

Yau algebras arising from two-dimensional rational hypersurface singularities

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Abstract. Let $\mathcal{O}_n = \mathbb{C}\{x_1, \dots, x_n\}$ denote the ring of convergent power series at the origin in $\mathbb{C}^n$. Let $(V, 0)$ be an isolated hypersurface singularity defined by $f \in \mathcal{O}_n$. The celebrated Mather-Yau theorem states that the complex structure of $(V, 0)$ determines and is determined by its moduli algebra $A(V)$. Later, Yau introduced the Lie algebra of derivations of $A(V)$, denoted as $L(V)$ and he proved that $L(V)$ is solvable. This Lie algebra is usually referred to as the Yau algebra.

This paper investigates Yau algebras arising from two-dimensional rational hypersurface singularities. We have determined the cohomology structure (including H^2 and H^3) of these Yau algebras. In particular, from the point of view of deformation of Lie algebra, we have identified all the rigid Yau algebras. *Mathematics Subject Classification 2010:* 14B05, 32S25.

Key Words and Phrases: rational hypersurface singularities, Yau algebras, rigid..

1. Introduction

Let $(V, 0)$ be an isolated hypersurface singularity in $(\mathbb{C}^n, 0)$ defined by the zero of a holomorphic function f . The moduli algebra $A(V)$ of V is defined as

$$A(V) := \mathcal{O}_n / (f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}),$$

where $\mathcal{O}_n = \mathbb{C}\{x_1, \dots, x_n\}$ denotes the ring of convergent power series at the origin in $\mathbb{C}^n$. Mather and Yau proved that the complex structure of $(V, 0)$ determines and is determined by its moduli algebra in [10]. Following this, Yau [15] introduced the Lie algebra $L(V)$ associated with $(V, 0)$, defined as the Lie algebra of derivations of $A(V)$, i.e., $L(V) = \text{Der}(A(V), A(V))$ (see Definition 2.6). He proved that $L(V)$ is solvable in [16]. In [17], $L(V) := \text{Der}_{\mathbb{C}}(A(V), A(V))$ is referred to as the Yau algebra of V . The dimension of $L(V)$, denoted by $\lambda(V)$, is termed the Yau number by Elashvili and Khimshiashvili in [5].

Yau and his collaborators have systematically studied the Lie algebras of isolated hypersurface singularities since 1980s ([2]-[4], [6]-[8], [13]-[16], [18], [19]). Among them, [2] introduces the Yau algebras arising from two-dimensional rational hypersurface singularities. In this article, we will investigate the cohomology structure of these Yau algebras and examine its rigidity properties.

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First of all, we present a new method to calculate cohomology of finite-dimensional Lie algebras. By using computer computations, we obtain the cohomology structure of the Yau algebras in small dimensions.

Theorem 1.1. *For the rational hypersurface singularity with small dimensions, we can compute the results of cohomology of Yau algebra as shown in the table below.*

Table 1:

$\searrow$	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
$\dim A(V)$	1	2	3	4	5	6	7	8	9
$\lambda(V)$	0	1	2	3	4	5	6	7	8
$\dim H^2(L, L)$	0	0	0	1	1	1	0	0	1
$\dim H^3(L, L)$	0	0	0	0	0	0	0	0	1
$\searrow$	A_{10}	A_{11}	A_{12}	A_{13}	D_4	D_5	D_6	D_7	D_8
$\dim A(V)$	10	11	12	13	4	5	6	7	8
$\lambda(V)$	9	10	11	12	4	5	6	7	8
$\dim H^2(L, L)$	1	1	1	1	0	1	2	1	1
$\dim H^3(L, L)$	1	1	1	1	0	0	0	0	1
$\searrow$	D_9	D_{10}	D_{11}	D_{12}	D_{13}	D_{14}	E_6	E_7	E_8
$\dim A(V)$	9	10	11	12	13	14	6	7	8
$\lambda(V)$	9	10	11	12	13	14	7	8	10
$\dim H^2(L, L)$	0	3	1	3	1	2	4	0	1
$\dim H^3(L, L)$	0	3	1	3	1	2	2	0	2

For rational hypersurface singularity A_{k+1} and D_k , we demonstrate that for sufficiently large k , the dimensions of both $B^2(L, L)$ and $Z^2(L, L)$ are $(k-1)k$. Thus, the dimension of $H^2(L, L)$ is 0. The crucial point is that in the proof process, we are able to determine a set of bases for $Z^2(L, L)$.

Theorem 1.2. *Let $V = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2 - z^{d+2} = 0\}$ be the A_{d+1} singularity, where $d \geq 13$, and $L(V) = \langle e_1, e_2, \dots, e_d \rangle$ be the Yau algebra of V . Then we have that $\dim Z^2(L, L) = (d-1)d$, $\dim B^2(L, L) = (d-1)d$, and $\dim H^2(L, L) = \dim Z^2(L, L) - \dim B^2(L, L) = 0$.*

Theorem 1.3. *Let $V = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2z + z^{d+1} = 0\}$ be the D_{d+2} singularity, where $d \geq 13$, and $L(V) = \langle e_1, e_2, \dots, e_{d+2} \rangle$ be the Yau algebra of V . Then we have that $\dim Z^2(L, L) = (d+1)(d+2)$, $\dim B^2(L, L) = (d+1)(d+2)$, and $\dim H^2(L, L) = \dim Z^2(L, L) - \dim B^2(L, L) = 0$.*

Let V be a finite-dimensional real vector space and let $\mathfrak{M}$ denote the real algebraic set of all Lie algebra multiplications on V . We consider $\mathfrak{M}$ as a topological space with the topology induced as a subset of $C^2(L, L)$. ($C^2(L, L)$ is given the usual (Hausdorff) topology of a finite-dimensional real vector space.) A Lie algebra $L = (V, \eta)$ is said to be rigid if its orbit $G(\eta)$ is an open subset of $\mathfrak{M}$, where $G(\eta) = \{g\eta(g^{-1}x, g^{-1}y) | g \in GL(V)\}$. Nijenhuis and Richardson established a connection between the rigidity of Lie algebras and their cohomology

structure in [11, 12]. Leveraging this relationship, we have verified that most Yau algebras arising from two-dimensional rational hypersurface singularities are indeed rigid. For the remaining cases, we have also verified their non-rigidity through other approaches.

Theorem 1.4. *For rational hypersurface singularities, the Yau algebra $L(V)$ is not rigid for the following cases: $A_4, A_5, A_6, A_9, A_{10}, A_{11}, A_{12}, A_{13}, D_5, D_6, D_7, D_8, D_{10}, D_{11}, D_{12}, D_{13}, D_{14}, E_6$, and E_8 . Conversely, $L(V)$ is rigid for all other cases.*

2. Preliminaries

2.1. Rational hypersurface singularity

Definition 2.1. A singularity $(V, 0)$ of dimension d is defined by:

$$(V, 0) = (\{f_1 = \dots = f_m = 0\}, 0) \subset (\mathbf{C}^N, 0),$$

where $f_i : (\mathbf{C}^N, 0) \rightarrow (\mathbf{C}, 0)$ are germs of analytic functions with

$$r(p) = \text{rank} \left[\frac{\partial f_i}{\partial z_j}(p) \right]_{i=1, \dots, m; j=1, \dots, N} \leq N - d$$

for any generic (or smooth) point p of V . If $r(0) = N - d$, then $(V, 0)$ is a smooth germ, which means it is analytically isomorphic to $(\mathbf{C}^d, 0)$. If $r(0) < N - d$ but $r(p) = N - d$ for all $p \in V - \{0\}$, we say that $(V, 0)$ has an isolated singularity at the origin. If $d = 2$, we say that $(V, 0)$ has a surface singularity. If $m = 1$, we say that $(V, 0)$ has a hypersurface singularity.

Definition 2.2. Let $(V, 0)$ be an isolated surface singularity at the origin. Let $\pi : X \rightarrow V$ be a resolution of singularity. We say that $(V, 0)$ is a rational singularity if $R^1\pi_*\mathcal{O}_X = 0$.

Theorem 2.3 ([1]). *If $(V, 0)$ is a rational hypersurface singularity, then $(V, 0)$ is a rational double point. Moreover these singularities in $\mathbf{C}^3$ are defined by one of following polynomial equations:*

$$(1) \ A_k : x^2 + y^2 - z^{k+1}, \text{ for } k \geq 1,$$

$$(2) \ D_k : x^2 + y^2z + z^{k-1}, \text{ for } k \geq 4,$$

$$(3) \ E_6 : x^2 + y^3 + z^4,$$

$$(4) \ E_7 : x^2 + y^3 + yz^3,$$

$$(5) \ E_8 : x^2 + y^3 + z^5.$$

2.2. Yau algebra

Definition 2.4. Let $\mathcal{O}_n = \mathbb{C}\{x\}$ denote the ring of convergent power series at the origin in $\mathbb{C}^n$. Let $(V, 0) = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid f(x_1, \dots, x_n) = 0\}$ be an isolated hypersurface singularity, where $f \in \mathcal{O}_n$. The moduli algebra of the singularity $(V, 0)$ is defined as

$$A(V) := \mathcal{O}_n / (f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}).$$

Definition 2.5. Let A be a commutative algebra over the complex field $\mathbb{C}$. We define a linear endomorphism D of A as a colorblackderivation of A if it satisfies the Leibniz rule

$$D(ab) = D(a)b + aD(b) \quad \forall a, b \in A.$$

Definition 2.6. Let $(V, 0)$ be an isolated hypersurface singularity and $A(V)$ be the moduli algebra. The set of all colorblackderivations of $A(V)$, denoted by $Der_{\mathbb{C}}(A(V), A(V))$, forms a complex Lie algebra, where the Lie bracket is defined as

$$[X, Y] := X \circ Y - Y \circ X \quad \forall X, Y \in Der_{\mathbb{C}}(A(V), A(V)).$$

In [17], $L(V) := Der_{\mathbb{C}}(A(V), A(V))$ is referred as the Yau algebra of V . The dimension of $L(V)$, denoted by $\lambda(V)$, is termed the Yau number by Elashvili and Khimshiashvili in [5].

2.3. Cohomology of Lie algebras For a general theory of the cohomology of Lie algebra, the interested readers can refer to [9].

Let L be a Lie algebra. We define the space of the n -linear alternating mappings from L^n to L as $C^n(L, L)$, and $C^*(L, L)$ denotes the direct sum $\bigoplus_{n \geq 0} C^n(L, L)$. The coboundary operator

$$\begin{aligned} \delta : C^*(L, L) &\rightarrow C^*(L, L) \\ \Phi &\mapsto \delta\Phi \end{aligned}$$

is defined as follows:

$$\delta\Phi(x_1, \dots, x_{n+1}) = \begin{cases} x_1\Phi & \text{if } n = 0, \\ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \Phi([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}) \\ + \sum_{1 \leq i \leq n+1} (-1)^{i+1} [x_i, \Phi(x_1, \dots, \hat{x}_i, \dots, x_{n+1})] & \text{if } n \geq 1, \end{cases}$$

where the sign $\hat{}$ indicates that the argument below it has been omitted.

It can be shown that $\delta(C^n(L, L)) \subset C^{n+1}(L, L)$ and $\delta \circ \delta = 0$, thus $(C^*(L, L), \delta)$ is a cochain complex. For $n \geq 1$, let $Z^n(L, L) = \text{Ker} \delta|_{C^n(L, L)}$ and $B^n(L, L) = \text{Im} \delta|_{C^n(L, L)}$. The quotient space $H^n(L, L) = Z^n(L, L) / B^n(L, L)$ is called the cohomology space of L of degree n . For $n = 0$, we put $B^0(L, L) = 0$ and $H^0(L, L) = Z^0(L, L)$.

3. Computation of the cohomology of Yau algebras

3.1. Yau algebras of rational surface singularities First, we need to calculate the moduli algebra and Yau algebra of a rational hypersurface singularity.

These results have already been provided by M. Benson and S. S.-T. Yau in [2], so we will directly refer to these results.

