivation of moduli algebras of weighted homogeneous singularities*, J. Algebra **172** (1995), no. 2, 243–254, DOI 10.1016/S0021-8693(05)80001-8. MR1322403

- [9] Hao Chen, Stephen S.-T. Yau, and Huaiqing Zuo, *Non-existence of negative weight derivations on positively graded Artinian algebras*, Trans. Amer. Math. Soc. **372** (2019), no. 4, 2493–2535, DOI 10.1090/tran/7628. MR3988584
- [10] Yves Félix, Stephen Halperin, and Jean-Claude Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001. MR1802847
- [11] Lee Kennard and Yantao Wu, *Halperin’s conjecture in formal dimensions up to 20*, Comm. Algebra **51** (2023), no. 8, 3606–3622, DOI 10.1080/00927872.2023.2186705. MR4585901
- [12] Gregory Lupton, *Note on a conjecture of Stephen Halperin’s*, Topology and combinatorial group theory (Hanover, NH, 1986/1987; Enfield, NH, 1988), Lecture Notes in Math., vol. 1440, Springer, Berlin, 1990, pp. 148–163, DOI 10.1007/BFb0084459. MR1082989
- [13] W. Meier, *Rational universal fibrations and flag manifolds*, Math. Ann. **258** (1981/82), no. 3, 329–340, DOI 10.1007/BF01450686. MR0649203
- [14] John N. Mather and Stephen S. T. Yau, *Classification of isolated hypersurface singularities by their moduli algebras*, Invent. Math. **69** (1982), no. 2, 243–251, DOI 10.1007/BF01399504. MR0674404
- [15] Naveed Hussain, Guorui Ma, Stephen S.-T. Yau, and Huaiqing Zuo, *Higher Nash blow-up local algebras of singularities and its derivation Lie algebras*, J. Algebra **618** (2023), 165–194, DOI 10.1016/j.jalgebra.2022.11.016. MR4527059
- [16] Naveed Hussain, Stephen S.-T. Yau, and Huaiqing Zuo, *On the new k -th Yau algebras of isolated hypersurface singularities*, Math. Z. **294** (2020), no. 1-2, 331–358, DOI 10.1007/s00209-019-02269-x. MR4050070
- [17] ———, *Inequality conjectures on derivations of local k -th Hessain algebras associated to isolated hypersurface singularities*, Math. Z. **298** (2021), no. 3-4, 1813–1829, DOI 10.1007/s00209-020-02688-1. MR4282149
- [18] ———, *New k -th Yau algebras of isolated hypersurface singularities and weak Torelli-type theorem*, Math. Res. Lett. **29** (2022), no. 2, 455–486, DOI 10.4310/mrl.2022.v29.n2.a7. MR4492225
- [19] Craig Seeley and Stephen S.-T. Yau, *Variation of complex structures and variation of Lie algebras*, Invent. Math. **99** (1990), no. 3, 545–565, DOI 10.1007/BF01234430. MR1032879
- [20] Jean-Claude Thomas, *Rational homotopy of Serre fibrations*, Ann. Inst. Fourier (Grenoble) **31** (1981), no. 3, v, 71–90 (English, with French summary). MR0638617
- [21] Jonathan M. Wahl, *Derivations, automorphisms and deformations of quasihomogeneous singularities*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 613–624, DOI 10.1090/pspum/040.2/713285. MR0713285
- [22] Stephen S.-T. Yau, *Solvable Lie algebras and generalized Cartan matrices arising from isolated singularities*, Math. Z. **191** (1986), no. 4, 489–506, DOI 10.1007/BF01162338. MR0832806
- [23] Stephen S.-T. Yau, *Solvability of Lie algebras arising from isolated singularities and nonisolatedness of singularities defined by $\mathfrak{sl}(2, \mathbb{C})$ invariant polynomials*, Amer. J. Math. **113** (1991), no. 5, 773–778, DOI 10.2307/2374785. MR1129292

SHANGHAI INSTITUTE FOR MATHEMATICS AND INTERDISCIPLINARY SCIENCES (SIMIS), SHANGHAI 200433, CHINA

RESEARCH INSTITUTE OF INTELLIGENT COMPLEX SYSTEMS, FUDAN UNIVERSITY, SHANGHAI 200433, CHINA

Email address: `maguorui@simis.cn`

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING, 100084, P. R. CHINA.

Email address: `taosy22@mails.tsinghua.edu.cn`

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING, 100084, P. R. CHINA.

Email address: `yau@uic.edu`

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING, 100084, P. R. CHINA.

Email address: `hqzuo@mail.tsinghua.edu.cn`