Proposition 3.1 ([2]). *Let $V = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2 - z^{k+1} = 0\}$ be the A_k singularity, $k \geq 1$. Then*

(1) $A(V) = \langle 1, z, z^2, \dots, z^{k-1} \rangle$ with multiplication rule $z^k = 0$.

(2) $L(V) = \begin{cases} \langle e_1 = z\partial_z, e_2 = z^2\partial_z, \dots, e_{k-1} = z^{k-1}\partial_z \rangle & \text{if } k \geq 2, \\ 0 & \text{if } k = 1, \end{cases}$

with the multiplication rule by the following table when $k \geq 2$.

Table 2:

$\searrow$	e_1	$\dots$	e_j	$\dots$	e_{k-1}
e_1	0	$\dots$	$(j-1)e_j$	$\dots$	$(k-2)e_{k-1}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
e_i	$(1-i)e_i$	$\dots$	$\begin{cases} (j-i)e_{i+j-1} & \text{if } i+j \leq k \\ 0 & \text{if } i+j \geq k+1 \end{cases}$	$\dots$	0
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
e_{k-1}	$(2-k)e_{k-1}$	$\dots$	0	$\dots$	0

Proposition 3.2 ([2]). *Let $V = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2z + z^{k-1} = 0\}$ be the D_k singularity, $k \geq 4$. Then*

(1) $A(V) = \langle 1, y, z, z^2, \dots, z^{k-2} \rangle$ with multiplication rule $y^2 = (1-k)z^{k-2}$, $yz = 0$, $z^{k-1} = 0$.

(2) $L(V) = \langle e_1 = \frac{k-2}{2}y\partial_y + z\partial_z, e_2 = z^2\partial_z, \dots, e_{k-2} = z^{k-2}\partial_z, e_{k-1} = y\partial_z + (k-1)z^{k-3}\partial_y, e_k = z^{k-2}\partial_y \rangle$ with the multiplication rule by the following table.

Table 3:

$\searrow$	$e_j (1 \leq j \leq k-2)$	e_{k-1}	e_k
e_i	$\begin{cases} (j-i)e_{i+j-1} & \text{if } i+j < k \\ 0 & \text{if } i+j \geq k \end{cases}$	$\begin{cases} \frac{k-4}{2}e_{k-1} & \text{if } i = 1 \\ (k-1)(k-3)e_k & \text{if } i = 2 \\ 0 & \text{else} \end{cases}$	$\begin{cases} \frac{k-2}{2}e_k & \text{if } i = 1 \\ 0 & \text{else} \end{cases}$
e_{k-1}	$\begin{cases} \frac{4-k}{2}e_{k-1} & \text{if } j = 1 \\ (1-k)(k-3)e_k & \text{if } j = 2 \\ 0 & \text{else} \end{cases}$	0	$-e_{k-2}$
e_k	$\begin{cases} \frac{2-k}{2}e_k & \text{if } j = 1 \\ 0 & \text{else} \end{cases}$	e_{k-2}	0

Proposition 3.3 ([2]). *Let $V = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^3 + z^4 = 0\}$ be the E_6 singularity, $k \geq 1$. Then*

(1) $A(V) = \langle 1, y, z, yz, z^2, yz^2 \rangle$ with multiplication rule $y^2 = 0$, $z^3 = 0$.

(2) $L(V) = \langle e_1 = y\partial_y, e_2 = z\partial_z, e_3 = yz\partial_y, e_4 = yz\partial_z, e_5 = z^2\partial_z, e_6 = yz^2\partial_y, e_7 = yz^2\partial_z \rangle$ with the multiplication rule by the following table.

Table 4:

$\searrow$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	0	0	e_4	0	0	e_7
e_2	0	0	e_3	0	e_5	$2e_6$	e_7
e_3	0	$-e_3$	0	e_7	$-e_6$	0	0
e_4	$-e_4$	0	$-e_7$	0	e_7	0	0
e_5	0	$-e_5$	e_6	$-e_7$	0	0	0
e_6	0	$-2e_6$	0	0	0	0	0
e_7	$-e_7$	$-e_7$	0	0	0	0	0

Proposition 3.4 ([2]). Let $V = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^3 + yz^3 = 0\}$ be the E_7 singularity, $k \geq 1$. Then

(1) $A(V) = \langle 1, z, y, z^2, yz, z^3, z^4 \rangle$ with multiplication rule $y^3 = 0$, $y^2 = -\frac{1}{3}z^3$, $yz^2 = 0$, $z^5 = 0$.

(2) $L(V) = \langle e_1 = 3y\partial_y + 2z\partial_z, e_2 = y^2\partial_y, e_3 = y^2\partial_z, e_4 = 3yz\partial_y + 2z^2\partial_z, e_5 = 3y\partial_z + 2z^2\partial_y, e_6 = yz\partial_z, e_7 = y^2z\partial_y, e_8 = y^2z\partial_z \rangle$ with the multiplication rule by the following table.

Table 5:

$\searrow$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	0	$3e_2$	$4e_3$	$2e_4$	e_5	$3e_6$	$5e_7$	$6e_8$
e_2	$-3e_2$	0	0	$-3e_7$	$3e_3$	e_8	0	0
e_3	$-4e_3$	0	0	$-2e_8$	$4e_7$	0	0	0
e_4	$-2e_4$	$3e_7$	$2e_8$	0	$-15e_2 - 3e_6$	$-3e_7$	0	0
e_5	$-e_5$	$-3e_3$	$-4e_7$	$15e_2 + 3e_6$	0	$-3e_3$	$-3e_8$	0
e_6	$-3e_6$	$-e_8$	0	$3e_7$	$3e_3$	0	0	0
e_7	$-5e_7$	0	0	0	$3e_8$	0	0	0
e_8	$-6e_8$	0	0	0	0	0	0	0

Proposition 3.5 ([2]). Let $V = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^3 + z^5 = 0\}$ be the E_8 singularity, $k \geq 1$. Then

(1) $A(V) = \langle 1, y, z, yz, z^2, yz^2, z^3, yz^3 \rangle$ with multiplication rule $y^2 = 0$, $z^4 = 0$.

(2) $L(V) = \langle e_1 = y\partial_y, e_2 = z\partial_z, e_3 = yz\partial_y, e_4 = yz\partial_z, e_5 = z^2\partial_z, e_6 = yz^2\partial_y, e_7 = yz^2\partial_z, e_8 = z^3\partial_z, e_9 = yz^3\partial_y, e_{10} = yz^3\partial_z \rangle$ with the multiplication rule by the following table.

Table 6:

$\setminus$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
e_1	0	0	0	e_4	0	0	e_7	0	0	e_{10}
e_2	0	0	e_3	0	e_5	$2e_6$	e_7	$2e_8$	$3e_9$	$2e_{10}$
e_3	0	$-e_3$	0	e_7	$-e_6$	0	e_{10}	$-e_9$	0	0
e_4	$-e_4$	0	$-e_7$	0	e_7	$-e_{10}$	0	$2e_{10}$	0	0
e_5	0	$-e_5$	e_6	$-e_7$	0	$2e_9$	0	0	0	0
e_6	0	$-2e_6$	0	e_{10}	$-2e_9$	0	0	0	0	0
e_7	$-e_7$	$-e_7$	$-e_{10}$	0	0	0	0	0	0	0
e_8	0	$-2e_8$	e_9	$-2e_{10}$	0	0	0	0	0	0
e_9	0	$-3e_9$	0	0	0	0	0	0	0	0
e_{10}	$-e_{10}$	$-2e_{10}$	0	0	0	0	0	0	0	0

3.2. Computation of cohomology for Yau algebras with small dimensions

For Yau algebras in small dimensions, we can calculate their cohomology using the methods mentioned in the following theorem.

Theorem 3.6 (i.e. Theorem 1.1). *For the rational hypersurface singularity with small dimensions, the cohomology of Yau algebra are shown in the table below.*

Table 7:

$\setminus$	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
$\dim A(V)$	1	2	3	4	5	6	7	8	9
$\lambda(V)$	0	1	2	3	4	5	6	7	8
$\dim H^2(L, L)$	0	0	0	1	1	1	0	0	1
$\dim H^3(L, L)$	0	0	0	0	0	0	0	0	1
$\setminus$	A_{10}	A_{11}	A_{12}	A_{13}	D_4	D_5	D_6	D_7	D_8
$\dim A(V)$	10	11	12	13	4	5	6	7	8
$\lambda(V)$	9	10	11	12	4	5	6	7	8
$\dim H^2(L, L)$	1	1	1	1	0	1	2	1	1
$\dim H^3(L, L)$	1	1	1	1	0	0	0	0	1
$\setminus$	D_9	D_{10}	D_{11}	D_{12}	D_{13}	D_{14}	E_6	E_7	E_8
$\dim A(V)$	9	10	11	12	13	14	6	7	8
$\lambda(V)$	9	10	11	12	13	14	7	8	10
$\dim H^2(L, L)$	0	3	1	3	1	2	4	0	1
$\dim H^3(L, L)$	0	3	1	3	1	2	2	0	2

Proof. Assuming $L(V) = \langle e_1, e_2, e_3, \dots, e_d \rangle$ represents the Yau algebra of a rational hypersurface singularity V with the multiplication rule $[e_i, e_j] = \sum_{k=1}^d x_{ij}^k e_k$ for $i, j = 1, \dots, d$.

We first consider the following mapping

$$\delta|_{C^1(L, L)} : C^1(L, L) \rightarrow C^2(L, L).$$

$$\alpha \mapsto \delta\alpha$$

Let $\alpha : V \rightarrow V$ be a linear map in $C^1(L, L)$ with $\alpha(e_i) = \sum_{k=1}^d a_i^k e_k$ for

$i = 1, \dots, d$. Then we have

$$\begin{aligned}
\delta\alpha(e_i, e_j) &= [e_i, \alpha(e_j)] - [e_j, \alpha(e_i)] - \alpha([e_i, e_j]) \\
&= [e_i, \sum_{s=1}^d a_j^s e_s] - [e_j, \sum_{s=1}^d a_i^s e_s] - \alpha(\sum_{s=1}^d x_{ij}^s e_s) \\
&= \sum_{s=1}^d \sum_{t=1}^d a_j^s x_{is}^t e_t - \sum_{s=1}^d \sum_{t=1}^d a_i^s x_{js}^t e_t - \sum_{s=1}^d \sum_{t=1}^d x_{ij}^s a_s^t e_t \\
&= \sum_{k=1}^d (\sum_{m=1}^d \sum_{n=1}^d (\delta_j^m x_{in}^k - \delta_i^m x_{jn}^k - \delta_n^k x_{ij}^m) a_m^n) e_k,
\end{aligned}$$

where δ_i^j represents the Kronecker delta function, defined as $\delta_i^j = 1$ when $i = j$ and $\delta_i^j = 0$ when $i \neq j$.

Note that $C^1(L, L)$ is a d^2 -dimensional vector space with a basis α_k^i for $i, k = 1, \dots, d$, where α_k^i is defined as $\alpha_k^i(e_i) = e_k$ and $\alpha_k^i(e_j) = 0$ when $j \neq i$. Then we have that $\alpha = \sum_{i=1}^d \sum_{k=1}^d a_i^k \alpha_k^i$. Since the map of $C^2(L, L)$ is alternating, we conclude that $C^2(L, L)$ is a $\frac{(d-1)d^2}{2}$ -dimensional vector space with a basis $\varphi_k^{ij} - \varphi_k^{ji}$ for $i, j, k = 1, \dots, d$ and $i < j$, where φ_k^{ij} is defined as follows: $\varphi_k^{ij}(e_i, e_j) = e_k$ and $\varphi_k^{ij}(e_m, e_n) = 0$ otherwise. Let $\delta\alpha = \sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^d b_{ij}^k \varphi_k^{ij}$. Then we have that $b_{ij}^k = \sum_{m=1}^d \sum_{n=1}^d (\delta_j^m x_{in}^k - \delta_i^m x_{jn}^k - \delta_n^k x_{ij}^m) a_m^n$.

Considering the $\frac{(d-1)d^2}{2} \times d^2$ matrix $D_1 = (\delta_j^m x_{in}^k - \delta_i^m x_{jn}^k - \delta_n^k x_{ij}^m)_{(i,j,k);(m,n)}$, it represents the matrix for $\delta|_{C^1(L,L)}$ where the domain $C^1(L, L)$ is under basis α_k^i , and the codomain $C^2(L, L)$ is under basis φ_k^{ij} . We can rely on computer assistance to calculate the rank of D_1 , which is equivalent to the dimension of $B^2(L, L)$.

Next, we consider the following mapping

$$\begin{aligned}
\delta|_{C^2(L,L)} : C^2(L, L) &\rightarrow C^3(L, L). \\
\varphi &\mapsto \delta\varphi
\end{aligned}$$

Let $\varphi \in C^2(L, L)$ with $\varphi(e_i, e_j) = \sum_{k=1}^d b_{ij}^k e_k$ for $i, j = 1, \dots, d$. Then we have

$$\begin{aligned}
\delta\varphi(e_i, e_j, e_k) &= [e_i, \varphi(e_j, e_k)] - [e_j, \varphi(e_i, e_k)] + [e_k, \varphi(e_i, e_j)] \\
&\quad - \varphi([e_i, e_j], e_k) + \varphi([e_i, e_k], e_j) - \varphi([e_j, e_k], e_i) \\
&= [e_i, \sum_{s=1}^d b_{jk}^s e_s] - [e_j, \sum_{s=1}^d b_{ik}^s e_s] + [e_k, \sum_{s=1}^d b_{ij}^s e_s] \\
&\quad - \varphi(\sum_{s=1}^d x_{ij}^s e_s, e_k) + \varphi(\sum_{s=1}^d x_{ik}^s e_s, e_j) - \varphi(\sum_{s=1}^d x_{jk}^s e_s, e_i) \\
&= \sum_{s=1}^d \sum_{t=1}^d (b_{jk}^s x_{is}^t - b_{ik}^s x_{js}^t + b_{ij}^s x_{ks}^t - x_{ij}^s b_{sk}^t + x_{ik}^s b_{sj}^t - x_{jk}^s b_{si}^t) e_t \\
&= \sum_{t=1}^d (\sum_{m=1}^d \sum_{n=1}^d \sum_{l=1}^d (\delta_j^m \delta_k^n x_{il}^t - \delta_i^m \delta_k^n x_{jl}^t + \delta_i^m \delta_j^n x_{kl}^t - \delta_k^n \delta_l^t x_{ij}^m \\
&\quad + \delta_j^n \delta_l^t x_{ik}^m - \delta_i^n \delta_l^t x_{jk}^m) b_{mn}^l) e_t.
\end{aligned}$$

Note that $C^3(L, L)$ is a $\frac{(d-2)(d-1)d^2}{6}$ -dimensional vector space with a basis μ_t^{ijk} for $i, j, k, t = 1, \dots, d$ and $i < j < k$, where μ_t^{ijk} is defined as follows: $\mu_t^{ijk}(e_i, e_j, e_k) = \mu_t^{ijk}(e_j, e_k, e_i) = \mu_t^{ijk}(e_k, e_i, e_j) = e_t$, $\mu_t^{ijk}(e_j, e_i, e_k) = \mu_t^{ijk}(e_i, e_k, e_j) = \mu_t^{ijk}(e_k, e_j, e_i) = -e_t$, and $\mu_t^{ijk}(e_m, e_n, e_l) = 0$ otherwise. Then we know that the $\frac{(d-2)(d-1)d^2}{6} \times \frac{(d-1)d^2}{2}$ matrix $D_2 = (\delta_j^m \delta_k^n x_{il}^t - \delta_i^m \delta_k^n x_{jl}^t + \delta_i^m \delta_j^n x_{kl}^t - \delta_k^n \delta_l^t x_{ij}^m + \delta_j^n \delta_l^t x_{ik}^m - \delta_i^n \delta_l^t x_{jk}^m)_{(i,j,k,t);(m,n,l)}$ represents the matrix for $\delta|_{C^2(L,L)}$, where the domain $C^2(L, L)$ is under basis φ_k^{ij} and the codomain $C^2(L, L)$ is under basis μ_t^{ijk} . We can rely on computer assistance to calculate the dimension of the null space of D_2 , which is equivalent to the dimension of $Z^2(L, L)$.

Finally, we can calculate the dimension of $H^2(L, L)$ as $\dim H^2(L, L) = \dim Z^2(L, L) - \dim B^2(L, L) = \frac{(d-1)d^2}{2} - \text{rank}(D_2) - \text{rank}(D_1)$. By using a similar method, we can also calculate the dimension of $H^3(L, L)$. ■

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Remark 3.7. All rank and nullity computations mentioned in the proof of Theorem 3.6 were performed using MATLAB (version R2021a or later). The structural constants x_{ij}^k of the Yau algebra were stored in a three-dimensional array X of size $d \times d \times d$, where $X(i, j, k) = x_{ij}^k$. To handle the alternating property of the cochains, we introduced auxiliary index arrays as follows:

```
O2=zeros(d,d);
S2=zeros(d,d);
a=0;
for i=1:d
    for j=1:d
        if i<j
            S2(i,j)=1;
            O2(i,j)=a;
            a=a+1;
        elseif i>j
            S2(i,j)=-S2(j,i);
            O2(i,j)=O2(j,i);
        end
    end
end
end
```

The array $O2$ assigns the same integer to ordered pairs that differ by swapping entries (e.g., (i, j) and (j, i)), while $S2$ records the sign $+1$ or -1 indicating the parity of the swap. The arrays $O3, S3, O4, S4$ are defined analogously for triples and quadruples, respectively.

The matrix D_1 (representing $\delta|_{C^1(L,L)} : C^1(L, L) \rightarrow C^2(L, L)$) was then constructed using the following MATLAB code:

```
D1=zeros(d^2*(d-1)/2, d^2);
for i=1:d
    for j=1:d
        if i<j
            for s=1:d
                for t=1:d
```

```

        b = d*02(i,j) + t;
        D1(b, d*j+s-d) = D1(b, d*j+s-d) + X(i,s,t);
        D1(b, d*i+s-d) = D1(b, d*i+s-d) - X(j,s,t);
        D1(b, d*s+t-d) = D1(b, d*s+t-d) - X(i,j,s);
    end
end
end
end
end
end

```

Its rank and nullity give $\dim B^2$ and $\dim Z^1$, respectively. The matrices D_2 (for $\delta|_{C^2}$) and D_3 (for $\delta|_{C^3}$) were built similarly; their ranks and nullities yield $\dim B^3 = \text{rank}(D_2)$, $\dim Z^2 = \text{nullity}(D_2)$, and $\dim Z^3 = \text{nullity}(D_3)$.

The values in Table 7 were obtained by running these routines for each listed singularity. The MATLAB scripts are available from the corresponding author upon request.

3.3. The cohomology of Yau algebras for the A_k and D_k In the previous subsection, we have already calculated the cohomology of Yau algebras for the A_k and D_k with small dimensions. However, the dimensions of A_k and D_k can be arbitrarily large. Therefore, for high-dimensional cases, we need to seek other methods for computation.

Theorem 3.8 (i.e. Theorem 1.2). *Let $V = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2 - z^{d+2} = 0\}$ be the A_{d+1} singularity, where $d \geq 13$, and $L(V) = \langle e_1, e_2, \dots, e_d \rangle$ be the Yau algebra of V . Then we have that $\dim Z^2(L, L) = (d-1)d$, $\dim B^2(L, L) = (d-1)d$, and $\dim H^2(L, L) = \dim Z^2(L, L) - \dim B^2(L, L) = 0$.*

Proof. Using the notations in Theorem 3.6, we have $\dim H^2(L, L) = \dim Z^2(L, L) - \dim B^2(L, L) = \dim Z^2(L, L) - \text{rank}(D_1) = \dim Z^2(L, L) - (d^2 - \dim Z^1(L, L))$. We claim that $\dim Z^2(L, L) \leq (d-1)d$ and $\dim Z^1(L, L) \leq d$. It implies that $\dim Z^2(L, L) = (d-1)d$ and $\dim Z^1(L, L) = d$ because $\dim H^2(L, L) \geq 0$.

For any $\alpha \in Z^1(L, L)$, we can represent α by $\alpha = \sum_{i=1}^d \sum_{k=1}^d a_i^k \alpha_k^i$. We assert that α is determined by $\{a_i^k | (i, k) = (2, 2), (1, 2), (1, 3), \dots, (1, d)\}$, then $\dim Z^1(L, L) \leq d$.

$$\text{By Proposition 3.1, we have } [e_i, e_j] = \begin{cases} (j-i)e_{i+j-1} & \text{if } i+j-1 \leq d \\ 0 & \text{else} \end{cases},$$

which implies

$$x_{ij}^k = \begin{cases} j-i & \text{if } i+j = k+1, \\ 0 & \text{else.} \end{cases} \quad (1)$$

Using the notations used in Theorem 3.6, we obtain $\delta\alpha = \sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^d b_{ij}^k \varphi_k^{ij}$, where $b_{ij}^k = \sum_{m=1}^d \sum_{n=1}^d (\delta_j^m x_{in}^k - \delta_i^m x_{jn}^k - \delta_n^k x_{ij}^m) a_m^n$. Since $\alpha \in Z^1(L, L)$, we have

$$0 = \sum_{m=1}^d \sum_{n=1}^d (\delta_j^m x_{in}^k - \delta_i^m x_{jn}^k - \delta_n^k x_{ij}^m) a_m^n = \sum_{n=1}^d (x_{in}^k a_j^n - x_{jn}^k a_i^n) - \sum_{m=1}^d x_{ij}^m a_m^k. \quad (2)$$

We will use $\{a_2^2, a_1^2, a_1^3, \dots, a_1^d\}$ to represent a_p^q for $p, q = 1, \dots, d$. Let $i = p$, $j = 1$, and $k = q$ in equation (2), we have

$$\sum_{n=1}^d (x_{pn}^q a_1^n - x_{1n}^q a_p^n) - \sum_{m=1}^d x_{p1}^m a_m^q = 0. \quad (3)$$

If $p < q$, according to (1), we can simplify (3) to

$$(q + 1 - 2p)a_1^{q+1-p} - (q - 1)a_p^q - (1 - p)a_p^q = 0. \quad (4)$$

(4) means

$$a_p^q = \frac{q + 1 - 2p}{q - p} a_1^{q+1-p} \quad (\text{when } p < q). \quad (5)$$

If $p > q$, according to (1), we can simplify (3) to

$$-(q - 1)a_p^q - (1 - p)a_p^q = 0. \quad (6)$$

(6) means

$$a_p^q = 0 \quad (\text{when } p > q). \quad (7)$$

Let $(i, j, k) = (1, 2, 2), (2, 3, 4), (2, 4, 5), (2, 5, 6), (3, 4, 6)$ in (2), and combine these with (1), we have

$$a_1^1 = 0, \quad (8)$$

$$a_4^4 = a_3^3 + a_2^2, \quad (9)$$

$$a_5^5 = a_4^4 + a_2^2, \quad (10)$$

$$a_6^6 = a_5^5 + a_2^2, \quad (11)$$

$$a_6^6 = a_4^4 + a_3^3. \quad (12)$$

Using the relations (9)-(12), we get

$$a_p^p = (p - 1)a_2^2 \quad (\text{when } p = 3, 4, 5, 6). \quad (13)$$

Let $(i, j, k) = (2, p - 1, p)$ in (2), and combine these with (1), we have

$$a_p^p = a_2^2 + a_{p-1}^{p-1}. \quad (14)$$

Using the relations (8), (13), and (14), we get

$$a_p^q = (p - 1)a_2^2 \quad (\text{when } p = q). \quad (15)$$

Using the relations (5), (7), and (15), we have shown that α is determined by $\{a_i^k | (i, k) = (2, 2), (1, 2), (1, 3), \dots, (1, d)\}$, which implies that $\dim Z^1(L, L) \leq d$.

For any $\varphi \in Z^2(L, L)$, we can represent φ by $\varphi = \sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^d b_{ij}^k \varphi_k^{ij}$. We assert that φ is determined by $\{b_{1i}^j | 2 \leq i \leq d, 1 \leq j \leq d, i \neq j\} \cup \{b_{12}^2\} \cup \{b_{2i}^{i+1} | 3 \leq i \leq d-1\} \cup \{b_{34}^6\}$, then $\dim Z^2(L, L) \leq (d-1)d$.

Since $\varphi \in Z^2(L, L)$, similar to (2), we can obtain

$$0 = \sum_{n=1}^d (x_{in}^t b_{jk}^n - x_{jn}^t b_{ik}^n + x_{kn}^t b_{ij}^n - x_{ij}^n b_{nk}^t + x_{ik}^n b_{nj}^t - x_{jk}^n b_{ni}^t) \quad (\forall i, j, k, t = 1, \dots, d). \quad (16)$$

Let $i = 1$ in (16), we have

$$0 = \sum_{n=1}^d (x_{1n}^t b_{jk}^n - x_{jn}^t b_{1k}^n + x_{kn}^t b_{1j}^n - x_{1j}^n b_{nk}^t + x_{1k}^n b_{nj}^t - x_{jk}^n b_{n1}^t). \quad (17)$$

By (1), we have

$$0 = (t+1-j-k)b_{jk}^t - x_{j,t+1-j}^t b_{1k}^{t+1-j} + x_{k,t+1-k}^t b_{1j}^{t+1-k} - x_{jk}^{j+k-1} b_{j+k-1,1}^t, \quad (18)$$

where $x_{i,j}^k = x_{ij}^k$ and $b_{i,j}^k = b_{ij}^k$. Additionally, define $x_{j,t+1-j}^t b_{1k}^{t+1-j} = 0$ when $t+1-j \leq 0$, $x_{k,t+1-k}^t b_{1j}^{t+1-k} = 0$ when $t+1-k \leq 0$, and $x_{jk}^{j+k-1} b_{j+k-1,1}^t = 0$ when $j+k-1 > d$.

If $j+k \neq t+1$, then $x_{j,t+1-j}^t b_{1k}^{t+1-j}$, $x_{k,t+1-k}^t b_{1j}^{t+1-k}$, and $x_{jk}^{j+k-1} b_{j+k-1,1}^t$ can be represented by $\{b_{1i}^j | 2 \leq i \leq d, 1 \leq j \leq d, i \neq j\}$. Therefore b_{pq}^r can also be represented by $\{b_{1i}^j | 2 \leq i \leq d, 1 \leq j \leq d, i \neq j\}$ when $p+q \neq r+1$.

If $p+q = r+1$, we can assume $p \leq q$, let $i = 2$, $j = p-1$, $k = q$, and $t = r$ in (16), we have

$$0 = (p+q-4)b_{p-1,q}^{r-1} - (q+2-p)b_{2q}^{q+1} + (p-q)b_{2,p-1}^p - (p-3)b_{pq}^r + (q-2)b_{q+1,p-1}^r - (q+1-p)b_{r-1,2}^r. \quad (19)$$

If $p = 1$, we can simplify (19) to

$$(q-3)b_{1,q}^q = (q-3)b_{1,q-1}^{q-1} + (q-3)b_{12}^2. \quad (20)$$

We can observe that (20) is similar as (14). By substituting $(i, j, k, t) = (1, 2, 3, 4)$, $(1, 2, 4, 5)$, $(1, 2, 5, 6)$, $(1, 3, 4, 6)$ into (16), we can reach a conclusion similar as (15), which implies that $\{b_{1i}^j | 1 \leq i \leq d\}$ can be represented by b_{12}^2 .

If $p > 3$, b_{pq}^r can be represented by $\{b_{12}^2\} \cup \{b_{2i}^{i+1} | 3 \leq i \leq d-1\} \cup \{b_{p-1,q}^{r-1}, b_{p-1,q+1}^r\}$. By induction, b_{pq}^r can be represented by $\{b_{12}^2\} \cup \{b_{2i}^{i+1} | 3 \leq i \leq d-1\} \cup \{b_{3i}^{i+2} | 4 \leq i \leq d-2\}$.

Now we just need to prove that $\{b_{3i}^{i+2} | 4 \leq i \leq d-2\}$ can be represented by $\{b_{12}^2\} \cup \{b_{2i}^{i+1} | 3 \leq i \leq d-1\} \cup \{b_{34}^6\}$. We don't care about the coefficients of $\{b_{12}^2\} \cup \{b_{2i}^{i+1} | 3 \leq i \leq d-1\}$ in $\{b_{3i}^{i+2} | 4 \leq i \leq d-2\}$, so we can assume that $b_{2i}^{i+1} = 0$ for $3 \leq i \leq d-1$ and $b_{12}^2 = 0$ in the following calculations.

Let $(i, j, k, t) = (3, 4, q, q + 5)$ in (16), we have

$$\begin{aligned} 0 &= qb_{4q}^{q+3} - (q-2)b_{3q}^{q+2} + (6-q)b_{34}^6 \\ &\quad - b_{6q}^{q+5} + (q-3)b_{q+2,4}^{q+5} - (q-4)b_{q+3,3}^{q+5}. \end{aligned} \quad (21)$$

Let $p = 4, 5, 6$ in (19), we have

$$3b_{6,q}^{q+5} = (q+2)b_{5q}^{q+4} - (q-2)b_{5,q+1}^{q+5}, \quad (22)$$

$$2b_{5q}^{q+4} = (q+1)b_{4q}^{q+3} - (q-2)b_{4,q+1}^{q+4}, \quad (23)$$

$$b_{4q}^{q+3} = qb_{3q}^{q+2} - (q-2)b_{3,q+1}^{q+3}. \quad (24)$$

Using the relations (21)-(24), we get

$$\begin{aligned} 0 &= \left(\frac{1}{6}(q-2)(q-1)q + (q-3)q + (q-4)\right)b_{3,q+3}^{q+5} \\ &\quad - ((q-3)(q+2) + \frac{1}{6}(q-2)(q-1)(q+2) + \frac{1}{3}(q-2)(q+2)(q-1))b_{3,q+2}^{q+4} \\ &\quad + \left(\frac{1}{3}(q-2)(q+2)(q+1) + \frac{1}{6}(q+2)(q+1)(q-2) - q(q-2)\right)b_{3,q+1}^{q+3} \\ &\quad - \left(\frac{1}{6}(q+2)(q+1)q - q^2 + (q-2)\right)b_{3,q}^{q+2} + (6-q)b_{34}^6. \end{aligned} \quad (25)$$

Let $q = 4$ in (24), we have

$$2b_{35}^7 = 4b_{34}^6 - b_{44}^7 = 4b_{34}^6. \quad (26)$$

Then we can conclude that $\{b_{3i}^{i+2} | 4 \leq i \leq d-2\}$ can be represented by both b_{34}^6 and b_{36}^8 according to (25) and (26). Additionally, when $3 \leq p < q$ and $p + q = r + 1$, b_{pq}^r can also be represented by both b_{34}^6 and b_{36}^8 .

Let $(i, j, k, t) = (4, 5, 6, 13)$ in (16), we have

$$0 = 6b_{5,6}^{10} - 4b_{4,6}^9 + 2b_{4,5}^8 - b_{8,6}^{13} + 2b_{9,5}^{13} - b_{10,4}^{13}. \quad (27)$$

We can represent each component in (26) as a linear combination of b_{34}^6 and b_{36}^8 . By solving it, we have

$$b_{36}^8 = 3b_{34}^6. \quad (28)$$

In conclusion, we have shown that φ is determined by $\{b_{1i}^j | 2 \leq i \leq d, 1 \leq j \leq d, i \neq j\} \cup \{b_{12}^2\} \cup \{b_{2i}^{i+1} | 3 \leq i \leq d-1\} \cup \{b_{34}^6\}$, which implies that $\dim Z^2(L, L) \leq (d-1)d$. $\blacksquare$

Remark 3.9. In order for (12) to hold, it is necessary that $d \geq 6$. This is why the $\dim H^2(L, L)$ of $A_{d+1}(d = 3, 4, 5)$ are not 0. In order for (27) to hold, it is necessary that $d \geq 13$. This is why the $\dim H^2(L, L)$ of $A_{d+1}(d = 8, 9, 10, 11, 12)$ are not 0.

Theorem 3.10 (i.e. Theorem 1.3). *Let $V = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2z + z^{d+1} = 0\}$ be the D_{d+2} singularity, where $d \geq 13$, and $L(V) = \langle e_1, e_2, \dots, e_{d+2} \rangle$ be the Yau algebra of V . Then we have that $\dim Z^2(L, L) = (d+1)(d+2)$, $\dim B^2(L, L) = (d+1)(d+2)$, and $\dim H^2(L, L) = \dim Z^2(L, L) - \dim B^2(L, L) = 0$.*

Proof. Similar to Theorem 3.8, using the notations used in Theorem 3.6, we have $\dim H^2(L, L) = \dim Z^2(L, L) + \dim Z^1(L, L) - (d+2)^2$. We claim that $\dim Z^2(L, L) \leq (d+1)(d+2)$ and $\dim Z^1(L, L) \leq d+2$. This implies that $\dim Z^2(L, L) = (d+1)(d+2)$ and $\dim Z^1(L, L) = d+2$ because $\dim H^2(L, L) \geq 0$.

By Proposition 3.2, we have

$$x_{ij}^k = \begin{cases} j-i & \text{if } k \leq d \text{ and } i+j = k+1, \\ \frac{d-2}{2} & \text{if } i=1 \text{ and } j=k=d+1, \\ \frac{d}{2} & \text{if } i=1 \text{ and } j=k=d+2, \\ (d+1)(d-1) & \text{if } i=2 \text{ and } j=d+1 \text{ and } k=d+2, \\ -1 & \text{if } i=d+1 \text{ and } j=d+2 \text{ and } k=d, \\ 0 & \text{else,} \end{cases} \quad (29)$$

where $1 \leq i < j \leq d+2$ and $1 \leq k \leq d+2$.

Let $\alpha = \sum_{i=1}^{d+2} \sum_{k=1}^{d+2} a_i^k \alpha_k^i \in Z^1(L, L)$ and $\varphi = \sum_{i=1}^{d+2} \sum_{j=1}^{d+2} \sum_{k=1}^{d+2} b_{ij}^k \varphi_k^{ij} \in Z^2(L, L)$. Note that if exactly one of i, j , and k is equal to either $d+1$ or $d+2$, then $x_{ij}^k = 0$. Additionally, when $1 \leq i, j, k \leq d$, the value of x_{ij}^k of A_{d+1} is equal to the value of x_{ij}^k of D_{d+2} . If $1 \leq i, j, k, t \leq d$, the relations (1)-(28) still hold on in D_{d+2} . This implies that the element of $\{a_i^k | 1 \leq i, k \leq d\}$ can be expressed as a linear combination of $\{a_i^k | (i, k) = (2, 2), (1, 2), (1, 3), \dots, (1, d)\}$ and the element of $\{b_{ij}^k | 1 \leq i, j, k \leq d\}$ can be expressed as a linear combination of $\{b_{1i}^j | 2 \leq i \leq d, 1 \leq j \leq d, i \neq j\} \cup \{b_{12}^2\} \cup \{b_{2i}^{i+1} | 3 \leq i \leq d-1\} \cup \{b_{34}^6\}$.

We assert that α is determined by $\{a_i^k | (i, k) = (2, 2), (1, 2), (1, 3), \dots, (1, d+2)\}$, then $\dim Z^1(L, L) \leq d+2$. According to the above discussion, we only need to prove $\{a_i^{d+1} | i = 2, \dots, d\} \cup \{a_i^{d+2} | i = 2, \dots, d\} \cup \{a_{d+1}^i | i = 1, \dots, d\} \cup \{a_{d+2}^i | i = 1, \dots, d\} \cup \{a_{d+1}^{d+1}, a_{d+2}^{d+2}, a_{d+1}^{d+2}, a_{d+2}^{d+1}\}$ can be expressed as a linear combination of $\{a_i^k | 1 \leq i, k \leq d\} \cup \{a_1^{d+1}, a_1^{d+2}\}$.

Since $\alpha \in Z^1(L, L)$, we have

$$0 = \sum_{n=1}^{d+2} (x_{in}^k a_j^n - x_{jn}^k a_i^n - x_{ij}^n a_n^k) \quad (\forall i, j, k = 1, \dots, d+2). \quad (30)$$

Let $i = 1$ and $k = d+1$ in (30), we have

$$0 = \frac{d-2}{2} a_j^{d+1} - \sum_{n=1}^{d+2} x_{jn}^{d+1} a_1^n - x_{1j}^j a_j^{d+1}. \quad (31)$$

When $2 \leq j \leq d$ and $j \neq \frac{d}{2}$, according to (31), we have

$$a_j^{d+1} = 0 \quad (2 \leq j \leq d, j \neq \frac{d}{2}). \quad (32)$$

Let $j = d + 2$ in (31), we have

$$a_{d+2}^{d+1} = 0. \quad (33)$$

Let $i = 3$, $j = \frac{d}{2} - 2$, and $k = d + 1$ in (30), we have

$$a_{\frac{d}{2}}^{d+1} = 0. \quad (34)$$

Let $i = 1$ and $k = d + 2$ in (30), we have

$$0 = \frac{d-2}{2}a_j^{d+2} - \sum_{n=1}^{d+2} x_{jn}^{d+2}a_1^n - x_{1j}^j a_j^{d+2}. \quad (35)$$

When $3 \leq j \leq d$ and $j \neq \frac{d}{2} + 1$, according to (35), we have

$$a_j^{d+2} = 0 \quad (3 \leq j \leq d, j \neq \frac{d}{2} + 1). \quad (36)$$

Let $j = d + 1$ in (35), we have

$$a_{d+1}^{d+2} = -(d+1)(d-1)a_1^2. \quad (37)$$

Let $j = 2$ in (35), we have

$$a_2^{d+2} = \frac{2(d+1)(d-1)}{d-2}a_1^{d+1}. \quad (38)$$

Let $i = 3$, $j = \frac{d}{2} - 1$, and $k = d + 2$ in (30), we have

$$a_{\frac{d}{2}+1}^{d+2} = 0. \quad (39)$$

Let $i = 1$ and $j = d + 1$ in (30), we have

$$0 = x_{1k}^k a_{d+1}^k - \sum_{n=1}^{d+2} x_{d+1,n}^k a_1^n - \frac{d-2}{2}a_{d+1}^k. \quad (40)$$

When $1 \leq k \leq d - 1$ and $k \neq \frac{d}{2}$, according to (40), we have

$$a_{d+1}^k = 0 \quad (1 \leq k \leq d - 1, k \neq \frac{d}{2}). \quad (41)$$

Let $k = d$ in (40), we have

$$a_{d+1}^d = -\frac{2}{d}a_1^{d+2}. \quad (42)$$

Let $i = 3$, $j = d + 1$, and $k = \frac{d}{2} + 2$ in (30), we have

$$a_{d+1}^{\frac{d}{2}} = 0. \quad (43)$$

Let $i = 1$ and $j = d + 2$ in (30), we have

$$0 = x_{1k}^k a_{d+2}^k - \sum_{n=1}^{d+2} x_{d+2,n}^k a_1^n - \frac{d}{2} a_{d+2}^k. \quad (44)$$

When $1 \leq k \leq d - 1$ and $k \neq \frac{d}{2} + 1$, according to (44), we have

$$a_{d+1}^k = 0 \quad (1 \leq k \leq d - 1, k \neq \frac{d}{2} + 1). \quad (45)$$

Let $k = d$ in (44), we have

$$a_{d+2}^d = \frac{2}{d-2} a_1^{d+1}. \quad (46)$$

Let $i = 3$, $j = d + 2$, and $k = \frac{d}{2} + 3$ in (30), we have

$$a_{d+2}^{\frac{d}{2}+1} = 0. \quad (47)$$

Let $i = d + 1$, $j = d + 2$, and $k = d$ in (30), we have

$$0 = -a_{d+2}^{d+2} - a_{d+1}^{d+1} + a_d^d. \quad (48)$$

Let $i = 2$, $j = d + 1$, and $k = d + 2$ in (30), we have

$$0 = (d+1)(d-1)a_{d+1}^{d+1} + (d+1)(d-1)a_2^2 - (d+1)(d-1)a_{d+2}^{d+2}. \quad (49)$$

Using the relations (48) and (49), we get

$$a_{d+1}^{d+1} = \frac{1}{2}(a_d^d - a_2^2), \quad (50)$$

$$a_{d+2}^{d+2} = \frac{1}{2}(a_d^d + a_2^2). \quad (51)$$

In conclusion, we have shown that α is determined by $\{a_i^k | (i, k) = (2, 2), (1, 2), (1, 3), \dots, (1, d+2)\}$, which implies that $\dim Z^1(L, L) \leq d + 2$.

We assert that φ is determined by $\{b_{1i}^j | 2 \leq i \leq d, 1 \leq j \leq d, i \neq j\} \cup \{b_{12}^2\} \cup \{b_{2i}^{i+1} | 3 \leq i \leq d-1\} \cup \{b_{34}^6\} \cup \{b_{1n}^{d+1} | n = 2, 3, \dots, d\} \cup \{b_{1n}^{d+2} | n = 2, 3, \dots, d\} \cup \{b_{1,d+1}^n | n = 1, 2, \dots, d\} \cup \{b_{1,d+2}^n | n = 1, 2, \dots, d\} \cup \{b_{1,d+1}^{d+2}, b_{1,d+2}^{d+1}, b_{2,d+1}^{d+2}, b_{d+1,d+2}^d\}$ when d is odd, and by $\{b_{1i}^j | 2 \leq i \leq d, 1 \leq j \leq d, i \neq j\} \cup \{b_{12}^2\} \cup \{b_{2i}^{i+1} | 3 \leq i \leq d-1\} \cup \{b_{34}^6\} \cup \{b_{1n}^{d+1} | n = 2, 3, \dots, \frac{d}{2}-1, \frac{d}{2}+1, \dots, d\} \cup \{b_{1n}^{d+2} | n = 2, 3, \dots, \frac{d}{2}, \frac{d}{2}+2, \dots, d\} \cup \{b_{1,d+1}^n | n = 1, 2, \dots, \frac{d}{2}-1, \frac{d}{2}+1, \dots, d\} \cup \{b_{1,d+2}^n | n = 1, 2, \dots, \frac{d}{2}, \frac{d}{2}+2, \dots, d\} \cup \{b_{2,\frac{d}{2}-1}^{d+1}, b_{2,\frac{d}{2}}^{d+2}, b_{2,d+1}^{\frac{d}{2}+1}, b_{2,d+2}^{\frac{d}{2}+2}, b_{1,d+1}^{d+2}, b_{1,d+2}^{d+1}, b_{2,d+1}^{d+2}, b_{d+1,d+2}^d\}$ when d is even. Given that the elements of $\{b_{ij}^k | 1 \leq i, j, k \leq d\}$ can be expressed as a linear combination of $\{b_{1i}^j | 2 \leq i \leq d, 1 \leq j \leq d, i \neq j\} \cup \{b_{12}^2\} \cup \{b_{2i}^{i+1} | 3 \leq i \leq d-1\} \cup \{b_{34}^6\}$, it suffices to focus on the values of $\{b_{ij}^{d+1} | 1 \leq i, j \leq d\} \cup \{b_{ij}^{d+2} | 1 \leq i, j \leq d\} \cup \{b_{i,d+1}^j | 1 \leq i, j \leq d\} \cup \{b_{i,d+2}^j | 1 \leq i, j \leq d\} \cup \{b_{i,d+1}^{d+1} | 1 \leq i \leq d\} \cup \{b_{i,d+1}^{d+2} | 1 \leq i \leq d\} \cup \{b_{i,d+2}^{d+1} | 1 \leq i \leq d\} \cup \{b_{i,d+2}^{d+2} | 1 \leq i \leq d\} \cup \{b_{d+1,d+2}^i | 1 \leq i \leq d+2\}$.

Since $\varphi \in Z^2(L, L)$, we have

$$0 = \sum_{n=1}^{d+2} (x_{in}^t b_{jk}^n - x_{jn}^t b_{ik}^n + x_{kn}^t b_{ij}^n - x_{ij}^n b_{nk}^t + x_{ik}^n b_{nj}^t - x_{jk}^n b_{ni}^t) \quad (\forall i, j, k, t = 1, \dots, d+2). \quad (52)$$

Let $i = 1$, $2 \leq j, k \leq d$ and $t = d+1$ in (52), we have

$$0 = x_{1,d+1}^{d+1} b_{jk}^{d+1} - \sum_{n=1}^{d+2} x_{jn}^{d+1} b_{1k}^n + \sum_{n=1}^{d+2} x_{kn}^{d+1} b_{1j}^n - x_{1j}^j b_{jk}^{d+1} + x_{1k}^k b_{kj}^{d+1} - \sum_{n=1}^{d+2} x_{jk}^n b_{n1}^{d+1}. \quad (53)$$

When $j+k \neq \frac{d+2}{2}$, according to (53), we have

$$b_{jk}^{d+1} = \frac{2}{d+2-2j-2k} \sum_{n=1}^{d+2} x_{jk}^n b_{n1}^{d+1}. \quad (54)$$

When d is odd, we have b_{jk}^{d+1} ($1 \leq j, k \leq d$) determined by $\{b_{1n}^{d+1} | n = 2, 3, \dots, d\}$ (note that $x_{jk}^n = 0$ when $1 \leq j, k \leq d$ and $n > d$).

When d is even. Let $i = 2$, $3 \leq j \leq \frac{d}{2} - 3$, $k = \frac{d}{2} - j$, and $t = d+1$ in (52), we have

$$0 = -(j-2)b_{j+1, \frac{d}{2}-j}^{d+1} + (\frac{d}{2} - 2 - j)b_{\frac{d}{2}+1-j, j}^{d+1} - (\frac{d}{2} - 2j)b_{\frac{d}{2}-1, 2}^{d+1} \quad (3 \leq j \leq \frac{d}{2} - 3). \quad (55)$$

Equation (55) indicates that $b_{j, \frac{d}{2}+1-j}^{d+1}$ ($4 \leq j \leq \frac{d}{4} + 1$) can be expressed as a linear combination of $b_{2, \frac{d}{2}-1}^{d+1}$ and $b_{3, \frac{d}{2}-2}^{d+1}$. Additionally, from Equation (55), it follows that $b_{1, \frac{d}{2}}^{d+1} = 0$. Then we let $i = 3$, $j = 4$, $k = \frac{d}{2} - 5$, and $t = d+1$ in (52), we have

$$0 = -b_{6, \frac{d}{2}-5}^{d+1} + (\frac{d}{2} - 8)b_{\frac{d}{2}-3, 4}^{d+1} - (\frac{d}{2} - 9)b_{\frac{d}{2}-2, 3}^{d+1}. \quad (56)$$

It can transform into

$$(d-2)(d-6)(d-16)b_{2, \frac{d}{2}-1}^{d+1} = (d-2)(d-10)(d-16)b_{3, \frac{d}{2}-2}^{d+1}. \quad (57)$$

When $d \neq 16$, the elements $b_{j, \frac{d}{2}+1-j}^{d+1}$ ($1 \geq j \geq \frac{d}{2}$) can be expressed as rational multiples of $b_{2, \frac{d}{2}-1}^{d+1}$. This conclusion remains valid when $d = 16$, and in this case, $Z^2(L, L)$ can be computed directly using the method outlined in Theorem 3.6. Consequently, the values of b_{jk}^{d+1} for $1 \leq j, k \leq d$ are determined by $\{b_{1n}^{d+1} | n = 2, 3, \dots, d\}$ when d is odd, and by $\{b_{1n}^{d+1} | n = 2, 3, \dots, \frac{d}{2} - 1, \frac{d}{2} + 1, \dots, d\} \cup \{b_{2, \frac{d}{2}-1}^{d+1}\}$ when d is even.

Let $i = 1$, $2 \leq j, k \leq d$ and $t = d+2$ in (52), we have

$$0 = x_{1,d+2}^{d+2} b_{jk}^{d+2} - x_{j,d+1}^{d+2} b_{1k}^{d+1} + x_{k,d+1}^{d+2} b_{1j}^{d+1} - x_{1j}^j b_{jk}^{d+2} + x_{1k}^k b_{kj}^{d+2} - \sum_{n=1}^{d+2} x_{jk}^n b_{n1}^{d+2}. \quad (58)$$

When $j + k \neq \frac{d+4}{2}$, according to (58), we have

$$b_{jk}^{d+2} = \frac{2}{d+4-2j-2k} (x_{j,d+1}^{d+2} b_{1k}^{d+1} - x_{k,d+1}^{d+2} b_{1j}^{d+1} + \sum_{n=1}^d x_{jk}^n b_{n1}^{d+2}). \quad (59)$$

When d is odd, we have $b_{jk}^{d+2} (1 \leq j, k \leq d)$ determined by $\{b_{1n}^{d+1} | n = 2, 3, \dots, d\} \cup \{b_{1n}^{d+2} | n = 2, 3, \dots, d\}$.

When d is even. Let $i = 2$, $3 \leq j \leq \frac{d}{2} - 2$, $k = \frac{d}{2} + 1 - j$, and $t = d + 2$ in (52), we have

$$0 = x_{2,d+1}^{d+2} b_{jk}^{d+1} - (j-2) b_{j+1, \frac{d}{2}+1-j}^{d+2} + (\frac{d}{2} - 1 - j) b_{\frac{d}{2}+2-j, j}^{d+2} - (\frac{d}{2} + 1 - 2j) b_{\frac{d}{2}, 2}^{d+1}. \quad (60)$$

Equation (60) indicates that $b_{j, \frac{d}{2}+2-j}^{d+2} (4 \leq j \leq \frac{d}{4} + 1)$ can be expressed as a linear combination of $b_{2, \frac{d}{2}}^{d+2}$, $b_{3, \frac{d}{2}-1}^{d+2}$, and b_{jk}^{d+1} for $1 \leq j, k \leq d$. By setting $i = 3$, $j = 4$, $k = \frac{d}{2} - 4$, and $t = d + 2$ in (52), similar to equation (57), we find that the elements $b_{j, \frac{d}{2}+1-j}^{d+2} (1 \geq j \geq \frac{d}{2})$ can be represented as a linear combination of $b_{2, \frac{d}{2}}^{d+2}$ and b_{jk}^{d+1} for $1 \leq j, k \leq d$ when $d \neq 14$. This conclusion remains valid when $d = 14$, and in this case, $Z^2(L, L)$ can be computed directly using the method outlined in Theorem 3.6. Consequently, the values of b_{jk}^{d+1} and b_{jk}^{d+2} for $1 \leq j, k \leq d$ are determined by $\{b_{1n}^{d+1} | n = 2, 3, \dots, d\} \cup \{b_{1n}^{d+2} | n = 2, 3, \dots, d\}$ when d is odd, and by $\{b_{1n}^{d+1} | n = 2, 3, \dots, \frac{d}{2} - 1, \frac{d}{2} + 1, \dots, d\} \cup \{b_{1n}^{d+2} | n = 2, 3, \dots, \frac{d}{2}, \frac{d}{2} + 2, \dots, d\} \cup \{b_{2, \frac{d}{2}-1}^{d+1}, b_{2, \frac{d}{2}}^{d+2}\}$ when d is even.

Let $i = 1$, $2 \leq j \leq d$, $k = d + 2$, and $1 \leq t \leq d + 2$ in (52), we have

$$0 = x_{1t}^t b_{j,d+2}^t - \sum_{n=1}^{d+2} x_{jn}^t b_{1,d+2}^n + \sum_{n=1}^{d+2} x_{d+2,n}^t b_{1j}^n - x_{1j}^j b_{j,d+2}^t + x_{1,d+2}^{d+2} b_{d+2,j}^t - \sum_{n=1}^{d+2} x_{j,d+2}^n b_{n,1}^t. \quad (61)$$

When $1 \leq t \leq d$ and $t - j \neq \frac{d}{2}$, according to (61), we have

$$b_{j,d+2}^t = \frac{2}{2t-2j-d} \left(\sum_{n=1}^d x_{jn}^t b_{1,d+2}^n - x_{d+2,d+1}^t b_{1j}^{d+1} \right). \quad (62)$$

When d is odd, we have $b_{j,d+2}^t (1 \leq j, t \leq d)$ determined by $\{b_{1,d+2}^n | n = 1, 2, \dots, d\} \cup \{b_{1j}^{d+1} | j = 2, 3, \dots, d\}$.

When d is even. Let $i = 2$, $3 \leq j \leq \frac{d}{2} - 1$, $k = d + 2$, and $t = j + \frac{d}{2} + 1$ in (52), we have

$$0 = (\frac{d}{2} - 2 + j) b_{j,d+2}^{j+\frac{d}{2}} - (\frac{d}{2} + 2 - j) b_{2,d+2}^{\frac{d}{2}+2} + x_{d+2,d+1}^{j+\frac{d}{2}+1} b_{2j}^{d+1} - (j-2) b_{j+1,d+2}^{j+\frac{d}{2}+1}. \quad (63)$$

Equation (63) indicates that $b_{j,d+2}^{j+\frac{d}{2}} (4 \leq j \leq \frac{d}{2})$ can be expressed as a linear combination of $b_{2,d+2}^{\frac{d}{2}+2}$, $b_{3,d+2}^{\frac{d}{2}+3}$ and $b_{2, \frac{d}{2}-1}^{d+1}$. Additionally, from Equation (61),

it follows that $b_{1,d+2}^{\frac{d}{2}+1} = 0$. Then we let $i = 3$, $j = 4$, $k = d + 2$, and $t = \frac{d}{2} + 6$ in (52), we have

$$(d-2)(d^2+2d+24)b_{3,d+2}^{\frac{d}{2}+3} = (d-4)(d^2+2d+24)b_{2,d+2}^{\frac{d}{2}+2}. \quad (64)$$

In summary, the values of $b_{j,d+2}^t$ for $1 \leq j, t \leq d$ are determined by $\{b_{1,d+2}^n | n = 1, 2, \dots, d\} \cup \{b_{1j}^{d+1} | j = 2, 3, \dots, d\}$ when d is odd, and by $\{b_{1,d+2}^n | n = 1, 2, \dots, \frac{d}{2}, \frac{d}{2} + 2, \dots, d\} \cup \{b_{1j}^{d+1} | j = 2, 3, \dots, \frac{d}{2} - 1, \frac{d}{2} + 1, \dots, d\} \cup \{b_{2,d+2}^{\frac{d}{2}+2}, b_{2,\frac{d}{2}-1}^{d+1}\}$ when d is even. By employing a comparable approach, we can derive the subsequent conclusion: the values of b_{jk}^{d+1} , b_{jk}^{d+2} , $b_{j,d+1}^t$, and $b_{j,d+2}^t$ for $1 \leq j, k, t \leq d$ are determined by $\{b_{1n}^{d+1} | n = 2, 3, \dots, d\} \cup \{b_{1n}^{d+2} | n = 2, 3, \dots, d\} \cup \{b_{1,d+1}^n | n = 1, 2, \dots, d\} \cup \{b_{1,d+2}^n | n = 1, 2, \dots, d\}$ when d is odd, and by $\{b_{1n}^{d+1} | n = 2, 3, \dots, \frac{d}{2} - 1, \frac{d}{2} + 1, \dots, d\} \cup \{b_{1n}^{d+2} | n = 2, 3, \dots, \frac{d}{2}, \frac{d}{2} + 2, \dots, d\} \cup \{b_{1,d+1}^n | n = 1, 2, \dots, \frac{d}{2} - 1, \frac{d}{2} + 1, \dots, d\} \cup \{b_{1,d+2}^n | n = 1, 2, \dots, \frac{d}{2}, \frac{d}{2} + 2, \dots, d\} \cup \{b_{2,\frac{d}{2}-1}^{d+1}, b_{2,\frac{d}{2}}^{d+2}, b_{2,d+1}^{\frac{d}{2}+1}, b_{2,d+2}^{\frac{d}{2}+2}\}$ when d is even.

Let $t = d + 1$ in (61), we have

$$0 = \frac{d-2}{2}b_{j,d+2}^{d+1} - (j-1)b_{j,d+2}^{d+1} + \frac{d}{2}b_{d+2,j}^{d+1}. \quad (65)$$

Let $t = d + 2$ in (61), we have

$$0 = \frac{d}{2}b_{j,d+2}^{d+2} - x_{j,d+1}^{d+2}b_{1,d+2}^{d+1} - \frac{d}{2}b_{1j}^1 - (j-1)b_{j,d+2}^{d+2} + \frac{d}{2}b_{d+2,j}^{d+2}. \quad (66)$$

For $k = d + 1$, we have similar equation

$$0 = -(j-1)b_{j,d+1}^{d+1} - \frac{d-2}{2}b_{1j}^1 - x_{j,d+1}^{d+2}b_{d+2,1}^{d+1}, \quad (67)$$

$$0 = -(j-2)b_{j,d+1}^{d+2} - x_{j,d+1}^{d+2}b_{1,d+1}^{d+1} + x_{d+1,2}^{d+2}b_{1j}^2 - x_{j,d+1}^{d+2}b_{d+2,1}^{d+2}. \quad (68)$$

Equation (65)-(68) indicates that the values of $b_{j,d+1}^{d+1}$, $b_{j,d+1}^{d+2}$, $b_{j,d+2}^{d+1}$, and $b_{j,d+2}^{d+2}$ for $1 \leq j \leq d$ are determined by $b_{1,d+1}^{d+1}$, $b_{1,d+1}^{d+2}$, $b_{1,d+2}^{d+1}$, $b_{1,d+2}^{d+2}$, $b_{2,d+1}^{d+2}$, b_{1j}^2 ($j = 2, 3, \dots, d$) and b_{1j}^1 ($j = 2, 3, \dots, d$).

Furthermore, let $j = 2$ in (68) and let $i = 1$, $j = d + 1$, $k = d + 2$, and $t = d$ in (52), we have

$$0 = b_{1,d+1}^{d+1} + b_{12}^2 - b_{1,d+2}^{d+2}, \quad (69)$$

$$0 = b_{1,d+1}^{d+1} + b_{1,d+2}^{d+2} - b_{1d}^d. \quad (70)$$

Equation (69) and (70) indicates that the values of $b_{1,d+1}^{d+1}$ and $b_{1,d+2}^{d+2}$ are determined by b_{12}^2 (note that the value of b_{1d}^d is also determined by b_{12}^2).

Finally, the only remaining undetermined values are those of $b_{d+1,d+2}^t$ for $1 \leq t \leq d + 2$. Let $i = 1$, $j = d + 1$ and $k = d + 2$ in (52), we have

$$0 = (x_{1t}^t - d + 1)b_{d+1,d+2}^t - \sum_{n=1}^{d+2} (x_{d+1,n}^t b_{1,d+2}^n - x_{d+2,n}^t b_{1,d+1}^n) - x_{d+1,d+2}^d b_{d1}^t \quad (1 \leq t \leq d + 2). \quad (71)$$

When $t \neq d$, the value of $b_{d+1,d+2}^t$ is determined by b_{1d}^t , $b_{1,d+1}^n$, and $b_{1,d+2}^n$ for $n = 1, 2, \dots, d+2$. In conclusion, we have proved the previous assertion that the values of $\{b_{ij}^{d+1} | 1 \leq i, j \leq d\} \cup \{b_{ij}^{d+2} | 1 \leq i, j \leq d\} \cup \{b_{i,d+1}^j | 1 \leq i, j \leq d\} \cup \{b_{i,d+2}^j | 1 \leq i, j \leq d\} \cup \{b_{i,d+1}^{d+1} | 1 \leq i \leq d\} \cup \{b_{i,d+1}^{d+2} | 1 \leq i \leq d\} \cup \{b_{i,d+2}^{d+1} | 1 \leq i \leq d\} \cup \{b_{i,d+2}^{d+2} | 1 \leq i \leq d\} \cup \{b_{d+1,d+2}^i | 1 \leq i \leq d+2\}$ are determined by $\{b_{ij}^k | 1 \leq i, j, k \leq d\} \cup \{b_{1n}^{d+1} | n = 2, 3, \dots, d\} \cup \{b_{1n}^{d+2} | n = 2, 3, \dots, d\} \cup \{b_{1,d+1}^n | n = 1, 2, \dots, d\} \cup \{b_{1,d+2}^n | n = 1, 2, \dots, d\} \cup \{b_{1,d+1}^{d+2}, b_{1,d+2}^{d+1}, b_{2,d+1}^{d+2}, b_{d+1,d+2}^d\}$ when d is odd, and by $\{b_{ij}^k | 1 \leq i, j, k \leq d\} \cup \{b_{1n}^{d+1} | n = 2, 3, \dots, \frac{d}{2} - 1, \frac{d}{2} + 1, \dots, d\} \cup \{b_{1n}^{d+2} | n = 2, 3, \dots, \frac{d}{2}, \frac{d}{2} + 2, \dots, d\} \cup \{b_{1,d+1}^n | n = 1, 2, \dots, \frac{d}{2} - 1, \frac{d}{2} + 1, \dots, d\} \cup \{b_{1,d+2}^n | n = 1, 2, \dots, \frac{d}{2}, \frac{d}{2} + 2, \dots, d\} \cup \{b_{2,\frac{d}{2}-1}^{d+1}, b_{2,\frac{d}{2}}^{d+2}, b_{2,d+1}^{\frac{d}{2}+1}, b_{2,d+2}^{\frac{d}{2}+2}, b_{1,d+1}^{d+2}, b_{1,d+2}^{d+1}, b_{2,d+1}^{d+2}, b_{d+1,d+2}^d\}$ when d is even, which implies that $\dim Z^2(L, L) \leq (d+1)(d+2)$. $\blacksquare$

Remark 3.11. The crucial point is that in the proof process, we are able to determine a set of bases for $Z^1(L, L)$ (resp. $Z^2(L, L)$). Specifically, for the identified elements, setting one element to 1 and the rest to 0, the corresponding α (resp. φ) form the bases of $Z^1(L, L)$ (resp. $Z^2(L, L)$).

4. Rigidity of Yau algebras

4.1. Deformations of Lie algebra structures For a general theory of the deformations of Lie algebra structures, the interested readers can refer to colorblack[11, 12].

Let $L = (V, \eta)$ be a Lie algebra and let $\varphi \in C^2(L, L)$ be a 2-cochain. Now, $\eta' = \eta + \varphi$ is also a Lie algebra multiplication if it satisfies the Jacobi identity:

$$\sum_{cyc} \eta'(x, \eta'(y, z)) = 0 \quad (72)$$

for any $x, y, z \in V$, where $\sum_{cyc}$ denotes the sum over all cyclic permutations of (x, y, z) .

Since $\sum_{cyc} \eta(x, \eta(y, z)) = 0$ and $\delta\varphi(x, y, z) = \sum_{cyc} \eta(x, \varphi(y, z)) + \sum_{cyc} \varphi(x, \eta(y, z))$, we can transform (72) into

$$\delta\varphi - [\varphi, \varphi]/2 = 0, \quad (73)$$

where $[\varphi, \varphi]$ is defined as:

$$[\varphi, \varphi](x, y, z) = 2 \sum_{cyc} \varphi(\varphi(x, y), z).$$

Equation (73) is called the deformation equation. Let $\eta_t = \eta + t\varphi_1 + t^2\varphi_2 + \dots$ be a one-parameter family of Lie algebra multiplications on V , where $\varphi_i \in C^2(L, L)$. Then $t\varphi_1 + t^2\varphi_2 + \dots$ satisfies the deformation equation, which implies that $\delta\varphi_1 = 0$. Hence $\varphi_1 \in Z^2(L, L)$, and we call φ_1 an infinitesimal deformation of η .

A one-parameter family of Lie algebra multiplications $\eta_t = \eta + t\varphi_1 + t^2\varphi_2 + \dots$ is said to be trivial if $(V, \eta_s) \cong (V, \eta_t)$ for any s, t . Then there exists a

one-parameter family of invertible linear maps $I_t = I + t\alpha_1 + t^2\alpha_2 + \dots$, where $\alpha_i \in (L, L)$ is a linear map from V to itself and I is the identity map, such that

$$\eta_t(x, y) = I_t \eta((I_t)^{-1}x, (I_t)^{-1}y) \quad (74)$$

for any $x, y \in V$. We call φ_1 a trivial infinitesimal deformation.

Definition 4.1. Let V be a finite-dimensional complex vector space. A Lie algebra $L = (V, \eta)$ is rigid if there not exist a non-trivial infinitesimal deformation. It means the orbit $G(\eta) = \{g\eta(g^{-1}x, g^{-1}y) | g \in GL(V)\}$ is an open subset of $\mathfrak{M}$, where $\mathfrak{M}$ is the real algebraic set of all Lie algebra multiplications on V .

Let $\alpha : V \rightarrow V$ be a linear map, and $I_t = I + t\alpha$ be a one-parameter family of invertible linear maps. Then equation (74) can transform into

$$\eta_t(x, y) = \eta(x, y) + t(\alpha(\eta(x, y))) - \eta(\alpha(x), y) - \eta(x, \alpha(y)) + (\text{higher order terms}).$$

This show that $\delta\alpha = \eta(x, \alpha(y)) + \eta(\alpha(x), y) - \alpha(\eta(x, y))$ is a trivial infinitesimal deformation. Thus the trivial infinitesimal deformations are precisely the 2-coboundaries, and every 2-coboundaries is the first order term of some one-parameter family of deformations of η . It is noted that we mentioned before that infinitesimal deformation $\varphi_1 \in Z^2(L, L)$, so we have the following theorem.

Theorem 4.2 (colorblackTheorem 7.1, [12]). *Let V be a finite-dimensional complex vector space, and let $L = (V, \eta)$ be a Lie algebra such that $H^2(L, L) = 0$. Then L is rigid.*

Equation (73) can be decomposed into the following equations

$$\delta\varphi_n - \frac{1}{2} \sum_{i=1}^{n-1} [\varphi_i, \varphi_{n-i}] = 0 \quad (n = 1, 2, \dots).$$

We can prove that $\sum_{i=1}^{n-1} [\varphi_i, \varphi_{n-i}] \in Z^3(L, L)$, leading to the following theorem. Detailed discussions can be found in colorblack[11].

Theorem 4.3 (colorblackTheorem 20.3, [11]). *Let V be a finite-dimensional complex vector space, and let $L = (V, \eta)$ be a Lie algebra such that $H^3(L, L) = 0$. Then each infinitesimal deformation is the first order term of a one-parameter family. Additionally, L isn't rigid when $H^2(L, L) \neq 0$.*

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Remark 4.4. The condition $H^3(L, L) = 0$ in Theorem 4.3 is essential. Indeed, without it, the conclusion that $H^2(L, L) \neq 0$ implies non-rigidity fails in general. As discussed after the statement of Theorem 7.2 in [12], there exist rigid Lie algebras with $H^2 \neq 0$.

4.2. Rigidity of Yau algebras arising from two-dimensional rational hypersurface singularities According to Theorem 3.6, Theorem 3.8, Theorem

3.10, Theorem 4.2 and Theorem 4.3, we can conclude that the Yau algebra $L(V)$ is rigid for $A_k(k \neq 4, 5, 6, 9, 10, 11, 12, 13)$, $D_k(k \neq 5, 6, 7, 8, 10, 11, 12, 13, 14)$ and for E_7 , however, the Yau algebra $L(V)$ is not rigid for A_4 , A_5 , A_6 , D_5 , D_6 , and D_7 . Our next goal is to prove that $L(V)$ is not rigid for $A_k(k = 9, 10, 11, 12, 13)$, $D_k(k = 8, 10, 11, 12, 13, 14)$, E_6 , and E_8 .

Since the trivial infinitesimal deformations are precisely the 2-coboundaries, we need to find an infinitesimal deformation φ_1 such that $\varphi_1 \notin B^2(L, L)$. This would imply that φ_1 is a non-trivial infinitesimal deformation, thereby showing that $L(V)$ is not rigid. For this, we need to figure out the structure of $Z^2(L, L)$ first.

Remark 4.5. In fact, we can directly apply the method in Theorem 3.6 to calculate $B^2(L, L)$ and $Z^2(L, L)$ with the assistance of a computer. However, as noted in mentioned in Remark 3.11, we can obtain the set of bases of $B^2(L, L)$ by using the processes described in Theorem 3.8 and Theorem 3.10.

In order for (27) to hold, it is necessary that $d \geq 13$. Therefore b_{36}^8 cannot be determined by the elements previously identified for $A_k(k = 9, 10, 11, 12, 13)$. Let b_{36}^8 be 1, and let the values of the elements we found before be 0. Then, the corresponding $\varphi \in Z^2(L, L)$ such that $\varphi \notin B^2(L, L)$, hence φ serves as a representative element of the cohomology group $H^2(L, L)$ for $A_k(k = 9, 10, 11, 12, 13)$. According to the relationship between A_k and D_{k+1} , this conclusion also holds true for $D_k(k = 10, 11, 12, 13, 14)$. This element will play an important role in the subsequent proof.

Proposition 4.6. *For the singularities A_9 , A_{10} , D_8 , D_{10} , D_{11} , E_6 , and E_8 , we construct a $\varphi \notin B^2(L, L)$ such that $\eta_t = \eta + t\varphi$ is a one-parameter family of Lie algebra multiplications. Consequently, the Yau algebra $L(V)$ is not rigid in these cases.*

Proof. When $\eta_t = \eta + t\varphi$, the deformation equation (73) can be transform into

$$\delta\varphi = 0, \quad (75)$$

$$[\varphi, \varphi] = 0. \quad (76)$$

So we only need to construct a $\varphi \in Z^2(L, L)$ such that $\varphi \notin B^2(L, L)$ and $[\varphi, \varphi] = 0$. The φ for each singularity is defined as follows:

- (1) A_9 : $\varphi(e_3, e_6) = -\varphi(e_6, e_3) = e_8$, $\varphi(e_4, e_5) = -\varphi(e_5, e_4) = -3e_8$, and $\varphi(e_i, e_j) = 0$ for all other pairs (i, j) .
- (2) A_{10} : $\varphi(e_3, e_6) = -\varphi(e_6, e_3) = e_8$, $\varphi(e_3, e_7) = -\varphi(e_7, e_3) = 3e_9$, $\varphi(e_4, e_5) = -\varphi(e_5, e_4) = -3e_8$, $\varphi(e_4, e_6) = -\varphi(e_6, e_4) = -6e_9$, and $\varphi(e_i, e_j) = 0$ for all other pairs (i, j) .
- (3) D_8 : $\varphi(e_3, e_7) = -\varphi(e_7, e_3) = e_5$, $\varphi(e_4, e_7) = -\varphi(e_7, e_4) = 3e_6$, and $\varphi(e_i, e_j) = 0$ for all other pairs (i, j) .

- (4) D_{10} : $\varphi(e_3, e_6) = -\varphi(e_6, e_3) = e_8$, $\varphi(e_4, e_5) = -\varphi(e_5, e_4) = -3e_8$, and $\varphi(e_i, e_j) = 0$ for all other pairs (i, j) .
- (5) D_{11} : $\varphi(e_3, e_6) = -\varphi(e_6, e_3) = e_8$, $\varphi(e_3, e_7) = -\varphi(e_7, e_3) = 3e_9$, $\varphi(e_4, e_5) = -\varphi(e_5, e_4) = -3e_8$, $\varphi(e_4, e_6) = -\varphi(e_6, e_4) = -6e_9$, and $\varphi(e_i, e_j) = 0$ for all other pairs (i, j) .
- (6) E_6 : $\varphi(e_1, e_3) = -\varphi(e_3, e_1) = e_5$, $\varphi(e_1, e_5) = -\varphi(e_5, e_1) = e_3$, $\varphi(e_1, e_7) = -\varphi(e_7, e_1) = -e_7$, and $\varphi(e_i, e_j) = 0$ for all other pairs (i, j) .
- (7) E_8 : $\varphi(e_3, e_4) = -\varphi(e_4, e_3) = e_7$ and $\varphi(e_i, e_j) = 0$ for all other pairs (i, j) .

We can rely on computer assistance to confirm that they meet the conditions. ■

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Remark 4.7. The verifications in Proposition 4.8 were carried out using the same MATLAB routines as in Remark 3.7. Each φ was encoded as a column vector using $O3$ and $S3$. To check $\varphi \in Z^2$, we verified $D_2 \cdot \text{vec}(\varphi) = 0$; to check $\varphi \notin B^2$, we appended the vector to D_1 and confirmed that the rank increased. The condition $[\varphi, \varphi] = 0$ was verified by direct substitution.

Proposition 4.8. *For the singularities A_{11} , A_{12} , A_{13} , D_{12} , D_{13} , and D_{14} , we construct a $\varphi_1 \notin B^2(L, L)$ that is the first-order term of some one-parameter family of deformations of η . Consequently, the Yau algebra $L(V)$ is not rigid in these cases.*

Proof. Let φ be a 2-linear alternating map in $C^2(L, L)$. We can decompose φ into $\sum_{1 \leq i < j \leq d} \sum_{k=1}^d b_{ij}^k (\varphi_k^{ij} - \varphi_k^{ji})$, where φ_k^{ij} is defined as follows: $\varphi_k^{ij}(e_i, e_j) = e_k$ and $\varphi_k^{ij}(e_m, e_n) = 0$ otherwise. This means that φ is determined by the coefficients b_{ij}^k where $1 \leq i < j \leq d$ and $1 \leq k \leq d$. Similarly, let μ be a 2-linear alternating map in $C^3(L, L)$. Then μ is determined by the coefficients c_{ijk}^t where $1 \leq i < j < k \leq d$ and $1 \leq t \leq d$. For simplicity, when describing φ (resp. μ) later, we will only give the non-zero values among these b_{ij}^k (resp. c_{ijk}^t).

The φ_1 for each singularity that we construct is defined as follows:

- (1) A_{11} and D_{12} : $b_{36}^8 = 1$, $b_{45}^8 = -3$, $b_{37}^9 = 3$, $b_{46}^9 = -6$, $b_{38}^{10} = \frac{40}{7}$, $b_{47}^{10} = -\frac{53}{7}$, $b_{56}^{10} = -\frac{41}{7}$.
- (2) A_{12} and D_{13} : $b_{36}^8 = 1$, $b_{45}^8 = -3$, $b_{37}^9 = 3$, $b_{46}^9 = -6$, $b_{38}^{10} = \frac{40}{7}$, $b_{47}^{10} = -\frac{53}{7}$, $b_{56}^{10} = -\frac{41}{7}$, $b_{39}^{11} = \frac{62}{7}$, $b_{48}^{11} = -\frac{52}{7}$, $b_{57}^{11} = -\frac{82}{7}$.
- (3) A_{13} and D_{14} : $b_{36}^8 = 1$, $b_{45}^8 = -3$, $b_{37}^9 = 3$, $b_{46}^9 = -6$, $b_{38}^{10} = \frac{40}{7}$, $b_{47}^{10} = -\frac{53}{7}$, $b_{56}^{10} = -\frac{41}{7}$, $b_{39}^{11} = \frac{62}{7}$, $b_{48}^{11} = -\frac{52}{7}$, $b_{57}^{11} = -\frac{82}{7}$, $b_{3,10}^{12} = \frac{403}{33}$, $b_{49}^{12} = -\frac{1333}{231}$.

$$b_{58}^{12} = -\frac{3723}{231}, \quad b_{67}^{12} = -\frac{1913}{231}.$$

The situation of A_k is identical to that of D_{k+1} . Moreover, the situation of A_{11} and A_{12} are simplified instances of the situation of A_{13} ; hence, it suffices to prove the case for A_{13} .

Let φ_a be a 2-linear alternating map in $C^2(L, L)$ such that $b_{38}^{10} = 1$, $b_{47}^{10} = -5$, $b_{56}^{10} = 10$, $b_{39}^{11} = 4$, $b_{48}^{11} = -16$, $b_{57}^{11} = 20$, $b_{3,10}^{12} = \frac{665}{66}$, $b_{49}^{12} = -\frac{2279}{66}$, $b_{58}^{12} = \frac{2085}{66}$, $b_{67}^{12} = \frac{485}{66}$.

Let φ_b be a 2-linear alternating map in $C^2(L, L)$ such that $b_{3,10}^{12} = 1$, $b_{49}^{12} = -7$, $b_{58}^{12} = 21$, $b_{67}^{12} = -35$.

Let μ_a be a 3-linear alternating map in $C^3(L, L)$ such that $c_{345}^{10} = 1$, $c_{346}^{11} = \frac{8}{3}$, $c_{347}^{12} = 6$.

Let μ_b be a 3-linear alternating map in $C^3(L, L)$ such that $c_{347}^{12} = 1$, $c_{356}^{12} = -2$.

Firstly, we have that $\varphi_1 \in Z^2(L, L)$ and $\varphi_1 \notin B^2(L, L)$. Based on the calculations, we have determined that

$$\delta\varphi_a = 21\mu_a, \quad (77)$$

$$\delta\varphi_b = 66\mu_b, \quad (78)$$

$$[\varphi_1, \varphi_1] = \frac{240}{7}\mu_a - \frac{12800}{231}\mu_b, \quad (79)$$

$$[\varphi_1, \varphi_a] = \frac{4030}{33}\mu_b, \quad (80)$$

$$[\varphi_1, \varphi_b] = 0, \quad (81)$$

$$[\varphi_a, \varphi_1] = 3\mu_a - \frac{2990}{33}\mu_b, \quad (82)$$

$$[\varphi_b, \varphi_1] = -\frac{188}{7}\mu_b, \quad (83)$$

$$[\varphi_a, \varphi_a] = \frac{6650}{66}\mu_b, \quad (84)$$

$$[\varphi_a, \varphi_b] = 0, \quad (85)$$

$$[\varphi_b, \varphi_a] = 10\mu_b, \quad (86)$$

$$[\varphi_b, \varphi_b] = 0. \quad (87)$$

By equation (77)-(79), we can find rational numbers a_2, b_2 , such that

$$\delta(a_2\varphi_a + b_2\varphi_b) - \frac{1}{2}[\varphi_1, \varphi_1] = 0.$$

We then set $\varphi_2 = a_2\varphi_a + b_2\varphi_b$. By equation (80)-(83), it follows that $\sum_{i=1}^2[\varphi_i, \varphi_{3-i}]$ can be expressed as a linear combination of μ_a and μ_b . Consequently, we can find rational numbers a_3, b_3 , such that

$$\delta(a_3\varphi_a + b_3\varphi_b) - \frac{1}{2}\sum_{i=1}^2[\varphi_i, \varphi_{3-i}] = 0.$$

We set $\varphi_3 = a_3\varphi_a + b_3\varphi_b$. By equation (80)-(87), it is shown that $\sum_{i=1}^3[\varphi_i, \varphi_{4-i}]$ can also be expressed as a linear combination of μ_a and μ_b . By induction, for any $n \geq 2$, we can always find rational numbers a_n, b_n satisfy

$$\delta(a_n\varphi_a + b_n\varphi_b) - \frac{1}{2}\sum_{i=1}^{n-1}[\varphi_i, \varphi_{n-i}] = 0.$$

Letting $\varphi_n = a_n\varphi_a + b_n\varphi_b$ for all $n \geq 2$, we conclude that $\eta_t = \eta + t\varphi_1 + t^2\varphi_2 + \dots$ forms a one-parameter family of deformations of η , with φ_1 being its first-order term. ■

Based on the above conclusions, we have completely clarified the rigidity of Yau algebras arising from two-dimensional rational hypersurface singularities.

Theorem 4.9 (i.e. Theorem 1.4). *For rational hypersurface singularities, the Yau algebra $L(V)$ is not rigid for the following cases: $A_4, A_5, A_6, A_9, A_{10}, A_{11}, A_{12}, A_{13}, D_5, D_6, D_7, D_8, D_{10}, D_{11}, D_{12}, D_{13}, D_{14}, E_6$, and E_8 . Conversely, $L(V)$ is rigid for all other cases.*

References

- [1] M. Artin, *On isolated rational singularities of surfaces*, Amer. J. Math. **88** (1966), 129–136.
- [2] M. Benson and S. S.-T. Yau, *Lie algebra and their representations arising from isolated singularities: Computer method in calculating the Lie algebras and their cohomology*, Advance studies in Pure Mathematics **8**, Complex Analytic Singularities, (1986), 3-58.
- [3] B. Chen, H. Chen, S. S.-T. Yau, and H. Zuo, *The Non-existence of Negative Weight Derivations on Positive Dimensional Isolated Singularities: Generalized Wahl Conjecture*, Journal of Differential Geometry, **115**(2)(2020), 195-224.
- [4] H. Chen, S. S.-T. Yau, and H. Zuo, *Non-existence of negative weight derivations on positively graded Artinian algebras*, Trans. Amer. Math. Soc. **372**(4)(2019), 2493-2535.

- [5] A. Elashvili and G. Khimshiashvili, *Lie algebras of simple hypersurface singularities*, J. Lie Theory **16**(2006), 621-649.
- [6] N. Hussain, S. S.-T. Yau, and H. Q. Zuo, *On the derivation Lie algebras of fewnomial singularities*, Bull. Aust. Math. Soc. **98**(1)(2018), 77-78.
- [7] N. Hussain, S. S.-T. Yau, and H. Q. Zuo, *On the new k -th Yau algebras of isolated hypersurface singularities*, Math. Z. **294**(1-2)(2020), 331-358.
- [8] N. Hussain, S. S.-T. Yau, and H. Q. Zuo, *k th Yau number of isolated hypersurface singularities and an inequality conjecture*, J. Aust. Math. Soc. **110** (2021), no. 1, 94-118.
- [9] J. L. Koszul, *Homologie et cohomologie des algèbres de Lie*, Bull. Soc. Math. France **78** (1950), 41-64.
- [10] J. Mather and S. S.-T. Yau, *Classification of isolated hypersurface singularities by their moduli algebras*, Invent. Math. **69** (1982), 243-251.
- [11] A. Nijenhuis and R. W. Richardson, *Cohomology and deformations in graded Lie algebras*, Bull. Amer. Math. Soc. **72**(1966), 1-29.
- [12] A. Nijenhuis and R. W. Richardson, *Deformations of Lie algebra structures*, J. Math. Mech. **17**(1) (1967), 89-105.
- [13] C. Seeley and S. S.-T. Yau, *Variation of complex structures and variation of Lie algebras*, Invent. Math. **99** (1990), 545-565.
- [14] S. S.-T. Yau, *Continuous family of finite-dimensional representations of a solvable Lie algebra arising from singularities*, Proc. Natl. Acad. Sci. **80** (1983), 7694-7696.
- [15] S. S.-T. Yau, *Solvable Lie algebras and generalized Cartan matrices arising from isolated singularities*, Math. Z. **191** (1986), 489-506.
- [16] S. S.-T. Yau, *Solvability of Lie algebras arising from isolated singularities and nonisolatedness of singularities defined by $sl(2, \mathbb{C})$ invariant polynomials*, Amer. J. Math. **113** (1991), 773-778.
- [17] Y. Yu, *On Jacobian ideals invariant by reducible $sl(2, \mathbb{C})$ action*, Trans. Amer. Math. Soc. **348** (1996), 2759-2791.
- [18] S. S.-T. Yau and H. Zuo, *Derivations of the moduli algebras of weighted homogeneous hypersurface singularities*, J. Algebra **457** (2016), 18-25.
- [19] S. S.-T. Yau and H. Zuo, *A Sharp upper estimate conjecture for the Yau number of weighted homogeneous isolated hypersurface singularity*, Pure Appl. Math. Q. **12**(1) (2016), 165-181.

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